mathematics

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Existence and Stability Results for Differential Equations with a Variable-Order Generalized Proportional Caputo Fractional Derivative

Donal O'Regan [1], Ravi P. Agarwal [2], Snezhana Hristova [3,*] and Mohamed I. Abbas [4]

[1] School of Mathematical and Statistical Sciences, University of Galway, H91 TK33 Galway, Ireland; donal.oregan@nuigalway.ie
[2] Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363, USA; agarwal@tamuk.edu
[3] Faculty of Mathematics and Informatics, Plovdiv University "P. Hilendarski", 4000 Plovdiv, Bulgaria
[4] Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt; miabbas@alexu.edu.eg
* Correspondence: snehri@uni-plovdiv.bg

Abstract: An initial value problem for a scalar nonlinear differential equation with a variable order for the generalized proportional Caputo fractional derivative is studied. We consider the case of a piecewise constant variable order of the fractional derivative. Since the order of the fractional integrals and derivatives depends on time, we will consider several different cases. The argument of the variable order could be equal to the current time or it could be equal to the variable of the integral determining the fractional derivative. We provide three different definitions of generalized proportional fractional integrals and Caputo-type derivatives, and the properties of the defined differentials/integrals are discussed and compared with what is known in the literature. Appropriate auxiliary systems with constant-order fractional derivatives are defined and used to construct solutions of the studied problem in the three cases of fractional derivatives. Existence and uniqueness are studied. Also, the Ulam-type stability is defined in the three cases, and sufficient conditions are obtained. The suggested approach is more broadly based, and the same methodology can be used in a number of additional issues.

Keywords: variable-order fractional differential equations; generalized proportional Caputo fractional derivatives; Hyers–Ulam stability

MSC: 34A34; 34A08; 34D20

Citation: O'Regan, D.; Agarwal, R.P.; Hristova, S.; Abbas, M.I. Existence and Stability Results for Differential Equations with a Variable-Order Generalized Proportional Caputo Fractional Derivative. *Mathematics* **2024**, *12*, 233. https://doi.org/10.3390/math12020233

Academic Editor: Rodica Luca

Received: 23 November 2023
Revised: 6 January 2024
Accepted: 9 January 2024
Published: 11 January 2024

1. Introduction

Differential systems of fractional order are important due to their many applications in numerous fields of science and engineering and their utilization in real-world models (see, for example, [1–4]).

Fractional-order differential operators with different definitions have been introduced in the literature, notably the Caputo, Grunwald–Letnikov, Hadamard, Riemann–Liouville, and Riesz operators. Fractional calculus operators are nonlocal operators, which makes them suitable for describing long-term memory or nonlocal effects prevalent in non-regular real-world phenomena (see, for example, [5] for several applications of fractional calculus in the fields of physics, mechanics, biology, engineering, and signal processing, [6] for several real-world applications in science and engineering, [7] for fractional models in bioengineering, and [8] for modeling of viscoelastic systems). Also, for fractional differential equations, many mathematical techniques via numerical methods have been used (see, for example, [9] for discretization of Caputo fractional derivatives using the L1 scheme, [10] for the orthogonal spline collocation method for differential equations with a sum of Caputo fractional derivatives of different constant orders, and [11] for the first-order fractional convolution quadrature scheme and backward Euler alternating direction implicit method for the Riemann–Liouville fractional integral term and temporal derivative).

Another form of differential operator, where the order is taken as a continuous function, was introduced in [12]. We note that selecting the order as a function depending on time makes the operator more flexible, offering more degrees of freedom. In the last two decades, researchers have considered variable-order fractional differential operators and established their existence, uniqueness, stability, and numerical results. Recently, a comprehensive review of variable fractional calculus was given in [13]. Also, fractional derivatives where the order is a random variable have been defined, studied, and applied to differential equations ([14]). One of the most studied and applied cases of variable order is when there is a partition of the interval of consideration, and the order is a constant on each subinterval, i.e., the order is a piecewise constant function. This kind of variable order of fractional derivatives has been applied and studied in several papers, and we will discuss these results in the main body of this paper and correct some of them. In this paper, we use the generalized proportional Caputo fractional derivative with a constant order (GPCFD) (see, for example, [15–18]). It should be noted that the classical Caputo fractional derivative is a partial case of GPCFD, and we will generalize GPCFD to a fractional derivative of a piecewise constant variable order. Motivated by the literature on Caputo fractional derivatives of variable order, we will define three different types of variable-order fractional integrals and Caputo-type fractional derivatives. The main characteristic of these types of derivatives is the time argument of the variable order, which could be the recent time or the same as the variable of the integral. We discuss the main characteristics and properties of these three types. Also, we compare them to what is known in the literature on piecewise constant variable-order fractional derivatives. We give algorithms for constructing solutions of the initial value problems of differential equations with any of the three defined fractional derivatives with a constant order. We prove the existence and uniqueness. Ulam-type stability is appropriately defined, and sufficient conditions are obtained. We note that the definitions of Ulam-type stability depend significantly on the applied derivatives.

2. Basic Definitions and Preliminaries

In the literature, there are several types of fractional integrals and derivatives of variable order. In [19], three types of Caputo derivatives with variable order were defined and applied, with the main difference between the fractional derivatives being related to the argument of the variable order.

In this paper, we extend the notions of the generalized proportional fractional derivatives and integrals of constant order [17,18] in three different ways. We consider the derivative order as $\delta(\cdot) : [0,T] \to (0,1), 0 < T \leq \infty$.

In the definitions of the variable order, unlike the constant order, the argument of the variable order $\delta(.)$ is important. It could be equal to the current time or it could be equal to the variable of the integral. This leads to different definitions of variable-order fractional differential/integrals, and we consider three different types of differentials/integrals. In these cases, we compare the definitions with existing ones in the literature.

2.1. Variable-Order Generalized Proportional Integrals and Caputo Derivatives of Type I

We consider the case when the argument of the variable order is equal to the current time.

Definition 1. *Let $\varrho \in (0,1]$, $\delta : [a,b] \to (0,1)$, $0 \leq a < b \leq \infty$. The variable-order generalized proportional fractional integral of type I of a function $g \in L([a,b])$ is defined by*

$$\mathfrak{I}_{a^+}^{\delta(t),\varrho} g(t) = \frac{1}{\varrho^{\delta(t)}\Gamma(\delta(t))} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta(t)-1} g(s)\, ds, \quad t \in (a,b]. \tag{1}$$

Definition 2. *Let $\varrho \in (0,1]$, $\delta : [a,b] \to (0,1)$, $0 \le a < b \le \infty$. The generalized proportional Caputo fractional derivative of type I (GPFI) of a function $g \in C^1[a,b]$ is defined by*

$$\begin{aligned} {}^C\mathfrak{D}_{a^+}^{\delta(t),\varrho} g(t) = \frac{1}{\varrho^{\delta(t)}\Gamma(1-\delta(t))} \Big((1-\varrho) \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{-\delta(t)} g(s)\, ds \\ + \varrho \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{-\delta(t)} g'(s)\, ds \Big), \ \ t \in (a,b]. \end{aligned} \tag{2}$$

In the case where $\varrho = 1$, Definitions 1 and 2 reduce to the variable-order fractional integrals and fractional Caputo derivative defined and successfully used in [12,20,21]. Also, the variable-order Hadamard fractional derivative was defined and applied in [22].

Remark 1. *In some papers, such as [23,24], the definitions of variable-order fractional differentials/integrals differ, incorrectly citing the definitions of [20,21].*

2.2. Variable-Order Generalized Proportional Integrals and Caputo Derivatives of Type II

We consider the case when the argument of the variable order is equal to the variable of the integral, but in the Gamma function, it is equal to the current time.

Definition 3. *Let $\varrho \in (0,1]$, $\delta : [a,b] \to (0,1)$, $0 \le a < b \le \infty$. The variable-order generalized proportional fractional integral of type II of a function $g \in L([a,b])$ is defined by*

$$\mathcal{I}_{a^+}^{\delta(t),\varrho} g(t) = \frac{1}{\varrho^{\delta(t)}\Gamma(\delta(t))} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta(s)-1} g(s)\, ds, \ \ t \in (a,b]. \tag{3}$$

Definition 4. *Let $\varrho \in (0,1]$, $\delta : [a,b] \to (0,1)$, $0 \le a < b \le \infty$. The variable-order generalized proportional Caputo fractional derivative of type II (GPFII) of a function $g \in C^1[a,b]$ is defined by*

$$\begin{aligned} {}^C\mathcal{D}_{a^+}^{\delta(t),\varrho} g(t) = \frac{1}{\varrho^{\delta(t)}\Gamma(1-\delta(t))} \Big((1-\varrho) \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{-\delta(s)} g(s) \\ + \varrho \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{-\delta(s)} g'(s)\, ds \Big), \ \ t \in (a,b]. \end{aligned} \tag{4}$$

In the case where $\varrho = 1$, Definitions 3 and 4 reduce to the variable-order fractional integrals and fractional derivative used in [25].

2.3. Variable-Order Generalized Proportional Integrals and Caputo Derivatives of Type III

We consider the case when the argument of the variable order is equal to the variable of the integral.

Definition 5. *Let $\varrho \in (0,1]$, $\delta : [a,b] \to (0,1)$, $0 \le a < b \le \infty$. The variable-order generalized proportional fractional integral of type III of a function $g \in L([a,b])$ is defined by*

$$\mathbb{I}_{a^+}^{\delta(t),\varrho} g(t) = \int_a^t \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta(s)}\Gamma(\delta(s))} (t-s)^{\delta(s)-1} g(s)\, ds, \ \ t \in (a,b]. \tag{5}$$

Definition 6. *Let $\varrho \in (0,1]$, $\delta : [a,b] \to (0,1)$, $0 \le a < b \le \infty$. The variable-order generalized proportional Caputo fractional derivative of type III (GPFIII) of a function $g \in C^1([a,b])$ is defined by*

$$\begin{aligned} {}^C\mathbb{D}_{a^+}^{\delta(t),\varrho} g(t) = (1-\varrho) \int_a^t \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta(s)}\Gamma(1-\delta(s))} (t-s)^{-\delta(s)} g(s) \\ + \varrho \int_a^t \frac{e^{\frac{\varrho-1}{\varrho}}(t-s)}{\varrho^{\delta(s)}\Gamma(1-\delta(s))} (t-s)^{-\delta(s)} g'(s)\, ds, \ \ t \in (a,b]. \end{aligned} \tag{6}$$

In the case where $\varrho = 1$, Definitions 5 and 6 reduce to the variable-order fractional integrals and fractional derivative used in [20,26].

Remark 2. *The partial case of the fractional derivative of type III (with $\varrho = 1$) was defined and applied in [27] for a particular type of differential equation.*

Remark 3. *In [20], the authors defined three types of fractional integrals and derivatives. The types C1 and C2, defined in [20], are similar to the special case of our type I and type II (with $\varrho = 1$).*

2.4. Constant-Order Generalized Proportional Diffintegrals

In the case of a constant order of the fractional integrals, the fractional integrals defined in Definitions 1, 3 and 5 coincide and reduce to the generalized proportional fractional integrals [17,18], i.e., if $\delta(t) \equiv \alpha = const$, $\varrho \in (0,1]$, then

$$\mathcal{I}_a^{\alpha,\varrho} g(t) = \mathfrak{I}_a^{\alpha,\varrho} g(t) = \mathbb{I}_a^{\alpha,\varrho} g(t) = I_a^{\alpha,\varrho} g(t), \tag{7}$$

where

$$I_{a+}^{\alpha,\varrho} g(t) = \frac{1}{\varrho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\alpha-1} g(s)\, ds, \quad t \in (a,b]. \tag{8}$$

In the case of a constant order, the three fractional derivatives defined in Definitions 2, 4 and 6 coincide and reduce to generalized proportional Caputo fractional derivative ([17,18]), i.e., if $\delta(t) \equiv \alpha = const$, then

$${}^C\mathcal{D}_a^{\alpha,\varrho} g(t) = {}^C\mathfrak{D}_a^{\alpha,\varrho} g(t) = {}^C\mathbb{D}_a^{\alpha,\varrho} g(t) = {}^C D_a^{\alpha,\varrho} g(t), \tag{9}$$

where

$$\begin{aligned} {}^C D_{a+}^{\alpha,\varrho} g(t) = & \frac{1}{\varrho^\alpha \Gamma(1-\alpha)} \Big((1-\varrho) \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{-\alpha} g(s)\, ds \\ & + \varrho \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{-\alpha} g'(s)\, ds \Big), \quad t \in (a,b]. \end{aligned} \tag{10}$$

In this section, we present some known results in the literature concerning generalized proportional integrals and derivatives of a constant order.

Lemma 1 (Gronwall inequality, Corollary 3 [15])**.** *Let $\varrho \in (0,1]$, $\alpha \in (0,1)$, $0 \le a < b \le \infty$ and $u(t)$ be nonnegative functions locally integrable on $[a,b]$, and $v(t), w(t)$ be nonnegative, nondecreasing, and continuous functions defined on $[a,b]$ such that $w(t) \le M$, $M > 0$. If*

$$u(t) \le v(t) + \varrho^\alpha \Gamma(\alpha) w(t) I_a^{\alpha,\varrho} u(t), \quad t \in [a,b],$$

then

$$u(t) \le v(t) E_\alpha\Big(w(t)\Gamma(\alpha)(t-a)^\alpha\Big),$$

where $E_\alpha(.)$ is the Mittag–Leffler function with one parameter α.

We use the following existence results for the generalized proportional Caputo fractional differential equation with a constant order α:

$${}^C D_a^{\alpha,\varrho} u(t) = f(t, u(t)), \quad t \in (a,b], \qquad u(a) = u_0. \tag{11}$$

Lemma 2 (Corollary 4.5 [18])**.** *Let $\varrho \in (0,1]$, $\alpha \in (0,1)$, $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$ be Lipschtz with a constant $L > 0$, $|f(t,0)| \le \Omega(t)$ with $\Omega \in C([a,b],[0,\infty))$ and $L < (1-\varrho)^\alpha$. Then, Problem (11) has a unique solution on $[a,b]$.*

We use the following integral presentation of the solution of the constant-order fractional differential equation:

Lemma 3 (Theorem 5.3 [17])**.** *Let $\varrho \in (0,1]$ and $\alpha \in (0,1)$. Then, the solution of the initial value problem (11) satisfies the integral presentation*

$$u(t) = u_0 e^{\frac{\varrho-1}{\varrho}(t-a)} + I_{a^+}^{\alpha,\varrho} f(t,u(t)),\ \ t \in (a,b].$$

Lemma 4. *Let $\varrho \in (0,1]$, $\alpha \in (0,1)$ and $a \in [0,\infty)$. Then,*

$$I_a^{\alpha,\varrho} 1 \leq \frac{(t-a)^\alpha}{\varrho^\alpha \Gamma(1+\alpha)},\ \ t > a.$$

Proof. From Equation (5), we have

$$I_a^{\alpha,\varrho} 1 = \int_a^t \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^\alpha \Gamma(\alpha)} (t-s)^{\alpha-1}\, ds \leq \frac{1}{\varrho^\alpha \Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} = \frac{(t-a)^\alpha}{\varrho^\alpha \Gamma(1+\alpha)}. \tag{12}$$

□

3. Statement of the Problem and Existence Results

Let n be a positive integer and $\{T_k\}_{k=0}^n$ be a finite sequence such that

$$0 = T_0 < T_{k-1} < T_k < T_n = T,\ \ k = 2, \cdots, n-1.$$

Denote $\mathcal{J}_k := [T_{k-1}, T_k),\ \ k = 1, 2, \cdots, n$. Then, $P = \{\mathcal{J}_k : k = 1, 2, \cdots, n\}$ is a partition of the interval $[0, T]$.

In this paper, we study the case of a piecewise constant order of the fractional integrals and derivatives, i.e., the variable order $\delta : \mathcal{J} \to (0,1)$ is given by

$$\delta(t) = \sum_{k=1}^n \delta_k I_k(t) = \begin{cases} \delta_1, & \text{if } \ t \in \mathcal{J}_1, \\ \delta_2, & \text{if } \ t \in \mathcal{J}_2, \\ \vdots \\ \delta_n, & \text{if } \ t \in \mathcal{J}_n, \end{cases} \tag{13}$$

where $\delta_k \in (0,1)$ are positive constants and the function I_k is the indicator of the interval $\mathcal{J}_k$, $k = 1, 2, \cdots, n$, with

$$I_k(t) = \begin{cases} 1, & \text{if } \ t \in \mathcal{J}_k, \\ 0, & \text{otherwise.} \end{cases}$$

Denote

$$PC([0,T]) = \{u : [0,T] \to \mathbb{R} :\ u \in C[T_{k-1}, T_k),\ k = 1, 2, \ldots, n\},$$

$$PC^1([0,T]) = \{u : [0,T] \to \mathbb{R} :\ u \in C^1[T_{k-1}, T_k),\ k = 1, 2, \ldots, n\}.$$

In this paper, we study the nonlinear fractional differential equation

$$\begin{aligned} &{}^C\mathbb{D}_{0^+}^{\delta(t),\varrho} \omega(t) = F(t, \omega(t)),\ \ t \in (0,T], \\ &\omega(0) = V_0, \end{aligned} \tag{14}$$

where the fractional derivative ${}^C\mathbb{D}_{0^+}^{\delta(t),\varrho}$ denotes the variable-order generalized proportional Caputo fractional derivatives of type I, ${}^C\mathfrak{D}_{0^+}^{\delta(t),\varrho}$, defined by (2), or the variable-order generalized proportional Caputo fractional derivatives of type II, ${}^C\mathcal{D}_{0^+}^{\delta(t),\varrho}$, defined by (4), or the variable-order generalized proportional Caputo fractional derivatives of type III, ${}^{C}\mathbb{D}_{0^+}^{\delta(t),\varrho}$, defined by (6).

We introduce the following assumptions:

Assumption 1. *The function* $\delta : [0,T] \to (0,1)$ *is defined by* (13) *and* $\varrho \in (0,1]$.

Assumption 2. *The function* $F \in C([0,T]\times\mathbb{R},\mathbb{R})$ *is Lipschitz with a constant* $L > 0$: $L < (1-\varrho)^{\min_{k=1,2,\dots,n}\delta_k}$, *and* $|F(t,0)| \le \Omega(t)$ *with* $\Omega \in C([0,T],[0,\infty))$.

The assumptions are deeply connected with the studied Equation (14), and they are natural and nonrestrictive. Assumption 1 is connected with the type of order of the fractional derivatives studied in this paper. Assumption 2 is about the right-hand side of the studied nonlinear differential Equation (14), and it is connected with its continuity and boundedness.

For example, if the interval of consideration is $[0,3]$, its partition is $\mathcal{J}_1 = [0,1)$, $\mathcal{J}_2 = [1,2)$, $\mathcal{J}_3 = [2,3]$, and then the variable order $\delta : [0,2] \to (0,1)$ could be defined by

$$\delta(t) = \begin{cases} 0.8, & \text{if } \ t \in \mathcal{J}_1, \\ 0.3, & \text{if } \ t \in \mathcal{J}_2, \\ 0.5, & \text{if } \ t \in \mathcal{J}_3, \end{cases} \tag{15}$$

$\varrho = 0.6$, $\min_{k=1,2,3}\delta_k = 0.3$, and the function $F(t,x) = -\frac{t}{t+1}e^{-x}$ satisfies Assumption 2 with $L = 0.75 < (1-0.6)^{0.3} \approx 0.759658$ and $\Omega(t) = t$.

Remark 4. *Note that the semi-group property is not fulfilled for the case of variable-order generalized proportional fractional integrals of type I, type II, and type III, i.e., for general functions* $\delta(t)$, $\eta(t)$. *Note:*

$$\mathcal{I}_a^{\delta(t),\varrho}\,\mathcal{I}_a^{\eta(t),\varrho} g(t) \neq \mathcal{I}_a^{\delta(t)+\eta(t),\varrho} g(t),$$

$$\mathfrak{I}_a^{\delta(t),\varrho}\,\mathfrak{I}_a^{\eta(t),\varrho} g(t) \neq \mathfrak{I}_a^{\delta(t)+\eta(t),\varrho} g(t),$$

and

$$\mathbb{I}_a^{\delta(t),\varrho}\,\mathbb{I}_a^{\eta(t),\varrho} g(t) \neq \mathbb{I}_a^{\delta(t)+\eta(t),\varrho} g(t)$$

hold.

Thus, we cannot transform a differential equation with a variable-order generalized proportional fractional derivative of type I, type II, and type III into an equivalent integral equation and apply it. As a result, we require a different approach to studying the properties of solutions of differential equations with GPFI, GPFII, and GPFIII.

3.1. Variable-Order Generalized Proportional Caputo Fractional Derivatives of Type I

Consider the initial value problem for the nonlinear differential equation with GPDI:

$$\begin{aligned} &{}^{C}\mathfrak{D}_{0^+}^{\delta(t),\varrho}\omega(t) = F(t,\omega(t)), \ \ t \in (0,T], \\ &\omega(0) = V_0, \end{aligned} \tag{16}$$

where $\varrho \in (0,1]$, $\delta(t)$ is defined by (13), ${}^{C}\mathfrak{D}_{0^+}^{\delta(t),\varrho}$ denotes the variable-order generalized proportional Caputo fractional derivatives of type I defined by (2), and $F : [0,T] \times C([0,T],\mathbb{R}) \to \mathbb{R}$ is a given function.

We discuss some properties of the applied fractional derivative.

Let $t \in (T_{k-1},T_k)$, $k = 1,2,\dots,n$. According to (13), $\delta(t) = \delta_k$ for $t \in (T_{k-1},T_k)$, and we could write

$$\int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta(t)}g(s)\,ds = \int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k}g(s)\,ds, \tag{17}$$

and according to (13) and $\delta(t) \neq \delta_j$, for $t \in (T_{k-1}, T_k)$, $j = 1,2,\ldots,k-1$, we have

$$\begin{aligned}
&\int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta(t)} g(s)\,ds = \int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k} g(s)\,ds \\
&\neq \sum_{j=1}^{k-1} \int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_j} g(s)\,ds + \int_{T_{k-1}}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k} g(s)\,ds
\end{aligned} \tag{18}$$

(compare (18) in the special case of $\varrho = 1$ with Equation (9) [28], Equation (9) [29], Equation (5) [30], Equation (30) [31], Equation (5) [32], Equation (5) [33], and Equation (3.1) [34]).

Also, from (17), we obtain

$$\begin{aligned}
&\int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta(t)} g(s)\,ds \\
&= \sum_{j=1}^{k-1} \int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k} g(s)\,ds + \int_{T_{k-1}}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k} g(s)\,ds,
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta(t)} g(s)\,ds \\
&\neq \sum_{j=1}^{k-1} \int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(T_j-s)}(T_j-s)^{-\delta_k} g(s)\,ds + \int_{T_{k-1}}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k} g(s)\,ds.
\end{aligned} \tag{20}$$

Therefore, from Definition 6 and (20), for any $t \in (T_{k-1}, T_k)$ $k = 1,2,\ldots,n$, we have

$$ {}^{C}\mathfrak{D}_0^{\delta(t),\varrho}\omega(t) \neq \sum_{j=1}^{k-1} {}^{C}\mathfrak{D}_{T_{j-1}}^{\delta(t),\varrho}\omega(t)\Big|_{t=T_j} + {}^{C}\mathfrak{D}_{T_{k-1}}^{\delta(t),\varrho}\omega(t). \tag{21}$$

As a result, we require a new approach to studying the behavior of the solutions of the IVP (16) (different from the ones applied in [29–34]).

From Equation (17), for any $t \in (T_{k-1}, T_k]$, $k = 1,2,\ldots,n$, and Equation (9), we have the equality

$$ {}^{C}\mathfrak{D}_0^{\delta(t),\varrho}\omega(t) = {}^{C}D_0^{\delta_k,\varrho}\omega(t). \tag{22}$$

Applying (22) to the initial value problem (16), we consider for every $k = 1,2,\ldots,n$ the auxiliary scalar differential equation with a generalized proportional Caputo fractional derivative of the constant order $\delta_k \in (0,1)$

$$\begin{aligned}
&{}^{C}\mathfrak{D}_{0+}^{\delta_k,\varrho}\omega(t) = F(t,\omega(t)), \ \ t \in (0, T_k], \\
&\omega(0) = V_0,
\end{aligned} \tag{23}$$

Theorem 1. *Let conditions Assumptions 1 and 2 be satisfied. Then, Problem (16) has a unique solution.*

Proof. According to Lemma 2, Problem (23) has a unique solution $\omega_k(t)$, $t \in (0, T_k]$ for any $k = 1,2,3,\ldots,n$.

Define the function

$$\omega(t) = \begin{cases} \omega_1(t), & \text{if} \ \ t \in [T_0, T_1), \\ \omega_2(t), & \text{if} \ \ t \in [T_1, T_2), \\ \vdots & \\ \omega_n(t), & \text{if} \ \ t \in [T_{n-1}, T_n]. \end{cases} \tag{24}$$

The function $\omega(\cdot) \in PC^1[0,T]$ is a solution of Problem (16). Indeed, $\omega(0) = \omega_1(0) = V_0$, and from Equations (22) and (24), for any $t \in [T_{k-1}, T_k)$, $k = 1,2,\dots,n$, we have

$$ {}^C\mathfrak{D}_0^{\delta(t),\varrho}\omega(t) = {}^C D_0^{\delta_k,\varrho}\omega(t) = {}^C D_0^{\delta_k,\varrho}\omega_k(t) = F(t,\omega_k(t)) = F(t,\omega(t)). $$

From Assumption 2, the function $F(t,x)$ is Lipschitz with a constant $L < 1$ and, therefore, the initial value problem (16) has a unique solution. □

3.2. Variable-Order Generalized Proportional Caputo Fractional Derivatives of Type II

Consider the initial value problem for the nonlinear differential equation with GPDII:

$$ \begin{aligned} &{}^C\mathcal{D}_{0^+}^{\delta(t),\varrho}\omega(t) = F(t,\omega(t)), \ \ t \in \mathcal{J}, \\ &\omega(0) = V_0, \end{aligned} \tag{25} $$

where $\varrho \in (0,1]$, $\delta(t)$ is defined by (13), ${}^C\mathcal{D}_{0^+}^{\delta(t),\varrho}$ denotes the variable-order generalized proportional Caputo fractional derivatives of type II defined by Definition 4, and $F : [0,T] \times PC^1([0,T]\,\mathbb{R}) \to \mathbb{R}$ is a given function.

We discuss some properties of the applied fractional derivative.

Let $t \in (T_{k-1}, T_k)$, $k = 1,2,\dots,n$. Then, from (13), $\delta(s) = \delta_j$ for $s \in (T_{j-1}, T_j]$, $j = 1,2,\dots,k-1$, and $\delta(s) = \delta_k$ for $s \in (T_{k-1}, t]$, and we obtain

$$ \begin{aligned} &\int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta(s)}g(s)\,ds \\ &= \sum_{j=1}^{k-1}\int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_j}g(s)\,ds + \int_{T_{k-1}}^{t} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_k}g(s)\,ds, \end{aligned} \tag{26} $$

and from Definition 2, we obtain

$$ \begin{aligned} &\int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta(s)}g(s)\,ds \neq \int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(T_j-s)}(T_j-s)^{-\delta_j}g(s)\,ds, \\ &\qquad j = 1,2,\dots,n \end{aligned} \tag{27} $$

(compare (27) in the special case of $\varrho = 1$ with Equation (5) [25]).

From Equation (26), the definition of the variable order $\delta(t)$, and the definition of the fractional derivative, we obtain for any $t \in (T_{k-1}, T_k)$,

$$ \begin{aligned} &{}^C\mathcal{D}_0^{\delta(t),\varrho}g(t) \\ &= \frac{1}{\varrho^{\delta_k}\Gamma(1-\delta_k)}\Big\{(1-\varrho)\sum_{j=1}^{k-1}\int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_j}g(s)\,ds \\ &\quad + \varrho\sum_{j=1}^{k-1}\int_{T_{j-1}}^{T_j} e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{-\delta_j}g'(s)\,ds\Big\} + {}^C\mathcal{D}_{T_{k-1}}^{\delta_k,\varrho}g(t). \end{aligned} \tag{28} $$

and

$$ {}^C\mathcal{D}_0^{\delta(t),\varrho}\omega(t) \neq \sum_{j=1}^{k-1}\frac{\Gamma(1-\delta_j)}{\Gamma(1-\delta_k)}\,{}^C D_{T_{j-1}}^{\delta_j,\varrho}\omega(t)\Big|_{t=T_j} + {}^C D_{T_{k-1}}^{\delta_k,\varrho}\omega(t). \tag{29} $$

As a result, we require a new approach to studying the behavior of the solutions of the IVP (25) (different from the ones applied in [25]).

Equation (28) does not directly allow us to define and use the corresponding auxiliary scalar fractional differential equations with a generalized proportional Caputo fractional derivative of constant order $\delta_k \in (0,1)$, $k = 1,2,\dots,n$, for the initial value problem (25) (as is done in the case of a variable-order fractional derivative of type I to obtain a solution to Problem (16)).

In this case, we use the following auxiliary scalar fractional differential equations with a generalized proportional Caputo fractional derivative of constant order $\delta_k \in (0,1)$

$$\begin{aligned} &{}^C D^{\delta_k,\varrho}_{T_{k-1}+}\omega(t) = G_k(t,\omega(t)), \ \ t \in (T_{k-1}, T_k], \\ &\omega(T_{k-1}) = A_{k-1}, \end{aligned} \tag{30}$$

where the function $G_k : [T_{k-1}, T_k] \times \mathbb{R} \to \mathbb{R}$ and the constant $A_{k-1} \in \mathbb{R}$ are defined in the proof of our next lemma.

As a consequence of Lemma 2, we obtain the following existence result for Problem (30):

Corollary 1. *Let $\varrho \in (0,1)$, $\delta_k \in (0,1)$, $G_k \in C([T_{k-1}, T_k] \times \mathbb{R}, \mathbb{R})$ be Lipschtz with a constant $L_k > 0$, $|G_k(t,0)| \le \Omega_k(t)$ with $\Omega_k \in C([T_{k-1}, T_k], [0,\infty))$ and $L_k < (1-\rho)^{\delta_k}$. Then, Problem (30) has a unique solution on $[T_{k-1}, T_k]$.*

Consider the following system of n differential equations with generalized proportional Caputo fractional derivatives of constant orders:

$${}^C D^{\delta_k,\varrho}_{T_{k-1}}\omega_k(t) = F(t,\omega_k(t)), \quad t \in (T_{k-1}, T_k], \ k = 1,2,\ldots,n \tag{31}$$

with iterative initial conditions

$$\begin{aligned} &\omega_1(0) = V_0, \\ &\omega_k(T_{k-1}) = \omega_{k-1}(T_{k-1}), \ \ k = 2,3,\ldots,n. \end{aligned} \tag{32}$$

We now prove an existence result for the initial value problem of the iterative system in (31) and (32).

Lemma 5. *Let Assumptions 1 and 2 be satisfied. Then, Problems (31) and (32) have a unique solution.*

Proof. Let $k = 1$. Consider the first equation ($k = 1$) in the system (31) and the first equation in the initial condition (32). Thus, we obtain an initial value problem for a scalar equation in the form (30) with $k = 1$, $A_0 = V_0$, $G_1(t,x) = F(t,x)$, $t \in [0,T_1], x \in \mathbb{R}$, $L_1 = L$. According to Corollary 1, the problem for $k = 1$ has a unique solution $\omega_1(t)$, $t \in [0,T_1]$.

Consider the second equation ($k = 2$) in the system (31) and the second equation in the initial condition (32). Thus, we obtain an initial value problem for a scalar equation in the form (30) with $k = 2$, $A_1 = \omega_1(T_1)$, $L_2 = L$, and $G_2(t,x) = F(t,x), t \in [T_1,T_2], x \in \mathbb{R}$. According to Corollary 1, this problem has a solution $\omega_2(t)$, $t \in [T_1,T_2]$.

Continue this process inductively by taking the j-th equation in the system (31) and the j-th equation in the initial condition (32). We obtain an initial value problem in the form (30) with $k = j$, $A_{j-1} = \omega_{j-1}(T_{j-1})$, $G_j(t,x) = F(t,x), t \in [T_{j-1}, T_j]$, and $L_j = L$. According to Corollary 1, this problem has a unique solution $\omega_j(t)$, $t \in [T_{j-1}, T_j]$ for all $j = 1,2,\ldots,n$.

Following this procedure, we obtain unique functions $\omega_k \in C([T_{k-1}, T_k]), k = 1,2,\ldots,n$, which satisfy Problems (31) and (32). □

We connect Problems (31) and (32) with the studied problem (25).

Definition 7. *The function $\omega \in PC^1([0,T])$ is called a* mild solution *of Problem (25) if it is defined by*

$$\omega(t) = \begin{cases} \omega_1(t), & \text{if} \ \ t \in [T_0, T_1), \\ \omega_2(t), & \text{if} \ \ t \in [T_1, T_2), \\ \vdots \\ \omega_n(t), & \text{if} \ \ t \in [T_{n-1}, T_n], \end{cases} \tag{33}$$

where the functions $\omega_k \in C^1([T_{k-1}, T_k])$, $k = 1, 2, \ldots, n$, *satisfy Problems* (31) *and* (32).

Theorem 2. *Let Assumptions 1 and 2 be satisfied. Then, Problem* (25) *has a unique mild solution.*

Proof. According to Lemma 5, for any $k = 1, 2, \ldots, n$, Problems (31) and (32) have a unique solution $\omega_k(t) \in C^1([T_{k-1}, T_k], \mathbb{R})$. Then, the function $\omega(t)$, defined by (33), is a mild solution of (25). Assumption 2 guarantees the uniqueness of the mild solution of (25). □

Some Changes in the Algorithm for Constructing a Solution

The procedure described in Section 3.2 can be changed slightly.

If $\{\omega_1(t), \omega_2(t), \ldots, \omega_n(t)\}$ with $\omega_j(t) \in C^1([T_{j-1}, T_j], \mathbb{R})$, $j = 1, 2, \ldots, n$, is the unique solution of Problems (31) and (32). Then, for any $k = 1, 2, \ldots, n$, we define the functions $h_k : [0, T_k] \to \mathbb{R}$ by the equalities

$$h_k(t) = \begin{cases} 0, & \text{if} \quad t \in [T_0, T_{k-1}), \\ \omega_k(t), & \text{if} \quad t \in [T_{k-1}, T_k], \end{cases} \tag{34}$$

and the function $\omega \in PC^1([0, T])$ by

$$\omega(t) = \begin{cases} h_1(t), & \text{if} \quad t \in [T_0, T_1], \\ h_2(t), & \text{if} \quad t \in (T_1, T_2], \\ \vdots & \\ h_n(t), & \text{if} \quad t \in (T_{n-1}, T_n]. \end{cases} \tag{35}$$

The function h_k, $k = 1, 2, \ldots, n$, is defined on $[0, T_k]$, it has a discontinuity at T_{k-1}, $h_k(T_{k-1} - 0) = 0$, $h_k(T_{k-1}) = h_k(T_{k-1} + 0) = \omega_{k-1}(T_{k-1}) = h_{k-1}(T_{k-1})$. According to (28), we obtain

$${}^C\mathcal{D}_0^{\delta(t),\varrho} h_k(t) = {}^C\mathcal{D}_{T_{k-1}}^{\delta_k,\varrho} \omega_k(t) = F(t, \omega_k(t)) = F(t, h_k(t)), \quad t \in (T_{k-1}, T_k]. \tag{36}$$

Weslightly modify Definition 7.

Definition 8. *The function* $\omega \in PC^1([0, T])$, *defined by* (35), *is called a* mild solution *of Problem* (25) *if for any* $k = 1, 2, \ldots, n$, *the equality* ${}^C\mathcal{D}_0^{\delta(t),\varrho}\omega(t) = {}^C\mathcal{D}_0^{\delta(t),\varrho} h_k(t)$, $t \in [0, T_k]$ *holds.*

According to (28) and the properties of the functions $h_k(t)$, it follows from Definition 8 that the function $\omega(t)$ is a mild solution of (25). Note that if Definition 8 is not applied in this procedure, we cannot conclude that the function defined by (35) is a solution of (25) (as done in [24]).

3.3. Variable-Order Generalized Proportional Caputo Fractional Derivatives of Type III

Consider the initial value problem for the nonlinear differential equation with GPDIII:

$$\begin{aligned} &{}^C\mathbb{D}_{0^+}^{\delta(t),\varrho}\omega(t) = F(t, \omega(t)), \quad t \in [0, T], \\ &\omega(0) = V_0, \end{aligned} \tag{37}$$

where $\varrho \in (0, 1]$, $\delta(t)$ is defined by (13), ${}^C\mathbb{D}_{0^+}^{\delta(t),\varrho}$ denotes the variable-order generalized proportional Caputo fractional derivatives of type III defined by (6), and $F : [0, T] \times PC^1([0, T], \mathbb{R}) \to \mathbb{R}$ is a given function.

We discuss some properties of the applied fractional derivative.

Let $t \in (T_{k-1}, T_k)$, $k = 1, 2, \ldots, n$. Then, from (13), $\delta(s) = \delta_j$ for $s \in (T_{j-1}, T_j]$, $j = 1, 2, \ldots, k-1$, and $\delta(s) = \delta_k$ for $s \in (T_{k-1}, t]$, and we obtain

$$\begin{aligned}
&\int_0^t \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta(s)}\Gamma(1-\delta(s))}(t-s)^{-\delta(s)}g(s)\,ds \\
&= \sum_{j=1}^{k-1}\int_{T_{j-1}}^{T_j} \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta_j}\Gamma(1-\delta_j)}(t-s)^{-\delta_j}g(s)\,ds \\
&\quad + \int_{T_{k-1}}^{t} \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta_k}\Gamma(1-\delta_k)}(t-s)^{-\delta_{k-1}}g(s)\,ds.
\end{aligned} \tag{38}$$

From Equation (38), the definition of the variable order $\delta(t)$, and the definition of the fractional derivative, we obtain for any $t \in (T_{k-1}, T_k)$,

$$\begin{aligned}
&{}^{C}\mathbb{D}_0^{\delta(t),\varrho} g(t) \\
&= (1-\varrho)\sum_{j=1}^{k-1}\int_{T_{j-1}}^{T_j} \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta_j}\Gamma(1-\delta_j)}(t-s)^{-\delta_j}g(s)\,ds \\
&\quad + \varrho\sum_{j=1}^{k-1}\int_{T_{j-1}}^{T_j} \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta_j}\Gamma(1-\delta_j)}(t-s)^{-\delta_j}g'(s)\,ds \Big\} + {}^{C}D_{T_{k-1}}^{\delta_k,\varrho} g(t).
\end{aligned} \tag{39}$$

Also, for any $j = 1, 2, \ldots, n$ we obtain

$$\int_{T_{j-1}}^{T_j} \frac{e^{\frac{\varrho-1}{\varrho}(t-s)}}{\varrho^{\delta_j}\Gamma(1-\delta_j)}(t-s)^{-\delta_j}g(s)\,ds \neq \int_{T_{j-1}}^{T_j} \frac{e^{\frac{\varrho-1}{\varrho}(T_{j-1}-s)}}{\varrho^{\delta_j}\Gamma(1-\delta_j)}(T_{j-1}-s)^{-\delta_j}g(s)\,ds.$$

Therefore, for any $t \in (T_{k-1}, T_k]$, $k = 1, 2, \ldots, n$, we have

$${}^{C}\mathbb{D}_0^{\delta(t),\varrho}\omega(t) \neq \sum_{j=1}^{k-1} {}^{C}D_{T_{j-1}}^{\delta_j,\varrho}\omega(t)\Big|_{t=T_j} + {}^{C}D_{T_{k-1}}^{\delta_k,\varrho}\omega(t). \tag{40}$$

Similar to the case of the fractional derivative of type II discussed in Section 3.2, we consider the scalar Equation (30) as an auxiliary scalar fractional differential equation with a generalized proportional Caputo fractional derivative of constant order $\delta_k \in (0,1)$. Furthermore, we define a mild solution of the nonlinear problem (37) and construct it as shown in Section 3.2.

Remark 5. *Note that in the case of the fractional derivative of type I, we could proceed directly and obtain a solution of the given problem* (3)*. However, this could not be done directly for problems with fractional derivatives of type II and type III. In this case, we had to define a special type of solution and study it.*

Remark 6. *In [35], a system similar to* (31) *was studied directly with the iterative initial conditions* (32)*.*

4. Ulam-Type Stability

We define and study Ulam-type stability for the differential equations with the three types of fractional derivatives.

4.1. Variable-Order Generalized Proportional Caputo Fractional Derivatives of Type I

Definition 9. *Equation (16) is said to be Ulam–Hyers stable if there exists a constant $C_0 > 0$ such that for any $\epsilon > 0$ and any function $\eta = (\eta_1, \eta_2, \dots, \eta_n) : \eta_k \in C^1([0, T_k], \mathbb{R}),\ k = 1, 2 \dots, n$, satisfying the inequalities*

$$\left| {}^C D_0^{\delta_k,\varrho} \eta_k(t) - F(t, \eta_k(t)) \right| \leq \epsilon, \quad t \in [0, T_k],\ k = 1, 2, \dots, n, \tag{41}$$

there exists a solution $\omega \in PC^1[0, T]$ of (16) with

$$|\omega(t) - \nu(t)| \leq C_0 \epsilon, \quad t \in [0, T],$$

where

$$\nu(t) = \begin{cases} \eta_1(t), & \text{if } \ t \in [T_0, T_1), \\ \eta_2(t), & \text{if } \ t \in [T_1, T_2), \\ \vdots \\ \eta_m(t), & \text{if } \ t \in [T_{n-1}, T_n]. \end{cases} \tag{42}$$

Theorem 3. *(UHS). Let conditions (A) be satisfied. Then, (16) is Ulam–Hyers stable with a constant $C_0 = P(1 + M)$, where $P = \max_{k=1,2,\dots,n} E_{\delta_k}(\frac{L}{\delta_k} T_k^{\delta_k})$ and $M = \max_{k=1,2,\dots,n} \frac{T_k^{\delta_k}}{\varrho^{\delta_k}\Gamma(1+\delta_k)}$.*

Proof. Let $\epsilon > 0$ be an arbitrary number. Let $\eta = (\eta_1, \eta_2, \dots, \eta_n) : \ \eta_k \in C^1([0, T_k], \mathbb{R})$, $k = 1, 2, \dots, n$, be a solution of the system of inequalities (41) and the function $\nu \in PC^1[0, T]$ be defined by (42). Then, for any $k = 1, 2, \dots, n$ there exists a function $g_k \in C([T_{k-1}, T_k], \mathbb{R}) :$ $|g_k(t)| \leq \epsilon$ such that the function $\eta_k(.)$ satisfies the fractional equation

$${}^C D_0^{\delta_k,\varrho} \eta_k(t) = F(t, \eta_k(t)) + g_k(t), \quad t \in [0, T_k]. \tag{43}$$

According to Lemma 3 with $a = 0$, $b = T_k$, $u(t) \equiv \eta_k(t)$, $\alpha = \delta_k$, $u_0 = \eta_k(0)$, $f(t, x) \equiv F(t, u) + g_k(t)$, for any $k = 1, 2, \dots, n$, from Equation (43), we have

$$\eta_k(t) = e^{\frac{\varrho-1}{\varrho}(t-0)} \eta_k(0) + I_{0+}^{\delta_k,\varrho}(F(t, \eta_k(t)) + g_k(t)), \quad t \in [0, T_k]. \tag{44}$$

Choose $V_0 \in [C - \epsilon, K + \epsilon]$, where $C = \min_{k=1,2,\dots,n} \eta_k(0)$ and $K = \max_{k=1,2,\dots,n} \eta_k(0)$. Thus, $|V_0 - \eta_k(0)| \leq \epsilon$, $k = 1, 2, \dots, n$, and consider the initial value problem (16). According to Theorem 1, the initial value problem (16) has a unique solution $\omega(t)$, defined by (24).

For any integer $k = 1, 2, \dots, n$ from Lemma 3 applied to (23) with $a = 0$, $b = T_k$, $\alpha = \delta_k$, we have

$$\omega_k(t) = V_0 e^{\frac{\varrho-1}{\varrho}t} + I_{0+}^{\delta_k,\varrho} F(t, \omega_k(t)), \ \ t \in (0, T_k]. \tag{45}$$

Thus, using Lemma 4, Condition (A2), and Equations (44) and (45), we obtain for $t \in (0, T_k]$

$$\begin{aligned} |\omega_k(t) - \eta_k(t)| &\leq \left| \eta_k(t) - e^{\frac{\varrho-1}{\varrho}t} \eta_k(0) - I_{0+}^{\delta_k,\varrho}\Big(F(t, \eta_k(t)) + g_k(t)\Big) \right| \\ &\quad + \left| \omega_k(t) - e^{\frac{\varrho-1}{\varrho}t} V_0 - I_{0+}^{\delta_1,\varrho} F(t, \omega_k(t)) \right| + |(V_0 - \eta_k(0))e^{\frac{\varrho-1}{\varrho}t}| + |I_{0+}^{\delta_k,\varrho} g_k(t)| \\ &\quad + I_{0+}^{\delta_k,\varrho} \left| F(t, \omega_k(t)) - F(t, \eta_k(t)) \right| \\ &\leq \epsilon \Big(1 + \frac{t^{\delta_k}}{\varrho^{\delta_k}\Gamma(1 + \delta_k)} \Big) + L I_{0+}^{\delta_k,\varrho} |\omega_k(t) - \eta_k(t)| \\ &\leq \epsilon(1 + M) + L I_{0+}^{\delta_k,\varrho} |\omega_k(t) - \eta_k(t)|, \quad t \in [0, T_k]. \end{aligned} \tag{46}$$

According to Lemma 1, from Inequality (46), we obtain

$$|\eta_k(t) - \omega_k(t)| \leq \epsilon E_{\delta_k}\left(\frac{L}{\delta_k}(T_k - T_0)^{\delta_k}\right)(1+M) \leq \epsilon P(1+M), \quad t \in [0, T_k].$$

Thus, $|\omega(t) - \eta(t)| \leq \epsilon P(1+M), \quad t \in [0,T]$. □

4.2. Variable-Order Generalized Proportional Caputo Fractional Derivatives of Type II and Type III

As mentioned in Sections 3.2 and 3.3, the study of differential equations with fractional derivatives of type II and type III is similar, so we combine the study of Ulam-type stability for both types of derivatives into one section.

Definition 10 (Equation (25) (respectively, (37))). *is said to be Ulam–Hyers stable if there exists a constant $C_0 > 0$ such that for any $\epsilon > 0$ and any function $\eta = (\eta_1, \eta_2, \ldots, \eta_n) : \eta_k \in C^1([T_{k-1}, T_k], \mathbb{R})$, $k = 1, 2 \ldots, n$, satisfying the inequalities*

$$\left| {}^C D^{\delta_k,\varrho}_{T_{k-1}} \eta_k(t) - F(t, \eta_k(t)) \right| \leq \epsilon, \quad t \in [T_{k-1}, T_k],\ k = 1, 2, \ldots, n, \tag{47}$$

there exists a mild solution $\omega \in PC^1[0,T]$ of (25) (respectively, (37)) such that

$$|\omega(t) - \nu(t)| \leq C_0 \epsilon, \quad t \in [0,T],$$

where ν is defined by (42).

Theorem 4. *(UHS). Let conditions (A) be satisfied. Then, (25) (respectively, (37)) is Ulam–Hyers stable with a constant $C_0 = nP^n M$, where*

$$M = \max_{k=1,2,\ldots,n} \frac{(T_k - T_{k-1})^{\delta_k}}{\varrho^{\delta_k}\Gamma(1+\delta_k)}, \quad P = \max_{k=1,2,\ldots,n} E_{\delta_K}\left(\frac{L}{\varrho^{\delta_k}}(T_k - T_{k-1})^{\delta_k}\right) > 1.$$

Proof. Let $\epsilon > 0$ be an arbitrary number. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_n) : \ \eta_k \in C^1([T_{k-1}, T_k], \mathbb{R})$, $k = 1, 2, \ldots, n$, be a solution of the system of inequalities (47). Then, for any $k = 1, 2, \ldots, n$, there exists a function $g_k \in C([T_{k-1}, T_k], \mathbb{R}) : \ |g_k(t)| \leq \epsilon$ such that the function $\eta_k(.)$ satisfies the fractional equation

$${}^C D^{\delta_k,\varrho}_{T_{k-1}} \eta_k(t) = F(t, \eta_k(t)) + g_k(t), \qquad t \in (T_{k-1}, T_k]. \tag{48}$$

Accordingto Lemma 3 with $a = T_{k-1},\ b = T_k,\ \ u(t) \equiv \eta_k(t),\ \alpha = \delta_k,\ u_0 = \eta_{k-1}(T_{k-1})$, and $f(t,x) \equiv F(t,u) + g_k(t)$, for any $k = 1, 2, \ldots, n$, we have

$$\eta_k(t) = e^{\frac{\varrho-1}{\varrho}(t - T_{k-1})}\eta_{k-1}(T_{k-1}) + I^{\delta_k,\varrho}_{T_{k-1}+}\Big(F(t, \eta_k(t)) + g_k(t)\Big), \quad t \in [T_{k-1}, T_k]. \tag{49}$$

Consider Problem (25) with the initial value $V_0 = \eta_1(0)$. According to Theorem 2, the initial value problem (25) has a unique mild solution ω, defined by (33).

Let $k = 1$. From Lemma 3 applied to (31) with $k = 1$ and the first equation of (32) with $a = 0$, $b = T_1$, $\alpha = \delta_1$, we have

$$\omega_1(t) = \eta_1(0)e^{\frac{\varrho-1}{\varrho}t} + I^{\delta_1,\varrho}_{0+}F(t, \omega_1(t)), \ \ t \in (0, T_1]. \tag{50}$$

From Condition (A2), Lemma 4, and Equalities (49) and (50), we obtain

$$\begin{aligned}|\omega_1(t)-\eta_1(t)| &\le \left|\eta_1(t)-e^{\frac{\varrho-1}{\varrho}(t-0)}\eta_1(0)-I_{0+}^{\delta_1,\varrho}\Big(F(t,\eta_1(t))+g_1(t)\Big)\right| \\ &\quad+\left|\omega_1(t)-e^{\frac{\varrho-1}{\varrho}(t-0)}\eta_1(0)-I_{0+}^{\delta_1,\varrho}F(t,\omega_1(t))\right|+|I_{0+}^{\delta_1,\varrho}g_1(t)| \\ &\quad+I_{0+}^{\delta_1,\varrho}\Big|F(t,\omega_1(t))-F(t,\eta_1(t))\Big| \\ &\le \epsilon\frac{t^{\delta_1}}{\varrho^{\delta_1}\Gamma(1+\delta_1)}+I_{0+}^{\delta_1,\varrho}|\omega_1(t)-\eta_1(t)|, \quad t\in[0,T_1].\end{aligned} \tag{51}$$

According to Lemma 1, from Inequality (51), we have

$$|\eta_1(t)-\omega_1(t)|\le\epsilon ME_{\delta_1}\left(\frac{L}{\varrho^{\delta_1}}(T_1-T_0)^{\delta_1}\right)\le\epsilon PM,\ \ t\in[T_0,T_1]. \tag{52}$$

Let $k=2$. From Lemma 3 applied to (31) with $k=2$ and the second equation of (32) with $a=T_1$, $b=T_2$, $\alpha=\delta_2$, we have

$$\omega_2(t)=\omega_1(T_1)^{\frac{\varrho-1}{\varrho}(t-T_1)}+I_{T_1^+}^{\delta_2,\varrho}F(t,\omega_2(t)),\ \ t\in(T_1,T_2]. \tag{53}$$

For any $t\in(T_1,T_2]$, from Assumption 2, Lemma 4, Equations (49) and (53), and Inequality (52) for $t=T_1$, we obtain

$$\begin{aligned}&|\omega_2(t)-\eta_2(t)| \\ &\le\left|\eta_2(t)-e^{\frac{\varrho-1}{\varrho}(t-T_{k-1})}\eta_1(T_1)-I_{T_1+}^{\delta_2,\varrho}F(t,\eta(t))\right| \\ &+\left|\omega_2(t)-e^{\frac{\varrho-1}{\varrho}(t-T_1)}\omega_1(T_1)-I_{T_1+}^{\delta_2,\varrho}F(t,\omega_2(t))\right| \\ &\quad+I_{T_1+}^{\delta_2,\varrho}|g_2(t)|+I_{T_1+}^{\delta_2,\varrho}\Big|F(t,\eta_2(t))-F(t,\omega_2(t))\Big| \\ &+|\eta_1(T_1)-\omega_1(T_1)| \\ &\le\epsilon I_{T_1+}^{\delta_2,\varrho}1+LI_{T_1+}^{\delta_2,\varrho}|\eta_2(t))-\omega_2(t))|+|\eta_1(T_1)-\omega_1(T_1)| \\ &\le LI_{T_1+}^{\delta_2,\varrho}|\eta_2(t))-\omega_2(t))|+\epsilon M+\epsilon PM \\ &\le LI_{T_1+}^{\delta_2,\varrho}|\eta_2(t))-\omega_2(t))|+2\epsilon PM.\end{aligned} \tag{54}$$

According to Lemma 1, from Inequality (54), we have

$$|\eta_2(t)-\omega_2(t)|\le 2\epsilon P^2M,\ \ t\in[T_1,T_2].$$

Inductively, for any k from Lemma 3 applied to (31) with k, and the second equation of (32) with $a=T_{k-1}$, $b=T_k$, $\alpha=\delta_k$, we have

$$\omega_k(t)=\omega_{k-1}(T_{k-1})^{\frac{\varrho-1}{\varrho}(t-T_1)}+I_{T_{k-1}^+}^{\delta_k,\varrho}F(t,\omega_k(t)),\ \ t\in(T_{k-1},T_k]. \tag{55}$$

Then, for $t\in(T_{k-1},T_k]$, from Assumption 2, Lemma 4, and Equalities (49) and (55), we obtain

$$\begin{aligned}&|\omega_k(t)-\eta_k(t)| \\ &\le\epsilon M+LI_{T_{K-1}+}^{\delta_k,\varrho}|\eta_k(t))-\omega_k(t))|+|\eta_{k-1}(T_{k-1})-\omega_{k-1}(T_{k-1})| \\ &\le LI_{T_{K-1}+}^{\delta_k,\varrho}|\eta_k(t))-\omega_k(t))|+\epsilon M+\epsilon(k-1)P^{k-1}M \\ &\le LI_{T_{K-1}+}^{\delta_k,\varrho}|\eta_k(t))-\omega_k(t))|+k\epsilon P^{k-1}M.\end{aligned} \tag{56}$$

Accordingto Lemma 1, from Inequality (54), it follows that

$$|\eta_k(t) - \omega_k(t)| \leq kP^k M, \quad t \in [T_{k-1}, T_k].$$

Thus, $|\omega(t) - \eta(t)| \leq \epsilon n P^n M, \quad t \in [0, T]$. □

Remark 7. *In the case of fractional differential equations with Caputo-type fractional derivatives with constant orders, the preservation of some important physical properties and physical structures, such as positivity preservation and the maximum principle, has been studied, for example, in [36] (for Nagumo-type differential equations with several Caputo fractional derivatives of constant orders) and in [37] (for subdiffusion equations on general non-conforming distorted meshes).*

5. Conclusions

In this paper, we investigate nonlinear scalar differential equations featuring the variable-order generalized proportional Caputo fractional derivative. We study the case where there is a partition of the finite interval of consideration, and the fractional order remains constant across the corresponding subintervals. Motivated by the literature on variable-order fractional differentials/integrals, we propose three types of fractional integrals and derivatives. The main difference between these derivatives is the argument of the variable order: it can either be equal to the current time or be equal to the variable of the integral in the definition. For these three types of corresponding nonlinear differential equations, we suggest algorithms for constructing solutions and establish their existence and Ulam-type stability.

In future work, we hope to propose various types of variable fractional derivatives and algorithms to obtain solutions of the corresponding nonlinear differential equations, aiming to model some real-world problems. Furthermore, we hope to study positivity preservation, the maximum principle, long-term behavior, singular solutions, etc.

Author Contributions: Conceptualization, D.O., R.P.A., S.H. and M.I.A.; methodology, D.O., R.P.A., S.H. and M.I.A.; formal analysis, D.O., R.P.A., S.H. and. M.I.A.; investigation, D.O., R.P.A., S.H. and M.I.A.; writing—original draft preparation, D.O., R.P.A., S.H. and M.I.A.; writing—review and editing, D.O., R.P.A., S.H. and M.I.A. All authors have read and agreed to the published version of the manuscript.

Funding: S.H. is partially supported by the Bulgarian National Science Fund under Project KP-06-PN62/1.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus Models and Numerical Methods*; World Scientific: Singapore, 2012.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: North-Holland, The Netherlands, 2006.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
4. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publisher: London, UK, 1993.
5. Jacob, J.S.; Priya, J.H.; Karthika, A. Applications of fractional calculus in Science and Engineering. *J. Critical. Rev.* **2020**, *7*, 4385–4394.
6. Sun, H.G.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y.Q. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 213–231. [CrossRef]
7. Magin, R.L. Fractional calculus in bioengineering. *Crit. Rev. Biomed. Eng.* **2004**, *32*, 1–104. [CrossRef]
8. Matlob, M.A.; Jamali, Y. The concepts and applications of fractional order differential calculus in modeling of viscoelastic systems: A primer. *Crit. Rev. Biomed. Eng.* **2019**, *47*, 249–276. [CrossRef] [PubMed]
9. Zhou, Z.; Zhang, H.; Yang, X. H^1-norm error analysis of a robust ADI method on graded mesh for three-dimensional subdiffusion problems. *Numer. Algor.* **2023**. [CrossRef]
10. Zhang, H.; Yang, X.; Tang, Q.; Xu, D. A robust error analysis of the OSC method for a multi-term fourth-order sub-diffusion equation. *Comput. Math. Appl.* **2022**, *109*, 180–190. [CrossRef]

11. Zhang, H.; Liu, Y.; Yang, X. An efficient ADI difference scheme for the nonlocal evolution problem in three-dimensional space. *J. Appl. Math. Comput.* **2023**, *69*, 651–674. [CrossRef]
12. Samko, S.; Ross, R. Integration and differentiation to a variable fractional order. *Integral Transform. Spec. Funct.* **1993**, *1*, 277–300. [CrossRef]
13. Patnaik, S.; Hollkamp, J.P.; Semperlotti, F. Applications of variable-order fractional operators: A review. *Proc. R. Soc. A* **2020**, *476*, 20190498. [CrossRef]
14. Lopez-Cresencio, O.U.; Ariza-Hernandez, F.J.; Sanchez-Ortiz, J.; Arciga-Alejandre, M.P. A boundary value problem for a random-order fractional differential equation. *Res. Appl. Math.* **2022**, *16*, 100328. [CrossRef]
15. Alzabut, J.; Abdeljawad, T.; Jarad, F.; Sudsutad, W. A Gronwall inequality via the generalized proportional fractional derivative with applications. *J. Inequalities Appl.* **2019**, *2019*, 101. [CrossRef]
16. Jarad, F.; Abdeljawad, T. Generalized fractional derivatives and Laplace transform. *Discr. Cont. Dynam. Syst. Ser. S* **2020**, *13*, 709–722. [CrossRef]
17. Jarad, F.; Abdeljawad, T.; Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3457–3471. [CrossRef]
18. Laadjal, Z.; Abdeljawad, T.; Jarad, F. On existence-uniqueness results for proportional fractional differential equations and incomplete gamma functions. *Adv. Difference Equ.* **2020**, *2020*, 641. [CrossRef]
19. Almeida, R.; Tavares D.; Torres, D.F.M. *The Variable-Order Fractional Calculus of Variations, Springer Briefs in Applied Sciences and Technology*; Springer: Cham, Switzerland, 2019.
20. Valerio, D.; Costa, J.S. Variable-order fractional derivatives and their numerical approximations. *Signal Process.* **2011**, *91*, 470–483. [CrossRef]
21. Samko, S.G. Fractional integration and differentiation of variable order. *Anal. Math.* **1995**, *21*, 213–236. [CrossRef]
22. Hristova, S.; Benkerrouche, A.; Souid, M.S.; Hakem, A. Boundary value problems of Hadamard fractional differential equations of variable order. *Symmetry* **2021**, *13*, 896. [CrossRef]
23. Telli, B.; Souid, M.S.; Alzabut, J.; Khan, H. Existence and Uniqueness Theorems for a Variable-Order Fractional Differential Equation with Delay. *Axioms* **2023**, *12*, 339. [CrossRef]
24. Telli, B.; Souid, M.S.; Stamova, I. Boundary-Value Problem for Nonlinear Fractional Differential Equations of Variable Order with Finite Delay via Kuratowski Measure of Noncompactness. *Axioms* **2023**, *12*, 80. [CrossRef]
25. Shah, K.; Ali, G.; Ansari, K.J.; Abdeljawad, T.; Meganathan, M.; Abdalla, B. On qualitative analysis of boundary value problem of variable order fractional delay differential equations. *Bound. Value Probl.* **2023**, *2023*, 55. [CrossRef]
26. Zhang, S.; Hu, L. The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order. *Aims Math.* **2019**, *5*, 2923–2943. [CrossRef]
27. Zhang, S. The uniqueness result of solutions to initial value problems of differential equations of variable-order. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2018**, *112*, 407–423. [CrossRef]
28. Wang, F.; Liu, L. The Existence and Uniqueness of Solutions for Variable-Order Fractional Differential Equations with Antiperiodic Fractional Boundary Conditions. *Funct. Spaces* **2022**, *2022*, 7663192. [CrossRef]
29. Sarwar, S. On the Existence and Stability of Variable Order Caputo Type Fractional Differential Equations. *Fractal Fract.* **2022**, *6*, 51. [CrossRef]
30. Benkerrouche, A.; Baleanu, D.; Souid, M.S.; Hakem, A.; Inc, M. Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique. *Adv. Diff. Equ.* **2021**, *2021*, 365. [CrossRef]
31. Benkerrouche, A.; Souid, M.S.; Chandok, S.; Hakem, A. Existence and Stability of a Caputo Variable-Order Boundary Value Problem. *J. Math.* **2021**, *2021*, 7967880. [CrossRef]
32. Souid, M.S.; Bouazza, Z.; Yakar, A. Existence, Uniqueness, and Stability of Solutions to Variable Fractional Order Boundary Value Problems. *J. New Theory* **2022**, *41*, 82–93. [CrossRef]
33. Refice, A.; Souid, M.S.; Yakar, A. Some qualitative properties of nonlinear fractional integro-differential equations of variable order. *Int. J. Optim. Control. Theor. Appl.* **2021**, *11*, 68–78. [CrossRef]
34. Benia, K.; Souid, M.S.; Jarad, F.; Alqudah, M.A.; Abdeljawad, T. Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order. *J. Inequalities Appl.* **2023**, *2023*, 127. [CrossRef]
35. Wu, G.C.; Gu, C.Y.; Huang, L.L.; Baleanu, D. Fractional defferential equations of variable order: Existence results, numerical method and asymptotic stability conditions. *Miskolc Math. Notes* **2022**, *23*, 485–493. [CrossRef]
36. Yang, X.; Zhang, Z. On conservative, positivity preserving, nonlinear FV scheme on distorted meshes for the multi-term nonlocal Nagumo-type equations. *Appl. Math. Lett.* **2024**, *150*, 108972. [CrossRef]
37. Yang, X.; Zhang, H.; Zhang, Q.; Yuan, G. Simple positivity-preserving nonlinear finite volume scheme for subdiffusion equations on general non-conforming distorted meshes. *Nonlin. Dyn.* **2022**, *108*, 3859–3886. [CrossRef]