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L^p Bounds for Rough Maximal Operators along Surfaces of Revolution on Product Domains

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Abstract: In this paper, we study the boundedness of rough Maximal integral operators along surfaces of revolution on product domains. For several classes of surfaces, we establish appropriate L^p bounds of these Maximal operators under the assumption $\Omega \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $q > 1$, and then we employ these bounds along with Yano's extrapolation argument to obtain the L^p boundedness of the aforementioned integral operators under a weaker condition in which Ω belongs to either the space $B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ or to the space $L(logL)^{2/\tau'}(\mathbb{S}^{m-} \times \mathbb{S}^{n-1})$. Our results extend and improve many previously known results.

Keywords: product spaces; surfaces of revolution; rough kernels; Maximal integrals

MSC: 42B20; 42B25

1. Introduction

Let $\mathbb{R}^s$ ($s = m$ or $s = n$) be the s-dimensional Euclidean space and $\mathbb{S}^{s-1}$ be the unit sphere in $\mathbb{R}^s$ equipped with the normalized Lebesgue surface measure $d\sigma_s(\cdot)$.

Let $\Omega \in L^1(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ be a function defined on $\mathbb{R}^m \times \mathbb{R}^n$ with the following properties:

$$\Omega(rw, tv) = \Omega(w, v), \quad \forall r, t > 0, \tag{1}$$

$$\int_{\mathbb{S}^{m-1}} \Omega(w, .)d\sigma_m(w) = \int_{\mathbb{S}^{n-1}} \Omega(., v)d\sigma_n(v) = 0. \tag{2}$$

Let $\overline{\mathcal{B}}_\tau$ ($\tau > 1$) be the class of all radial functions $h : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that

$$\|h\|_{L^\tau(\mathbb{R}_+ \times \mathbb{R}_+, \frac{drdt}{rt})} = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |h(r,t)|^\tau \frac{drdt}{rt} \right)^{1/\tau} \leq 1.$$

For an appropriate mapping $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, we define the maximal operator $\mathfrak{J}^{(\tau)}_{\Omega,\Psi}$ along the surface of revolution $\Lambda_\Psi(w, v) = (w, v, \Psi(|w|, |v|))$ given by

$$\mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f)(x, y, z) = \sup_{h \in \overline{\mathcal{B}}_\tau} \left| T_{\Omega,\Psi,h}(f)(x, y, z) \right|,$$

where $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ and

$$T_{\Omega,\Psi,h}(f)(x, y, z) = \iint_{\mathbb{R}^m \times \mathbb{R}^n} f(x - w, y - v, z - \Psi(|w|, |v|)) \frac{h(|w|, |v|)\Omega(w, v)}{|w|^m |v|^n} dwdv.$$

When $\Psi \equiv 0$ and $\tau = 2$, we denote $\mathfrak{J}^{(\tau)}_{\Omega,\Psi}$ by $\mathfrak{J}_\Omega$. In this case, the operator $\mathfrak{J}_\Omega$ is just the classical maximal integral operator on product domains, which was proven to be bounded

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on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ provided that $\Omega \in L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ (see [1]). Thereafter, the study of the L^p boundedness of $\mathfrak{J}_\Omega$ has attracted the attention of many authors. Let us recall some known results relevant to our current study. In [2], the author improved and extended the result in [1]. In fact, he proved the L^p boundedness of $\mathfrak{J}_\Omega$ for all $p \geq 2$ under the condition $\Omega \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ and the condition that $\Omega \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is the best possible in the sense that the space $L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ cannot be replaced by a larger space $L(\log L)^\kappa(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for any $\kappa \in (0,1)$. On the other hand, if Ω lies in the space $B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $q > 1$, the author of [3] proved the L^p boundedness of $\mathfrak{J}_\Omega$ for all $p \geq 2$ and the condition $\Omega \in B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is the best possible in the sense that it cannot be replaced by any weaker condition $\Omega \in B_q^{(0,\kappa)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $-1 < \kappa < 0$. Subsequently, the investigation of the L^p boundedness of $\mathfrak{J}_\Omega$ and some of its extensions has attracted the attention of many authors in the last two decades. For more results we advice the readers to refer to [4,5], and the references therein. For the importance of studying the integral operators with oscillating kernels, the readers are advised to consult [6–15].

We point out our maximal operator $\mathfrak{J}_{\Omega,\Psi}^{(\tau)}$ is an extension of the maximal operator (in the one parameter setting) $\mathfrak{M}_\Omega^{(\tau),\psi}$ given by

$$\mathfrak{M}_\Omega^{(\tau),\psi}(f)(x, x_{m+1}) = \sup_{g \in U} \left| \int_{\mathbb{R}^{m+1}} f(x-w, x_{m+1} - \psi(|w|)) g(|w|) \Omega(w) |x|^{-m} dw \right|,$$

where $\tau \geq 1$,

$$U = \left\{ g : g \text{ is a radial function on } \mathbb{R}_+ \text{ and } \left(\int_{\mathbb{R}_+} |g(r)|^\tau \frac{dr}{r} \right)^{1/\tau} \leq 1 \right\},$$

and Ω is homogeneous function of degree zero, integrable over $\mathbb{S}^{m-1}$ and satisfies the cancellation condition

$$\int_{\mathbb{S}^{m-1}} \Omega(w) d\sigma_m(w) \quad = \quad 0.$$

The maximal operator $\mathfrak{M}_\Omega^{(\tau),\psi}$ and its extensions has attracted the attention of many authors, see for example [16–24].

We remark that recently, the authors of [25] studied the L^p boundedness of the singular integral operators $T_{\Omega,\Psi,h}$ under various conditions on Ψ, Ω and h. Indeed, they established the L^p boundedness of $T_{\Omega,\Psi,h}$ whenever Ω belongs to either $L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ or to $B_q^{(1,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$.

In light of the results in [18,20] concerning the boundedness of the maximal integral $\mathfrak{M}_\Omega^{(\tau),\psi}$ in the one-parameter setting and the results in [25] concerning the boundedness of $T_{\Omega,\Psi,h}$ in the product spaces, a question arises naturally as follows:

Question: Under the assumptions imposed on Ψ in [25], does the operator $\mathfrak{J}_{\Omega,\Psi}^{(\tau)}$ satisfy the boundedness if $1 < \tau \leq 2$ and $\Omega \in L(\log L)^{2/\tau'}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $q > 1$?

The purpose of this work is to answer this question in the affirmative. In fact, we have the following:

Theorem 1. *Let $\Psi \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ such that for any fixed $r, t > 0$, we have $\varphi_{1,r}(.) = \Psi(r,.)$, $\varphi_{2,t}(.) = \Psi(.,t)$ are in $C^2(\mathbb{R}_+)$, convex and increasing functions with $\varphi_{1,r}(0) = \varphi_{2,t}(0) = 0$. Assume that $1 \leq \tau \leq 2$ and $\Omega \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $1 < q \leq 2$. Then, there exists a constant $C_p > 0$ such that*

$$\left\| \mathfrak{J}_{\Omega,\Psi}^{(\tau)}(f) \right\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \leq C_p \left(\frac{1}{q-1} \right)^{2/\tau'} \|\Omega\|_{L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \tag{3}$$

for all $\tau' \le p < \infty$ *if* $1 < \tau \le 2$, *and*

$$\left\| \mathfrak{J}^{(1)}_{\Omega,\Psi}(f) \right\|_{L^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \le C \|f\|_{L^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})}. \tag{4}$$

Theorem 2. *Suppose that* Ω *and* τ *are given as in Theorem 1. Let* $\Psi(r,t) = \sum_{k=0}^{\kappa_1} \sum_{j=0}^{\kappa_2} a_{j,k} r^{\alpha_k} t^{\gamma_j}$ *with* $\alpha_k, \gamma_j > 0$ *is a generalized polynomial on* $\mathbb{R}^2$. *Then the estimate (3) holds for all* $\tau' \le p < \infty$ *if* $1 < \tau \le 2$, *and (4) holds if* $\tau = 1$.

Theorem 3. *Suppose that* Ω *and* τ *are given as in Theorem 1. Let* $\Psi(r,t) = \varphi(r)P(t)$, *where* φ *is in* $C^2(\mathbb{R}_+)$, *convex and increasing function with* $\varphi(0) = 0$, *and* P *is a generalized polynomial given by* $P(t) = \sum_{j=0}^{\kappa_2} c_j t^{\gamma_j}$ *with* $\gamma_j > 0$. *Then the estimate (3) holds for all* $\tau' \le p < \infty$ *if* $1 < \tau \le 2$, *and (4) holds if* $\tau = 1$.

Theorem 4. *Suppose that* Ω *and* τ *are given as in Theorem 1. Let* $\Psi(r,t) = \varphi_1(r) + \varphi_2(t)$, *where* $\varphi_k(\cdot)$ $(k = 1, 2)$ *is either a generalized polynomial or is in* $C^2(\mathbb{R}_+)$, *convex and increasing function with* $\varphi_k(0) = 0$. *Then the estimate (3) holds for all* $\tau' \le p < \infty$ *if* $1 < \tau \le 2$, *and (4) holds if* $\tau = 1$.

By the estimates obtained in Theorems 1–4 and employing an extrapolation argument (see [26,27]) we get the following results.

Theorem 5. *Let* Ψ *be given as in any of Theorems 1–4,* Ω *satisfy the conditions (1) and (2) and* $1 < \tau \le 2$.

(i) If $\Omega \in B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ *for some* $q > 1$, *then we have*

$$\left\| \mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f) \right\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \left(1 + \|\Omega\|_{B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})} \right)$$

for all $\tau' \le p < \infty$ *with* $1 < \tau \le 2$;

(ii) If $\Omega \in L(\log L)^{2/\tau'}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$, *then we have*

$$\left\| \mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f) \right\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \left(1 + \|\Omega\|_{L(logL)^{2/\tau'}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})} \right)$$

for all $\tau' \le p < \infty$ *with* $1 < \tau \le 2$.

By using the estimates in Theorem 5, employing the fact that

$$\left| T_{\Omega,\Psi,h}(f)(x,y,z) \right| \le \|h\|_{L^\tau(\mathbb{R}_+ \times \mathbb{R}_+, \frac{drdt}{rt})} \mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f)(x,y,z)$$

and using a standard duality argument, we directly get the following:

Theorem 6. *Suppose that* Ψ *is given as in Theorem 5. Let* $\Omega \in B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup L(logL)^{2/\tau'}(\mathbb{S}^{m-} \times \mathbb{S}^{n-1})$ *with* $1 < q \le 2$ *and* $h \in L^\tau(\mathbb{R}_+ \times \mathbb{R}_+, \frac{drdt}{rt})$ *with* $1 < \tau \le 2$. *Then the operator* $T_{\Omega,\Psi,h}$ *is bounded on* $L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ *for all* $p \in (1, \infty)$.

Remark 1. *For the special cases* $\Psi \equiv 0$ *and* $\tau = 2$, *the conditions on* Ω *in Theorem 5 are the weakest possible conditions among their respective classes, (see [2,3]).*

Remark 2. *For the special cases* $\Psi \equiv 0$ *and* $\tau = 2$*, the author of [1] showed that* $\mathfrak{J}^{(\tau)}_{\Omega,\Psi}$ *is bounded on* $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ *provided that* $\Omega \in L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$*. This result is improved in Theorem 5 since* $L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \subset L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$*.*

Remark 3. *For the case* $1 < \tau \leq 2$*, it was proved in [25] that* $T_{\Omega,\Psi,h}$ *is bounded on* $L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ *for all* $p \in (1,\infty)$ *provided that* $\Omega \in B_q^{(0,1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup L(logL)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \subset B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup L(logL)^{2/\tau'}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$*. Hence, the results in Theorem 6 improve the results in [25] whenever* $h \in L^\tau(\mathbb{R}_+ \times \mathbb{R}_+, \frac{drdt}{rt})$ *with* $1 < \tau \leq 2$*.*

Remark 4. *In Theorem 6, we proved the* L^p *boundedness of the operator* $T_{\Omega,\Psi,h}$ *for all* $p \in (1,\infty)$*, which is the full range for* p*.*

Remark 5. *The surfaces of revolutions* $\Lambda_\Psi(w,v) = (w,v,\Psi(|w|,|v|))$ *considered in this paper cover various natural classical surfaces such as those in [28–34]. For instance, our main results allow surfaces of the type* Λ_Ψ *with* $\Psi(r,t) = r^2t^2(e^{-1/r} + e^{-1/t}), (r,t > 0)$*;* $\Psi(r,t) = r^\gamma s^\mu$ *with* $\gamma, \mu > 0$*;* $\Psi(r,t) = \varphi_1(r)\varphi_2(t)$*, where each* $\varphi_k \in C^2(\mathbb{R}_+)$ *is a convex increasing function with* $\varphi_k(0) = 0$*;* $\Psi(r,t) = P(r,t)$ *is a polynomial.*

Henceforward, the constant C denotes a positive real constant which may not necessarily be the same at each occurrence, but it is independent of all the essential variables.

2. Preliminary Lemmas

In this section, we introduce some notations and establish some lemmas. The class $L(\log L)^\kappa(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ (for $\kappa > 0$) denotes the class of all measurable functions Ω on $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$ which satisfy

$$\|\Omega\|_{L(\log L)^\kappa(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} |\Omega(w,v)| \log^\kappa(2 + |\Omega(w,v)|) d\sigma_m(w) d\sigma_n(v) < \infty.$$

Now we recall the definition of the class of $B_q^{(0,\kappa-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. A q-block on $\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}$ is an L^q $(1 < q \leq \infty)$ function $b(w,v)$ that satisfies $b \subset I$ and $\|b\|_{L^q} \leq |I|^{-1/q'}$, where $|\cdot|$ denotes the product measure on $\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}$ and I is an interval on $\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}$, i.e.,

$$I = \left\{w' \in \mathbb{S}^{m-1} : |w' - w_0'| < \alpha\right\} \times \left\{v' \in \mathbb{S}^{n-1} : |v' - v_0'| < \beta\right\}$$

for some $\alpha, \beta > 0$, $w_0' \in \mathbb{S}^{m-1}$ and $v_0' \in \mathbb{S}^{n-1}$. The block space $B_q^{(0,\kappa)} = B_q^{(0,\kappa)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is defined by

$$B_q^{(0,\kappa)} = \left\{\Omega \in L^1(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu,\ M_q^{(0,\kappa)}\left(\{\lambda_\mu\}\right) < \infty\right\},$$

where each λ_μ is a complex number, each b_μ is a q-block supported on an interval I_μ on $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$, $\kappa > -1$, and

$$M_q^{(0,\kappa)}\left(\{\lambda_\mu\}\right) = \sum_{\mu=1}^{\infty} \left|\lambda_\mu\right| \left\{1 + \log^{(\kappa+1)}\left(\left|I_\mu\right|^{-1}\right)\right\}.$$

Let $\|\Omega\|_{B_q^{(0,\kappa)}(\mathbf{S}^{n-1}\times\mathbb{S}^{n-1})} = \inf\{M_q^{(0,\kappa)}\left(\{\lambda_\mu\}\right) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\}$.

We remark here that for any $q > 1$ and $0 < \kappa \leq 1$, the following inclusions hold and are proper:

$$\begin{aligned} L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) &\subset L(\log L)^{\kappa}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \subset L^1(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \text{ for } \kappa > 0, \\ \bigcup_{r>1} L^r(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) &\subset B_q^{(0,\kappa)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \text{ for any } \kappa > -1 \text{ and } q > 1, \\ L(\log L)^{\kappa_1}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) &\subset L(\log L)^{\kappa_2}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \text{ if } 0 < \kappa_2 < \kappa_1, \\ B_q^{(0,\kappa_1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) &\subset B_q^{(0,\kappa_2)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \text{ if } -1 < \kappa_2 < \kappa_1. \end{aligned}$$

The question concerning the relationship between the spaces $B_q^{(0,\kappa-1)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ and $L(\log^+ L)^{\kappa}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ (for $\kappa > 0$) remains open.

For $\theta \geq 2$, an arbitrary mapping Ψ on $\mathbb{R}_+ \times \mathbb{R}_+$ and $\Omega : \mathbb{S}^{m-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$, we consider the family of measures $\{\rho_{\theta,\Omega,\Psi,k,j} := \rho_{k,j} : k, j \in \mathbb{Z}\}$ and its related maximal operator ρ^* on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ by

$$\begin{aligned} \iiint_{\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}} f d\rho_{k,j} &= \int_{\theta^j \leq |v| \leq \theta^{j+1}} \int_{\theta^k \leq |w| \leq \theta^{k+1}} f(w, v, \Psi(|w|, |v|)) \\ &\times \frac{\Omega(w,v)}{|w|^m |v|^n} dw dv, \end{aligned}$$

and

$$\rho^*(f) = \sup_{j,k \in \mathbb{Z}} ||\rho_{k,j}| * f|,$$

where $|\rho_{k,j}|$ is defined similarly to ρ_{kj} but with replacing Ω by $|\Omega|$.

Lemma 1. *Suppose that $\Psi \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\Omega \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $q \in (1, 2]$ satisfies the conditions (1) and (2). For $\theta \geq 2$, $(\xi, \zeta, \varkappa) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+$ and $k, j \in \mathbb{Z}$, let*

$$\mathcal{W}(r,t) = \iint_{\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} e^{-i\{r\theta^k w \cdot \xi + t\theta^j u \cdot \zeta + \Phi(r,t)\varkappa\}} \Omega(w,v) d\sigma_m(w) d\sigma_n(v).$$

Then we have

$$\int_1^{\theta} \int_1^{\theta} |\mathcal{W}(r,t)|^2 \frac{dr dt}{rt} \leq C(\ln \theta)^2 \|\Omega\|^2_{L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})} \left|\theta^k \xi\right|^{\pm \frac{\varepsilon}{q'}} \left|\theta^j \zeta\right|^{\pm \frac{\varepsilon}{q'}}, \tag{5}$$

where $0 < \varepsilon < \frac{1}{2q'}$ and $a^{\pm b} = \min\{a^b, a^{-b}\}$.

Proof. By Schwartz inequality, we get

$$\begin{aligned} \int_1^{\theta} \int_1^{\theta} |\mathcal{W}(r,t)|^2 \frac{dr dt}{rt} &\leq C \iint_{(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})^2} \mathcal{H}(w,b,\xi) \mathcal{G}(v,c,\zeta) \\ &\times \Omega(w,v) \overline{\Omega(b,c)} d\sigma_m(w) d\sigma_n(v) d\sigma_m(b) d\sigma_n(c), \end{aligned}$$

where $\mathcal{H}(w,b,\xi) = \int_1^{\theta} e^{-i\theta^k r \xi \cdot (w-b)} \frac{dr}{r}$ and $\mathcal{G}(v,c,\zeta) = \int_1^{\theta} e^{-i\theta^j t \zeta \cdot (v-c)} \frac{dt}{t}$. Hence, by Van der Corput's lemma we deduce that

$$|\mathcal{H}(w,b,\xi)| \leq C |\theta^k \xi \cdot (w - \tau)|^{-1} \leq C |\theta^k \xi|^{-1} |\xi' \cdot (w - b)|^{-1},$$

where $\xi' = \xi/|\ \xi\ |$. By combining the last estimate with the trivial estimate $|\ \mathcal{H}(w,b,\xi)\ | \leq C(\ln\theta)$, we obtain that

$$|\ \mathcal{H}(w,b,\xi)\ | \leq C(\ln\theta)\ |\ \theta^k\xi\ |^{-\varepsilon} |\ \xi'\cdot(w-b)\ |^{-\varepsilon} \tag{6}$$

for any $\varepsilon \in (0,1)$. Thus, by Hölder's inequality, we have that

$$\begin{aligned}\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} &\leq C(\ln\theta)^2\left|\theta^k\xi\right|^{-\frac{\varepsilon}{q'}}\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\\ &\times \left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}}\left|\xi'\cdot(w-b)\right|^{-\varepsilon q'}d\sigma_m(w)d\sigma_m(b)\right)^{1/q'}.\end{aligned}$$

Choosing ε small enough such that $0<\varepsilon<\frac{1}{2q'}$ gives that the last integral is finite. Therefore,

$$\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} \leq C(\ln\theta)^2\left|\theta^k\xi\right|^{-\frac{\varepsilon}{q'}}\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}. \tag{7}$$

In the same manner, we derive that

$$\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} \leq C(\ln\theta)^2\left|\theta^j\zeta\right|^{-\frac{\varepsilon}{q'}}\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}. \tag{8}$$

On the other side, the cancellation condition (2) gives that

$$\begin{aligned}\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} &\leq C\int_1^\theta\int_1^\theta\left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|e^{-i\theta^k r\xi\cdot w}-1||\Omega(w,v)|d\sigma_m(w)d\sigma_n(v)\right)^2\frac{drdt}{rt}\\ &\leq C(\ln\theta)^2|\theta^k\xi|^2\|\Omega\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\end{aligned}$$

which when combined with the trivial estimate $\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} \leq C(\ln\theta)^2\|\Omega\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}$ imply

$$\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} C(\ln\theta)^2|\theta^k\xi|^{\frac{\varepsilon}{q'}}\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}. \tag{9}$$

Similarly, we acquire that

$$\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2\frac{drdt}{rt} C(\ln\theta)^2|\theta^j\zeta|^{\frac{\varepsilon}{q'}}\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}. \tag{10}$$

Consequently, by combining the inequalities (7)–(10), the estimate (5) is satisfied. The proof of this lemma is complete. □

We need the following lemma from [25] which will play a key role in proving our main results.

Lemma 2. *Let $\theta \geq 2$ and $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $1<q\leq 2$ satisfy (1) and (2). Suppose that Φ is given as in any of Theorems 1–4. Then the inequality*

$$\|\rho^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p(\ln\theta)^2\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{11}$$

holds for $f\in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(1,\infty]$.

3. Proof of the Main Results

Let Φ be given as in any of Theorems 1–4, $\Omega \in L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ with $1<q\le 2$ and $h\in \overline{\mathcal{B}}_\tau$ for some $1\le\tau\le 2$. By duality, we get

$$\begin{aligned}\mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f)(x,y,z) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|T_{\Omega,\Psi,r,t}f(x,y,z)|^{\tau'}\frac{drdt}{rt}\right)^{1/\tau'}\\ &= \left(\sum_{j,k\in\mathbb{Z}}^{\infty}\iint_{[\theta^j,\theta^{j+1})\times[\theta^k,\theta^{k+1})}|T_{\Omega,\Psi,r,t}f(x,y,z)|^{\tau'}\frac{drdt}{rt}\right)^{1/\tau'},\end{aligned} \tag{12}$$

where

$$T_{\Omega,\Psi,r,t}f(x,y,z)=\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} f(x-rw,y-tv,z-\Phi(r,t))\Omega(w,v)d\sigma_m(w)d\sigma_n(v). \tag{13}$$

To prove our the main results, we consider three cases:

Case 1. $\tau=2$. Let $\{\Delta_k\}$ be a set of smooth partition of unity defined on $(0,\infty)$ and adapted to the interval $[\theta^{-1-k},\theta^{1-k}]$. Precisely, we have the following:

$$\begin{aligned}\Delta_k &\in C^\infty,\ 0\le\Delta_k\le 1,\ \sum_{k\in\mathbb{Z}}\Delta_k(r)=1,\\ \text{supp}\,(\Delta_k) &\subseteq [\theta^{-1-k},\theta^{1-k}],\ \textit{and}\ \left|\frac{d^\nu\Delta_k(r)}{dr^\nu}\right|\le\frac{C_\nu}{r^\nu},\end{aligned}$$

where the constant C_ν does not depend on the lacunary sequence $\{\theta^k;k\in\mathbb{Z}\}$. For $k,j\in\mathbb{Z}$, let $\{\mathsf{M}_{j,k}\}$ be the multiplier operators defined on $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$ by

$$\widehat{(\mathsf{M}_{j,k}(f))}(\zeta,\xi,\varkappa)=\Delta_j(|\zeta|)\Delta_k(|\xi|)\hat{f}(\zeta,\xi,\varkappa).$$

So, for any $f\in C_0^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ and $\eta_1,\eta_2\in\mathbb{Z}$, we have

$$f(x,y,z)=\sum_{j,k\in\mathbb{Z}}(\mathsf{M}_{j+\eta_2,k+\eta_1}(f))(x,y,z).$$

By invoking Minkowski's inequality, it is easy to see that

$$\mathfrak{J}^{(2)}_{\Omega,\Psi}(f)(x,y,z)\le\sum_{\eta_1,\eta_2\in\mathbb{Z}}\mathcal{Q}_{\eta_2,\eta_1}(f)(x,y,z), \tag{14}$$

where

$$\mathcal{Q}_{\eta_2,\eta_1}(f)(x,y,z)=\left(\sum_{j,k\in\mathbb{Z}}^{\infty}\iint_{[\theta^j,\theta^{j+1})\times[\theta^k,\theta^{k+1})}\left|\mathcal{U}^{r,t}_{j+\eta_2,k+\eta_1}(f)(x,y,z)\right|^2\frac{drdt}{rt}\right)^{1/2}$$

and

$$\mathcal{U}^{r,t}_{\eta_2,\eta_1}(f)(x,y,z)=\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}\Omega(w,v)(\mathsf{M}_{\eta_2,\eta_1}f)(x-rw,y-tv,z-\Phi(r,t))d\sigma_m(w)d\sigma_n(v).$$

Therefore, to prove our results, it is enough to show that the estimate

$$\begin{aligned}&\left\|\mathcal{Q}_{\eta_2,\eta_1}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\ &\le C_p(\ln\theta)\ 2^{-\epsilon(|\eta_1|+|\eta_2|)}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.\end{aligned} \tag{15}$$

holds for all $p\ge 2$ for some $\epsilon>0$.

First, the L^2-norm for $\mathcal{Q}_{\eta_2,\eta_1}(f)$ can be obtained by employing Plancherel's Theorem and Fubini's Theorem together with Lemma 1. Precisely, we have

$$
\begin{aligned}
&\left\|\mathcal{Q}_{\eta_2,\eta_1}(f)\right\|^2_{L^2(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
&\le \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \left(\iint_{[\theta^j,\theta^{j+1})\times[\theta^k,\theta^{k+1})} \left|\mathcal{U}^{r,t}_{j+\eta_2,k+\eta_1}(f)(x,y,z)\right|^2 \frac{drdt}{rt}\right) dxdydz\\
&\le \sum_{j,k\in\mathbb{Z}} \iiint_{\mathfrak{V}_{j+\eta_2,k+\eta_1}} \left(\int_1^\theta\int_1^\theta |\mathcal{W}(r,t)|^2 \frac{drdt}{rt}\right) \left|\hat{f}(\xi,\zeta,\varkappa)\right|^2 d\xi d\zeta d\varkappa\\
&\le C_p(\ln\theta)^2\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \sum_{j,k\in\mathbb{Z}} \iiint_{\mathfrak{V}_{j+\eta_2,k+\eta_1}} \left|\theta^k\xi\right|^{\pm\frac{2\epsilon}{q'}} \left|\theta^j\zeta\right|^{\pm\frac{2\epsilon}{q'}} \left|\hat{f}(\xi,\zeta,\varkappa)\right|^2 d\xi d\zeta d\varkappa\\
&\le C_p(\ln\theta)^2\, 2^{-2\epsilon(|\eta_1|+|\eta_2|)}\|\mathfrak{V}\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \sum_{j,k\in\mathbb{Z}} \iiint_{\mathfrak{V}_{j+\eta_2,k+\eta_1}} \left|\hat{f}(\xi,\zeta,\varkappa)\right|^2 d\xi d\zeta d\varkappa\\
&\le C_p(\ln\theta)^2\, 2^{-2\epsilon(|\eta_1|+|\eta_2|)}\|\mathfrak{V}\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})}\|f\|^2_{L^2(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})},
\end{aligned}
\tag{16}
$$

where $\mathfrak{V}_{j,k} = \left\{(\xi,\zeta,\varkappa)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R} : (|\xi|,|\zeta|)\in[\theta^j,\theta^{j+1})\times[\theta^k,\theta^{k+1})\right\}$ and $\epsilon\in(0,1)$.

Next, the L^p-norm of $\mathcal{Q}_{\eta_2,\eta_1}(f)$ is estimated as follows: by duality, there is a non-negative function $Z\in L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ such that $\|Z\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\le 1$ and

$$
\begin{aligned}
&\left\|\mathcal{Q}_{\eta_2,\eta_1}(f)\right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
&= \sum_{j,k\in\mathbb{Z}} \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \left(\iint_{[\theta^j,\theta^{j+1})\times[\theta^k,\theta^{k+1})} \left|\mathcal{U}^{r,t}_{j+\eta_2,k+\eta_1}(f)(x,y,z)\right|^2 \frac{drdt}{rt}\right) Z(x,y,z)dxdydz\\
&\le C\|\Omega\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \sum_{j,k\in\mathbb{Z}} \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \iint_{[\theta^j,\theta^{j+1})\times[\theta^k,\theta^{k+1})} \iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} |\Omega(w,v)|\\
&\times \left|\mathsf{M}_{j+\eta_2,k+\eta_1}(f)(x,y,z)\right|^2 |Z(x+rw,y+tv,z+\Psi(r,t))| d\sigma_m(w)d\sigma_n(v)\frac{drdt}{rt}dxdydz\\
&\le C\|\Omega\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \sum_{j,k\in\mathbb{Z}} \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \left|\mathsf{M}_{j+\eta_2,k+\eta_1}(f)(x,y,z)\right|^2 \rho^*(Z)(-x,-y,-z)dxdydz\\
&\le C_p\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \left\|\sum_{j,k\in\mathbb{Z}} \left|\mathsf{M}_{j+\eta_2,k+\eta_1} * f\right|^2\right\|_{L^{(p/2)}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \left\|\rho^*(\overline{Z})\right\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})},
\end{aligned}
\tag{17}
$$

where $\overline{Z}(x,y,z) = Z(-x,-y,-z)$. Hence, by Lemma 2 and Littlewood–Paley theory (see [35]), we get

$$
\left\|\mathcal{Q}_{\eta_2,\eta_1}(f)\right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} C_p \le (\ln\theta)^2\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|f\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{18}
$$

for all $p>2$. Therefore, by interpolating (16) with (18), we obtain (15). Therefore, use (14) and take $\theta = 2^{q'}$, we directly obtain that

$$
\left\|\mathfrak{J}^{(2)}_{\Omega,\Psi}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\frac{1}{q-1}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{19}
$$

for all $2\le p<\infty$.

Case 2. $\tau = 1$. In this case, we have $f\in L^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$. Hence,

$$
\left|\iint_{\mathbb{R}_+\times\mathbb{R}_+} h(r,t)\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} f(x-rw,y-tv,z-\Psi(r,t))\Omega(w,v)d\sigma_m(w)d\sigma_n(v)\frac{drdt}{rt}\right|.
$$

$$
\le C\|h\|_{L^1(\mathbb{R}_+\times\mathbb{R}_+,\frac{drdt}{rt})}\|f\|_{L^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}
$$

for every $(x,y,z)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$. Thus, when we take the supremum to the both sides of the last inequality over all radial functions h with $\|h\|_{L^1(\mathbb{R}_+\times\mathbb{R}_+,\frac{drdt}{rt})}\le 1$, we get

$$
\mathfrak{J}^{(1)}_{\Omega,\Psi}(f)(x,y,z) \le C\|f\|_{L^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}
$$

for almost every $(x,y,z)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$. Therefore,

$$\left\|\mathfrak{J}^{(1)}_{\Omega,\Psi}(f)\right\|_{L^{\infty}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C\|f\|_{L^{\infty}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \tag{20}$$

Case 3. $1<\tau<2$. Notice that the operator $T_{\Omega,\Psi,r,t}: L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}) \to L^p(L^{\tau'}(\mathbb{R}_+\times\mathbb{R}_+, \frac{drdt}{rt}), \mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ given in (13) is linear and

$$\left\|\mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} = \|T_{\Omega,\Psi,r,t}(f)\|_{L^p(L^{\tau'}(\mathbb{R}_+\times\mathbb{R}_+,\frac{drdt}{rt}),\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

Thus, by using the interpolation theorem for the Lebesgue mixed normed spaces between (19) and (20) (see [36]), we deduce that

$$\left\|\mathfrak{J}^{(\tau)}_{\Omega,\Psi}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p(\ln\theta)^{2/\tau'}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{21}$$

for all $p\in[\tau',\infty)$ with $1<\tau\leq 2$. This completes the proof of the main results.

4. Conclusions

In this article, we obtained sharp L^p bounds for several classes of Maximal integral operators related to surfaces of revolution on product spaces whenever $\Omega\in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ and $h\in\overline{\mathcal{B}}_\tau$ for some $q>1$, $\tau\in[1,2]$. By the virtue of these bounds and the extrapolation argument of Yano, we proved that the operator $\mathfrak{J}^{(\tau)}_{\Omega,\Psi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ provided that $\Omega\in B_q^{(0,\frac{2}{\tau'}-1)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\cup L(logL)^{2/\tau'}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$. In addition, under specific assumptions, we confirmed the boundedness of $\mathfrak{J}^{(\tau)}_{\Omega,\Psi}$ for the full range of $p\in(1,\infty)$. Further, the boundedness of the aforesaid operator is proven when the conditions on Ω are the weakest possible conditions among their respective classes. The results in this article extend or improve many known results on maximal operators as the results in [1–3,25].

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References

1. Ding, Y. A note on a class of rough maximal operators on product domains. *J. Math. Anal. Appl.* **1999**, *232*, 222–228. [CrossRef]
2. Al-Salman, A. Maximal operators with rough kernels on product domains. *J. Math. Anal. Appl.* **2005**, *311*, 338–351. [CrossRef]
3. Al-Qassem, H. L^p estimates for a rough maximal operator on product spaces. *J. Korean Math. Soc.* **2005**, *42*, 405–434. [CrossRef]
4. Xu, H.; Fan, D.; Wang, M. Some maximal operators related to families of singular integral operators. *Acta. Math. Sin.* **2004**, *20*, 441–452. [CrossRef]
5. Al-Qassem, H.; Cheng, L.; Pan, P. On the boundedness of a class of rough maximal operators on product spaces. *Hokkaido Math. J.* **2011**, *40*, 1–32. [CrossRef]
6. Fan, D.; Pan, Y. Boundedness of certain oscillatory singular integrals. *Studia Math.* **1995**, *114*, 105–116. [CrossRef]
7. Kim, W.; Wainger, S.; Wright, J.; Ziesler, S. Singular Integrals and Maximal Functions Associated to Surfaces of Revolution. *Bull. Lond. Math. Soc.* **1996**, *28*, 291–296. [CrossRef]
8. Jiang, Y.; Lu, S. Oscillatory singular integrals with rough kernel. In *Harmonic Analysis in China*; Cheng, M.D., Deng, D.G., Gong, S., Yang, C.-C., Eds.; Kluwer Academic: Dordrecht, The Netherlands, 1995; Volume 327, pp. 135–145.
9. Stein, E. On the functions of Littlewood-Paley, Lusin and Marcinkiewicz. *Trans. Am. Math. Soc.* **1958**, *88*, 430–466. [CrossRef]
10. Stein, E. Problems in harmonic analysis related to curvature and oscillatory integrals. *Proc. Internat. Congr. Math.* **1986**, *1*, 196–221.
11. Stein, E. Some geometrical concepts arising in harmonic analysis. *Geom. Funct. Anal.* **2000**, 434–453. [CrossRef]
12. Lu, S.; Zhang, Y. Criterion on L^p-boundedness for a class of oscillatory singular integrals with rough kernels. *Rev. Mat. Iber.* **1992**, *8*, 201–219.

13. Pan, P. L^2 estimates for convolution operators with oscillating kernels. *Math. Proc. Camb. Phil. Soc.* **1993**, *113*, 179–193. [CrossRef]
14. Ricci, F.; Stein, E. Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals. *J. Func. Anal.* **1987**, *73*, 179–194. [CrossRef]
15. Sjolin, P. Convolution with oscillating kernels. *Indiana Univ. Math. J.* **1981**, *30*, 47–55. [CrossRef]
16. Chen, L.; Lin, H. A maximal operator related to a class of singular integral. *Illinois J. Math.* **1990**, *34*, 120–126. [CrossRef]
17. Al-Salman, A. On maximal functions with rough kernels in $L(logL)^{1/2}(\mathbb{S}^{n-1})$. *Collec. Math.* **2005**, *56*, 47–56.
18. Al-Qassem, H. On the boundedness of maximal operators and singular operators with kernels in $L(logL)^{\alpha}(\mathbb{S}^{m-1})$. *J. Ineq. Apll.* **2006**, *1*, 1–16.
19. Jiang, Y.; Lu, S. A class of singular integral operators with rough kernel on product domains. *Hokkaido Math. J.* **1995**, *24*, 1–7. [CrossRef]
20. Al-Qassem, H.; Pan, Y. Singular integrals along surfaces of revolution with rough kernels. *SUT J. Math.* **2003**, *39*, 55–70. [CrossRef]
21. Ding, Y.; He, Q. Weighted boundedness of a rough maximal operator. *Acta Math. Sci. (Engl. Ed.)* **2000**, *20*, 417–422. [CrossRef]
22. Le, H. Maximal operators and singular integral operators along submanifolds. *J. Math. Anal. Appl.* **2004**, *296*, 44–64. [CrossRef]
23. Duoandikoetxea, J.; Rubio de Francia, J. Maximal and singular integral operators via Fourier transform estimates. *Invent Math.* **1986**, *84*, 541–561. [CrossRef]
24. Ali, M.; Al-Mohammed, O. Boundedness of a class of rough maximal functions. *J. Ineq. Appl.* **2018**, *2018*, 305. [CrossRef] [PubMed]
25. Al-Qassem, H.; Cheng, L.; Pan, Y. On singular integrals and maximal operators along surfaces of revolution on product domains. *J. Math. Ineq.* **2023**, *17*, 739–759. [CrossRef]
26. Yano, S. Notes on Fourier analysis. XXIX. An extrapolation theorem. *J. Math. Soc. Jpn.* **1951**, *3*, 296–305. [CrossRef]
27. Sato, S. Estimates for singular integrals and extrapolation. *Studia Math.* **2009**, *192*, 219–233. [CrossRef]
28. Carbery, A.; Wainger, S.; Wright, J. Double Hilbert transforms along polynomial surfaces in $\mathbb{R}^3$. *Duke Math. J.* **2000**, *101*, 499–513. [CrossRef]
29. Carlsson, H.; Sjōgren, P. Estimates for maximal functions along hypersurfaces. *Ark. Math.* **1987**, *25*, 1–14. [CrossRef]
30. Carlsson, H.; Sjōgren, P.; Stromberg, J. Multiparameter maximal functions along dilationinvariant hypersurfaces. *Trans. Am. Math. Soc.* **1985**, *292*, 335–343. [CrossRef]
31. Duoandikoetxea, J. Multiple singular integrals and maximal functions along hypersurfaces. *Ann. Ins. Four.* **1986**, *36*, 185–206. [CrossRef]
32. Fan, D.; Guo, K.; Pan, Y. Singular integrals with rough kernels on product spaces. *Hokkaido Math. J.* **1999**, *28*, 435–460. [CrossRef]
33. Le, H. A note on singular integrals with dominating mixed smoothness in Triebel-Lizorkinspaces. *Acta Math. Scientia.* **2014**, *34*, 1331–1344. [CrossRef]
34. Patel, S. Double Hilbert Transforms along Polynomial Surfaces in $\mathbb{R}^3$. *Glasgow Math. J.* **2008**, *50*, 395–428. [CrossRef]
35. Stein, E. *Singular Integrals and Differentiability Properties of Functions*; Princeton University Press: Princeton, NJ, USA, 1970.
36. Benedek, A.; Panzone, R. The spaces L^p with mixed norm. *Duke Math. J.* **1961**, *28*, 301–324. [CrossRef]

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