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Global Existence and Uniqueness of Solutions of Integral Equations with Multiple Variable Delays and Integro Differential Equations: Progressive Contractions

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Abstract: In this work, we delve into a nonlinear integral equation (IEq) with multiple variable time delays and a nonlinear integro-differential equation (IDEq) without delay. Global existence and uniqueness (GEU) of solutions of that IEq with multiple variable time delays and IDEq are investigated by the fixed point method using progressive contractions, which are due to T.A. Burton. We prove four new theorems including sufficient conditions with regard to GEU of solutions of the equations. The results generalize and improve some related published results of the relevant literature.

Keywords: GEU of solutions; integral equation; integro-differential equation; variable time delay; fixed point; progressive contraction

MSC: 45D05; 45G10; 45J05; 47H09; 47H10

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1. Introduction

T.A. Burton [1–4] constructed an interesting technique called progressive contractions to investigate GEU of solutions of various kinds of differential equations, integral equations, and integro-differential equations. As a consequence of this technique, researchers avoided the process of establishing existence on a possibly short interval, translating the equation to a later starting time, and then connecting a solution on the new interval to the previous one. Indeed, this technique is very powerful, flexible, and simple for investigating GEU of solutions of integral equations, integro-differential equations, fractional differential equations, etc. (see Burton [1–3,5,6], Burton and Purnaras [4,7,8], Ilea and Otrocol [9,10], and references in these papers).

We will now briefly outline some earlier results with regard to GEU of solutions of various IEqs with and without delay and an IDEq.

In 2017 and 2018, Burton and Purnaras ([7,8]) delved into the following IEqs including a variable delay and without delay:

$$x(t) = L(t) + g(t, x(t)) + \int_0^t A(t-s)[f(s, x(s)) + f(s, x(s - r(s)))]ds \tag{1}$$

and

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds,$$

respectively. The authors obtained some new and attractive results with regard to GEU of solutions of these IEqs according to progressive contractions. In light of the available data of the present literature, for the first time, Burton and Purnaras [7] achieved the application of progressive contractions from IEqs without delay to the delay IEq (1) to obtain global unique solutions. Next, when we convert IEq (1) to the corresponding delay differential equation, the derivative of the function $L(t)$ of IEq (1) can be considered as a nonhomogeneous term.

Later, in 2019, Burton (Theorems 2.1 and 2.2 in [4]) constructed sufficient conditions guaranteeing a unique solution of the above last IEq on $\mathbb{R}^+, \mathbb{R}^+ = [0, \infty)$, by virtue of a simple way of progressive contractions.

In 2020, Ilea and Otrocol [9] delved into the nonlinear IEqs

$$x(t) = \int_0^t K(t, s, x(s)) ds$$

and

$$x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s)) ds.$$

Ilea and Otrocol [9] extended and improved the Burton method to the case where instead of scalar equations, they discussed GEU of solutions of these IEqs in a Banach space.

Recently, Tunç et al. [11] dealt with the following nonlinear Hammerstein-type functional integral equation (HTFIE):

$$x(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))] ds,$$

where $x \in C([0, b], B)$, $t, s \in [0, b]$, $s \le t$, $F, q \in C(B, B)$, $G, H \in C([0, b] \times B, B)$, $K, Q \in C([0, b] \times [0, b] \times B, B)$, $b \in \mathbb{R}$, $b > 0$, and B is a Banach space. Tunç et al. [11] applied and extended Burton's method to this general and nonlinear HTFIE in a Banach space using the Chebyshev norm and complete metric. In Tunç et al. [11], two new results consisting of sufficient conditions have been proved with regard to existence and uniqueness of solutions. Hence, the authors extended Burton's progressive contraction method to general and nonlinear HTFIEs in Banach spaces.

As for some other fixed point results, applications of fixed point methods, etc., one can find several interesting results in the papers of Abbas and Benchohra [12], Banaś and Rzepka [13], Becker et al. [5], Burton and Purnaras [14–16], Burton and Zhang [17], Chauhan et al. [18], Ilea and Otrocol [10], Khan et al. ([19]), Petruşel et al. ([20,21]), Tunç and Tunç ([22–24]), the books of Burton [6], Smart [25], and the references therein. On the other hand, recently, Assari et al. [26] and Assari and Dehghan [27] presented a numerical method for solving logarithmic Fredholm integral equations, which occur as a reformulation of two-dimensional Helmholtz equations over the unit circle with the Robin boundary conditions, and a computational scheme to solve nonlinear logarithmic singular boundary integral equations, which arise from boundary value problems of Laplace equations with nonlinear Robin boundary conditions, respectively.

The first key work for our paper is the paper of Burton and Purnaras [7] and their results (Theorems 2.2 and 2.3 in [7]). In our work, we delve into a more general IEq than an IEq studied for GEU of solutions by progressive contractions in Burton and Purnaras [7]. Indeed, we will delve into a nonlinear IEq with multiple variable time delays. Hence, motivated by the results of Burton [4], Burton and Purnaras ([7,8]), and Ilea and Otrocol [9], we first delve into the following IEq including multiple variable delays:

$$\begin{aligned} x(t) =& q(t) + r(x(t)) + h(t, x(t)) + g(t, x(t)) \int_0^t A(t-s) f(s, x(s)) ds \\ &+ \sum_{i=1}^{N} \int_0^t A_i(t-s) f_i(s, x(s), x(s-\tau_i(s))) ds, \end{aligned} \tag{2}$$

where $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $x \in \mathbb{R}$, $\tau_i \in C(\mathbb{R}^+, (0, \infty))$, $q \in C(\mathbb{R}^+, \mathbb{R})$, $r \in C(\mathbb{R}, \mathbb{R})$, h, $g \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $f_i \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and A, $A_i \in C((0, \infty), \mathbb{R})$, $i = 1, 2, \ldots, N$.

Let $\tau_i(t) \geq \alpha > 0$, $\alpha \in \mathbb{R}$, such that $\alpha = \max\{\alpha_i\}$, $i = 1, 2, \ldots, N$.

It should be noted that qualitative behaviors of solutions such as existence, continuability, and uniqueness of solutions of retarded differential equations, which include multiple constant delays or variable delays, are very attractive concepts in the recent literature. Hence, to the best of our information, it is natural to consider IEq (2) as a new mathematical model.

The first purpose of this paper is to broaden the application of progressive contractions of Burton and Purnaras (Theorems 2.2 and 2.3 in [7]) to obtain GEU of solutions of IEq (2).

Each of the above problems in the papers of Burton [1–4], Burton and Purnaras ([7,8]), Ilea and Otrocol [22], and Tunç et al. [25] is an essentially different type and the title of each paper is chosen to allow interested readers to detect which subject is being treated.

The second key work for our paper is the paper of Burton [1] and his results (Theorems 2.1 and 2.2 in [1]). In 2016, Burton [1] considered a nonlinear scalar IDEq of the form

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds. \tag{3}$$

Burton [1] used the method of "direct fixed point mappings" by considering progressive contractions and obtained some sufficient conditions that guarantee the GEU of solutions of scalar IDEq (3). To the best of our information, we should also state that there is no other paper with regard to GEU of solutions of IDEqs where progressive contractions are used as a basic tool to achieve proofs. In this paper, we secondly delve into a more general IDEq than IDEq (3), which has been studied for GEU of solutions by progressive contractions in the paper of Burton [1]. Hence, motivated by the results of Burton [1], the works mentioned above, and the references of this paper, we will delve into the following IDEq:

$$x'(t) = r(x(t)) + g(t, x(t)) + h(t, x(t)) \int_0^t A(t-s) f(s, x(s)) ds, x(0) = a, a \in \mathbb{R}, \tag{4}$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $r(x(0)) = 0$, $r \in C(\mathbb{R}, \mathbb{R})$, h, $g \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $g(0, x) = 0$, $f \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ and $A \in C((0, \infty), \mathbb{R})$.

The remaining sections of this work include the following contents. Some settings with regard to IEq (2) are put forward in Section 2. Afterwards, Section 3 includes a lemma which extends progressive contractions to nonlinear delay IEq (2), and this section also includes four new theorems as the findings of this paper with regard to IEq (2) and IDEq (4), respectively. As for the remaining two sections, called Sections 4 and 5, they consist of the discussion and conclusion of this work, respectively.

2. The Setting

We now point out some basic assumptions with regard to IEq (2), which will be used in the proofs of the results of this paper.

We assume that

$$\lim_{t\to 0}\int_0^t |A(s)|ds = 0.$$

Let $\beta_1 < 1$ and T be positive constants such that

$$0 < T < \alpha \ \text{ and } \ r_0 + h_0 + f_0 g_0 \int_0^T |A(s)|ds < \beta_1. \tag{5}$$

For $E > 0$, there exist H_i, $i = 1, 2, \ldots, N$, such that $-H_i \leq t - \tau_i(t)$, $0 \leq t \leq E$, where $H_i \in \mathbb{R}$, $H_i > 0$. Let $H = \max(H_i)$. Then $-H \leq t - \tau_i(t)$.

Hence, there is an initial function $\omega \in C([-H, 0], \mathbb{R})$ with $\omega(0) = q(0)$ such that

$$x(t - \tau_i(t)) = \omega(t - \tau_i(t)), -H \leq t - \tau_i(t) \leq 0, i = 1, 2, \ldots, N. \tag{6}$$

Progressive contractions allow the following conditions: K can be $f_0 g_0$, and H grows while E grows and is unbounded when α tends to zero.

We will divide the interval $[0, E]$ into n equal splits such that the length of each part is denoted by S, $0 < S < T$, and we represent the terminal marks by

$$0 = T_0 < T_1 < T_2 < \ldots < T_n = E$$

with $T_i - T_{i-1} = S$ and $nS = T$. Next, on each n equal segment, the mapping derived from IEq (2) will be a contraction giving a unique segment of the solution of IEq (2) and each of these segments will allow us to ignore $f_i(t, x(t), x(t - \tau_i(t)))$, $i = 1, 2, \ldots, N$ in the future contractions steps.

3. The Main Results

We will turn now to our main results with regard to GEU of solutions and we will call the method of proofs a progressive contraction; see Burton [1–4]. Next, the basic information with regard to complete metric space of this paper can be found in Burton [4] and Ilea and Otrocol [9], respectively.

The following Lemma 1 extends progressive contractions to nonlinear IEqs including multiple variable time delays.

Lemma 1. *If $T_{i-1} \leq t \leq T_i$ and $\phi(t) = \psi(t)$ for $-H \leq t \leq T_i$, then*

$$f_i(t, \phi(t), \phi(t - \tau_i(t))) - f_i(t, \psi(t), \psi(t - \tau_i(t))) = 0, i = 1, 2, \ldots, N. \tag{7}$$

Proof. For $T_{i-1} \leq t \leq T_i$, we have the relations:

$$t - \tau_i(t) \leq t - \alpha < T_i - T < T_i - S = T_{i-1}, i = 1, 2, \ldots, N.$$

Hence, it follows that the arguments of (7) are equal. This result completes the proof of Lemma 1.

The first result of this paper is given in Theorem 1. □

Theorem 1. *Let f_0, g_0, h_0, r_0, α, β_1, E, H, and T be positive constants such that the below conditions (As1) and (As2) hold:*
(As1)

$$\tau_i \in C\big(\mathbb{R}^+, (0, \infty)\big), q \in C\big(\mathbb{R}^+, \mathbb{R}\big), \omega \in C([-H, 0], \mathbb{R}),$$
$$r(x(0)) = 0, r \in C(\mathbb{R}, \mathbb{R}), h,\ g \in C\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big),$$
$$g(0, x) = 0, f \in C\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big), f_i \in C\big(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big), i = 1, 2, \ldots, N,$$

$$
\begin{aligned}
&\tau_i(t) \geq \alpha, \forall t \in \mathbb{R}^+, \\
&-H \leq t - \tau_i(t) \leq 0, \forall t \in [0, E], \\
&q(0) = \omega(0), \\
&x(t - \tau_i(t)) = \omega(t - \tau_i(t)), \\
&|r(x) - r(y)| \leq r_0|x - y|, \forall x,\ y \in \mathbb{R}\,, \\
&|h(t, x) - h(t, y)| \leq h_0|x - y|, \forall t \in \mathbb{R}^+, \forall x,\ y \in \mathbb{R}\,, \\
&|g(t, x)| \leq g_0, \forall t \in \mathbb{R}^+, \forall x \in \mathbb{R}, \\
&|f(t, x) - f(t, y)| \leq f_0|x - y|, \forall t \in \mathbb{R}^+, \forall x,\ y \in \mathbb{R}\,;
\end{aligned}
$$

(As2)

$$
\begin{aligned}
&A,\ A_i \in C((0, \infty), \mathbb{R}), i = 1, 2, \ldots, N, \\
&\lim_{t \to 0} \int_0^t |A(s)| ds = 0, \\
&0 < T < \alpha \text{ and } r_0 + h_0 + f_0 g_0 \int_0^T |A(s)| ds < \beta_1 < 1.
\end{aligned}
$$

Then, for every $E > 0$*, IEq* (2) *with multiple variable time delays admits a unique solution on* $[0, E]$.

Proof. We will divide the interval $[0, E]$ into n equal splits such that the length of each part is denoted by S, $S < T$, and we represent the terminal points by

$$0 = T_0, T_1,\ T_2, \ldots, T_n = E.$$

We will provide the proof step by step as in the following steps, Steps (1a)–(3a), respectively.

Step (1a). Let $(\aleph_1,\ \|.\|_1)$ be a complete metric space including the functions $\phi \in C([-H, T_1], \mathbb{R})$ with the supremum metric and $\phi(t) = \omega(t)$, $t \in [-H_1, 0]$. We define a transformation

$$P_1 : \aleph_1 \to \aleph_1,\ \phi \in \aleph_1 \text{ and } t \in [-H_1, 0],$$

which implies that $(P_1\phi)(t) = \omega(t)$. Next, let $t \in (0, T_1]$. Then, we have

$$
\begin{aligned}
(P_1\phi)(t) =& q(t) + r(\phi(t)) + h(t, \phi(t)) + g(t, \phi(t)) \int_0^t A(t - s) f(s, \phi(s)) ds \\
&+ \sum_{i=1}^{N} \int_0^t A_i(t - s) f_i(s, \phi(s), \phi(s - \tau_i(s))) ds.
\end{aligned}
$$

Since $\omega(0) = q(0)$ in (6), $(P_1\phi)$ is continuous. After that, letting $\phi,\ \psi \in \aleph_1$, $t \in [-H, T_1]$, using (As1), (As2), and Lemma 1, we derive that

$$
\begin{aligned}
|(P_1\phi)(t)-(P_1\psi)(t)| \leq & |r(\phi(t))-r(\psi(t))|+|h(t,\phi(t))-h(t,\psi(t))| \\
& + g_0\int_0^t |A(t-s)|\,|f(s,\phi(s))-f(s,\psi(s))|ds \\
& + \sum_{i=1}^{N}\int_0^t |A_i(t-s)|\,|f_i(s,\phi(s),\phi(s-\tau_i(s)))-f_i(s,\psi(s),\psi(s-\tau_i(s)))|ds \\
\leq & (r_0+h_0)|\phi(t)-\psi(t)| \\
& + g_0\int_0^t |A(t-s)|\,|f(t,s,\phi(s))-f(t,s,\psi(s))|ds \\
\leq & (r_0+h_0)|\phi(t)-\psi(t)|+f_0g_0\int_0^t |A(t-s)|\,|\phi(s)-\psi(s)|ds \\
\leq & (r_0+h_0)|\phi(t)-\psi(t)|_1+f_0g_0|\phi(t)-\psi(t)|_1\int_0^{T_1}|A(s)|\,ds \\
= & \left(r_0+h_0+f_0g_0\int_0^{T_1}|A(s)|ds\right)|\phi(t)-\psi(t)|_1 \\
< & \beta_1|\phi-\psi|_1,
\end{aligned}
$$

which is a contraction with a unique fixed point ξ_1 on the interval $[-H,T_1]$, and for $t\in[0,T_1]$ it satisfies that

$$
\begin{aligned}
(P_1\xi_1)(t) = & \xi_1(t) = q(t)+r(\xi_1(t)+h(t,\xi_1(t)) \\
& + g(t,\xi_1(t))\int_0^t A(t-s)f(s,\xi_1(s))ds \\
& + \sum_{i=1}^{N}\int_0^t A_i(t-s)f_i(s,\xi_1(s),\xi_1(s-\tau_i(s)))ds.
\end{aligned}
$$

We note that $\xi_1(0)=q(0)$ and $\xi_1(t)=\omega(t)$, $t\in[-H,0]$.

Step (2a). Let $(\aleph_2,\ \|.\|_2)$ be the complete metric space, which includes the functions $\phi\in C([-H,T_2],\mathbb{R})$ with the supremum metric such that

$$\phi(t)=\xi_1 \text{ on } [-H,T_1].$$

We define the mapping $P_2:\aleph_2\to\aleph_2$ with $\phi\in\aleph_2$ and $t\in[-H,T_1]$, which implies that $(P_2\phi)(t)=\xi_1(t)$, and when $t\in(T_1,T_2]$, it implies that

$$
\begin{aligned}
(P_2\phi)(t) = & q(t)+r(\phi(t))+h(t,\phi(t))+g(t,\phi(t))\int_0^t A(t-s)f(s,\phi(s))ds \\
& + \sum_{i=1}^{N}\int_0^t A_i(t-s)f_i(s,\phi(s),\phi(s-\tau_i(s)))ds. \qquad (8)
\end{aligned}
$$

We will now prove that $P_2\phi$ is continuous for all $t\in[-H,T_2)$. Since $(P_2\phi)(t)=\xi_1(t)$ on the interval when $t\in[-H,T_1]$, the operator P_2 is continuous for all $t\in[-H,T_1)$. Next, since the functions $q,\ r, h,\ g, f, f_i, A$ and A_i are continuous, $P_2\phi$ is also continuous for all

$t \in (T_1, T_2]$. Hence, it only remains to verify that $P_2\phi$ is continuous at end point T_1. Next, for $t = T_1$, from (11) we can derive that

$$\begin{aligned}(P_2\phi)(T_1) =&\xi_1(T_1) = q(T_1) + r(\phi(T_1)) + h(T_1, \phi(T_1)) \\ &+ g(T_1, \phi(T_1)) \int_0^{T_1} A(T_1 - s) f(s, \phi(s)) ds \\ &+ \sum_{i=1}^{N} \int_0^{T_1} A_i(T_1 - s) f_i(s, \phi(s), \phi(s - \tau_i(s))) ds = \lim_{t \downarrow T_1} (P_2\phi)(t).\end{aligned}$$

Hence, $P_2\phi$ agrees with ξ_1 on $[-H, T_1]$ (by definition) and it is also continuous on the entire interval $[-H, T_2]$. This means that $P_2 : \aleph_2 \to \aleph_2$ is continuous on $[-H, T_2]$.

We now need a change of variable for $T_1 \le t \le T_2$. Hence, from (5) we have

$$r_0 + h_0 + f_0 g_0 \int_{T_1}^{t} |A(t - s)| ds < \beta_1.$$

Letting ϕ, $\psi \in \aleph_2$, using $\phi(t) = \psi(t) = \xi_1(t)$ on $[-H, T_1]$, Lemma 1, and later taking $t > T_1$, we obtain

$$\begin{aligned}|(P_2\phi)(t) - (P_2\psi)(t)| \le& |r(\phi(t)) - r(\psi(t))| + |h(t, \phi(t)) - h(t, \psi(t))| \\ &+ g_0 \int_0^t |A(t - s)| \, |f(s, \phi(s)) - f(s, \psi(s))| ds \\ &+ \sum_{i=1}^{N} \int_0^t |A_i(t - s)| \, |f_i(s, \phi(s), \phi(s - \tau_i(s))) - f_i(s, \psi(s), \psi(s - \tau_i(s)))| ds \\ \le& (r_0 + h_0) |\phi(t) - \psi(t)|_2 \\ &+ f_0 g_0 \int_0^t |A(t - s)| \, |\phi(s) - \psi(s)| ds \\ \le& (r_0 + h_0) |\phi(t) - \psi(t)|_2 + f_0 g_0 |\phi(t) - \psi(t)|_2 \int_{T_1}^{t} |A(t - s)| \, ds \\ =& \left(r_0 + h_0 + f_0 g_0 \int_{T_1}^{t} |A(t - s)| ds \right) |\phi(t) - \psi(t)|_2 \\ <& \beta_1 |\phi - \psi|_2,\end{aligned}$$

which is a contraction on the interval $[-H, T_2]$ with a unique fixed point ξ_2 on the entire interval $[-H, T_2]$. Hence, it follows that ξ_2 is a unique solution of IEq (2) on $[0, T_2]$. It also agrees with ξ_1 on the entire interval $[-H, T_1]$ by contraction.

Step (3a). Assume that $(\aleph_3, \|.\|_3)$ is the complete metric space, which includes the functions $\phi \in C([-H, T_3], \mathbb{R})$ with the supremum metric and

$$\phi(t) = \xi_2 \text{ on } [-H, T_2].$$

We define the mapping

$$P_3 : \aleph_3 \to \aleph_3 \text{ with } \phi \in \aleph_3,$$

which implies that

$$(P_3\phi)(t) = q(t) + r(\phi(t)) + h(t,\phi(t)) + g(t,\phi(t)) \int_0^t A(t-s) f(s,\phi(s)) ds$$
$$+ \sum_{i=1}^{N} \int_0^t A_i(t-s) f_i(s,\phi(s),\phi(s-\tau_i(s))) ds.$$

We note that $\phi(t) = \xi_2$ is a fixed point of P_3 on $[-H, T_2]$. Next, similarly as in Step (2a), according to (As1), (As2), and Lemma 1, we can obtain a continuous function ξ_3 on the interval $[0, T_3]$.

Afterwards, using mathematical induction, we will obtain a unique solution on the interval $[0, E]$, and while this is sufficient for a complete understanding, here the induction details are given in the following.

For the case $2 < i \le n-1$, let ξ_{i-1} be the unique solution of IEq (2) on the interval $[0, T_{i-1}]$. Next, let $(\aleph_i, \|.\|_i)$ be the complete metric, which includes the functions $\phi \in C([-H, T_i], \mathbb{R})$ with the supremum metric such that

$$\phi(t) = \xi_{i-1} \text{ on } [-H, T_{i-1}].$$

We define the mapping

$$P_i : \aleph_i \to \aleph_i \text{ with } \phi \in \aleph_i,$$

which implies that $(P_i\phi(t)) = \xi_{i-1}$ on $[-H, T_{i-1}]$, and when $t \in [0, T_i]$, let

$$(P_i\phi)(t) = q(t) + r(\phi(t)) + h(t,\phi(t)) + g(t,\phi(t)) \int_0^t A(t-s) f(s,\phi(s)) ds$$
$$+ \sum_{i=1}^{N} \int_0^t A_i(t-s) f_i(s,\phi(s),\phi(s-\tau_i(s))) ds.$$

The continuity of the function $P_i\phi$ can be shown as in Step (2a).We will now prove that the operator P_i is a contraction. Then, letting $\phi,\ \psi \in \aleph_i$ and $t \in [-H, T_i]$, according to (As1), (As2), and Lemma 1, we obtain

$$|(P_i\phi)(t) - (P_i\psi)(t)| \le (r_0 + h_0)|\phi(t) - \psi(t)| + f_0 g_0 \int_0^t |A(t-s)|\, |\phi(s) - \psi(s)| ds.$$

We note that $\phi(t) = \psi(t) = \xi_{i-1}$ on $[0, T_{i-1}]$. Hence, T_{i-1} is the lower limit for the next step. Then, letting $t > T_{i-1}$ and using a change of variable as given above, we derive that

$$\begin{aligned} |(P_i\phi)(t) - (P_i\psi)(t)| &\le \left(r_0 + h_0 + f_0 g_0 \int_{T_{i-1}}^{t} |A(t-s)| ds \right) |\phi(t) - \psi(t)|_i \\ &\le \left(r_0 + h_0 + f_0 g_0 \int_0^{T_i} |A(s)| ds \right) |\phi(t) - \psi(t)|_i \\ &< \beta_1 |\phi - \psi|_i, \end{aligned}$$

which is a contraction including a unique fixed point ξ_i on $[-H, T_i]$. This result is the end of the proof of Theorem 1.

For the next result, let $K = f_0 g_0$. We note that if $E \to \infty$, then the present $K \in \mathbb{R}$ may also tend to infinity. We still establish from the relation that as K increases, T decreases. This process operates for any $E > 0$. This is an important idea for the following result such that we can take $E \to \infty$ and always obtain a solution on $[0, E]$. □

We will now prove that a well-defined function on the interval $[0, \infty)$ can be chosen such that it is a unique solution of IEq (2) and also involves no translations or unfinished steps on the road to a solution on the interval $[0, \infty)$. Hence, the second result of this paper is presented in Theorem 2.

Theorem 2. *We assume that (As1) and (As2) of Theorem 1 with $\tau_i(t) > 0$, $i = 1, 2, \dots, N$ hold. Then, there is a unique solution of IEq (2) on the interval $[0, \infty)$.*

Proof. In light of (As1), (As2), and Lemma 1, following the proof of Theorem 1, we can obtain a sequence of uniformly continuous functions on the interval $[0, \infty)$, which converges uniformly on compact sets to a continuous function such that this function is the unique solution of IEq (2).

In fact, for each positive n, we benefit from Theorem 1 to obtain a solution of IEq (2) on the interval $[0, n]$. For the next step, we denote by $x_n(t)$ the solution on the interval $[0, n]$, which is extended to a function on the interval $[0, \infty)$ such that $x_n(t) = x_n(n)$, $t \geq n$. Then, the sequence (x_n) converges uniformly to a continuous function $x(t)$, which is a solution of IEq (2), since at every t, the function $x(t)$ agrees with a solution $x_n(t)$, $n > t$. Thus, the proof of Theorem 2 is completed. □

We will now give our main results with regard to IDEq (4). The third result of this paper is given in Theorem 3.

Theorem 3. *We assume that the following conditions hold for IDEq (4):*
(C1) Let $r \in C(\mathbb{R}, \mathbb{R})$, $h,\ g \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, and for each $E > 0$, there are positive constants $r_L = r_L(E)$, $g_L = g_L(E)$, $h_L = h_L(E)$, $f_L = f_L(E)$ such that

$$|r(x) - r(y)| \leq r_L |x - y|, \forall x,\ y \in \mathbb{R},$$

$$|g(t, x) - g(t, y)| \leq g_L |x - y|, \forall t \in [0, E], \forall x,\ y \in \mathbb{R},$$
$$|h(t, x)| \leq h_L, \forall t \in [0, E], \forall x \in \mathbb{R},$$
$$|f(t, x) - f(t, y)| \leq f_L |x - y|, \forall t \in [0, E], \forall x,\ y \in \mathbb{R};$$

(C2) Let
$$A \in C((0, \infty), \mathbb{R}), \phi \in C\big(\mathbb{R}^+, \mathbb{R}\big),$$
which imply that the integrals

$$\int_0^t A(t - s)\phi(s) ds \ \ and \ \int_0^t |A(s)| ds$$

are continuous, respectively, and the last integral converges to zero as $t \to 0$, and for the constants E, r_L, g_L, h_L, and f_L, pick $\alpha \in (0, 1)$ then choose a positive constant $T^* < 1$ and let $T \in \mathbb{R}$, $T > 0$ such that

$$0 < T < T^* < 1, (r_L + g_L) T^* < \alpha, (h_L f_L) \int_0^T |A(s)| ds < \frac{1 - \alpha}{2}. \tag{9}$$

Then, for every $E > 0$ and $a \in \mathbb{R}$, IDEq (4) admits a unique solution on $[0, E]$.

We will now start with a solution on $[0, E]$ and extend it to $[0, \infty)$.

Proof. For the given $E > 0$, let positive constants r_L, g_L, h_L, and f_L be positive constants satisfying (C1) and (C2), when T satisfies (9) with

$$(r_L + g_L)T^* < \alpha < 1, 0 < T < T^* < 1.$$

We will divide the interval $[0, E]$ into n equal pieces such that the length of each piece is denoted by S, $S < T$, and we represent the terminal points of that pieces by

$$0 = T_0, T_1,\ T_2, \ldots,\ T_n = E$$

such that

$$S = T_i - T_{i-1} < T < 1.$$

We will take two steps leading to a mathematical induction which generalizes the second step. The first step takes place in a Banach space; however, the subsequent step takes place in a complete metric space.

Step (1b). We assume that $(\Xi_1,\ \|.\|_1)$ is a Banach space, which includes the functions $\phi \in C([0, T_1], \mathbb{R})$ with the supremum metric. We define a transformation

$$P_1 : \Xi_1 \to \Xi_1 \ \text{ with } \phi \in \Xi_1,$$

which implies that

$$\begin{aligned}(P_1\phi)(t) =& r\left(a + \int_0^t \phi(s)ds\right) + g\left(t, a + \int_0^t \phi(s)ds\right) \\ &+ h\left(t, a + \int_0^t \xi(s)ds\right) \int_0^t A(t-s) f\left(s, a + \int_0^t \xi(s)ds\right) ds.\end{aligned}$$

If the operator P_1 has a fixed point such as ξ_1, then

$$\frac{d}{dt}\left(a + \int_0^t \xi_1(s)ds\right) = \xi_1(t)$$

and

$$x(t) = a + \int_0^t \xi_1(s)ds$$

satisfies IDEq (4) with $x(0) = a$.

We will now show that the operator P_1 is a contraction. First, letting $\phi,\ \psi \in \Xi_1$ and using (9), we have

$$\int_0^t |\phi(s) - \psi(s)|ds \le T^*|\phi - \psi|_1 \le |\phi - \psi|_1, 0 < T < T^* < 1.$$

Hence, according to the results above, (C1) and (C2), we derive that

$$\begin{aligned}|(P_1\phi)(t) - (P_1\psi)(t)| \le& r_L\left|a + \int_0^t \phi(s)ds - a - \int_0^t \psi(s)ds\right| \\ &+ g_L\left|a + \int_0^t \phi(s)ds - a - \int_0^t \psi(s)ds\right|\end{aligned}$$

$$
\begin{aligned}
&+ (h_L f_L) \int_0^t |A(t-s)| \int_0^s |\phi(s) - \psi(s)| du ds \\
\leq &(r_L + g_L) T^* |\phi(t) - \psi(t)|_1 \\
&+ (h_L f_L) |\phi - \psi|_1 \int_0^t |A(s)| ds \\
\leq &|\phi - \psi|_1 \left(\alpha + \frac{1-\alpha}{2} \right) = \frac{(1+\alpha)}{2} |\phi - \psi|_1,
\end{aligned}
$$

which is a contraction including a unique fixed point ξ_1 on the interval $[0, T_1]$.

Step (2b). We assume that $(\Xi_2, \ \|.\|_2)$ is a complete metric space, which includes the functions $\phi \in C([T_0, T_2], \mathbb{R})$ with the supremum metric and $\phi(t) = \xi_1(t)$ for $t \in [T_0, T_1]$. We define a transformation

$$
P_2 : \Xi_2 \to \Xi_2 \ \text{ with } \phi \in \Xi_2,
$$

which implies that

$$
\begin{aligned}
(P_2\phi)(t) = &r\left(a + \int_0^t \phi(s) ds \right) + g\left(t, a + \int_0^t \phi(s) ds \right) \\
&+ h\left(t, a + \int_0^t \phi(s) ds \right) \int_0^t A(t-s) f\left(s, a + \int_0^s \phi(u) du \right) ds.
\end{aligned}
$$

Since ξ_1 is a fixed point of P_1 on $[T_0, T_1]$ for $0 \leq t \leq T_1$, we obtain for any $\phi \in \Xi_2$ that

$$
\begin{aligned}
(P_2\phi)(t) = &r\left(a + \int_0^t \xi_1(s) ds \right) + g\left(t, a + \int_0^t \xi_1(s) ds \right) \\
&+ h\left(t, a + \int_0^t \xi_1(s) ds \right) \int_0^t A(t-s) f\left(t, s, a + \int_0^s \xi_1(u) du \right) ds = \xi_1(t).
\end{aligned}
$$

Hence, P_2 is a map from Ξ_2 to Ξ_2.

We will now show that P_2 is a contraction. Then, letting $\phi, \ \psi \in \Xi_2$ and using (C1), (C2), we have

$$
\begin{aligned}
|(P_2\phi)(t) - (P_2\psi)(t)| \leq &r_L \left| \int_0^t \phi(s) ds - \int_0^t \psi(s) ds \right| \\
&+ g_L \left| \int_0^t \phi(s) ds - \int_0^t \psi(s) ds \right| \\
&+ (h_L f_L) \int_0^t |A(t-s)| \int_0^s |[\phi(u) - \psi(u)] du| ds. \qquad (10)
\end{aligned}
$$

Let $t \in [T_1, T_2]$ and fix s at any value in $[0, T_1]$. Since $0 \le s \le T_1$, $u \in [0, T_1]$ and $\phi(u) = \psi(u)$. Hence, we have $\int_0^s |\phi(u) - \psi(u)|du = 0$. This result holds for every value of s in $0 \le s \le T_1$. Next, if $|\phi|^{[T_1,T_2]}$ denotes the sup with $S = T_2 - T_1 < T^*$, then

$$\int_{T_1}^{T_2} |\phi(s) - \psi(s)|ds = T^*|\phi - \psi|^{[T_1,T_2]} \le |\phi - \psi|^{[T_1,T_2]} = |\phi - \psi|_2.$$

Hence, using the above discussion, by changing a variable and $|\phi - \psi|_2 = |\phi - \psi|^{[T_1,T_2]}$, we obtain from (10) that

$$\begin{aligned} |(P_2\phi)(t) - (P_2\psi)(t)| \le & r_L\left|\int_{T_1}^{t} [\phi(s) - \psi(s)]ds\right| + g_L\left|\int_{T_1}^{t} [\phi(s) - \psi(s)]ds\right| \\ & + (h_L f_L)\int_{T_1}^{t} |A(t-s)| \int_{T_1}^{s} |\phi(u) - \psi(u)|duds \\ \le & (r_L + g_L)T^*|\phi - \psi|^{[T_1,T_2]} + (h_L f_L)\int_{T_1}^{t} |A(t-s)||\phi - \psi|^{[T_1,T_2]}ds \\ \le & |\phi - \psi|_2\left(\alpha + \frac{1-\alpha}{2}\right) = \frac{(1+\alpha)}{2}|\phi - \psi|_2, \end{aligned}$$

which is a contraction including a unique fixed point ξ_2 on the interval $[0, T_2]$. We also note that $\xi_1 = \xi_2$ on $[0, T_1]$, since both are unique and the definition of the space demands it.

As for Step (3b), we note that $\phi(t) = \xi_2(t)$ is a fixed point of P_3 on $[0, T_2]$. Next, as in Step (2b), we can obtain a continuous function ξ_3 on the interval $[0, T_3]$. The remaining of the mathematical calculations of Step (3b) are similar to that of Step (2b). We ignore the details of the calculations for this step.

Hence, using mathematical induction, we would obtain a unique solution on the interval $[0, E]$. While this is sufficient for a complete understanding, here the induction details are given in the following lines.

For the case $2 < i < n-1$, let ξ_{i-1} be the unique solution of IDEq (4) on the interval $[0, T_{i-1}]$ for $i \ge 2$. Next, let $(\Xi_i, \ \|.\|_i)$ be the complete metric, which includes the functions $\phi \in C([0, T_i], \mathbb{R})$ with the supremum metric such that

$$(P_i\phi(t)) = \xi_{i-1}(t) \ \text{ for } t \in [0, T_{i-1}].$$

We define the mapping

$$P_i : \Xi_i \to \Xi_i \text{ with } \phi \in \Xi_i$$

such that

$$\begin{aligned} (P_i\phi)(t) = & r\left(a + \int_0^t \phi(s)ds\right) + g\left(t, a + \int_0^t \phi(s)ds\right) \\ & + h\left(t, a + \int_0^t \phi(s)ds\right)\int_0^t A(t-s)f\left(s, a + \int_0^t \phi(u)du\right)ds. \end{aligned}$$

Since ξ_{i-1} is a solution on $[0, T_{i-1}]$ when $0 \le t \le T_{i-1}$, $(P_i\xi_{i-1})(t) = \xi_{i-1}(t)$, and so the mapping is into Ξ_i.

We will now show that P_i is a contraction. Then, letting $\phi,\ \psi \in \Xi_i$ and using (C1), (C2), we have

$$\begin{aligned}|(P_i\phi)(t)-(P_i\psi)(t)| \leq & r_L\left|\int_0^t \phi(s)ds - \int_0^t \psi(s)ds\right| \\ & + g_L\left|\int_0^t \phi(s)ds - \int_0^t \psi(s)ds\right| \\ & + (h_L f_L)\int_0^t |A(t-s)|\int_0^s |\phi(u)-\psi(u)|duds.\end{aligned}$$

Following a similar procedure as in Step (2b), we have $\int_0^{T_{i-1}} |\phi(u)-\psi(u)|du = 0$. This result is true for every value of s in $0 \leq s \leq T_{i-1}$. Next, if $|\phi|^{[T_{i-1},T_i]}$ denotes the sup with $s = T_i - T_{i-1} < T^*$, then

$$\int_{T_{i-1}}^{T_i} |\phi(s)-\psi(s)|ds = T^*|\phi-\psi|^{[T_{i-1},T_i]} \leq |\phi-\psi|^{[T_{i-1},T_i]} = |\phi-\psi|_i.$$

Hence, using the above discussion, by changing a variable and $|\phi-\psi|_i = |\phi-\psi|^{[T_{i-1},T_i]}$, we obtain

$$\begin{aligned}|(P_i\phi)(t)-(P_i\psi)(t)| \leq & r_L\left|\int_{T_{i-1}}^t [\phi(s)-\psi(s)]ds\right| + g_L\left|\int_{T_{i-1}}^t [\phi(s)-\psi(s)]ds\right| \\ & + (h_L f_L)\int_{T_{i-1}}^t |A(t-s)|\int_{T_{i-1}}^s |\phi(u)-\psi(u)|duds \\ \leq & (r_L+g_L)T^*|\phi-\psi|^{[T_{i-1},T_i]} \\ & + (h_L f_L)\int_{T_{i-1}}^t |A(s)||\phi(s)-\psi(s)|^{[T_{i-1},T_i]}ds \\ \leq & |\phi-\psi|_i\left(\alpha+\frac{1-\alpha}{2}\right) = \frac{(1+\alpha)}{2}|\phi-\psi|_i,\end{aligned}$$

which is a contraction including a unique fixed point ξ_i on the interval $[0, T_i]$. We also note that $\xi_{i-1} = \xi_i$ on $[0, T_{i-1}]$, since both are unique and the definition of the space demands it. This is the final step of the proof. □

Example 1. *Consider the following IDEq:*

$$x' = \sin x + \frac{1}{1+t^2}\sin x + \frac{1}{1+t^2+x^2}\int_0^t \exp[-(t-s)]\frac{x(s)}{1+s^4}ds, \tag{11}$$

where x' and x denote $x'(t)$ and $x(t)$, respectively.

We note that IDEq (11) is in the form of IDEq (4), with the data as follows:

$$r(x) = \sin x,$$

$$g(t,x) = \frac{1}{1+t^2}\sin x,$$
$$h(t,x) = \frac{1}{1+t^2+x^2},$$
$$f(t,x) = \frac{x}{1+t^4},$$
$$A(t-s) = \exp[-(t-s)],\ 0 \le s \le t.$$

Now we will check the assumptions (C1) and (C2) of Theorem 3. We verify that (C1) and (C2) hold. For this, we let $r_L = 1$, $g_L = 1$, $h_L = 1$, $f_L = 1$ and calculate

$$|r(x) - r(y)| = |\sin(x) - \sin(y)|$$
$$= 2\left|\frac{\cos(x+y)}{2}\right|\left|\frac{\sin(x-y)}{2}\right|$$
$$\le |x-y|, \forall x,\ y \in \mathbb{R},$$
$$|g(t,x) - g(t,y)| = \frac{1}{1+t^2}|\sin(x) - \sin(y)|$$
$$\le |\sin(x) - \sin(y)|$$
$$\le |x-y|, \forall t \in [0,E], \forall x,\ y \in \mathbb{R},$$
$$|h(t,x)| = \frac{1}{1+t^2+x^2} \le 1, \forall t \in [0,E], \forall x \in \mathbb{R},$$
$$|f(t,x) - f(t,y)| = \frac{1}{1+t^4}|x-y| \le |x-y|, \forall t \in [0,E], \forall x,\ y \in \mathbb{R},$$
$$\int_0^t |A(s)|ds = \int_0^t \exp(\text{-}s)ds = 1 - \exp(-t),$$
$$1 - \exp(-t)\ \text{ is continuous and } 1 - \exp(-t)\ \text{ converges to zero as } t \to 0.$$

Hence, (C1) and (C2) hold. Thus, the application of the result of Theorem 3 is valid.

Finally, our last result with regard to GEU of solutions of IDEq (4) is given in Theorem 4.

Theorem 4. *If (C1) and (C2) of Theorem 3 hold, then there is a unique solution ξ of IDEq (4) on $\mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$.*

Proof. Using (C1), (C2) and pursuing the proof of Theorem 3, we can construct a unique solution ξ_n on every interval $[0,n]$ for every positive integer n. As the next step, we extend each of the solutions to the interval $[0,\infty)$ by defining functions ξ_n past n such that ${\xi_n}^* = \xi_n(n)$ for $t > n$. Then, we have a sequence of uniformly continuous functions such as (ξ_n). Hence, this sequence converges uniformly on compact sets to a continuous function ξ on $\mathbb{R}^+$, which is a solution of IDEq (4), since at every value of t, the function ξ on $[0,t]$ coincides with any ξ_n for $n > t$. Thus, the proof of Theorem 4 is completed. □

4. Discussion

We will now present some information with regard to the results of this article.

(1^0) According to the related published results in the relevant literature (see Burton [1–4], Burton and Purnaras ([7,8]), Ilea and Otrocol [22]), the GEU of solutions of nonlinear IEq (2) and IDEq (4) has not been discussed in the relevant literature up to now. Hence, IEq (2) and IDEq (4) are new mathematical models to investigate the GEU of solutions of them. Therefore, this article includes new results with regard to the GEU of solutions of IEq (2) and IDEq (4).

(2^0) In this article, for the first time, we verified the GEU of solutions of an IEq having N-multiple variable time delays such that we extended the progressive contraction approach, which is due to T.A. Burton, for an IEq with the N-multiple variable time delays.

(3^0) We improved Lemma 1 to apply progressive contractions to nonlinear IEqs including multiple variable delays. This is a new contribution to IEqs with multiple variable time delays.

(4^0) If $A_i(t-s) = A(t-s)$, $f_i(s, x(s), x(s-\tau_i(s)) = f(s, x(s), x(s-r(s)))$, $g(t, x(t)) = 1$, $r(x(t)) = 0$, and $q(t) = 0$, then IEq (2) can be reduced to IEq (1) of Burton and Purnaras [7]. In this case, the conditions of Theorem 1 and Theorem 2 can be converted to that of Theorem 2.1 and Theorem 2.2 of Burton and Purnaras [7], respectively.

(5^0) If $A_i(t-s) = 0$, $f_i(s, x(s), x(s-\tau_i(s)) = 0$, $g(t, x(t)) = 1$, $r(x(t)) = 0$ and $q(t) = 0$, then IEq (2) can be reduced to the IEq of Burton and Purnaras [8]. Similar results as in 4^0 can also be obtained for Theorem 2.1 of Burton and Purnaras [8].

(6^0) Finally, as for the possible open problems in future advancements, GEU of solutions of delay IEq (2) with Caputo fractional derivative, and IDEq (4) with Caputo fractional derivative or with Riemann–Liouville fractional derivative are suggested for future work.

5. Conclusions

In this article, a nonlinear IEq including multiple variable time delays and a nonlinear IDEq without delay has been considered. GEU of solutions of the delay IEq and IDEq without delay have been investigated. We have constructed sufficient conditions through four new theorems with regard to GEU of solutions of the delay IEq and IDEq without delay. The technique of the proofs of the theorems is based on a fixed point method consisting of progressive contractions, which is due to T.A. Burton. The new results of this article generalize and improve the results of Burton and Purnaras (Theorems 2.2 and 2.3 in [7]) from the case with one variable time delay to the more general case and N-times multiple variable time delays (see Theorems 1 and 2). Next, the other new results of this article, called Theorems 3 and 4, improve and include the results of Burton (Theorems 2.1 and 2.2 in [1]).

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