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Introducing the Third-Order Fuzzy Superordination Concept and Related Results

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Abstract: Third-order fuzzy differential subordination studies were recently initiated by developing the main concepts necessary for obtaining new results on this topic. The present paper introduces the dual concept of third-order fuzzy differential superordination by building on the known results that are valid for second-order fuzzy differential superordination. The outcome of this study offers necessary and sufficient conditions for determining subordinants of a third-order fuzzy differential superordination and, furthermore, for finding the best subordinant for such fuzzy differential superordination, when it can be obtained. An example to suggest further uses of the new outcome reported in this work is enclosed to conclude this study.

Keywords: fuzzy set; third-order fuzzy differential subordination; third-order fuzzy differential superordination; fuzzy subordinant; best fuzzy subordinant

MSC: 30C45; 30A10



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1. Introduction

Examples of differential inequalities that can be used to generate constraints on a function using its derivatives of a given order are frequently seen in the theory of differential equations. A new theory known as differential subordination theory was developed when the concepts concerning differential inequalities for real functions were extended to complex functions. S.S. Miller and P.T. Mocanu proposed this theory in two papers that were published in 1978 [1] and 1981 [2]. The dual notion of differential superordination was introduced by the same authors in 2003 [3].

The present paper deals with the dual concept of the special form of fuzzy differential subordination, namely, fuzzy differential superordination. The fuzzy differential subordination concept was introduced in 2012 [4], embedding the concept of the fuzzy set introduced by Lotfi A. Zadeh in 1965 [5]. The theory of fuzzy differential subordination follows the general theory of differential subordination as seen in [6]. Geometric function theory scholars quickly adopted the concept, and all of the established research lines in this field have started to take into account the new fuzzy elements. The first published works on this topic were included in the references of a review paper released in 2017 [7], supporting the topic's development. Following the ideas of the classical theory of differential superordination initiated by S.S. Miller and P.T. Mocanu in [3], the dual notion of fuzzy differential superordination was further developed in 2017 [8]. The idea of a fuzzy set was one that mathematicians were eager to incorporate into their work, and numerous mathematical fields accomplished it. In honor of Zadeh's 100th birthday, a review paper [9] highlights the development of fuzzy set theory in relation to certain scientific fields and emphasizes the role of Professor I. Dzitac, one of Zadeh's disciples, in the advancement of

soft computing techniques related to fuzzy set theory. In honor of their friendship with the renowned scientist Lotfi A. Zadeh, Professor I. Dzitac wrote the preface for a fuzzy logic special issue that was released in association with Zadeh's 100th birthday [10].

The notion of the fuzzy set is applied in some fuzzy linear fractional programming cases in [11]. PID (Proportional Integral Derivative) and fuzzy-PID control data analysis for quadcopter movement and control is investigated in [12], and a development of the notion of fuzzy normed linear spaces is provided in [13]. Applications in quantum calculus of the notion of fuzzy differential superordination can be seen in [14–16]. The study [17] describes the latest developments in ordinary fuzzy sets and then reviews the literature on how fuzzy sets might be integrated with other artificial intelligence methods. A multi-attribute approach to decision making based on derived operators for intuitionistic fuzzy soft numbers, with the goal of applying machine learning or artificial intelligence to the evaluation of particular industrial situations, is presented in [18].

The concepts of fuzzy differential subordination and superordination continue to generate interesting outcomes as seen in recent publications regarding fuzzy differential subordination [19–21] or combining the two dual concepts [15,22].

The concept of third-order fuzzy differential subordination was introduced and developed in [23]. In their paper, the authors developed the concept of third-order fuzzy differential subordination building on the idea established by José A. Antonino and Sanford S. Miller in 2011 [24] and that continues to be studied by researchers to this day. The core ideas of the third-order fuzzy subordination approach and fundamental concepts to be employed for the studies concerning this part of the fuzzy differential subordination theory were presented, such as admissible functions class and certain fundamental theorems.

As said before, the differential superordination concept was introduced as a dual notion to that of differential subordination. Considering this idea, the generalizations of the classical theories of differential subordination and superordination, namely, strong differential subordination and superordination and fuzzy differential subordination and superordination, have discussed both dual theories. Furthermore, studies involving the concepts of differential subordination and superordination, separate or together, have been developed. With this motivation, since the concept of third-order fuzzy differential subordination was introduced in [23], in this study, we address of the dual problem of the third-order fuzzy differential superordination by introducing the basic notions related to the concept of third-order fuzzy differential superordination. Further motivation for the present study is provided by the fact that results on the classical third-order differential superordination are in trend nowadays, with studies like [25–28] being recently published. Hence, there are many results obtained that could be adapted to the third-order fuzzy differential superordination theory once the fundamentals are established here. In Section 3 of this paper, the definition of a third-order fuzzy differential superordination is given, the admissibility condition and the class of admissible functions are established, and also specific methods are developed to be applied for finding fuzzy subordinants and, furthermore, the best fuzzy subordinant when it exists.

2. Preliminaries

This section contains the terminology, notations, and previously established findings that were applied in the investigation.

$\mathcal{H}(U)$ indicates the class of holomorphic functions in U , where $U = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc of the complex plane, with $\bar{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial U = \{z \in \mathbb{C} : |z| = 1\}$.

The following subclasses of $\mathcal{H}(U)$ are familiar to geometric function theory researchers:

$$A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\},$$

where $A_1 = A$,

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\},$$

where $\mathcal{H}_0 = \mathcal{H}[0, 1]$, with $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, and

$$K = \left\{ f \in \mathcal{H}(U) : \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0, f'(0) \neq 0, z \in U \right\},$$

denoting the class of convex functions.

Furthermore,

$$S = \{f \in A : f(z) = z + a_2 z^2 + \dots, f(0) = 1, f'(0) = 1, f \text{ is univalent for } z \in U\}.$$

Definition 1 ([6,29]). Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g or g is said to be superordinate to f , written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function w analytic in U , with conditions $w(0) = 0$ and $|w(z)| < 1, z \in U$ and such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Following Definition 1, we can write the following:

Proposition 1. Let $f, g \in \mathcal{H}(U)$. If $f \prec g$, then $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 2 ([6]). Let Q denote the set of functions q that are analytic and univalent on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U; \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and are such that $\min |q'(\zeta)| = \rho > 0$ for $\zeta \in \partial U \setminus E(q)$. $E(q)$ is called the exception set, and the subclass of Q when $q(0) = a$ is denoted by $Q(a)$.

Remark 1. The set Q is not empty. The function $q(z) = \frac{z+1}{z-1} \in Q$, since $\min |q'(\zeta)| = \frac{1}{2} > 0$ for $\zeta \in \partial U \setminus E(1)$.

In order to prove the results related to third-order differential superordination, the following lemma must be employed:

Lemma 1 ([24,30]). Let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \in \mathcal{H}(U)$, with $q(z) \neq a, n \geq 2$ and let $p \in Q(a)$. If q is not subordinate to p , then there exist points $z_0 \in U, z_0 = r_0 e^{i\theta_0}$ and $\zeta_0 \in \partial U \setminus E(p)$ such that the following conditions are satisfied:

- (i) $q(z_0) = p(\zeta_0), q(\overline{U_{r_0}}) \subset p(U)$;
- (ii) $\operatorname{Re} \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} \geq 0$ and $\left| \frac{z p'(z)}{q'(\zeta)} \right| \leq n$, where $z \in \overline{U_{r_0}}, \zeta \in \partial U \setminus E(p)$, then there exists a real $m \geq n \geq 2$, such that:
- (iii) $\zeta_0 p'(\zeta_0) = \frac{z_0 q'(z_0)}{m}$;
- (iv) $\operatorname{Re} \left(\frac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1 \right) \leq \frac{1}{m} \operatorname{Re} \left(\frac{z_0 q''(z_0)}{q'(z_0)} + 1 \right)$;
- (v) $\operatorname{Re} \frac{\zeta_0^2 p'''(\zeta_0)}{p'(\zeta_0)} \leq \frac{1}{m^2} \operatorname{Re} \frac{z_0^2 q'''(z_0)}{p'(z_0)}$.

Next, certain notions and definitions regarding the fuzzy differential subordination and superordination theories are recalled.

In [31], the following definitions were used for introducing the notion of fuzzy subordination as a generalization of the classical notion of subordination:

Definition 3 ([31]). Let X be a non-empty set. An application $F : X \rightarrow [0, 1]$ is called a fuzzy subset.

A more accurate alternative definition is as follows:

A pair (A, F_A) , where $F_A : X \rightarrow [0, 1]$ and

$$A = \{x \in X : 0 < F_A(x) \leq 1\},$$

is called a fuzzy subset of X .

The set A is called the support of the fuzzy set (A, F_A) , and F_A is called the membership function of the fuzzy set (A, F_A) .

One can also denote $A = \text{supp}(A, F_A)$.

The notion of fuzzy subordination as a generalization of the classical notion of subordination was given in [31] as:

Definition 4 ([31]). Let $D \subset \mathbb{C}$ and let $z_0 \in D$ be a fixed point. We take the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g or g is said to be fuzzy superordinate to f and we write $f \prec_F g$ or $f(z) \prec_F g(z)$, if there exists a function $F : \mathbb{C} \rightarrow [0, 1]$, such that:

- (i) $f(z_0) = g(z_0)$;
- (ii) $F_{f(D)}f(z) \leq F_{g(D)}g(z), z \in D$.

Remark 2.

(a) The following definitions are considered:

$$f(D) = \{f(z) : 0 < F_{f(D)}f(z) \leq 1, z \in D\}$$

and

$$g(D) = \{g(z) : 0 < F_{g(D)}g(z) \leq 1, z \in D\}.$$

(b) Relation (ii) given by Definition 4 is written equivalently as

$$f(D) \subseteq g(D) \text{ and } \partial g(D) \not\subseteq f(D), \quad (1)$$

$\partial g(D)$ being the border of the domain $g(D)$.

(c) Such a function $F : \mathbb{C} \rightarrow [0, 1]$ can be considered as

$$F_1(z) = \frac{|z|}{1 + |z|}, \quad F_2(z) = \frac{1}{1 + |z|}, \quad \text{or } F_3(z) = \frac{1}{|z|}, z \in U.$$

(d) If $D = U$ then inequalities (i) and (ii) become:

- (i') $f(0) = g(0)$;
- (ii') $f(U) \subseteq g(U)$,

which is equivalent to the classical definitions of subordination and superordination, respectively.

3. Results

For obtaining the original results, the following notations are used.

Let $F : \mathbb{C} \rightarrow [0, 1]$.

Denote by

$$\Omega = (\Omega, F_\Omega) = \text{supp}(\Omega, F_\Omega) = \{z \in \mathbb{C} : 0 < F_\Omega(z) \leq 1\},$$

$$\Delta = (\Delta, F_\Delta) = \text{supp}(\Delta, F_\Delta) = \{z \in \mathbb{C} : 0 < F_\Delta(z) \leq 1\},$$

$p \in \mathcal{H}(U)$,

$$p(U) = \text{supp}(p(U), F_{p(U)}) = \{p(z) : 0 < F_{p(U)}(p(z)) \leq 1\}, z \in U,$$

and $\psi : \mathbb{C}^4 \times \bar{U} \rightarrow \mathbb{C}$

$$\psi(\mathbb{C}^4 \times \bar{U}) = \text{supp}(\mathbb{C}^4 \times U, F_{\mathbb{C}^4 \times \bar{U}}) =$$

$$\{0 < F_{\psi(\mathbb{C}^4 \times \bar{U})}\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \leq 1\}, z \in U.$$

Next, the class of admissible functions is introduced regarding the third-order fuzzy differential superordinations.

Definition 5. Let $\Omega \subset \mathbb{C}$ and let $q \in Q, n \in \mathbb{N}, n \geq 2$. Denote by $\Phi_n[\Omega, q]$ the set of functions $\phi : \mathbb{C}^4 \times \overline{U} \rightarrow \mathbb{C}$, satisfying the condition

$$F_{\phi(\mathbb{C}^4 \times \overline{U})} \phi(r, s, t, u; \zeta) < F_{\Omega}(z), \quad 0 < F_{\Omega}(z) \leq 1, \quad (2)$$

where

$$r = q(z), s = \frac{zq'(z)}{m}, \operatorname{Re} \left(\frac{t}{s} + 1 \right) \leq \frac{1}{m} \operatorname{Re} \left[\frac{zq''(z)}{q'(z)} + 1 \right]$$

and

$$\operatorname{Re} \frac{u}{s} \leq \frac{1}{m^2} \operatorname{Re} \frac{z^2 q'''(z)}{q'(z)}, z \in U, \zeta \in \partial U \setminus E(q) \text{ and } m \geq n \geq 2.$$

Condition (2) is called the admissibility condition.

In the special case when h is an analytic mapping of U onto $\Omega \neq \mathbb{C}$, the class $\Phi_n[h(U), q]$ is denoted by $\Phi_n[h, q]$.

Third-order fuzzy differential superordination can be seen as the dual problem of the third-order fuzzy differential subordination in the following form:

Consider $\Omega \subset \mathbb{C}, \Delta \subset \mathbb{C}$ and let $p \in \mathcal{H}(U)$ with $p(0) = a, a \in \mathbb{C}$. For a function $\phi(r, s, t, u; z) : \mathbb{C}^4 \times \overline{U} \rightarrow \mathbb{C}$, the problem consists of investigating the following implication:

$$F_{\Omega}(z) \leq F_{\phi(\mathbb{C}^4 \times \overline{U})} \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \implies F_{\Delta}(z) \leq F_{p(U)}(p(z)). \quad (3)$$

If $\Delta \neq \mathbb{C}$ is a simply connected domain with $a \in \Delta$, there exists a function q which maps conformally the unit disc satisfying the condition $q(0) = a$. Then, implication (3) can be rewritten as

$$F_{\Omega}(z) \leq F_{\phi(\mathbb{C}^4 \times \overline{U})} \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \implies F_{q(U)}(q(z)) \leq F_{p(U)}(p(z)), \quad (4)$$

i.e.,

$$q(z) \prec_F p(z), \quad z \in U.$$

If $\Omega \neq \mathbb{C}$ is also a simply connected domain, there exist a function h which maps conformally the unit disc satisfying the condition $h(0) = \phi(a, 0, 0, 0; 0)$. If we also have that $\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in \mathcal{H}(U)$, then implication (3) can be rewritten as:

$$h(z) \prec_F \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), \quad (5)$$

or

$$F_{h(U)}(h(z)) \leq F_{\phi(\mathbb{C}^4 \times \overline{U})} \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z),$$

which implies

$$q(z) \prec_F p(z) \quad \text{or} \quad F_{q(U)}(q(z)) \leq F_{p(U)}(p(z)), z \in U.$$

Remark 3. Relation (5) is valid even if functions h and q are only analytic functions and not necessarily univalent. Relation (5) leads to the definition of third-order fuzzy differential superordination.

Definition 6.

(1) Let $\phi : \mathbb{C}^4 \times \overline{U} \rightarrow \mathbb{C}$ and consider function h analytic in U .

If functions p and $\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)$ are univalent in U , satisfying

$$h(z) \prec_F \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \quad (6)$$

or

$$F_{h(U)}(h(z)) \leq F_{\phi(\mathbb{C}^4 \times \overline{U})} \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), z \in U,$$

then function p is called the solution of fuzzy differential superordination (6).

- (2) Fuzzy differential superordination (6) is called third-order fuzzy differential superordination and a univalent function q is called fuzzy subordinant of the third-order fuzzy differential superordination (6) if $q(z) \prec_F p(z)$ or it is equivalently written as $F_{q(U)}(q(z)) \leq F_{p(U)}(p(z))$, $z \in U$.
- (3) A fuzzy subordinant $\tilde{q}$ which is univalent and satisfies $q(z) \prec_F \tilde{q}(z)$ or is equivalently written $F_{q(U)}(q(z)) \leq F_{\tilde{q}(U)}(\tilde{q}(z))$, $z \in U$ for all fuzzy subordinants q of the third-order fuzzy differential superordination (6) is called the best fuzzy subordinant of the third-order fuzzy differential superordination (6).

Remark 4.

- (1) The best fuzzy subordinant of the third-order fuzzy differential superordination is unique up to a rotation in U .
- (2) If $\Omega \subset \mathbb{C}$ and functions ϕ and p are univalent in U , then relation (6) can be replaced by

$$\Omega \subset \left\{ \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z); z \in U \right\}, \quad (7)$$

which is a differential inclusion. In this case, the terms of third-order fuzzy differential superordination and the solution of the third-order fuzzy differential superordination are also used.

In the text theorem, an important result established for the second-order fuzzy differential subordinations is extended for the case of third-order fuzzy differential superordination.

Theorem 1. Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{H}[a, n]$, $n \geq 2$, functions $\phi \in \Phi_n[\Omega, q]$, $F : \mathbb{C} \rightarrow [0, 1]$ given by $F(z) = \frac{1}{1+|z|}$ and $p \in Q(a)$, $p(0) = q(0) = a$, satisfying the following conditions:

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left| \frac{zp'(z)}{q'(\zeta)} \right| \leq m, \quad (8)$$

where $z \in U$, $\zeta \in \partial U \setminus E(p)$, $m \geq n \geq 2$.

If $p(z)$ and $\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)$ are univalent functions in U , then

$$\Omega \subset \left\{ \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z); z \in U \right\} \quad (9)$$

or, equivalently,

$$\frac{1}{1+|z|} \leq \frac{1}{1+|\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)|}, \quad z \in U,$$

which implies

$$q(z) \prec_F p(z) \quad \text{or} \quad \frac{1}{1+|q(z)|} \leq \frac{1}{1+|p(z)|}, \quad z \in U.$$

Proof. For a fixed $z = z_0 \in U$, relation (9) becomes

$$\frac{1}{1+|z_0|} \leq \frac{1}{1+|\phi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0))|}. \quad (10)$$

Using relation (1), for $\zeta \in \partial U \setminus E(p)$, we obtain

$$\frac{1}{1+|\phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta), \zeta^3 p'''(\zeta))|} > \frac{1}{1+|z|}, \quad z \in U, \quad (11)$$

since $\partial\phi(U) = \phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta), \zeta^3 p'''(\zeta)), \zeta \in \partial U$. For a fixed $z = z_0 \in U$ and for $\zeta = \zeta_0 \in \partial U \setminus E(p)$, relation (11) becomes

$$\frac{1}{1 + |\phi(p(\zeta_0), \zeta_0 p'(\zeta_0), \zeta_0^2 p''(\zeta_0), \zeta_0^3 p'''(\zeta_0))|} > \frac{1}{1 + |z_0|}. \quad (12)$$

In order to prove this theorem, Lemma 1 is applied and the admissibility condition (2) is also used. We proceed by assuming that $q(z) \not\prec_F p(z)$. In this case, from Lemma 1, we deduce that there exist points $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$ and $\zeta_0 \in \partial U \setminus E(p)$ such that the conditions (i)–(v) are satisfied, and we write

$$q(z_0) = p(\zeta_0), \quad \zeta_0 p'(\zeta_0) = \frac{z_0 q'(z_0)}{m}, \quad \operatorname{Re} \left(\frac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1 \right) \leq \frac{1}{m} \operatorname{Re} \left(\frac{z_0 q''(z_0)}{q'(z_0)} + 1 \right), \quad (13)$$

and

$$\operatorname{Re} \frac{\zeta_0^2 p'''(\zeta_0)}{p'(\zeta_0)} \leq \frac{1}{m^2} \operatorname{Re} \frac{z_0^2 q'''(z_0)}{p'(z_0)}.$$

Using the conditions given by (13), for fixed $\zeta = \zeta_0$ and $z = z_0$ and choosing $r = q(z_0) = p(\zeta_0)$, $s = \frac{z_0 q'(z_0)}{m} = \zeta_0 p'(\zeta_0)$, $t = z_0^2 p''(z_0)$, $u = \zeta_0^3 p'''(\zeta_0)$ in the admissibility condition (2), we obtain

$$\frac{1}{1 + |\phi(p(\zeta_0), \zeta_0 p'(\zeta_0), \zeta_0^2 p''(\zeta_0), \zeta_0^3 p'''(\zeta_0))|} \leq \frac{1}{1 + |z_0|}. \quad (14)$$

Since relation (14) contradicts relation (12), we conclude that the assumption made is false, hence we must have $q(z) \prec_F p(z)$ or $\frac{1}{1+|q(z)|} \leq \frac{1}{1+|p(z)|}$, $z \in U$. $\square$

We next consider h a conformal mapping of U , $h(U) = \Omega \subset \mathbb{C}$. In this case, the class $\Phi_n[h(U), q]$ is denoted as $\Phi_n[h, q]$ and the next theorem follows directly from Theorem 1.

Theorem 2. Let $q \in \mathcal{H}[a, n]$, $n \geq 2$, let h be analytic in U , and consider $\phi \in \Phi_n[h, q]$, $F : \mathbb{C} \rightarrow [0, 1]$ given by $F(z) = \frac{1}{1+|z|}$ and $p \in Q(a)$, $p(0) = q(0) = a$, satisfying the following conditions:

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left| \frac{z p'(z)}{q'(\zeta)} \right| \leq m, \quad (15)$$

where $z \in U$, $\zeta \in \partial U \setminus E(p)$, $m \geq n \geq 2$.

If $p \in Q(a)$ and $\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)$ are univalent functions in U , then

$$h(z) \prec_F \phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z), z \in U, \quad (16)$$

or, equivalently,

$$\frac{1}{1 + |h(z)|} \leq \frac{1}{1 + |\phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)|}, \quad z \in U,$$

which implies

$$q(z) \prec_F p(z) \quad \text{or} \quad \frac{1}{1 + |q(z)|} \leq \frac{1}{1 + |p(z)|}, \quad z \in U.$$

Remark 5. Theorems 1 and 2 show that the fuzzy subordinants of a third-order fuzzy differential superordination can be established simply by finding functions that satisfy the admissibility condition (2).

The next theorem provides the sufficient conditions for a fuzzy subordinant of a third-order fuzzy differential superordination to be the best fuzzy subordinant for the third-order fuzzy differential superordination considered.

Theorem 3. Let $h \in \mathcal{H}(U)$, $\phi \in \Phi_n[h, q]$ and $F : \mathbb{C} \rightarrow [0, 1]$ given by $F(z) = \frac{1}{1+|z|}$. Assume that the differential equation,

$$\phi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z \in U) = h(z), \quad (17)$$

has a solution $q \in Q(a)$. If functions $p \in Q(a)$ and $\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in U)$ are univalent functions in U satisfying the conditions

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left| \frac{zp'(z)}{q'(\zeta)} \right| \leq m,$$

and the following third-order fuzzy differential superordination holds,

$$h(z) \prec_F \phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), z \in U, \quad (18)$$

or, equivalently,

$$\frac{1}{1+|h(z)|} \leq \frac{1}{1+|\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)|}, \quad z \in U,$$

this implies

$$q(z) \prec_F p(z) \quad \text{or} \quad \frac{1}{1+|q(z)|} \leq \frac{1}{1+|p(z)|}, \quad z \in U,$$

and q is the best fuzzy subordinant for the third-order fuzzy differential superordination (18).

Proof. According to Definition 4 and condition (1), using the third-order fuzzy differential superordination (18), for a fixed $z = z_0 \in U$ we have

$$\frac{1}{1+|h(z_0)|} \leq \frac{1}{1+|\phi(p(z_0), z_0p'(z_0), z_0^2p''(z_0), z_0^3p'''(z_0))|}, \quad (19)$$

and, at the same time,

$$\frac{1}{1+|\phi(p(\zeta), \zeta p'(\zeta), \zeta^2p''(\zeta), \zeta^3p'''(\zeta))|} > \frac{1}{1+|h(z)|}, \quad \zeta \in \partial U \setminus E(p), z \in U. \quad (20)$$

For a fixed $\zeta = \zeta_0$ and $z = z_0 \in U$, relation (20) becomes

$$\frac{1}{1+|\phi(p(\zeta_0), \zeta_0p'(\zeta_0), \zeta_0^2p''(\zeta_0), \zeta_0^3p'''(\zeta_0))|} > \frac{1}{1+|h(z_0)|}. \quad (21)$$

For completing the proof, Lemma 1 is necessary, alongside relation (2). Assume that $q \not\prec_F p$. Then, from Lemma 1 we can write

$$q(z_0) = p(\zeta_0), \quad \zeta_0p'(\zeta_0) = \frac{z_0q'(z_0)}{m}, \quad \operatorname{Re} \left(\frac{\zeta_0p''(\zeta_0)}{p'(\zeta_0)} + 1 \right) \leq \frac{1}{m} \operatorname{Re} \left(\frac{z_0q''(z_0)}{q'(z_0)} + 1 \right), \quad (22)$$

and

$$\operatorname{Re} \frac{\zeta_0^2p'''(\zeta_0)}{p'(\zeta_0)} \leq \frac{1}{m^2} \operatorname{Re} \frac{z_0^2q'''(z_0)}{p'(z_0)}.$$

Using the conditions given by (22), for fixed $\zeta = \zeta_0$ and $z = z_0$ and choosing $r = q(z_0) = p(\zeta_0)$, $s = \frac{z_0q'(z_0)}{m} = \zeta_0p'(\zeta_0)$, $t = z_0^2p''(z_0)$, $u = \zeta_0^3p'''(\zeta_0)$ in the admissibility condition (2), we obtain

$$\frac{1}{1+|\phi(p(\zeta_0), \zeta_0p'(\zeta_0), \zeta_0^2p''(\zeta_0), \zeta_0^3p'''(\zeta_0))|} \leq \frac{1}{1+|h(z_0)|}. \quad (23)$$

Since relation (23) contradicts relation (20), we conclude that the assumption made is false, hence we must have $q(z) \prec_F p(z)$ or $\frac{1}{1+|q(z)|} \leq \frac{1}{1+|p(z)|}, z \in U$. Since q is the solution for the differential Equation (17), we conclude that q is the best fuzzy subordination for the third-order fuzzy differential superordination (18). $\square$

Remark 6.

- (1) Theorem 3 shows that finding the best fuzzy subordination of a third-order fuzzy differential superordination reduces to finding a univalent solution for the differential equation associated with the third-order fuzzy differential superordination.
- (2) The conclusion of Theorem 3 involving the third-order fuzzy differential superordination (18) can be written in the equivalent form:

$$\phi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z \in U) \prec_F \phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in U).$$

- (3) In all three proved theorems, the function ϕ can be given, for example, as $\phi((r, s, t, u) = r + s + 2t + 3u$.

Example 1. Let $\phi(r, s, t, u; z) : \mathbb{C}^4 \times \bar{U} \rightarrow \mathbb{C}$ given by $\phi((r, s, t, u) = r + s + 2t + 3u$ and let $q(z) = 1 + \frac{z^4}{4}$ a solution of the differential equation

$$h(z) = \phi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z \in U) = \phi(1 + \frac{z^4}{4}, z^4, 3z^4, 6z^4) = 1 + \frac{101z^4}{4}.$$

Consider the functions $p(z) = 1 + 2z + z^2$ and $\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in U) = 1 + 2z + z^2 + z(2 + 2z) + 2z^2 = 1 + 4z + 7z^2$ univalent in U . Furthermore, take $F : \mathbb{C} \rightarrow [0, 1]$ given by $F(z) = \frac{1}{1+|z|}$.

Using the findings given by Theorem 3, the following numerical illustration is obtained.

Let $q(z) = 1 + \frac{z^4}{4}$, a univalent solution of the differential equation

$$h(z) = \phi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z \in U) = \phi(1 + \frac{z^4}{4}, z^4, 3z^4, 6z^4) = 1 + \frac{101z^4}{4}.$$

Consider the functions $p(z) = 1 + 2z + z^2$ and $\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in U) = 1 + 4z + 7z^2$ univalent in U . If the following conditions are satisfied:

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left| \frac{z(2+2z)}{\zeta^3} \right| \leq m,$$

then

$$\frac{1 + 101z^4}{4} \prec_F 1 + 4z + 7z^2, \text{ or, } \frac{1}{1 + \left| 1 + \frac{101z^4}{4} \right|} \leq \frac{1}{1 + |1 + 4z + 7z^2|}, \quad z \in U,$$

which implies

$$1 + \frac{z^4}{4} \prec_F 1 + 2z + z^2, \text{ or, } \frac{1}{1 + \left| 1 + \frac{z^4}{4} \right|} \leq \frac{1}{1 + |2z + z^2|}, \quad z \in U,$$

and $q(z) = 1 + \frac{z^4}{4}$ is the best fuzzy subordination.

Indeed, we prove that the conditions of Theorem 3 are satisfied.

We evaluate

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} = \operatorname{Re} \frac{3\zeta^3}{z^3} = 3 \geq 0.$$

$$\left| \frac{2(1+z)}{\zeta^3} \right| = \frac{2|z||1+z|}{|\zeta^3|} = 2|1+z| = 2\sqrt{1+\rho^2+2\rho\cos\alpha}.$$

For $\rho \rightarrow 1$, we have

$$\left| \frac{2(1+z)}{\zeta^3} \right| = \lim_{\rho \rightarrow 1} 2\sqrt{1+2\cos\alpha} \geq 2 = m.$$

Next it is proved that the function $p(z) = 1 + 2z + z^2$ is convex, hence univalent.

$$\operatorname{Re}\left(\frac{zp''(z)}{p'(z)} + 1\right) = \operatorname{Re}\left(\frac{2z}{2+2z} + 1\right) = 1 + \operatorname{Re}\frac{1}{1+z} = 1 + \frac{1}{2} = \frac{3}{2} > 0.$$

Since $p'(0) = 2 \neq 0$ and $\operatorname{Re}\left(\frac{zp''(z)}{p'(z)} + 1\right) > 0$, we conclude that p is a convex function, hence univalent.

We next prove that the function $\phi(z) = 1 + 4z + 7z^2$ is convex. For that, we evaluate

$$\operatorname{Re}\left(\frac{z\phi''(z)}{\phi'(z)} + 1\right) = \operatorname{Re}\left(\frac{14z}{4+14z} + 1\right) = 1 + 7\operatorname{Re}\frac{2}{2+7z} = 1 + 7\frac{2\rho\cos\alpha + 7\rho^2}{4+28\rho\cos\alpha + 49\rho^2}.$$

For $\rho \rightarrow 1$, we obtain:

$$\operatorname{Re}\left(\frac{z\phi''(z)}{\phi'(z)} + 1\right) = 1 + 7\lim_{\rho \rightarrow 1} \frac{2\rho\cos\alpha + 7\rho^2}{4+28\rho\cos\alpha + 49\rho^2} = 1 + 7\frac{5+2(1+\cos\alpha)}{15+28(1+\cos\alpha)} > 0.$$

Since $\phi'(0) = 4 \neq 0$ and $\operatorname{Re}\left(\frac{z\phi''(z)}{\phi'(z)} + 1\right) > 0$, we conclude that ϕ is a convex function, hence univalent.

Having all the conditions required by Theorem 3 satisfied, we obtain that

$$1 + \frac{z^4}{4} \prec_F 1 + 2z + z^2$$

and $q(z) = 1 + \frac{z^4}{4}$ is the best fuzzy subdominant.

4. Discussion

The new results obtained as part of the investigation contained in this paper contribute to the development of the theory of differential superordination. The original results presented in Section 3 of the paper add knowledge regarding the topic of fuzzy differential superordination, a relatively new area of research in geometric function theory initiated in 2017 [8] that incorporates fuzzy set aspects into the study of differential subordinations. This study introduces the concept of third-order fuzzy differential superordination in Definition 6, the class of admissible functions in Definition 5, and the necessary tools for conducting investigations for obtaining fuzzy subordinants are given in Theorems 1 and 2, furthermore, the best fuzzy subdominant of the third-order fuzzy differential superordination that allows it in Theorem 3. The introduction of the concept was inspired by the successful extensions concerning the case of the classical third-order differential superordination theory [27] and the study proposed for the dual notion of third-order fuzzy differential subordination [23]. The foundations of the present work are given by the previously published papers [4,8,23,31] concerning the fuzzy aspects incorporated in the differential subordination and superordination studies as well as by the results established in [24,30] for the classical third-order differential subordination and superordination theories. A numerical example is developed based on the findings of Theorem 3.

5. Conclusions

The outcome of this study aims to lay the foundations for future studies regarding the newly highlighted notion of third-order fuzzy differential superordination which has not been explored so far. The topic could be extended, building on the concepts presented here by involving different types of operators such as quantum calculus operators previously used in very recent studies involving second-order fuzzy differential superordination

like [15,22,32], linear operators used in studies like [33,34], or fractional calculus operators, as seen in [14,35,36].

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