mathematics

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Uncovering Hidden Patterns: Approximate Resurgent Resummation from Truncated Series

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Abstract: We analyze truncated series generated as divergent formal solutions of non-linear ordinary differential equations. Motivating the study is a specific non-linear, first-order differential equation, which is the basis of the resurgent formulation of renormalized perturbation theory in quantum field theory. We use the Borel–Padé approximant and classical analysis to determine the analytic structure of the solution using the first few terms of its asymptotic series. Afterward, we build an approximant, consistent with the resurgent properties of the equation. The procedure gives an approximate expression for the Borel–Ecalle resummation of the solution useful for practical applications. Connections with other physical applications are also discussed.

Keywords: asymptotic series; quantum field theory; resummation; renormalons; ordinary differential equations

MSC: 81T13; 34E05; 34A45

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Citation: Maiezza, A.; Vasquez, J.C. Uncovering Hidden Patterns: Approximate Resurgent Resummation from Truncated Series. *Mathematics* **2024**, *12*, 3087. https://doi.org/10.3390/math12193087

Academic Editors: Hovik Matevossian and Gaetano Fiore

Received: 30 July 2024
Revised: 18 September 2024
Accepted: 30 September 2024
Published: 2 October 2024

1. Introduction

Many physical phenomena have non-linear dynamics associated with non-linear differential equations. One of the most common analytic tools to describe its solution is perturbation theory, often leading to asymptotic, divergent series. Even if one knows the generic term in the asymptotic series, it does not capture the complete solution of the underlying differential equation, and the series are formal solutions, at the most, that must be resummed to make sense out of them.

One well-known branch of physics where asymptotic series are commonplace is quantum field theory (QFT). Formal arguments show that these perturbative approaches lead to divergent series [1–3]. These series are non-Borel–Laplace resummable, and one reason is the presence of large-order $n!$ divergences for which one source is often referred to as renormalons [3]. Renormalons are regularly spaced singularities in the Borel transform of Green functions along the real positive axis, which render the Laplace integral ambiguous, thus hampering a unique Borel–Laplace resummation.

The theory of Resurgence [4,5] enables one to overcome problems as the one just introduced, generalizing the concept of Borel–Laplace resummation to the one of Borel–Ecalle resummation. It can solve the perturbative renormalization problem in QFT, provided that an underlying differential equation exists, leading to Ecalle's bridge equation. This result was recently achieved in [6] under the name of Resurgence of the renormalization group equation.

In particular, the Resurgence of the renormalization group equation consists of an ordinary differential equation (ODE), upon which a resurgent resummation isomorphism is built exploiting the mathematics of [7–9]. The latter case is a special, detailed instance of the more general Ecalle's Resurgence. Due to the underlying non-linear ODE, the resummed result is a trans-series (in general, a trans-series is an irreducible concatenation of symbols $(+, \times, \circ, \exp, \log)$ and coefficients—see [10] for a primer on the topic) with only one unknown parameter. This sharply contrasts with the historical point-of-view in QFT to tackle the renormalons, which effectively implements a solution by adding infinitely many arbitrary constants in the form of an operator product expansion [11], de facto rendering the model in question non-renormalizable (see, for example, [12]).

Although Resurgence formally solves the problem of resumming series with infinitely many singularities in the positive real axis of its Borel transform, in practice, it would require the knowledge of the entire power series. On the other hand, in many instances of practical interest, one only knows a few terms of the truncated series. One must then resort to large-order-behavior estimates that model the generic term in the series, such as the one proposed in [13] under the name of "Naive Non-Abelianization". Although grounded in QFT-educated guesses, these ad-hoc methods lack control over the uncertainties and thus are of limited use.

One promising alternative is to try reconstructing a function (for example, a Green function in QFT) from asymptotic data, if available. This approach has been dubbed "numerical resurgent analysis" [14], for the Painlevé I equation, or "physical resurgent extrapolations" [15] for study cases from quantum mechanics and quantum field theory.

Conceptually, this article belongs to the latter kind of approach. In particular, it studies a truncated series generated from non-linear ODEs, in a "normal form" defined in [9], which are expected to apply to perturbative calculations in QFT. As we shall see, the reason why one can reliably resum these truncated series is that one expects the renormalons to dominate the large-order behavior [12] and, in turn, the analytic structure of the renormalon singularities emerges from a normal-form ODE [6].

We shall trace a roadmap starting from a truncated asymptotic series and leading to an approximated Borel–Ecalle resummation of it through suitable approximants. One feature of the approach is that it explicitly leverages Ecalle's bridge equation and medianization while implementing some approximants. This approach is valid not only in quantum field theory but for any truncated series, as long as it comes from a first-order, non-resonant, non-linear ODE.

In particular, the article proceeds as follows. In Section 2, as a motivation of this study, we highlight the resurgent description of the renormalization group in QFT; in Section 3, we summarize the main elements of Resurgence and alien calculus; in Section 4, we exploit the classical Borel–Padé and Darboux's analysis to obtain information about the analytic structure of the Borel transform of the truncated series; in Section 6, we use such information to build an approximant consistent with the resurgent properties of the non-linear ODEs. Finally, we summarize and discuss our findings in Section 9. Further details are presented in Appendices A and B.

2. Motivation: Ordinary Differential Equations and the Singularities in the Borel Transform of Green Functions

It is well-known that, in QFT, there are singularities in the semi-positive axis of the Borel plane due to the $n!$ large-order behavior in perturbation theory. In this section, we highlight the connection of these singularities, known as "renormalon ", with a non-linear ODE coming from the renormalization group equation [6]. This non-linear ODE then provides the basis for applying the Borel–Ecalle resummation.

The large-order $n!$ behavior due to renormalons leads to the failure of the perturbative renormalization in QFT, which technically means that the expressions in renormalized perturbation theory are not Borel–Laplace resummable. We define the renormalons as infinitely many ambiguities in the Laplace integral of Green functions, due to singularities located in the Borel transform variable z at

$$z = \frac{2\,n}{|\beta_1|}, \tag{1}$$

where n is a positive integer, β_1 is the one-loop coefficient of the beta function. The absolute value takes into account whether one works in an asymptotically free model or not: in the former case, $\beta_1 < 0$, and one has the ambiguities due to the infrared renormalons; in the latter case, $\beta_1 > 0$, and one has the ambiguities due to the ultraviolet renormalons.

In what follows, we aim to highlight the results on the resurgent structure of perturbation theory with its associated non-linear ODE, as presented in [6] and elaborated in [16], which here we follow.

For concreteness, assume one works in a Yang–Mills field theory. In particular, consider the one-particle-irreducible Green function in the Landau gauge with Euclidean momentum,

$$\Gamma^{(2)}_{\mu\nu} = \left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) p^2\right] \Pi(p^2, \mu^2), \tag{2}$$

where p is the four-momentum.

Defining $L := \log(\mu_0^2/\mu^2)$, with $p^2 := \mu_0^2$, the so-called vacuum polarization function Π satisfies the equation

$$\left[-2\frac{\partial}{\partial L} + \beta(\alpha)\frac{\partial}{\partial_\alpha} - 2\gamma(\alpha)\right]\Pi(L) = 0, \tag{3}$$

where α is the coupling constant, $\beta(\alpha) = \mu\frac{d\alpha}{d\mu} = \beta_1\alpha^2 + \mathcal{O}(\alpha^3)$ and $\gamma(\alpha) = \gamma_1\alpha + \mathcal{O}(\alpha^2)$ is the anomalous dimension.

The vacuum polarization function $\Pi(L)$ has an expansion of the form [17–20]

$$\Pi(L) = 1 + R(\alpha) + \sum_{k=1}^{\infty} \pi_k(\alpha) L^k, \tag{4}$$

Replacing (4) in (3) yields a system of infinitely many ODEs [20], whose first equation enables one to write a non-linear ODE for the re-scaled function $R(x) = \frac{U(x)}{x}$ with $x = 1/\alpha \in \mathbb{R}$ [16]

$$U(x)' = -Q\,U(x) + A\frac{U(x)}{x} + \sum_{n>2,m} k_{n\,m}\,U(x)^n x^{-m}, \tag{5}$$

where the expression for A given in (6) is specific to the "minimal setup" discussed in [16]; in general, additional contributions can appear in A [6,21], and these contributions are unrelated to the loop expansion)

$$Q = -\frac{2}{\beta_1}, \qquad \text{and} \qquad A = \frac{\beta_1^2 - 2\beta_1\gamma_1 + 2\beta_2}{\beta_1^2}, \tag{6}$$

and $k_{n\,m}$ are some coefficients.

One shifts $\bar{U}(x) = U(x) + \mathcal{O}(1/x^N)$, with N being sufficiently large to have a formally small shift. This yields a normal form ODE

$$\bar{U}(x)' = -\,Q\,\bar{U}(x) + A\frac{\bar{U}(x)}{x} + \sum_{n\geq 2,m} k_{n\,m}\,\bar{U}(x)^n x^{-m} + \mathcal{O}(1/x^N). \tag{7}$$

The coefficient Q is fixed from the one-loop Landau pole structure [21]. The coefficient A gives the type of singularities in the Borel transform. The presence of non-linear terms in $\bar{U}$ is crucial for the structure of the actual solution, and the equation can describe the renormalon singularities in (1). We discuss this in detail in the next sections, focusing on the ODEs in the form of (7).

The scope of this work is to study the truncated perturbative solution of ODEs in the normal form, as (7), and approximately reconstruct the complete solution from the truncated series. The methods elaborated in what follows will be useful in quantum field theory once a sufficiently large perturbative input is known, such that one can perform a resurgent extrapolation.

3. Elements of ODEs, Resurgence, and Alien Calculus

This section highlights the main elements of Resurgence, which shall be useful for the rest of the discussion—for the seminal works of Ecalle and Costin, see [5,9]. A clear exposition of Resurgence and alien calculus is in [22]. Other reviews are [23,24].

3.1. ODEs Setup

Following Costin [9], consider the first-order, non-linear differential equation in the real domain

$$y'(x) = F[y(x), x], \qquad y(x) \in \mathbb{R}, x > 0 \tag{8}$$

where $y'(x) = dy(x)/dx$, and assume that F is analytic for formally small y and large x, such that we can expand as

$$y'(x) = -\lambda y(x) - A\frac{y(x)}{x} + f(x) + \sum_{n\geq 2,m} k_{n\,m}\, y^n x^{-m}, \tag{9}$$

where $f(x)$ is an analytic function as $x \to \infty$; the sign of the linear term is placed for convenience. The formal, asymptotic series solution of (9) is of the form

$$y(x) \sim \sum_{n=1}^{\infty} a_n x^{-n} \tag{10}$$

and its Borel transform of $y(x)$ is given by

$$B[y](z) := Y(z) := \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} z^n. \tag{11}$$

where $B_n \equiv a_n/(n-1)!$.

Due to the non-linear nature of the ODE, a fundamental property of (9) is that its solution has infinitely many, equally-spaced singularities in its Borel transform; namely, $Y(z)$ is singular at

$$z_{sing} = \lambda\, n \qquad n \in \mathbb{N}^+ = \{1, 2, 3, \ldots\}. \tag{12}$$

when λ is positive, the Laplace integral,

$$\int_0^{\infty} e^{-x\,z}\, Y(z) dz, \tag{13}$$

is ill-defined, with infinitely many ambiguities, due to (12). The term A, $\propto y/x$, gives the type of singularities, namely, around z_{sing}

$$Y(z) \simeq \frac{c}{(\lambda\, n - z)^{1+A}} + \text{analytic}, \tag{14}$$

being c some coefficient. Rescaling and changing variables in the ODE, one can always make $A \in [-1, 0]$. When $A = 0$, one deals with simple-pole singularities, which is technically the simplest case.

Now, let us explicitly point out that the structure of (7), extracted from RGE, matches with the one of (9): the Borel transform of the solution of (9) has singularities spaced as $-2/\beta_1$ ($\beta_1 < 0$ for Yang–Mills models), with these being singularities as in (14), with A determined in (6). Notice that the asymptotic series, the formal solution of (7), starts from

$a_N x^{-N}$, and this means that the renormalons start dominating the growth of perturbation theory at some unknown order N.

3.2. Definitions

Some basic definitions are as follows.

Definition 1. *The lateral Borel–Laplace summation $\mathcal{A}^{\pm}$ is defined as*

$$y^{\pm} := \mathcal{A}^{\pm} \circ y(x) := \int_0^{\infty \pm i\epsilon} dz\, e^{-xz}\, Y(z)$$

Definition 2. *Along the Stokes line ($z > 0$), the discontinuity operator δ is defined as*

$$\mathcal{A}^{-} - \mathcal{A}^{+} = \delta \mathcal{A}^{+}$$

thus

$$\delta = 0 \qquad \Leftrightarrow \qquad y^{+} = y^{-}.$$

Definition 3. *The Stokes automorphism is defined as $G := 1 + \delta$*

$$y^{-}(x) = G\, y^{+}(x)$$

G is a morphism since it preserves products: $G(f\,g) = G(f)G(g)$

Definition 4. *The alien derivative is defined via the identity*

$$G = e^{\log G} := e^{\dot{\Delta}}, \text{ or } \quad \dot{\Delta} = \log G = \log(1+\delta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \delta^n$$

$\dot{\Delta}$ is called the alien derivative. The alien derivative satisfies the same rules as the standard derivatives, namely

$$\dot{\Delta}(f\,g) = g\dot{\Delta}f + f\dot{\Delta}g \tag{15}$$

and

$$\dot{\Delta}(\lambda f) = \lambda \dot{\Delta}(f) \tag{16}$$

A fundamental property of the alien derivative is that it commutes with the standard one:

$$\left[\dot{\Delta}, \frac{d}{dx}\right] = 0. \tag{17}$$

The heuristic reason is that $\dot{\Delta}$ is related to the discontinuity of a function (δ, in Definition 2) and the ordinary derivative ($\frac{d}{dx}$) applied to a function does not alter the discontinuity structure and location of the singularities.

3.3. Bridge Equation

The general solution of (9) is in the form of the one-parameter trans-series

$$y(x) = \sum_{n=0}^{\infty} C^n e^{-\lambda\, n\, x} y_n(x), \tag{18}$$

where C is an arbitrary constant fixed by the initial condition. However, (18) has a priori infinitely many unknown functions $y_n(x)$. Resurgence enables us to calculate each of them from y_0, where this is the formal (asymptotic) series solution of (9).

To understand how resurgence comes into play, it is sufficient to apply $\dot{\Delta}$ and d/dC to (9). The crucial point is that $\dot{\Delta}$ commutes with d/dx, due to (17), and also d/dC commutes with d/dx. The application of $\dot{\Delta}$ and d/dC to (9) gives the same two differential equations for $\dot{\Delta}y(x)$ and $dy(x)/dC$. Hence, $\dot{\Delta}\,y(x) \propto dy(x)/dC$ up to a function of C, namely

$$\dot{\Delta}y(x) = S(C)\frac{d\,y(x)}{dC}, \tag{19}$$

Performing the Taylor expansion for $S(C)$

$$S(C) = \sum_{m=0}^{\infty} S_m C^m, \tag{20}$$

and replacing it in (19), one sees only S_0 in the non-zero case (we rename it S), leading to

$$\dot{\Delta}y_n(x) = S(n+1)e^{-\lambda\,x}y_{n+1}(x). \tag{21}$$

S is called a holomorphic invariant and is purely imaginary. This recursion can be solved in the form,

$$(\dot{\Delta})^n y_0(x) = n!\,S^n\,e^{-\lambda\,n\,x}\,y_n(x), \tag{22}$$

which tells us that by applying n-times the alien derivative to y_0, coming from perturbation theory, one calculates any of the functions y_n in the trans-series (18). This is precisely the objective of the Resurgence theory.

Finally, to keep contact between Ecalle's alien calculus and Costin's formalism, let us rewrite $y_n(x)$ in terms of $\delta y_0(x), y_1(x), \ldots, y_{n-1}(x)$, such that one recasts (22) as (see [21])

$$y_n(x) = \frac{e^{\lambda\,n\,x}}{S^n}\left(\delta y_0(x) - \sum_{j=1}^{n-1} S^j e^{-\lambda\,j\,x} y_j(x)\right), n \geq 1, \tag{23}$$

which corresponds to equation (5.116) in [9] but, here, in the multiplicative space x.

3.4. Medianization

The resurgent equation (22) is thus far more formal since one starts with y_0, which is an asymptotic series that needs to be properly resummed. Due to the singularities in (12), one cannot perform the Borel–Laplace resummation of y_0 in the usual sense. Naively, one can rely on the two analytic continuations in Definition 1. Nevertheless, these are not reality-preserving.

A resummation procedure must have the two properties:

1. It must be an algebra of homomorphism, namely, it must turn convolutions in Borel space (convolutive model) into standard multiplication of the Laplace transform (multiplicative model). The aforementioned analytic continuations $\mathcal{A}^{\pm}$ would preserve this property but break the realness condition discussed next.
2. It must preserve realness, namely, if the asymptotic series in (10) is real with real coefficients a_n, its resummed expression must also be real. The Cauchy principal value would ensure this reality condition, but it would break the homomorphism structure discussed above.

Medianization takes into account the above requirements [5]

$$\mathcal{A}_{med} := G^{-1/2} \circ \mathcal{A}^- = G^{1/2} \circ \mathcal{A}^+. \tag{24}$$

Unlike $y^{\pm}$, $y_{med} := \mathcal{A}_{med} \circ y(x)$ resums real asymptotic series into real functions while preserving the homomorphism structure. Due to Equation (17), y_{med} is a solution of the initial ODE. It is worth commenting that medianization in the context of non-linear ODE here discussed coincides with the balanced average proposed in [9]—see appendix A for more details.

One can check that (24) is satisfied order-by-order in powers of δ. To achieve this, one first expands $G^{\pm 1/2}$ in (24) in powers of δ using Definition 4. Then, repeatedly applying δ to Definition 3, one obtains

$$\delta^n y^- - \delta^n y^+ = \delta^{n+1} y^+, \tag{25}$$

which one solves as

$$\begin{aligned} \delta y^+ &= y^- - y^+ \\ \delta^n y^- &= \delta^{n+1} y^+ + \delta^n y^+. \end{aligned} \tag{26}$$

Replacing (26) into the expansion of (24), one sees that the latter holds at any order in δ.

3.5. Medianized Trans-Series

What remains is to consider (22) and (24) together. Note that one can apply either $G^{-1/2}$ or $G^{1/2}$ to (22). Thus, applying $G^{1/2} \circ \mathcal{A}^+$ one finds

$$(\dot{\Delta}_0)^n G^{1/2} y_0^+(x) = n!\, S^n\, e^{-\lambda\, n\, x}\, G^{1/2}\, y_n^+(x). \tag{27}$$

We proceed exactly as in the previous subsection, namely, we expand and consistently replace (26), but now for any y_n. In doing this, one calculates the most general, reality-preserving solution to Equation (8)

$$y(x)_{med} = \mathcal{A}_{med}\, y(x) = \sum_{n=0}^{\infty} C^n e^{-\lambda\, n\, x} y_n(x)_{med}. \tag{28}$$

In practice, one uses the medianization equation truncated to a given order in δ^n; we consider four for illustration, and the formula for medianization reduces to

$$\mathcal{A}_{med} = PV - \frac{\delta^2}{8}\mathcal{A}^+ + \frac{\delta^3}{16}\mathcal{A}^+ - \frac{5}{128}\delta^4\mathcal{A}^+ + \mathcal{O}(\delta^5). \tag{29}$$

where PV denotes the Cauchy principal value. Note that $y_0(x)_{med}$ is not simply the Cauchy principal value (PV) of y_0, but a generalization of it. In what follows, we give an explicit example, properly truncating at order δ^4 and y_4, and the Equation (27) implies:

$$\begin{aligned} &\frac{1}{24}\delta^4 y_0^+ - \frac{1}{24}\delta^3 y_0^+ + \delta y_0^+ = \frac{1}{16} S e^{-\lambda x}\delta^3 y_1^+ - \frac{1}{8} S e^{-\lambda x}\delta^2 y_1^+ + \frac{1}{2} S e^{-x} y_1^- \\ &+ \frac{1}{2} S e^{-\lambda x} y_1^+, \\ &\frac{7}{24}\delta^4 y_0^+ - \frac{1}{2}\delta^3 y_0^+ + \delta^2 y_0^+ = -\frac{1}{4} S^2 e^{-2x}\delta^2 y_2^+ + S^2 e^{-2\lambda x} y_2^- + S^2 e^{-2\lambda x} y_2^+, \\ &\delta^3 y_0^+ - \delta^4 y_0^+ = 3S^3 e^{-3\lambda x} y_3^- + 3S^3 e^{-3\lambda x} y_3^+, \\ &\delta^4 y_0^+ = 12 S^4 e^{-4\lambda x} y_4^- + 12 S^4 e^{-4\lambda x} y_4^+. \end{aligned} \tag{30}$$

Finally, after using (26) to remove all terms with $\mathcal{A}^-$, one solves this complex system consisting in seven independent equations for the seven variables $(y_1^+, \delta^2 y_1^+, \delta^3 y_1^+, y_2^+, \delta^2 y_2^+, y_3^+, y_4^+)$ in terms of $\delta^n y_0^+$ for $n \leq 4$ (taking into account that the discontinuities δs and S are imaginary, and defining the real objects $\delta' = \delta/i$, $S' = S/i$), leading to $y(x)_{med}$ in (28)

$$\begin{aligned} y(x)_{med} = &\left(PV(y_0) - \frac{1}{8}\delta'^2 y_0^+ + \frac{1}{16}\delta'^3 y_0^+\right) + \frac{C\left(24\delta' y_0^+ - \delta'^3 y_0^+\right)}{24 S'} \\ &+ \frac{C^2\left(\delta'^3 y_0^+ - 2\delta'^2 y_0^+\right)}{4 S'^2} - \frac{C^3\, \delta'^3 y_0^+}{6 S'^3} + \mathcal{O}\left(\delta'^4\right), \end{aligned} \tag{31}$$

The expression in (31) is real, as it must be, and is of practical interest once one needs to approximately resum a divergent series to a resurgent trans-series. Moreover, the real constant S' can be re-absorbed into the trans-series parameter C, which is real, and (31) has only one consistent parameter considering the fact that it is the (approximate) solution of a first-order ODE. At this order, $y_4(x)_{med}$ does not receive any contribution.

In Appendix B, we illustrate the concepts of this section for a simple example, where the exact solution is known.

4. Borel–Padé and Darboux's Analysis of the Asymptotic Series

Borel–Padé analysis allows one to reconstruct exactly the resummation of y_0 in the trivial cases of linear ODE. For example, the formal series solution of Euler ODE (recall $x > 0$),

$$y'(x) = -y(x) + \frac{1}{x}, \tag{32}$$

is

$$\sum_{n=0}^{\infty} a_n x^{-n}, \qquad a_n = (n-1)! \tag{33}$$

Its Borel transform is

$$\sum_{n=0}^{\infty} \frac{a_n}{(n-1)!} z^n = \sum_{n=0}^{\infty} z^n. \tag{34}$$

Taking a few terms of the truncation of the above series (in z), and building the (diagonal) Padé approximant, one easily obtains convergence to

$$\frac{1}{1-z}.$$

Thus, Borel–Padé is capable of reconstructing the correct structure.

For a further setup of what we are going to deal with later, we also consider a variation of Euler ODE,

$$y'(x) = -y(x) - \frac{1}{2}\frac{y(x)}{x} + \frac{1}{x}. \tag{35}$$

The only logical point for considering this equation is that it leads to a square-root branch point in Borel space:

$$\frac{1}{\sqrt{1-z}}, \tag{36}$$

therefore, Padé approximant, being rational, does not reproduce it. Nevertheless, taking the derivative of the log, one has

$$\frac{d}{dz}\left[\log\left(\frac{1}{\sqrt{1-z}}\right)\right] = \frac{1}{2}\frac{1}{1-z}. \tag{37}$$

Therefore, in this case, it is useful to perform the Padé approximant of the derivative of the log of the (truncated) Borel series. Doing so, the Padé approximant quickly converges to the expression on the right side of (37).

Borel–Padé and Non-Linear ODE

We leverage the previous setup based on linear ODE to obtain information on non-linear cases of interest. Consider the ODE with simple poles in Borel space at $z_{sing} = n$,

$$y'(x) = -y(x) + y(x)^2 + \frac{1}{x}, \tag{38}$$

and find a truncated, approximated solution in the form

$$y(x) \approx \sum_{n=0}^{N} a_n x^{-n}.$$

We plug the above equation into (38) and solve for the coefficients a_n up to the finite order $n = N$. We proceed by calculating many (from order N equals 5 to 25) diagonal Borel–Padé approximants of the truncated series and accumulating the values of the poles in a histogram. Therefore, unlike the linear case above, we proceed statistically. The result is shown in the left panel of Figure 1. The non-linearity manifests itself by the appearance of many poles. Not all the poles are well resolved, but there is a dominant peak in $z \simeq 1$. In particular, this is resolved by increasing the bins, with a precision of one in one thousand in the inset of the left panel in Figure 1. Now, invoking the knowledge of the general properties of the supposed underlying ODE, one infers that the poles of the Borel transform of the solution are at $z \simeq n$. In summary, the first location is manifested, and the others are deduced by non-linearity.

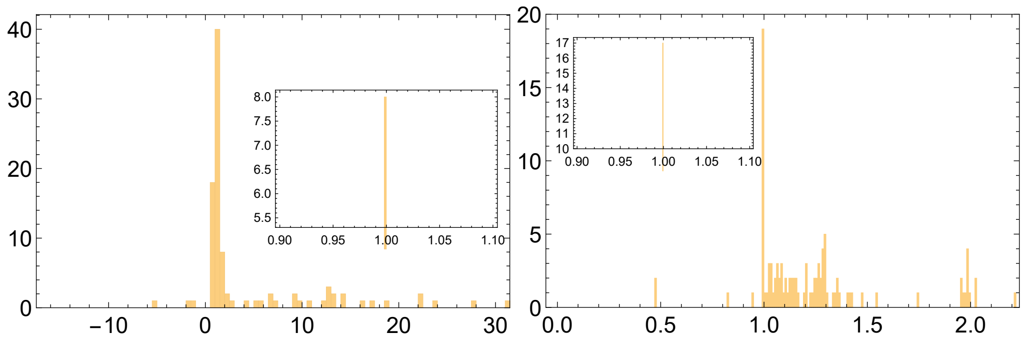

Figure 1. (**Left**): accumulation of poles of the Borel–Padé approximants for the asymptotic series related to (38); the inset shows the same but with finer bins, zooming on the dominant peak. (**Right**): accumulation of the Borel–Padé approximants poles of the derivative-of-log applied to the truncated Borel series from (39), as in (37); the inset shows the same but with finer bins, zooming on the dominant peak.

Next, consider a non-linear ODE with a square-root branch-point at $z_{sing} = n$ (recall (14))

$$y'(x) = -y(x) - \frac{1}{2}\frac{y(x)}{x} + y(x)^2 + \frac{1}{x}. \tag{39}$$

As usual, one finds by inspection the truncated series solution and Borel transforms. At this point, it is useful to implement the trick in (37): we build the Padé approximants of the derivative of log of the Borel transform. Except for this trick, the logic proceeds as in the previous example, yielding the right panel of Figure 1. After the use of (37), the comments are also conceptually the same as the case with simple poles: one resolves the first singularity in the inset accurately ($z \simeq 1$) and then deduces the others by non-linearity.

5. The Nature of Singularities

Let us check whether a Borel–Padé analysis can shed light even on the type of square-root branch points.

Consider a function around a singular point p

$$\frac{g(z)}{(p-z)^b}. \tag{40}$$

Once again, by taking the derivative of log as in (37), and then extracting the residue, one obtains $-b$. This offers a method to understand the kind of branch (say $|b|$). Thus, the aim is to build Padé approximants of the derivative of the log of the truncated Borel series of a non-linear ODE. Differently from the previous example, we now study the (absolute values of the) residues.

For the sake of generality and to exclude bias due to specific choices of coefficients in the non-linear ODE, we consider a different example,

$$y'(x) = -y(x) - \frac{2}{3}\frac{y(x)}{x} + \frac{y(x)^2}{x} + \frac{3}{x} + \frac{1}{4x^2}. \tag{41}$$

From (12) and (14), we know that the Borel transform has singularities at $z = n$, being of this kind $(n-z)^{-1/3}$. We saw above that the former information can be obtained from the generated asymptotic series, so now we discuss whether the latter is also attainable. The answer is partially positive: we can roughly estimate the expected value $|b| = 1/3$, at the price of using many terms of the series and evaluating very large Padé approximants. This requires a substantial computational effort and a very high machine precision evaluation to extract the residues (which we implement within the software Mathematca 11.0.1). The reason is that one has to compute the poles of the derivative of the log of the Borel series with high precision; otherwise, Mathematica does not recognize them as poles and fails to correctly evaluate the residues.

Our result is presented in Figure 2: we focus on the nearest to $z = 1$ pole of the approximant. One sees that the plot slowly converges to $1/3$. In particular, the last point (Padé approximant order 110) estimates $|b| \approx 0.38$. Beyond this order, numerical computation becomes even slower, and the Borel–Padé approach is not so effective for practical purposes.

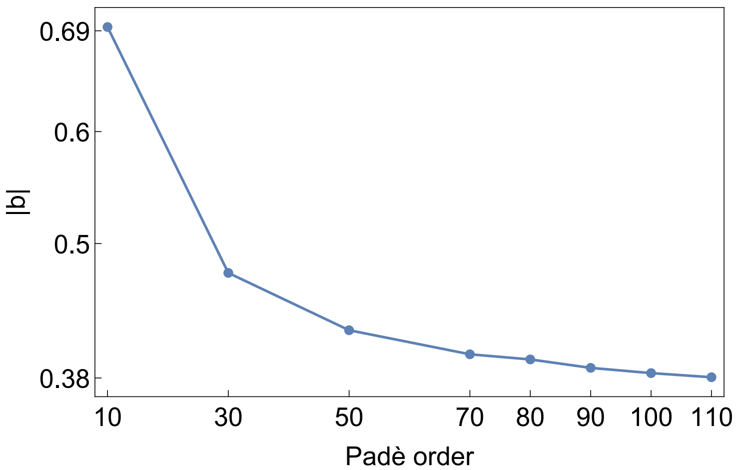

Figure 2. Padé-based estimate of the root branch point ($|b|$ from (40)) coming from the Borel transform of the truncated series from (39).

Fortunately, there is a method, based on Darboux's analysis [25,26], to extrapolate the nature of singularities from a few, $\mathcal{O}(10)$, terms of the truncated series. Writing the Borel transform around the singularity at $z = 1$ as

$$B(z) = (1-z)^{-b} H(z) + K(z) \tag{42}$$

$$H(z) = \sum_{k=0}^{\infty} c_k (z-1)^k, \tag{43}$$

and recalling that the B_n are the coefficients of the Borel transform of the truncated series, one has [27,28]

$$\frac{B_n}{B_{n-1}} = 1 + \frac{b-1}{n} + \frac{(b-1)s}{n(b-n-2)} + \mathcal{O}(1/n^3), \tag{44}$$

where $s = c_1/c_0$. Solving (44) for two different n's gives b and s, with the former being our objective since it characterizes the nature of the singularity. For instance, choosing $n = 12$ and $n = 11$ yields

$$b = 0.321045, \qquad s = -9.37312.$$

Notice that exploiting just a modest number of perturbative inputs, the value of b is remarkably close to the actual one that is $1/3$ (recall this comes from the Borel transform of (41)). As a test, increasing the value of n ($n = 100$ and $n = 99$) shows a good convergence:

$$b = 0.333096, \qquad s = -9.06805.$$

6. The Resurgent Approximant

In this section, we exploit both the information coming from the Resurgence theory of ODE and the above numerical analysis. The aim is to build an approximant with the correct analytic structure in Borel space, invoking Resurgence theory. In other words, we exploit all the information we have to approximate the true solution of the ODEs from the truncated asymptotic series. We shall deal with examples already introduced above, featuring simple poles and square-root branch points in Borel space.

Non-Linear ODE Leading to Simple Poles

Suppose one has estimated from the above Padé analysis that the ODE features simple poles in Borel space and, for simplicity but with no lack of generality, suppose that the poles lie at $z = n$, with $n = 1, 2, 3, \ldots$. For example, one may deal with the series generated from the ODE in (38).

One can try to approximate its solution, i.e., the resummed series via the approximant,

$$P = \frac{\sum_{n=0}^{N} c_n z^n}{\prod_{n=1}^{N}(n - z)}. \tag{45}$$

There is no necessity for both the summation and the production to run up to the same value N. For the moment, consider this as an illustration. Here, N is a finite natural number determined from how many terms of asymptotic series one knows. Unlike the Padé approximant, P in (45) has the correct analytical structure, i.e., the correct location and type of the singularities in Borel space.

Now, we have to calculate the coefficient c_i by matching with the truncated Borel series. For the sake of clarity, let us write this explicitly for the series emergent from (38). Replacing

$$\sum_{n=1}^{N} a_n x^{-n} \tag{46}$$

into (38), one finds for $N = 9$

$$\begin{aligned} &\{a(0) = 0, a(1) = 1, a(2) = 2, a(3) = 8, a(4) = 44, a(5) = 296,\\ &a(6) = 2312, a(7) = 20384, a(8) = 199376, a(9) = 2138336\}. \end{aligned} \tag{47}$$

The first term is trivial since the solution must go to zero for $x \to \infty$. Next, one Borel transforms (46), expands P for small z (i.e., Taylor expansion (45) around $z = 0$) and equates terms of the same order in z. This leads to an algebraic system for the c_i, thus determining P.

It is interesting to evaluate the goodness of the proposed approximant. This can be achieved by comparing some predicted coefficients of the Borel series, so the ones beyond those can be used to build P in (45). Specifically, one uses N terms in the truncated series, then one extrapolates the $N+1, N+2, \ldots.$ terms given by expanding the approximant P for small z and, finally, compares them with the exact Borel series calculated using more terms in the truncated series. Thus, the closer the ratio is to 1 between the predicted coefficients and the actual ones, the better the approximant. This is illustrated in the left panel of Figure 3.

As already mentioned, the summation and the production do not need to run up to the same N. Indeed, consider now

$$P' = \frac{\sum_{n=0}^{N} c_n z^n}{\prod_{n=1}^{N'}(n-z)}. \tag{48}$$

Note that if $N' > N$, the approximant P' does not require more perturbative inputs since the number of the coefficient c_n is fixed by the numerator of P'. Therefore, the logic is that using the same perturbative information, one can implement more poles, a notion that one knows from the resurgent structure of the non-linear ODE in question. In the right panel of Figure 3, we choose $N' = 8N$ for illustration and show the equivalent ratios described above (for the case with $N' = N$). One can appreciate a drastic improvement in the goodness of the approximant.

However, in choosing $N' = 8N$, we have only augmented the numbers of poles considered, but we have not yet proposed any rationale. The logic can be improved by fixing N' by using an additional coefficient of the actual truncated Borel series as a test. We find N' by demanding that the ratio between the first-predicted and the actual coefficients is ≈1 for any N. Computationally, we minimize the difference between the unit and that ratio for a given N. Thus, the resulting N' is, in general, a function of N. The result is shown in Figure 4.

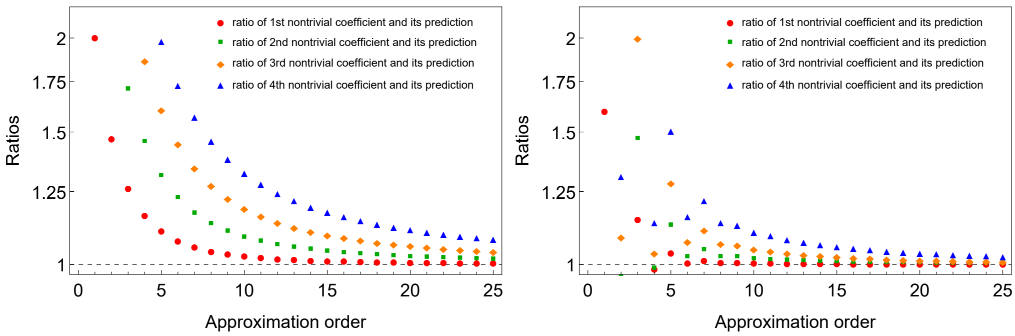

Figure 3. (**Left**) panel: goodness of the approximant in (45) in terms of the ratios of the predicted and actual coefficients of the Borel transform. (**Right**) panel: same thing but referring to the approximant in (48) with $N' = 8N$.

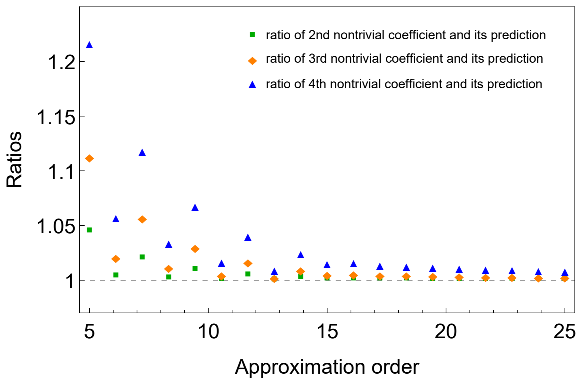

Figure 4. Ratios of the predicted and actual coefficients of the Borel transform for a varying N'. The first ratio is not present now since it is used to evaluate N'. The predicted coefficients have a better convergence than the one in Figure 3.

7. Resummation of the Series

To approximately resum the series, we first need to calculate the Cauchy principal value of the Laplace integral of (45). Consistent with the notation of (31), we have

$$PV(y_0(x)) = \int_0^\infty e^{-x\,z}\,P(z)\,dz, \tag{49}$$

which is an operation that we perform numerically.

Locally, around $z = \lambda$, the Borel transform of the solution of (9) is $Y(z) \sim c\,(\lambda - z)^{-1}$, and this holds for any x multiple of λ. In our example, we have $\lambda = 1$. Denoting the Laplace integration of a given function $h(z)$ as $\mathcal{L}[h]$, the discontinuity δ of the simple pole is

$$\mathcal{L}\left[\delta \frac{c}{1-z}\right] = \lim_{\epsilon\to 0} \mathcal{L}\left[\frac{c}{1-(z+i\epsilon)} - \frac{c}{1-(z-i\epsilon)}\right] = 2\pi i\,c\,e^{-x}. \tag{50}$$

The operator δ picks up the residue of all the poles in the Borel transform of y_0. The coefficient $2\pi i$, which is the residue, is a holomorphic constant, and thus, $\delta^n y_0 = 0$ when $n > 1$. This greatly simplifies (31), where only the linear term in C survives. In other words, only a non-perturbative sector (y_1 in the notation of (18)) contributes to the resummation.

In summary, fixing a determined order in P in (45), and using (49) and (50), one calculates the resummation of the truncated series (whose coefficients are in (47)). To compare the resummation with the exact, numerical solution of the ODE in (38), one has to fix an initial condition for both (38) and (31). In the latter, the condition shall fix the one-parameter transseries. The result is shown in Figure 5: we consider eight poles in P, (45), and fix $y(10) = 0.12$. In physical problems, of course, an equivalent condition must be established from some phenomenological data or physical insight.

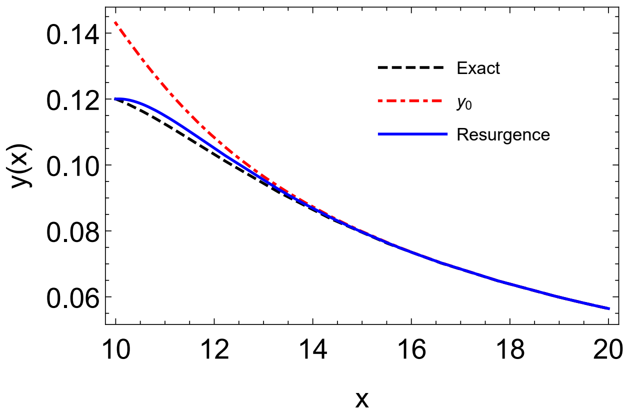

Figure 5. Comparison between the exact solution of (38), approximate resurgent resummation, and the Cauchy principal value of P (denoted as y_0) in (45).

To give a variety of examples and an illustration where we require a better approximation, we now consider the asymptotic series generated by the ODE,

$$y'(x) = -y(x) + y(x)^3 + \frac{1}{x}. \tag{51}$$

The modification concerning (38) is just in the non-linear term. Re-doing the same as discussed above, we now obtain the result in Figure 6.

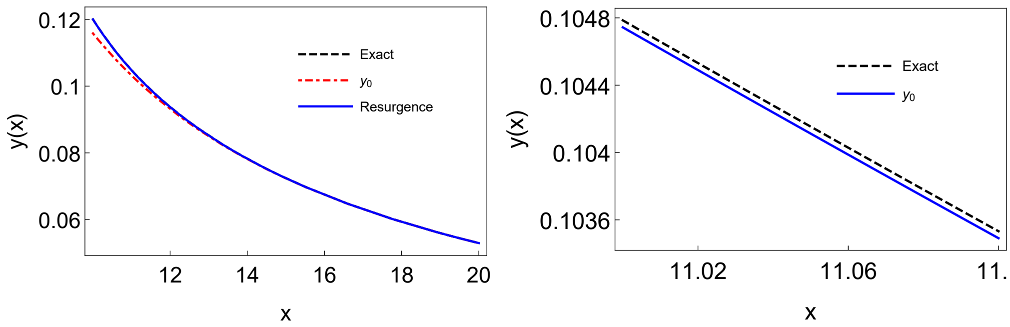

Figure 6. (**Left**) panel is the equivalent of Figure 5, but one cannot appreciate the difference between the actual result (black) and the resummation (blue); the (**Right**) panel is magnified to appreciate this difference.

Non-Linear ODE Leading to Branch Points

Suppose now that one has estimated via Darboux's analysis that one is dealing with square-root branch points in Borel space, instead of simple poles. Therefore, for definiteness, let us work with the asymptotic series generated from (39). In this case, we have to modify (45) as

$$B = \frac{\sum_{n=0}^{N} c_n z^n}{\prod_{n=1}^{N} \sqrt{(n-z)}}, \tag{52}$$

which is real for small z but not in general. Thus, we also define

$$B' = Re(B) + Im(B). \tag{53}$$

Here, we do not need to repeat all the discussions on simple poles in the previous Subsection; instead, we focus on the differences between the two cases—differences that appear in the resummation of the series.

8. Resummation of the Series

One calculates the Laplace integral, the equivalent of (49) but now using the approximant B' in (53). Then, one generalizes (50) as (recall that the operator $\mathcal{L}$ is a shorthand notation for the Laplace integral)

$$\mathcal{L}\left[\delta \frac{c}{\sqrt{1-z}}\right] = \lim_{\epsilon \to 0} \mathcal{L}\left[\frac{c}{\sqrt{1-(z+i\epsilon)}} - \frac{c}{\sqrt{1-(z-i\epsilon)}}\right] = 2\pi i\, c\, \frac{e^{-\lambda x}}{\sqrt{x}}. \tag{54}$$

Unlike (50), now the coefficient $1/\sqrt{x}$ is not holomorphic. In Borel space, for example, one has that the discontinuity of a square-root singularity is again a square-root singularity. The effect is that now all the δ^n are non-zero.

In particular, invoking Resurgence again, we know the discontinuities scale such that the dominant contribution for $\delta^n y_0 \propto e^{-nx}$. This follows directly from (22) and $\Delta = \delta + \mathcal{O}(\delta^2)$. Thus, one calculates from B, locally around the singularities, all the discontinuities δ^n (and so, δ'^n) in (31).

Finally, fixing an initial condition for (39), specifically $y(10) = 0.12$, gives a unique exact solution for (39) and a unique result for (31). The result is visible in Figure 7, where we see that the approach provides a function that converges to the true solution with good accuracy.

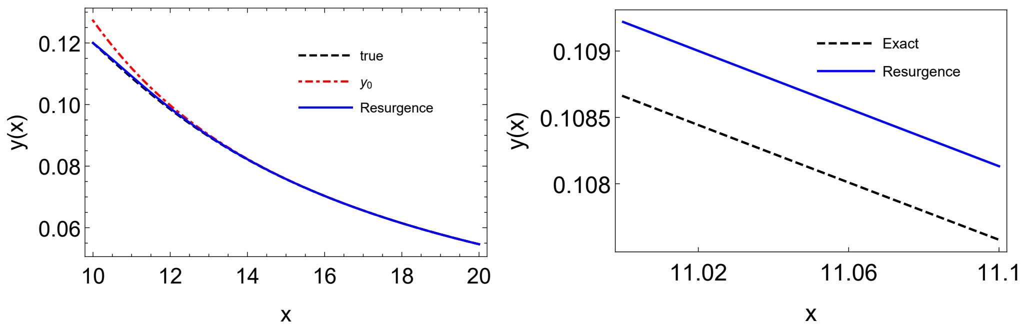

Figure 7. (**Left**) panel shows the exact result, the approximated resurgent resummation, and y_0 (the leading contribution, C^0, in (31)) Again, one cannot appreciate the difference between the actual result (black) and the resummation (blue); the (**Right**) panel is a zoom to see this difference.

9. Summary and Discussion

Motivated by a recent resurgent approach to the renormalization program in QFT, we built a procedure to extract non-perturbative information from truncated series. To achieve it, we first relied on the well-known Borel–Padé approximants and Darboux's analysis to decode the analytic structure solutions in Borel space using their truncated expression. Specifically, we have seen that one can pin down the position of the singularities (in Borel space) and also extrapolate the nature of the singularities, naming them simple poles or specific branch points.

Second, we implement this information in an educated approximation from Resurgence, setting the correct analytic structure emerging from the non-linear ODEs. In the case of ODEs featuring simple poles in Borel space, our approximant can be seen as a sort of Borel–Padé approximant in which we force the poles to lie in specific points. Similar logic holds for the ODEs leading to square-root branch points, but, in this case, there is no direct analogy with the Padé approximants.

In summary, we have implemented an approximated Borel–Ecalle resummation to a truncated asymptotic series—potentially coming from realistic physical calculations. As a result, we have seen that it is possible to reconstruct with good precision the actual solution of the ODEs from $\mathcal{O}(10)$ terms of the truncated series.

Due to (7), the results of this paper may be readily applicable to the truncated renormalized perturbation theory in QFT, where it may provide better approximations to the true non-perturbative solution, as long as one has sufficient orders in the expansion parameter (counplig constant), which is not yet the status of the current perturbative QFT. Interestingly, in lattice regularization (perturbation theory on the lattice), it is possible to find a truncated series with more terms, and in this case, one may directly employ the results found here. With a truncated series at hand, one can estimate the Borel transform's analytic structure. If this matches the one predicted from (7), then one is confident that the resummation procedure is robust.

Author Contributions: Conceptualization, A.M. and J.C.V.; Formal analysis, A.M. and J.C.V.; Writing—original draft, A.M. and J.C.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Medianization and Balanced Average

This appendix aims to make the relationship between the balanced average in [9], for non-linear ODE in the normal form, and the more general medianization in [5] transparent.

Defining the averaging weights [5],

$$\lambda_{p,q} = \frac{(2\,p)!(2\,q)!}{4^{p+q} p!\, q!\, (p+q)!}, \tag{A1}$$

with p, q denoting the number of circumventions of the singularities in Borel space above and below the real axis, respectively, the relationship is

$$\begin{aligned} \mathcal{A}_{bal} = G^{1/2} \circ \mathcal{A}^+ = \mathcal{A}_{med} &= \frac{1}{2}\left[\sum_{n=1}^{\infty}(-1)^n \lambda_{n,1} \delta^n \mathcal{A}^+ + \sum_{n=0}^{\infty}(-1)^n \lambda_{n,0} \delta^{n+1} \mathcal{A}^-\right] \\ &= \frac{1}{2}\left[\sum_{n=1}^{\infty}(-1)^n \lambda_{n,1} \delta^n \mathcal{A}^+ + \sum_{n=1}^{\infty}(-1)^{n-1} \lambda_{n-1,0} \delta^{n} \mathcal{A}^-\right] \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \left(\lambda_{n,1} \delta^n \mathcal{A}^+ - \lambda_{n-1,0} \delta^{n} \mathcal{A}^-\right). \end{aligned} \tag{A2}$$

Indeed, the above term $G^{1/2} \circ \mathcal{A}^+$ can be easily identified with the balanced average defined in Proposition 5.77 of [9], while Ecalle's medianiation is given in terms of the averaging weights. Finally, note that (A1) also connects with the topological interpretation of the medianization as paths near the singularities and disconnected from the origin—see Figure 5.1 of [9].

Appendix B. Example of Resurgence Applied to a Simple ODE

In Section 4, as a simple prototype to start with, we have discussed the effects of the truncation on the Euler's series,

$$y(x) = \sum_{n=1}^{\infty} (n-1)!\, x^{-n}, \tag{A3}$$

and the formal solution of the ODE,

$$y'(x) = -y(x) + \frac{1}{x}. \tag{A4}$$

For completeness, here, we use the same example to illustrate the concepts summarized in Section 3.

The Borel transform of (A3) gives

$$B[y](z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \tag{A5}$$

This expression is singular (simple pole) at $z = 1$, thus rendering it ambiguous on the integration path of the Laplace integral. As a consequence, the usual Borel–Laplace resummation is not applicable, but the Borel–Ecalle resummation of Section 3 enables overcoming the difficulty.

Starting from (A5) and evaluating the integrals in Definitions 1 and 2, one obtains

$$\delta y^-(x) - \delta y^+(x) = \delta y^+(x) = 2\pi i\, e^{-x}. \tag{A6}$$

The above (nonperturbative) contribution corresponds to the only singularity—in this specific case. Due to the absence of further singularities and the holomorphic nature of the coefficient of the non-analytic term e^{-x}, successive applications of the discontinuity operator give no contributions: $\delta^n y^+(x) = 0$ for $n \geq 2$.

Notice that $y^\pm$ and δy^+ in (A6) are not real. The last step to obtain a real-valued resummation is the medianization. Since we have worked out in the text a generic expression for the medianized $y(x)$ that is valid for the type of ODE studied here, namely (31), it is sufficient to apply this formula:

$$y_{res}(x) = PV[\int_0^\infty e^{-x\,z} \frac{1}{1-z} dz] + C\, e^{-x}, \tag{A7}$$

where we use the label "res" to recall "resummed", PV denotes the principal value, and the single-parameter trans-series C is real. Therefore, y_{res} is real, as it must be. The evaluation of the integral gives

$$PV[\int_0^\infty e^{-x\,z} \frac{1}{1-z} dz] = e^{-x}\, Ei(x), \tag{A8}$$

where $Ei(x)$ is the exponential integral function. Finally,

$$y_{res}(x) = e^{-x}\, Ei(x) + C\, e^{-x}, \tag{A9}$$

which is the general solution of (A4).

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