

$\mathcal{C}$ -Semigroups and Their Induced Order

Daniel Marín-Aragón ^{*,†}  and Raquel Tapia-Ramos [†] 

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cádiz, E-11510 Puerto Real, Spain; raquel.tapia@uca.es

* Correspondence: daniel.marin@uca.es

† Avda. Universidad de Cádiz, nº 10, 11519 Campus Universitario de Puerto Real Dirección Postal, Cádiz, Spain.

Abstract: Let $\mathcal{C} \subset \mathbb{N}^p$ be an integer polyhedral cone. An affine semigroup $S \subset \mathcal{C}$ is a $\mathcal{C}$ -semigroup if $|\mathcal{C} \setminus S| < +\infty$. This structure has always been studied using a monomial order. The main issue is that the choice of these orders is arbitrary. In the present work, we choose the order given by the semigroup itself, which is a more natural order. This allows us to generalise some of the definitions and results known from numerical semigroup theory to $\mathcal{C}$ -semigroups.

Keywords: $\mathcal{C}$ -semigroup; Wilf's conjecture; commutative monoids

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1. Introduction

Let $\mathbb{N}$ be the set of non-negative integers; we say that $S \subset \mathbb{N}$ is a numerical semigroup if it is an additive monoid and its complementary is finite. This structure has been studied broadly in the literature (see, for example, [1–3]). In [2], it is proven that if $a_1, \dots, a_e \in \mathbb{N}$ are coprimes, then $\langle a_1, \dots, a_e \rangle = \{\lambda_1 a_1 + \dots + \lambda_e a_e \mid \lambda_1, \dots, \lambda_e \in \mathbb{N}\}$ is a numerical semigroup. The number of the generator, i.e., e , is called the embedding dimension of S and the number $|\mathbb{N} \setminus S| = g \in \mathbb{N}$ is called the genus of S . Other relevant invariants are the Frobenius number, defined as $F(S) = \max\{n \in \mathbb{Z} \mid n \notin S\}$; the conductor, defined as $c(S) = F(S) + 1$; and the left elements, defined as $L(S) = |\{x \in S \mid x < F(S)\}|$ (see [4,5]).

There are still several open problems related to these invariants, such as Bras-Amorós's conjecture (see [6]) and Wilf's conjecture (see [7]) and, consequently, a lot of papers have been published that are trying to solve them (see [8–11]). The first conjecture, put forward 15 years ago, states that if S_g is the set of all numerical semigroups with genus g , then $|S_{g+1}| \geq |S_g|$ for all $g \geq 0$. This conjecture is true for the numerical semigroups with a genus less than 67 (see [10]) and with a genus greater than an unknown g (see [12]). On the other hand, Wilf's conjecture was raised in 1978 and establishes that $e(S)L(S) \geq c(S)$.

In order to try to solve these conjectures, in [13], the concept of a numerical semigroup was generalised, as follows: let $\mathcal{C} \subset \mathbb{N}^p$ be an integer polyhedral cone, then a $\mathcal{C}$ -semigroup is an additive affine semigroup $S \subset \mathcal{C}$ with finite complementary set. Thus, we transfer the problem from $\mathbb{N}$ to $\mathbb{N}^p$. Within this new structure, the classical invariants of the commutative monoids theory cited previously have been studied (see, for example, [14] or [15]).

Note that this kind of semigroup is very useful in the study of toric geometry (see [16,17]), as there is a bijection between the affine varieties parametrized by a finite set of monomials and the affine semigroups (see [18]).

There is a huge issue in all these papers: unlike in $\mathbb{N}$, in $\mathbb{N}^p$ there is no canonical total order. Therefore, researchers have to choose a total order (as the graded lexicographical order) in order to define invariants as the Frobenius number, but it is not always clear what happens when this order is changed. Our main goal is to show that this generalisation can be performed in a different way that does not depend of the researcher's choice.

In this work, we use the order induced by the semigroup itself to carry out the generalisation, i.e., given $a, b \in S$, we say that $a \leq b$ if, and only if, $a - b \in S$. This order



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has already been applied to numerical semigroups (see, for example, [19]), but never to $\mathcal{C}$ -semigroups. With this choice, the order is always clear and does not depend on an arbitrary choice. Thanks to this order, we open a new avenue for obtaining results that can be applied to numerical semigroups. In order to provide the examples shown, we have used a computer with a CPU Inter Core i7 8th Gen and codes which are available at <https://github.com/D-marina/CommutativeMonoids> (accessed on 8 July 2024).

The content of this work is organised as follows: In Section 2, we give the main definitions and some general results in order to provide a background for the reader. We show our generalisation and how apply the changes to the main invariants. Then, in Section 3, we introduce a new invariant, quasi-elasticity (based on the concept of elasticity in numerical semigroups; see [20]), and show its properties. Section 4 is devoted to idempotential semigroups and we show the boundaries for computing some invariants in this family. Finally, in Section 5, we generalize Wilf's conjecture by means of this partial order.

2. Preliminaries

Let $\mathcal{C} \subset \mathbb{N}^p$ be an integer polyhedral cone, with $\tau_1, \dots, \tau_q$ as its extremal rays and $h_1, \dots, h_r$ as its supporting hyperplanes. We say that S is a $\mathcal{C}$ -semigroup if it is an affine semigroup (i.e., S is finitely generated, cancellative, reduced and torsion-free), $S \subset \mathcal{C}$ and $|\mathcal{C} \setminus S| < +\infty$. The set $\mathcal{H}(S) = \mathcal{C} \setminus S$ is called the set of gaps, or the gap set, of S .

We define the induced order of S as

$$x \leq_S y \iff y - x \in S.$$

We use the symbol $\leq$ instead of $\leq_S$ when there is no risk of misunderstanding. Note that if $x \leq_S y$, then $x \leq_{\mathbb{N}^p} y$. The converse, trivially, is not true.

We recall that in a numerical semigroup, the multiplicity is the smallest element that is not zero of S and the Frobenius number is the greatest element of $\mathbb{Z}$ which is not in S . We generalise these definitions as follows.

Definition 1. We define the set of multiplicities of S as $\text{minimals}_{\leq}(S)$ and denote this set as $\mathfrak{m}(S)$. We define the set of Frobenius as $\mathfrak{F}(S) = \text{maximals}_{\leq_{\mathcal{C}}}(\mathcal{H}(S))$.

As before, we use $\mathfrak{m}$ and $\mathfrak{F}$ when there is no risk of confusion. Since $\mathcal{H}(S)$ is finite and $\mathfrak{F} \subset \mathcal{H}(S)$, then $\mathfrak{F}$ is also finite. Now, we prove that $\mathfrak{m}$ verifies this finiteness.

Proposition 1. Let S be a $\mathcal{C}$ -semigroup, then the set $\mathfrak{m}(S)$ is finite.

Proof. For each supporting hyperplane $h_i \equiv a_1x_1 + \dots + a_px_p = 0$, we define $h'_i(\alpha) \equiv a_1x_1 + \dots + a_px_p = \alpha$. We pick $\alpha_1, \dots, \alpha_r$ such that if $f \in \mathfrak{F}$ and $x \in h'_i(\alpha_i)$, then $fx \leq 0$.

We define $D = \{d \in \mathcal{C} \mid dx \leq 0, \forall x \in \cup_1^r h'_1(\alpha_i)\}$. Let $c \in \mathcal{C} \setminus 3D$, and then $c \in \alpha\mathcal{C}$ for some $\alpha \in \mathbb{N}$, $\alpha > 4$. Since there exists $d \in 3D \cap \mathcal{C}$ such that $c - d \in \mathcal{C}$, $\mathfrak{m} \subset 3D$, $\mathfrak{m}$ is finite. $\square$

In [14], the authors present an algorithm for computing the gap set of a $\mathcal{C}$ -semigroup; this allows us to create Algorithm 1 for computing $\mathfrak{m}$.

Given a $\mathcal{C}$ -semigroup, the algorithm admits a unique minimal system of generators, denoted by $\text{msg}(S) = \{a_1, \dots, a_e\}$. That means that $S = \{n_1a_1 + \dots + n_ea_e \mid n_i \in \mathbb{N}, a_i \in \text{msg}(S), i = 1, \dots, e\}$ and there is not a proper subset of $\text{msg}(S)$ verifying this condition. The following result proves that the minimal system of generators is in fact the minimals of S and its induced order.

Algorithm 1 Computing the set $m(S)$ **Input:** Set of gaps of S .**Output:** $m(S)$.Compute $\alpha_1, \dots, \alpha_r$.Define $2D$. $X \leftarrow \{First(D)\}$ **for** $d \in 2D$ **do** **for** $d \in 2D$ **do** **if** $d \leq x$ **then** $X \leftarrow X \setminus \{x\} \cup d$ **end if** **end for****end for****return** X .**Proposition 2.** Let S be a $\mathcal{C}$ -semigroup; then $msg(S) = m(S)$.**Proof.** $msg(S) \subset m(S)$: Let $x \in msg(S)$ and assume that $x \notin m(S)$. Then, there exists $y \in S$ such that $y \leq x$, i.e., $x - y = z \in S$. So $x = y + z$, which contradicts the fact that $x \in msg(S)$. $m(S) \subset msg(S)$: Let $x \in m(S)$ and assume that $x \notin msg(S)$. Then, there exists $y, z \in S$ such that $x = y + z$, so $x - y = z \in S$, which contradicts the fact that $x \in m(S)$. $\square$ In [21], the authors define the pseudo-Frobenius number of a $\mathcal{C}$ -semigroup as $a \in \mathcal{H}(S)$, verifying $a + (S \setminus \{0\}) \subset S$. The set of all the pseudo-Frobenius numbers of S is denoted by $PF(S)$.The result that follows relates $PF(S)$ to $\mathfrak{F}$.**Lemma 1.** If S is a $\mathcal{C}$ -semigroup, then $\mathfrak{F} \subset PF(S)$.**Proof.** If $f \in \mathfrak{F}$ and $f \notin PF(S)$, then $f + s = h \in \mathcal{H}(S)$ for some $s \in S$. But that means $h - f \in S$, so $h \geq_C f$, and this is a contradiction. $\square$

The next example shows that the converse is not true.

Example 1. Let $\mathcal{C}$ be the cone spanned by $(1,0)$ and $(1,1)$ and $S = \mathcal{C} \setminus \{(1,1), (2,2)\}$. The element $(1,1) \in PF(S)$, but since $(2,2) - (1,1) \in \mathcal{C}$ we know that $(1,1) \notin \mathfrak{F}$.

In [21], the following definition for the Apéry set is given.

Definition 2. Given $b \in S$, $Ap(S, b) = \{a \in S \mid a - b \in \mathcal{H}(S)\}$.Clearly, this set is finite ($|Ap(S, b)| \leq |\mathcal{H}(S)| \leq +\infty$) and $Ap(S, b) - b \in \mathcal{H}(S)$. The following question arises: is there a relationship between $Ap(S, b) - b$ and $\mathfrak{F}$?**Lemma 2.** Let S be a $\mathcal{C}$ -semigroup; then $\mathfrak{F} \subset Ap(S, b) - b$ for all $b \in S \setminus \{0\}$.**Proof.** Let f be in $\mathfrak{F}$; then $f + b = s \in S$. Note that if $f + b = h \in \mathcal{H}(S)$, then $h > f$ and this is a contradiction with $f \in \mathfrak{F}$. Therefore, $s - b \in \mathcal{H}(S)$ and $f \in Ap(S, b) - b$. $\square$ In [21], the authors define the Frobenius elements, denoted by $F(S)$, as gaps such that they are the maximum of the set of gaps for some term order of $\mathbb{N}^p$. We study the relation of this set with $\mathfrak{F}$. Firstly, based on ([22], pp. 72–73), every monomial ordering, $\preceq$, can be considered a weight order for some $a = (a_1, \dots, a_d) \in \mathbb{R}_{\geq}^d$, i.e., $v \preceq w$ if, and only if, $v \cdot a \leq w \cdot a$, with $\cdot$ the inner product. Note that if these inner products are the same, we can choose another vector $\tilde{a}$ as a tiebreaker. This is the same as saying that there exists a

Example 3. Let $\mathcal{C}$ be a cone spanned by $(1, 0)$ and $(1, 1)$ and let $S = \mathcal{C} \setminus \{(1, 1), (2, 2)\}$. In this case, $\mathfrak{F}(S) = \{(2, 2)\}$ and $W = \mathbb{N} \setminus \{4\}$.

Therefore, W is not a numerical semigroup. We are interested in the study of $w(\mathfrak{F})$. In particular, we want to be able to compute the following invariant, which is inspired by the elasticity studied in [20].

Definition 4. Given S a $\mathcal{C}$ -semigroup, we define the quasi-elasticity of $w(\mathfrak{F})$ as $\rho(S) = \frac{\max(w(\mathfrak{F}))}{\min(w(\mathfrak{F}))}$.

Our first questions are: Given a fixed cone $\mathcal{C}$, can a $\mathcal{C}$ -semigroup be found such that ρ is as big as we want? If not, what value bounds it?

Proposition 4. Let $\mathcal{C}$ be a cone; then ρ is not bounded.

Proof. Let S be a $\mathcal{C}$ -semigroup with $\rho(S) = M \in \mathbb{N}$. Let $f_1, f_2 \in \mathfrak{F}$ such that $\omega(f_1) = \min(\omega(\mathfrak{F}))$ and $\omega(f_2) = \max(\omega(\mathfrak{F}))$. By denoting $\mathcal{C}_i = f_i + \mathcal{C}$, we have two options:

- If $f_1 \neq f_2$, then clearly $f_2 \notin \mathcal{C}_1$. We choose $f_3 \in \mathcal{C}_2 \setminus \mathcal{C}_1$. If we consider the $\mathcal{C}$ -semigroup $(\mathcal{C}_1 \setminus f_1) \cup (\mathcal{C}_3 \setminus f_3)$, we obtain this result.
- If $f_1 = f_2$, then we only have to choose f_2 and f_3 such that they are not comparable and $f_2 \neq f_3$. Then, we built the semigroup $(\mathcal{C}_2 \setminus f_2) \cup (\mathcal{C}_3 \setminus f_3)$ and apply the previous option.

Therefore, ρ is unbounded. $\square$

We have the following corollary.

Corollary 1. Given a cone $\mathcal{C}$, we can find a sequence S_k , $k \in \mathbb{N}$ of $\mathcal{C}$ -semigroups such that $\lim_{k \rightarrow \infty} \rho(S_k) = \infty$.

4. Idemaxial Semigroups

In this section, we introduce a new family of $\mathcal{C}$ -semigroups, the idemaxial semigroups. As we recalled in the previous section, if S is a $\mathcal{C}$ -semigroup with extremal rays τ_i with $1 \leq i \leq k$, then $S \cap \tau_i$ is isomorphic to a numerical semigroup. We denote this numerical semigroup as S_i . In this section, we denote the isomorphism as ϕ_i such that $\phi_i(S \cap \tau_i) = S_i$.

Definition 5. We use π_j to denote the hyperplane containing the j -th elements of each $S \cap \tau_i$; based on F_i , the Frobenius number of each S_i and by π_F the hyperplane contains $\phi_i^{-1}(F_i)$. We say that S is an idemaxial semigroup if $S_1 \approx \dots \approx S_k$ and $S = (\cup_{i \geq 1} (\mathcal{C} \cap \pi_i)) \cup \{x \in \mathcal{C} : x \cdot y > 0, y \in \pi_F\}$.

A graphical example of this kind of semigroup can be found in Figure 2. In this case, S_1 and S_2 are isomorphic to $\langle 3, 5 \rangle$. This family is useful because it has good properties. We can, for example, find a bound where we can compute their Frobenius and pseudo-Frobenius set.

Proposition 5. Let S be an idemaxial semigroup such that $\Phi(S \cap \tau_i) = S_i$ for some isomorphism Φ . Let m_i and c_i be the multiplicity and the conductor of S_i , respectively. Let a_i be $\Phi^{-1}(m_i)$ and b_i be $\Phi^{-1}(c_i)$, while π_1 is the hyperplane that contains all $b_i - a_i$ and π_2 is the hyperplane that contains all b_i . Then, $\mathfrak{F} \subset \{x \in \mathcal{C} \setminus S : x \geq_S \pi_1, x \leq_S \pi_2\}$.

Proof. Let $f \in \mathfrak{F}$. Clearly, $f \leq_S \pi_2$. If $f <_S \pi_1$, then $f + a_i <_S \pi_2$ and $f + a_i \notin S$, and this is a contradiction. $\square$

The following proposition establishes the relationship between the Frobenius and pseudo-Frobenius numbers and the hyperplanes associated with an idemaxial semigroup.

Proposition 6. Let S be an idemaxial semigroup, PF_i be the set of pseudo-Frobenius numbers of S_i , and π_j be the hyperplane containing the j -th pseudo-Frobenius number of each S_i . Then, the set of pseudo-Frobenius numbers of S is $\{\cup(\pi_j \cap \mathcal{C})\} \subset PF$.

Proof. By definition, $\{\cup(\pi_j \cap \mathcal{C})\} \subset \mathcal{H}(S)$. Moreover, if $s \in S$ and $f \in PF$, then $f + S \in \pi$ with π -containing elements of S_i for all i , so $f + s \in S$. $\square$

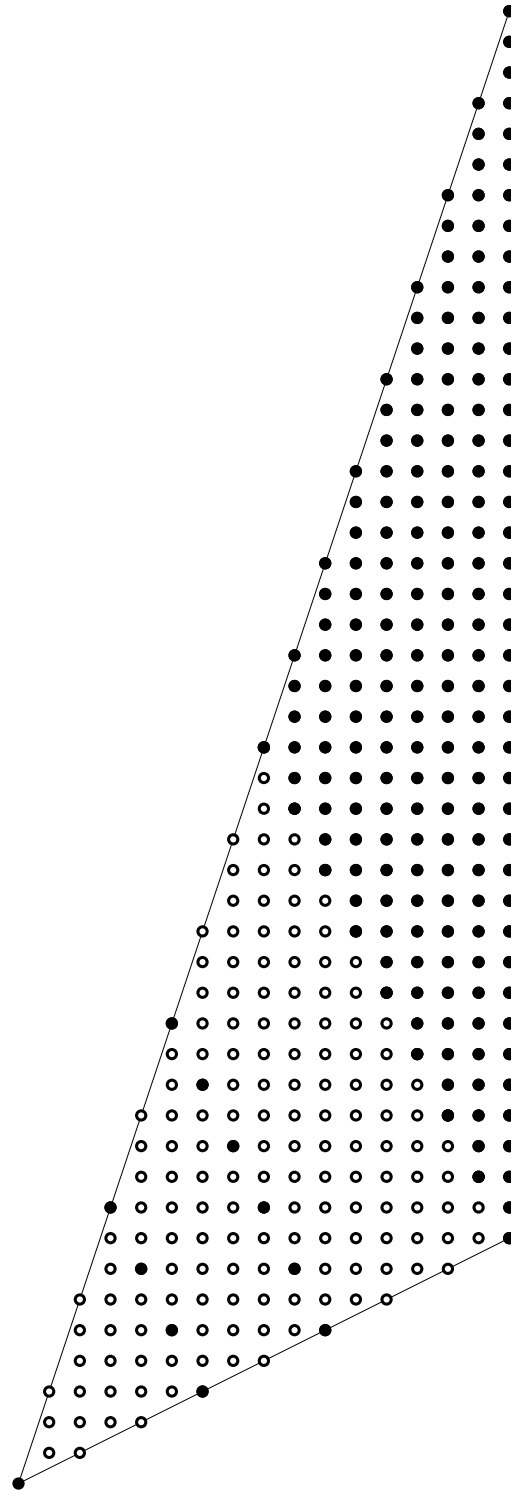


Figure 2. Example of an idemaxial semigroup with $S_1 \approx S_2 \approx \langle 3, 5 \rangle$.

5. Wilf's Conjecture

We cannot end an article about semigroups without a brief mention of Wilf's conjecture. Let S be a numerical semigroup; Wilf's conjecture claims that $e(S)n(S) \geq c(S)$, where $e(S)$, $n(S)$, and $c(S)$ are its embedding dimension, the cardinal of its sporadic elements, and its conductor (see [7]). This conjecture has been generalized in several ways (see [13,24]). In these works, the authors generalise this conjecture for $\mathcal{C}$ -semigroups using monomial total orders. On the other hand, in [25], Wilf's conjecture is extended to a generalized numerical semigroup using partial orders. In this section, we are going to give an even more general conjecture using the induced order of a $\mathcal{C}$ -semigroup.

Let S be a $\mathcal{C}$ -semigroup. We use the notation in [25]:

$$c(S) = |\{a \in \mathcal{C} : a \leq b \text{ for some } b \in H(S)\}|,$$

$$n(S) = |\{a \in S : a \leq b \text{ for some } b \in H(S)\}|,$$

where $H(S) = \mathcal{C} \setminus S$.

Therefore we can pose the General Extended Wilf's conjecture:

$$e(S)n(S) \geq pc(S).$$

This conjecture has been checked in a computational way and no counterexamples have been found.

6. Discussion

The traditional approach to the study of $\mathcal{C}$ -semigroups is to set a total order in a completely arbitrary way and then study the desired properties. This approach has the following issue: if the order is changed, the results obtained may not be true.

In this paper, we replace the total order chosen by researchers with the partial order induced by the semigroup itself. As this order depends exclusively on the semigroup, it does not depend on the researcher, so the results obtained are less artificial. For example, when classical invariants were generalised in the study of semigroups, such as the Frobenius number or multiplicity, as these were completely dependent on the order; they changed as the order changed.

Moreover, since numerical semigroups have only one Frobenius element and only one multiplicity, $\mathcal{C}$ -semigroups were forced to have only one of these elements. However, with our method of study, we have created a set of elements for each of these invariants which is more natural as we are working in a higher dimension. In addition to this, in this work, we have presented a family of $\mathcal{C}$ -semigroups with good properties.

This work aims to lay the foundation for future work on $\mathcal{C}$ -semigroups.

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References

1. Chapman, S.T.; García-Sánchez, P.A.; Llena, D.; Marshall, J. Elements in a numerical semigroup with factorizations of the same length. *Can. Math. Bull.* **2011**, *54*, 39–43. [CrossRef]
2. Rosales, J.C.; García-Sánchez, P.A. *Numerical Semigroup*; Springer: New York, NY, USA, 2009.
3. Rosales, J.C.; García-Sánchez, P.A.; García-García, J.I. Every positive integer is the Frobenius number of a numerical semigroup with three generators. *Math. Scand.* **2008**, *94*, 5–12. [CrossRef]
4. Eliahou, S. Wilf's conjecture and Macaulay's theorem. *J. Eur. Math. Soc.* **2018**, *20*, 2105–2129. [CrossRef]

5. Ramírez Alfonsín, J.L. *The Diophantine Frobenius Problem, Oxford Lectures Series in Mathematics and Its Applications*; Oxford University Press: Oxford, UK, 2005.
6. Bras-Amorós, M. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum* **2008**, *76*, 379–384. [[CrossRef](#)]
7. Wilf, H.S. A circle-of-lights algorithm for the “money-changing problem”. *Am. Math. Mon.* **1978**, *85*, 562–565.
8. Delgado, M. On a question of Eliahou and a conjecture of Wilf. *Math. Z.* **2018**, *288*, 595–627. [[CrossRef](#)]
9. Dhayni, M. Wilf’s conjecture for numerical semigroups. *Palest. J. Math.* **2018**, *7*, 385–396.
10. Fromentin, J.; Hivert, F. Exploring the tree of numerical semigroups. *Math. Comput.* **2016**, *85*, 2553–2568. [[CrossRef](#)]
11. Moreno-Frías, M.A.; Rosales, J.C. Counting the Ideals with a Given Genus of a Numerical Semigroup with Multiplicity Two. *Symmetry* **2023**, *13*, 794. [[CrossRef](#)]
12. Zhai, A. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum* **2013**, *86*, 634–662. [[CrossRef](#)]
13. García-García, J.I.; Marín-Aragón, D.; Vigneron-Tenorio, A. An extension of Wilf’s conjecture to affine semigroup. *Semigroup Forum* **2018**, *96*, 396–408. [[CrossRef](#)]
14. Díaz-Ramírez, J.D.; García-García, J.I.; Marín-Aragón, D.; Vigneron-Tenorio, A. Characterizing affine $\mathcal{C}$ -semigroups. *Ric. Mat.* **2022**, *71*, 283–296. [[CrossRef](#)]
15. García-García, J.I.; Marín-Aragón, D.; Sánchez-Loureiro, A.; Vigneron-Tenorio, A. Some Properties of Affine $\mathcal{C}$ -semigroups. *Results Math.* **2023**, *79*. [10.1007/s00025-023-02056-5](#) [[CrossRef](#)]
16. Cox, D.A.; Little, J.B.; Schenck, H.K. *Toric Varieties*; American Mathematical Society: Providence, RI, USA, 2011.
17. Fulton, W. *Introduction to Toric Varieties*; Princeton University Press: Princeton, NJ, USA, 1993.
18. Trung, N.V.; Hoa, L.T. Affine semigroups and cohen–macaulay rings generated by monomials. *Trans. Am. Math. Soc.* **1986**, *298*, 145–167. [[CrossRef](#)]
19. Delgado, M.; García-Sánchez, P.A.; Robles-Pérez, A.M. Numerical semigroups with a given set of pseudo-Frobenius numbers. *LMS J. Comput. Math.* **2016**, *19*, 186–205. [[CrossRef](#)]
20. Chapman, S.T.; García-García, J.I.; García-Sánchez, P.A.; Rosales, J.C. Computing the elasticity of a Krull monoid. *Linear Algebra Its Appl.* **2001**, *336*, 191–200. [[CrossRef](#)]
21. García-García, J.I.; Ojeda, I.; Rosales, J.C.; Vigneron-Tenorio, A. On pseudo-Frobenius elements of submonoids of $\mathbb{N}^d$. *Collect. Math.* **2020**, *71*, 189–204. [[CrossRef](#)]
22. Cox, D.; Little, J.; O’Shea, D. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*; Springer: Cham, Switzerland, 2015.
23. García-García, J.I.; Moreno-Frías, M.A.; Vigneron-Tenorio, A. Computation of delta sets of numerical monoids. *Monatshefte Math.* **2015**, *178*, 457–472. [[CrossRef](#)]
24. Cisto, C.; DiPasquale, M.; Failla, G.; Flores, F.; Peterson, C.; Utano, R. A generalization of Wilf’s conjecture for generalized numerical semigroups. *Semigroup Forum* **2020**, *101*, 303–325. [[CrossRef](#)]
25. Bilen, M.; Sakran, N. On Generalized Wilf Conjectures. *arXiv* **2023**, arXiv:2306.05530.

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