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Some Equations in Rings Involving Semiprime Ideals and Multiplicative Generalized Semiderivations

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Abstract: This paper examines the commutativity of the quotient ring $\mathcal{F}/\mathrm{Y}$ by utilizing specific differential identities in a general ring $\mathcal{F}$ that contains a semiprime ideal Y. This study particularly focuses on the role of a multiplicative generalized semiderivation ψ, which is associated with a map θ, in determining the commutative nature of the quotient ring.

Keywords: semiprime ideal; semiderivation; generalized semiderivation; multiplicative derivation

MSC: 16W20; 16W25; 16U70; 16U80; 16N60

1. Introduction

In the study of associative rings, the structure and properties of ideals play a crucial role. Prime and semiprime ideals, in particular, are significant due to their ability to characterize the ring's behavior.

Let $\mathcal{F}$ be an associative ring with center Z. A **prime ideal** in an associative ring $\mathcal{F}$ is a proper ideal Y such that for any $v_1, v_2 \in \mathcal{F}$, if $v_1\mathcal{F}v_2 \subseteq \mathrm{Y}$, then either $v_1 \in \mathrm{Y}$ or $v_2 \in \mathrm{Y}$. A ring $\mathcal{F}$ is called a **prime ring** if the zero ideal (0) is a prime ideal, meaning that for any $v_1, v_2 \in \mathcal{F}$, $v_1\mathcal{F}v_2 = (0)$ implies either $v_1 = 0$ or $v_2 = 0$.

A **semiprime ideal** is defined as a proper ideal Y where for any $v_1 \in \mathcal{F}$, if $v_1\mathcal{F}v_1 \subseteq \mathrm{Y}$, then $v_1 \in \mathrm{Y}$. A ring $\mathcal{F}$ is termed a **semiprime ring** if the zero ideal (0) is a semiprime ideal, which indicates that the ideal generated by any element squared lies in the ideal if the element itself does.

An additive mapping θ on $\mathcal{F}$ is a derivation if $\theta(v_1v_2) = \theta(v_1)v_2 + v_1\theta(v_2)$ for all $v_1, v_2 \in \mathcal{F}$. The first study on derivations of prime rings was published in 1957 by Posner [1]. Several mathematicians have since generalized the notion of derivation in various ways, leading to the introduction of different kinds of derivations in the literature.

One such generalization is the concept of semiderivations, introduced by Bergen [2]. An additive mapping $\theta : \mathcal{F} \to \mathcal{F}$ is a semiderivation if there exists a function $\gamma : \mathcal{F} \to \mathcal{F}$, such that

(i) $\theta(v_1v_2) = \theta(v_1)\gamma(v_2) + v_1\theta(v_2) = \theta(v_1)v_2 + \gamma(v_1)\theta(v_2)$ and
(ii) $\theta(\gamma(v_1)) = \gamma(\theta(v_1))$

for all $v_1, v_2 \in \mathcal{F}$. If γ is the identity map, semiderivations reduce to derivations, making semiderivations a broader concept.

In 1991, Bresar [3] introduced the notion of generalized derivations: an additive mapping ψ on $\mathcal{F}$ is called a generalized derivation associated with a derivation θ if $\psi(v_1v_2) = \psi(v_1)v_2 + v_1\theta(v_2)$ for all $v_1, v_2 \in \mathcal{F}$. Similarly, Daif [4] introduced multiplicative derivations by removing the additive condition, defining θ as a multiplicative derivation

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if $\theta(v_1v_2) = \theta(v_1)v_2 + v_1\theta(v_2)$ for all $v_1, v_2 \in \mathcal{F}$. Daif and Tammam El-Sayiad [5] further extended this to multiplicative generalized derivations.

Let S be a nonempty subset of $\mathcal{F}$. A mapping ψ from $\mathcal{F}$ to $\mathcal{F}$ is called centralizing on S if $[\psi(v_1), v_1] \in Z$ for all $v_1 \in S$, and commuting on S if $[\psi(v_1), v_1] = 0$ for all $v_1 \in S$. This concept has been generalized to θ-commuting maps: $\psi : \mathcal{F} \to \mathcal{F}$ is a θ-commuting map on S if $[\psi(v_1), v_1] \in \theta$ for all $v_1 \in S$ and some $\theta \subseteq \mathcal{F}$.

Daif and Bell in 1992 [6] showed that an ideal I of a semiprime ring $\mathcal{F}$ is contained in the center of $\mathcal{F}$ if any of the following conditions are satisfied for all $v_1, v_2 \in I$:

$$\theta([v_1, v_2]) = [v_1, v_2] \text{ and } \theta([v_1, v_2]) = -[v_1, v_2].$$

In particular, if $I = \mathcal{F}$, then $\mathcal{F}$ is commutative. This result has been extended by many authors, who have also obtained commutativity results for prime or semiprime rings with derivations satisfying certain polynomial identities (see, e.g., [7–12]).

The commutativity of prime and semiprime rings admitting derivations remains an active area of research. Recent approaches involve examining commutativity conditions in quotient rings $\mathcal{F}/\mathrm{Y}$ rather than assuming the ring is prime (see, e.g., [13–16]).

The principal aim of this study is to explore the definition of semiderivation given by Bergen. Inspired by the notion of multiplicative generalized derivation [17], we introduce the concept of a multiplicative generalized semiderivation. A mapping ψ on $\mathcal{F}$ is a multiplicative generalized semiderivation if there exists a multiplicative semiderivation θ associated with a map γ on $\mathcal{F}$, such that

(i) $\psi(v_1v_2) = \psi(v_1)v_2 + \gamma(v_1)\theta(v_2) = \theta(v_1)\gamma(v_2) + v_1\psi(v_2)$ and
(ii) $\psi(\gamma(v_1)) = \gamma(\psi(v_1))$

for all $v_1, v_2 \in \mathcal{F}$.

This study aims to investigate identities involving multiplicative generalized semiderivations in semiprime ideals. Since every prime ideal is semiprime, but the converse is not true, examining identities involving multiplicative generalized semiderivations in semiprime ideals is appropriate. We will prove Daif's theorem and discuss some functional identities, extending and unifying several known results.

2. Main Results

Throughout this paper, $\mathcal{F}$ will be an arbitrary ring, Y will be a semiprime ideal of $\mathcal{F}$, ψ will be a multiplicative generalized semiderivation associated with a map θ of $\mathcal{F}$, and γ is an epimorphism on $\mathcal{F}$. For any $v_1, v_2 \in \mathcal{F}$, as usual, $[v_1, v_2] = v_1v_2 - v_2v_1$ and $v_1ov_2 = v_1v_2 + v_2v_1$ will show the associated Lie and Jordan product, respectively. The following expressions will make our work easier for us:

$$\begin{aligned}
[v_1, v_2v_3] &= v_2[v_1, v_3] + [v_1, v_2]v_3 \\
[v_1v_2, v_3] &= [v_1, v_3]v_2 + v_1[v_2, v_3] \\
v_1o(v_2v_3) &= (v_1ov_2)v_3 - v_2[v_1, v_3] = v_2(v_1ov_3) + [v_1, v_2]v_3 \\
(v_1v_2)ov_3 &= v_1(v_2ov_3) - [v_1, v_3]v_2 = (v_1ov_3)v_2 + v_1[v_2, v_3].
\end{aligned}$$

Theorem 1. *Let $\mathcal{F}$ be a ring admitting a semiprime ideal* Y *and a multiplicative generalized semiderivation ψ associated with a map θ, and γ be an epimorphism on $\mathcal{F}$. If any one of the following conditions is satisfied for all $v_1, v_2 \in \mathcal{F}$, then $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$.*

(i) $\psi([v_1, v_2]) \in \mathrm{Y}$;
(ii) $\psi(v_1ov_2) \in \mathrm{Y}$;
(iii) $\psi([v_1, v_2]) \pm v_1v_2 \in \mathrm{Y}$;
(iv) $\psi([v_1, v_2]) \pm v_2v_1 \in \mathrm{Y}$;
(v) $\psi(v_1ov_2) \pm v_1v_2 \in \mathrm{Y}$;
(vi) $\psi(v_1ov_2) \pm v_2v_1 \in \mathrm{Y}$.

Proof. (i) We are assuming that

$$\psi([v_1, v_2]) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{1}$$

Substituting v_2v_1 in place of v_2 in (1), we obtain

$$\psi([v_1, v_2]v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This implies that

$$\psi([v_1, v_2])v_1 + \gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Using our hypothesis, we have

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{2}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathcal{F}$ in the last expression, we obtain

$$\gamma([v_1, v_2v_3])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Since γ is an epimorphism of $\mathcal{F}$ and using (2), we can write this equation as

$$\gamma([v_1, v_2])\gamma(v_3)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}. \tag{3}$$

Again replacing v_3 with v_3v_1 in (3), we obtain

$$\gamma([v_1, v_2])\gamma(v_3)\gamma(v_1)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}. \tag{4}$$

By right-multiplying (3) by $\gamma(v_1)$, we obtain

$$\gamma([v_1, v_2])\gamma(v_3)\theta(v_1)\gamma(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}. \tag{5}$$

Subtracting (4) from (5), we arrive at

$$\gamma([v_1, v_2])\gamma(v_3)[\gamma(v_1), \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Since γ is an epimorphism of $\mathcal{F}$, we have

$$[\gamma(v_1), t]\mathcal{F}[\gamma(v_1), \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1, t \in \mathcal{F}. \tag{6}$$

Writing $\theta(v_1)$ instead of t in the last equation, we obtain

$$[\gamma(v_1), \theta(v_1)]\mathcal{F}[\gamma(v_1), \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

Based on the semiprimeness of Y, we arrive at

$$[\gamma(v_1), \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}. \tag{7}$$

On the other hand, by using ψ as a multiplicative generalized semiderivation of $\mathcal{F}$ and this equation, we obtain

$$\begin{aligned} \psi(v_1) &= \psi(v_1)v_1 + \gamma(v_1)\theta(v_1) = \theta(v_1)\gamma(v_1) + v_1\psi(v_1) \\ &= \psi(v_1)v_1 - v_1\psi(v_1) = \theta(v_1)\gamma(v_1) - \gamma(v_1)\theta(v_1) \in \mathrm{Y}, \end{aligned}$$

and so

$$[\psi(v_1), v_1] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

This completes the proof.

(ii) Assume that

$$\psi(v_1 o v_2) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{8}$$

Writing $v_2 v_1$ instead of v_2 in (8), we have

$$\psi((v_1 o v_2)v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Since ψ is a multiplicative generalized semiderivation of $\mathcal{F}$, we obtain

$$\psi(v_1 o v_2)v_1 + \gamma(v_1 o v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Using (8), we obtain

$$\gamma(v_1 o v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{9}$$

If we put $v_2 v_3$ instead of v_2 in the last equation, we have

$$\gamma(v_1 o v_2 v_3)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Expanding this equation and using (9), we arrive at

$$\gamma([v_1, v_2])\gamma(v_3)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Furthermore, the proof follows directly from Theorem 1, after Equation (3). By applying the same technique, we obtain that $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$.

(iii) Assume that $\psi = 0$. According to the hypothesis, $v_1 v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. We can easily obtain $v_1 \mathcal{F} v_1 \in \mathrm{Y}$. Based on the semiprimeness of $\mathcal{F}$, we arrive at $v_1 \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$, and so $\mathrm{Y} = \mathcal{F}$, a contradiction.
Now, we assume that $\psi \neq 0$. We also assume that

$$\psi([v_1, v_2]) \pm v_1 v_2 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{10}$$

Replacing v_2 with $v_2 v_1$ in (10), we obtain

$$\psi([v_1, v_2]v_1) \pm v_1 v_2 v_1 \in \mathrm{Y},$$

which can be expanded as

$$\psi([v_1, v_2])v_1 \pm \gamma([v_1, v_2])\theta(v_1) \pm v_1 v_2 v_1 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This can be written as

$$(\psi([v_1, v_2]) \pm v_1 v_2)v_1 \pm \gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Using the hypothesis, this equation reduces to

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This expression is identical to Equation (2) in the proof of Theorem 1. By employing similar expressions within the proof of Theorem 1, we can obtain the desired result.

(iv) Assume that $\psi = 0$. If we apply similar operations to those applied in Theorem 1 (iii), a contradiction is obtained. Alternatively, assume that $\psi \neq 0$. Based on the hypothesis, we then have

$$\psi([v_1, v_2]) \pm v_2 v_1 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{11}$$

Replacing v_2 with $v_2 v_1$ in the above equation and using it accordingly, we can easily obtain

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

By applying the same techniques used after Equation (2) in the proof of Theorem 1, we obtain the required result.

(v) Assume that $\psi = 0$. If we perform the same operations as in Theorem 1 (iii), a contradiction arises. Thus, assume now that $\psi \neq 0$. Based on the hypothesis, we then have

$$\psi(v_1 o v_2) \pm v_1 v_2 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{12}$$

Replacing v_2 with $v_2 v_1$ in (12) and using this equation accordingly, we find that

$$\gamma(v_1 o v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This expression is identical to Equation (9) in the proof of Theorem 1. By using expressions similar to those used in the proof of Theorem 1, we can obtain the desired result.

(vi) Assume that $\psi = 0$. If similar operations are performed in Theorem 1 (iii), a contradiction is reached. Now assume that $\psi \neq 0$. Based the hypothesis, we then have

$$\psi(v_1 o v_2) \pm v_2 v_1 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{13}$$

Replacing v_2 with $v_2 v_1$ in (13) and using this equation, we arrive at the following conclusion

$$\gamma(v_1 o v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Again using the same lines as those used in the proof of Theorem 1, we conclude that $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$. □

Theorem 2. *Let $\mathcal{F}$ be a ring admitting a semiprime ideal* Y *and ψ be a multiplicative generalized semiderivation associated with a map θ, with γ being an epimorphism on $\mathcal{F}$. If any one of the following conditions is satisfied for all $v_1, v_2 \in \mathcal{F}$, then $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$.*

(i) $\psi(v_1 v_2) \pm v_1 v_2 \in \mathrm{Y}$;
(ii) $\psi(v_1 v_2) \pm v_2 v_1 \in \mathrm{Y}$;
(iii) $\psi(v_1 v_2) \pm [v_1, v_2] \in \mathrm{Y}$;
(iv) $\psi(v_1 v_2) \pm (v_1 o v_2) \in \mathrm{Y}$.

Proof. (i) Assume that $\psi = 0$. Based on the hypothesis, $v_1 v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. This expression is identical to the one in Theorem 1 (iii). We now assume that $\psi \neq 0$. Using our hypothesis, we obtain

$$\psi(v_1 v_2) \pm v_1 v_2 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{14}$$

Taking $v_2 v_3$ instead of v_2 within (14), we have

$$\psi(v_1 v_2 v_3) \pm v_1 v_2 v_3 \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Since ψ is a multiplicative generalized semiderivation of $\mathcal{F}$, we obtain

$$\psi(v_1 v_2) v_3 + \gamma(v_1 v_2)\theta(v_3) \pm v_1 v_2 v_3 \in \mathrm{Y},$$

which can be written as

$$(\psi(v_1 v_2) \pm v_1 v_2) v_3 + \gamma(v_1 v_2)\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Using the hypothesis, we arrive at

$$\gamma(v_1 v_2)\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Since γ is an epimorphism of $\mathcal{F}$, we have

$$\gamma(v_1)\gamma(v_2)\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}, \tag{15}$$

and so

$$\gamma(v_2)\theta(v_3)\mathcal{F}\gamma(v_2)\theta(v_3) \in \mathrm{Y}, \text{ for all } v_2, v_3 \in \mathcal{F}.$$

Due to the semiprimeness of Y, we arrive at

$$\gamma(v_2)\theta(v_3) \in \mathrm{Y}, \text{ for all } v_2, v_3 \in \mathcal{F}.$$

On the other hand, we know that $\psi(v_1v_2) \pm v_1v_2 \in \mathrm{Y}$ according to our hypothesis, and so $\psi(v_1)v_2 + \gamma(v_1)\theta(v_2) \pm v_1v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. Using the $\gamma(v_1)\theta(v_2) \in \mathrm{Y}$ in the last expression, we understand that

$$(\psi(v_1) \pm v_1)v_2 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}, \tag{16}$$

and so

$$v_1(\psi(v_1) \pm v_1)\mathcal{F}v_1(\psi(v_1) \pm v_1) \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

The semiprimeness of Y forces us to conclude that

$$v_1(\psi(v_1) \pm v_1) \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}. \tag{17}$$

Taking v_1 instead of v_2 in (16) and subtracting the relations (16) and (17), we find that

$$[(\psi(v_1) \pm v_1), v_1] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F},$$

and so

$$[\psi(v_1), v_1] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

This completes the proof.

(ii) Assume that $\psi = 0$. Based on the hypothesis, $v_1v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. This expression is the same as the expression in Theorem 1 (iii). We now assume that $\psi \neq 0$. Using our hypothesis, we obtain

$$\psi(v_1(v_2v_3)) \pm (v_2v_3)v_1 \in \mathrm{Y}$$

and

$$\psi((v_1v_2)v_3) \pm v_3(v_1v_2) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Subtracting these two equations, we have

$$v_2v_3v_1 \mp v_3v_1v_2 \in \mathrm{Y},$$

and so

$$[v_2, v_3v_1] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Writing v_3 instead of v_2 in this equation and using it accordingly, we find that

$$v_3[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}. \tag{18}$$

Replacing v_1 with v_2v_1 in (18) and using (18), we obtain

$$v_3v_2[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F},$$

and so

$$v_3v_2v_3[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3, v_3 \in \mathcal{F}. \tag{19}$$

Similarly, (18) reveals that

$$v_2v_3v_3[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3, v_3 \in \mathcal{F}. \tag{20}$$

Subtracting (19) from (20), we find that

$$[v_2, v_3]v_3[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3, v_3 \in \mathcal{F},$$

and so

$$[v_1, v_3]\mathcal{F}[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Due to the semiprimeness of Y, we obtain

$$[v_1, v_3] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}.$$

Replacing $\psi(v_1)v_3$ with v_3 in this equation and using it accordingly, we obtain

$$[v_1, \psi(v_1)]v_3 \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F},$$

and so

$$[v_1, \psi(v_1)]\mathcal{F}[v_1, \psi(v_1)] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

Again using the semiprimeness of Y, we obtain the required result.

(iii) Assume that

$$\psi(v_1v_2) \pm [v_1, v_2] \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Define the map $\gamma : \mathcal{F} \to \mathcal{F}, \gamma(\mathcal{F}) = \psi(\mathcal{F}) \pm \mathcal{F}$, for all $\mathcal{F} \in \mathcal{F}$. γ is a multiplicative generalized semiderivation associated with a nonzero map θ of $\mathcal{F}$. Based on the hypothesis, we have $\gamma(v_1v_2) \pm v_1v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. Hence, the conclusion is obtained via Theorem 2 (i).

(iv) Using our hypothesis, we obtain

$$\psi(v_1v_2) \pm (v_1ov_2) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Define the map $\gamma : \mathcal{F} \to \mathcal{F}, \gamma(\mathcal{F}) = \psi(\mathcal{F}) \pm \mathcal{F}$, for all $\mathcal{F} \in \mathcal{F}$. γ is a multiplicative generalized semiderivation associated with a nonzero map θ of $\mathcal{F}$, such that $\gamma(v_1v_2) \pm v_2v_1 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. Using Theorem 2 (ii), we find that $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$. □

Theorem 3. *Let $\mathcal{F}$ be a ring admitting a semiprime ideal* Y *and ψ be a multiplicative generalized semiderivation associated with a map θ, with γ being an epimorphism on $\mathcal{F}$. If any one of the following conditions is satisfied for all $v_1, v_2 \in \mathcal{F}$, then $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$.*

(i) $\psi([v_1, v_2]) \pm [v_1, v_2] \in \mathrm{Y}$;
(ii) $\psi([v_1, v_2]) \pm (v_1ov_2) \in \mathrm{Y}$;
(iii) $\psi(v_1ov_2) \pm (v_1ov_2) \in \mathrm{Y}$;
(iv) $\psi(v_1ov_2) \pm [v_1, v_2] \in \mathrm{Y}$.

Proof. (i) We assume that

$$\psi([v_1, v_2]) \pm [v_1, v_2] \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{21}$$

Writing v_2 instead of v_2v_1 in (21), we obtain

$$\psi([v_1, v_2]v_1) \pm [v_1, v_2]v_1 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Using the definition of ψ, we have

$$\psi([v_1, v_2])v_1 + \gamma([v_1, v_2])\theta(v_1) \pm [v_1, v_2]v_1 \in \mathrm{Y}.$$

Based on the hypothesis, we find that

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This expression is exactly the same as Equation (2) in the proof of Theorem 1. By using expressions similar to those used in the proof of Theorem 1, we obtain the desired result.

(ii) By using our hypothesis, we obtain

$$\psi([v_1, v_2]) \pm (v_1 o v_2) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{22}$$

Replacing v_2 with v_2v_1 in (22) and using this equation, we find that

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This expression is exactly the same as Equation (2) in the proof of Theorem 1. By using expressions similar to those used in that proof, we obtain the desired result.

(iii) Using our hypothesis, we obtain

$$\psi(v_1 o v_2) \pm (v_1 o v_2) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{23}$$

Replacing v_2 with v_2v_1 in (23), we obtain

$$\psi((v_1 o v_2)v_1) \pm (v_1 o v_2)v_1 \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Since ψ is a multiplicative generalized semiderivation of $\mathcal{F}$, we obtain

$$\psi(v_1 o v_2)v_1 + \gamma(v_1 o v_2)\theta(v_1) \pm (v_1 o v_2)v_1 \in \mathrm{Y}.$$

Based on the hypothesis, we find that

$$\gamma(v_1 o v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

This expression is exactly the same as Equation (9) in the proof of Theorem 1. By using expressions similar to those used in that proof, we obtain the desired result.

(iv) Using our hypothesis, we obtain

$$\psi(v_1 o v_2) \pm [v_1, v_2] \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{24}$$

Replacing v_2 with v_2v_1 in (24) and using (24), we obtain

$$\gamma(v_1 o v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Using the same reasoning as applied after Equation (9) in the proof of Theorem 1, we arrive at the desired result □

Theorem 4. *Let $\mathcal{F}$ be a ring admitting a semiprime ideal* Y *and ψ be a multiplicative generalized semiderivation associated with a map θ, with γ being an epimorphism on $\mathcal{F}$. If any of the following conditions are satisfied for all $v_1, v_2 \in \mathcal{F}$, $m, n \in \mathbb{N}$, then $[\psi(v_1), v_1] \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$.*

(i) $\psi([v_1, v_2]) \pm v_1^m [v_1, v_2] v_1^n \in \mathrm{Y}$;
(ii) $\psi([v_1, v_2]) \pm v_1^m (v_1 o v_2) v_1^n \in \mathrm{Y}$;
(iii) $\psi(v_1 o v_2) \pm v_1^m (v_1 \circ v_2) v_1^n \in \mathrm{Y}$;
(iv) $\psi(v_1 o v_2) \pm v_1^m [v_1, v_2] v_1^n \in \mathrm{Y}$.

Proof. (i) Based on our hypothesis

$$\psi([v_1, v_2]) \pm v_1^m[v_1, v_2]v_1^n \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{25}$$

Taking v_2 instead of v_2v_1 in (25), we obtain

$$\psi([v_1, v_2v_1]) \pm v_1^m[v_1, v_2v_1]v_1^n \in \mathrm{Y},$$

and so

$$\psi([v_1, v_2]v_1) \pm v_1^m[v_1, v_2]v_1^{n+1} \in \mathrm{Y},$$

which can be expanded as

$$\psi([v_1, v_2])v_1 + \gamma([v_1, v_2])\theta(v_1) \pm v_1^m[v_1, v_2]v_1^{n+1} \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Using (25), we find that

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{26}$$

Equation (2) yields the same expression. Similar expressions can be used in the proof of Theorem 1. Thus, the proof is complete.

(ii) Let us assume that

$$\psi([v_1, v_2]) \pm v_1^m(v_1ov_2)v_1^n \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{27}$$

Writing v_2v_1 instead of v_2 in (27) and using (27), we reveal that

$$\gamma([v_1, v_2])\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Equation (2) yields the same expression. Similar expressions can be used in the proof of Theorem 1. Therefore, the proof is complete.

(iii) We assume that

$$\psi(v_1ov_2) \pm v_1^m(v_1 \circ v_2)v_1^n \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{28}$$

Taking v_2 instead of v_2v_1 in (28) and using this, we find that

$$\gamma(v_1ov_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{29}$$

By applying the same arguments used after Equation (9) in the proof of Theorem 1, we obtain the required result.

(iv) Let us assume that

$$\psi(v_1ov_2) \pm v_1^m[v_1, v_2]v_1^n \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{30}$$

Replacing v_2 with v_2v_1 in (30) and using (30), we find that

$$\gamma(v_1ov_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

By using the same arguments as those following Equation (9) in the proof of Theorem 1, we obtain the required result. □

Theorem 5. *Let $\mathcal{F}$ be a* 2*-torsion-free ring with* Y *being a semiprime ideal of $\mathcal{F}$. Suppose that $\mathcal{F}$ admits a multiplicative generalized semiderivation ψ associated with a map θ. If any of the following conditions are satisfied for all $v_1, v_2 \in \mathcal{F}$, then θ is a* Y*-commuting map on $\mathcal{F}$.*

(i) $\psi(v_1)\psi(v_2) \pm v_1v_2 \in \mathrm{Y}$;
(ii) $\psi(v_1)\psi(v_2) \pm v_2v_1 \in \mathrm{Y}$.

Proof. (i) Assume that $\psi = 0$. Based on the hypothesis, $v_1v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. This expression is the same as the one in Theorem 1 (iii). We now assume that $\psi \neq 0$. Using the hypothesis, we have

$$\psi(v_1)\psi(v_2) \pm v_1v_2 \in \mathrm{Y} \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Replacing v_2 with v_2v_3 in the hypothesis and using the hypothesis, we find that

$$\begin{aligned}\psi(v_1)\psi(v_2v_3) \mp v_1v_2v_3 &= \psi(v_1)(\psi(v_2)v_3 + \gamma(v_2)\theta(v_3)) \mp v_1v_2v_3 \\ &= (\psi(v_1)\psi(v_2) \mp v_1v_2)v_3 + \psi(v_1)\gamma(v_2)\theta(v_3) \in \mathrm{Y}.\end{aligned}$$

Again using the hypothesis, we have

$$\psi(v_1)\gamma(v_2)\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}. \tag{31}$$

Since γ is an epimorphism of $\mathcal{F}$, we have

$$\psi(v_1)v_2\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}. \tag{32}$$

Replacing v_1 with tv_1, $t \in \mathcal{F}$, we have

$$\psi(tv_1)v_2\theta(v_3) = (\psi(t)v_1 + \gamma(t)\theta(v_1))v_2\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3, t \in \mathcal{F}.$$

Applying (32), we have

$$\gamma(t)\theta(v_1)v_2\theta(v_3) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3, t \in L.$$

Since γ is an epimorphism of $\mathcal{F}$, we have $v_2\theta(v_1)v_3v_2\theta(v_1) \in \mathrm{Y}$, for all $v_1, v_2, v_3 \in \mathcal{F}$. This implies that $v_2\theta(v_1) \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$, and so $\theta(v_1) \in \mathrm{Y}$, for all $v_1 \in \mathcal{F}$. Hence, we obtain $[\theta(v_1), v_1] \in \mathrm{Y}$.

(ii) Assume that $\psi = 0$. Based on the hypothesis, $v_1v_2 \in \mathrm{Y}$, for all $v_1, v_2 \in \mathcal{F}$. This expression is the same as the expression in Theorem 1 (iii). We now assume that $\psi \neq 0$. Using the hypothesis, we have

$$\psi(v_1)\psi(v_2) - v_2v_1 \in \mathrm{Y} \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{33}$$

Replacing v_2 with v_2v_1 in (33), we obtain

$$\begin{aligned}\psi(v_1)\psi(v_2v_1) - v_2v_1 &= \psi(v_1)(\psi(v_2)v_1 + \gamma(v_2)\theta(v_1)) - v_2v_1 \\ &= (\psi(v_1)\psi(v_2) - v_2v_1)v_1 + \psi(v_1)\gamma(v_2)\theta(v_1) \in \mathrm{Y}.\end{aligned}$$

Using the hypothesis, we obtain

$$\psi(v_1)\gamma(v_2)\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Since γ is an epimorphism of $\mathcal{F}$, we have

$$\psi(v_1)v_2\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Left-multiplying this equation by $\psi(v_3)$ for $v_3 \in \mathcal{F}$ and using it accordingly, we obtain

$$\psi(v_3)\psi(v_1)v_2\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Based on the hypothesis, we can express the last equation as follows:

$$v_1v_3v_2\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}. \tag{34}$$

Replacing v_3 with $\theta(v_1)$, we have

$$v_1\theta(v_1)v_2\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Taking v_2 instead of v_2v_1, we obtain

$$v_1\theta(v_1)v_2v_1\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}.$$

Since Y is a semiprime ideal of $\mathcal{F}$, we conclude that

$$v_1\theta(v_1) \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}. \tag{35}$$

Right-multiplying by v_1 in (34), we obtain

$$v_1v_3v_2\theta(v_1)v_1 \in \mathrm{Y}, \text{ for all } v_1,, v_2, v_3 \in \mathcal{F}.$$

Left-multiplying this equation by $\theta(v_1)$, we obtain

$$\theta(v_1)v_1v_3v_2\theta(v_1)v_1 \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Since Y is a semiprime ideal of $\mathcal{F}$, we conclude that

$$\theta(v_1)v_1 \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}. \tag{36}$$

Subtracting (35) from (36), we find that

$$[v_1, \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

This completes the proof. □

Theorem 6. *Let $\mathcal{F}$ be a 2-torsion-free ring with Y being a semiprime ideal of $\mathcal{F}$. Suppose that $\mathcal{F}$ admits a multiplicative generalized semiderivation ψ associated with a nonzero multiplicative semiderivation θ. If any of the following conditions are satisfied for all $v_1, v_2 \in \mathcal{F}$, then θ is a Y-commuting map on $\mathcal{F}$.*

(i) $\theta(v_1)\psi(v_2) \pm [v_1, v_2] \in \mathrm{Y}$;
(ii) $\theta(v_1)\psi(v_2) \pm v_1ov_2 \in \mathrm{Y}$.

Proof. (i) We obtain

$$\theta(v_1)\psi(v_2) \pm [v_1, v_2] \in \mathrm{Y}, \text{ for all } v_1, v_2 \in \mathcal{F}. \tag{37}$$

Replacing v_1 with $v_1v_3, v_3 \in \mathcal{F}$ in (37), we have

$$\theta(v_1)\gamma(v_3)\psi(v_2) + v_1\theta(v_3)\psi(v_2) \pm v_1[v_3, v_2] \pm [v_1, v_2]v_3 \in \mathrm{Y} \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Using (37), we see that

$$\theta(v_1)\gamma(v_3)\psi(v_2) \pm [v_1, v_2]v_3 \in \mathrm{Y}, \text{ for all } v_1, v_2, v_3 \in \mathcal{F}.$$

Substituting v_2 for v_1 in the last expression, we have

$$\theta(v_1)\gamma(v_3)\psi(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}.$$

Since γ is an epimorphism of $\mathcal{F}$, we have

$$\theta(v_1)v_3\psi(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}. \tag{38}$$

Replacing v_3 with $\theta(v_3)$, we obtain

$$\theta(v_1)\theta(v_3)\psi(v_1) \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}.$$

Using the hypothesis and $\mathrm{Y} \in \mathrm{Y}$, we have

$$\theta(v_1)([v_3, v_1] + \mathrm{Y}) \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F},$$

and so

$$\theta(v_1)[v_3, v_1] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}. \tag{39}$$

Writing $v_3\theta(v_1)$ instead of v_3 in (39) and using (39), we find that

$$\theta(v_1)v_3[v_1, \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}. \tag{40}$$

Replacing v_3 with v_1v_3 in (40), we find that

$$\theta(v_1)v_1v_3[v_1, \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}. \tag{41}$$

Multiplying (40) on the left by v_1, we have

$$v_1\theta(v_1)v_3[v_1, \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F}. \tag{42}$$

Subtracting (41) from (42), we determine that

$$[v_1, \theta(v_1)]v_2[v_1, \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1, v_3 \in \mathcal{F};$$

that is,

$$[v_1, \theta(v_1)]\mathcal{F}[v_1, \theta(v_1)] \subseteq \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F}.$$

Since Y is a semiprime ideal of $\mathcal{F}$, we conclude that

$$[v_1, \theta(v_1)] \in \mathrm{Y}, \text{ for all } v_1 \in \mathcal{F},$$

and so θ is a Y-commuting map on $\mathcal{F}$.

(ii) We have

$$\theta(v_1)\psi(v_2) \pm v_1ov_2 \in \mathrm{Y}. \tag{43}$$

If we write $v_1v_3, v_3 \in \mathcal{F}$ instead of v_1 in (43), then, since $\mathcal{F}$ is 2-torsion-free, we find that

$$\theta(v_1)\gamma(v_3)\psi(v_2) + v_1\theta(v_3)\psi(v_2) \pm (v_1(v_3ov_2) - [v_1, v_2]v_3) \in \mathrm{Y}.$$

Using (43), we obtain

$$\theta(v_1)\gamma(v_3)\psi(v_2) \mp [v_1, v_2]v_3 \in \mathrm{Y}. \tag{44}$$

Replacing v_2 with v_1 in (44), we have

$$\theta(v_1)\gamma(v_3)\psi(v_2) \in \mathrm{Y},$$

and so

$$\theta(v_1)v_3\psi(v_2) \in \mathrm{Y}.$$

Since γ is an epimorphism of $\mathcal{F}$, we can use the same arguments as those following Equation (38) to show that $[\theta(v_1), v_1] \in \mathrm{Y}$. This completes the proof. □

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