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# Unique Solutions for Caputo Fractional Differential Equations with Several Delays Using Progressive Contractions

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**Abstract:** We take into account a nonlinear Caputo fractional-order differential equation including several variable delays. We examine whether the solutions to the Caputo fractional-order differential equation taken under consideration, which has numerous variable delays, are unique. In the present study, first, we will apply the method of progressive contractions, which belongs to T.A. Burton, to Caputo fractional-order differential equation, including multiple variable delays, which has not yet appeared in the relevant literature by this time. The significant point of the method of progressive contractions consists of a very flexible idea to discuss the uniqueness of solutions for various mathematical models. Lastly, we provide two examples to demonstrate how this paper's primary outcome can be applied.

**Keywords:** unique solution; Caputo fractional derivative; differential equation; several variable delays; progressive contractions

**MSC:** 34K20; 45D05; 45G05; 45K05; 47H09; 47H10

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## 1. Introduction

In the majority of the current literature, authors prove the existence and uniqueness of solutions of differential equations by writing them as integral equations and applying some type of fixed-point theorem that can be tedious and challenging, often patching together solutions on short intervals after making complicated translations. Examples of direct fixed-point mappings can be seen in Anderson and Avery [1], Burton [2], Burton and Zhang [3], and Mureşan and Nica [4]. In each case, there are excellent reasons for not first converting DEs to IEs. Across these difficulties, in 2016, Burton [5] developed a fascinating method known as progressive contractions to investigate the global existence and uniqueness of solutions of an integro-differential equation. As a result of the technique of progressive contractions, researchers avoided the process of establishing existence on a little interval, translating the equation to a later starting time, and then connecting a solution on the new interval to the preceding one. Later, the technique of progressive contractions was extended and applied to delay IEs, scalar nonlinear IEs, fractional differential equations of the Riemann–Liouville type, and nonlinear IEs in a Banach space by Burton and Purnaras [6], Burton [7], Burton [8], Ilea and Otrocol [9], respectively. Indeed, this technique is quite effective and adaptable, and easy to investigate the global existence and uniqueness of solutions that of IEs, IDEs, FDEs, etc.; see Burton [5,7,8], Burton and Purnaras [6], Ilea and Otrocol [9], Tunç et al. [10,11], and the references of these papers. In fact, it is highly desirable for researchers to have an appropriate technique, like the progressive contractions technique, for analyzing the uniqueness of solutions in an infinite interval case. We also

direct readers to the construction of a fractional non-polynomial spline to approximate solutions of the KdV equation (Yousif and Hamasalh [12]).

We will now briefly review a few works related to different mathematical models where progressive contractions were used as the main techniques in the proofs of the outcomes of those works.

In 2016, Burton [5] presented a straightforward proof regarding the global existence and uniqueness of a solution of the following IDE:

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds.$$

The proofs of Burton [5], Theorem 2.1, Theorem 2.2 employed the concept of progressive contractions, which is attributed to T. A. Burton. It is an extensive fixed-point theorem for differential equations.

Subsequently, in 2017, Burton and Purnaras [6] expanded the use of progressive contractions to obtain global unique solutions of delay IEs of the following type:

$$x(t) = L(t) + g(t, x(t)) + \int_0^t A(t-s)[f(s, x(s)) + f(s, x(s - r(s)))] ds.$$

In Burton and Purnaras [6], Lemma 2.1 is the key result, which extends progressive contractions to delay IEs. Therefore, in Burton and Purnaras [6], considering Lemma 2.1, subsequently utilizing the method of progressive contractions, sufficient conditions were constructed regarding global unique solutions of this delay IE.

In 2017, the following scalar FDE of the Riemann–Liouville type was also taken into consideration by Burton [8]:

$$D^q x(t) = f(t, x(t)) \text{ with } \lim_{t \to} t^{1-q} x(t) = x^0.$$

Burton [8] presented a simple proof of the global existence and uniqueness of a solution of this FDE and then parlayed this into a solution on $(0, \infty)$. The proof of [8] utilized the concept of progressive contractions. This technique stays away from the traditional techniques that imply endless repeated translations.

Following that, in 2019, Burton [7] showed a very easy and direct method of obtaining a unique solution on $[0, \infty)$ for the scalar nonlinear IE that is as follows:

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds.$$

In Burton [7], once more, the author used the progressive contraction technique to obtain new qualitative outcomes. Burton [7] concluded that there is only one function $\xi(t)$ that can satisfy this IE on $[0, \infty)$ without resorting to any of the classical translations and extensions of solutions, which in actuality must invoke Zorn's lemma and can encounter challenges.

The progressive contractions method of Burton in [5] was expanded upon in 2020 by Ilea and Otrocol [9] regarding the subsequent IEs in a Banach space:

$$x(t) = \int_0^t K(t, s, x(s)) ds$$

and

$$x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s)) ds.$$

Ilea and Otrocol [9] obtained sufficient conditions to guarantee aforementioned IEs have a unique solution in a Banach space.

In 2023, Tunç et al. [13] examined the subsequent nonlinear and Hammerstein-type FIE in the Banach space:

$$x(t) = F(x(t)) + G(t, x(t)) + H(t, x(t)) \int_0^t [K(t, s, x(s)) + Q(t, s, q(x(s)))]ds.$$

In [13], utilizing the complete metric and the Chebyshev norm, Tunç et al. enhanced Burton's progressive contractions technique to the aforementioned nonlinear Hammerstein-type FIE in a Banach space. Hence, the authors acquired and enhanced new outcomes for this nonlinear Hammerstein-type FIE.

Motivated by Burton [6,7], Ilea and Otrocol [9] and that given above, Graef et al. [14] considered the Hammerstein-type IE of the form:

$$x(t) = p(t) + g(t, x(t)) + h(t, x(t)) \int_0^t A(t - s)f(t, s, x(s))ds.$$

Graef et al. [14] applied Burton's method to this Hammerstein-type IE and obtained two new results regarding the existence of solutions.

Recently, Tunç et al. [10] also considered the following nonlinear IE with several variable time delays

$$\begin{aligned} x(t) =& q(t) + r(x(t)) + h(t, x(t)) + g(t, x(t)) \int_0^t A(t - s)f(s, x(s))ds \\ &+ \sum_{i=1}^{N} \int_0^t A_i(t - s)f_i(s, x(s), x(s - \tau_i(s)))ds \end{aligned}$$

and the nonlinear IDE without delay

$$x'(t) = r(x(t)) + g(t, x(t)) + h(t, x(t)) \int_0^t A(t - s)f(s, x(s))ds.$$

Tunç et al. [10] studied the global existence of solutions of this IE with several variable time delays and the nonlinear IDE without delay by the fixed-point method using progressive contractions. The results of Tunç et al. [10] expanded upon and enhanced a few associated outcomes from the pertinent literature.

On the other hand, we also direct readers to a thorough discussion for some qualitative behaviors of solutions of various mathematical models as follows: existence and stability of implicit FDEs and IEs (Abbas et al. [15], Benchohra et al. [16]), existence of solutions to boundary value problems (Anderson and Avery [1]), theory and applications of FDEs (Balachandran [17], Diethelm [18], Kilbas et al. [19], Miller and Ross [20], Podlubny [21], Zhou [22]), existence of solutions of nonlinear FDEs of Riemann–Liouville type (Becker et al. [23]), fixed-point theory and applications (Burton [2]), periodicity in DDEs by fixed-point mappings (Burton and Zhang [3]), existence of solutions of IEs (Chauhan et al. [24], Deep et al. [25], Jung [26]), theory of functional differential equations (Hale [27]), existence of solutions of hybrid FDEs (Khan et al. [28,29]), the UHML stability of FDEs with delay (Tunç [30]), solution estimates for CFDEs with delay (Tunç and Tunç [11]), exponential discrete form for Caputo–Fabrizio fuzzy BAM neural networks (Zhang and Li [31]) and the references cited therein.

Inspired by the aforementioned works, especially by Burton [8], we will examine the subsequent CFDE with multiple variable time delays:

$$ {}_0^C D_t^q \vartheta(t) = \sum_{i=1}^{N} G_i(t, \vartheta(t), \vartheta(\theta_i(t))), \tag{1} $$

$$ x(t) = \psi(t), t \in [-\omega, 0], \tag{2} $$

where $t \in \Re^+$, $\Re^+ = [0, \infty)$, $\vartheta(t) \in \Re$, ${}^C D_t^q \vartheta(t)$ is the Caputo derivative of $q$ order, $q \in (0, 1)$. We assume that $G_i \in C\left(\Re^+ \times \Re^2, \Re\right)$, $\theta_i \in C(\Re^+, \Re^+)$, $\theta_i(t) \leq t$ with $0 \leq \theta_i(t) \leq \omega_i$, $\omega_i \in \Re$, $\omega_i > 0$, $\omega = \max(\omega_i)$, $i = 1, 2, \ldots, N$.

Regarding the reason behind the motivation of this study, it is well known that classical calculus is a special case of fractional calculus. For example, any ordinary differential equation describing an electrical circuit has exactly one time-dependent current function as the solution. Nevertheless, depending on the order of the fractional derivative, we obtain many time-dependent current functions when solving the associated differential equation of fractional order. This fact regarding fractional derivatives offers a substantial advantage in real-world applications of physics and electronics, where achieving the desired current function is crucial. Next, there is extensive literature on the existence of solutions of delay systems; however there are extensively open problems for delay systems of fractional order, especially in the case of multiple variable time delays, see the books of Benchohra et al. [16], Balachandran [17], Diethelm [18], Kilbas et al. [19], Miller and Ross [20], Podlubny [21] and the sources in the references of this study. To the best of our knowledge, the database of pertinent literature does not contain any paper or work on the uniqueness of solutions of CFDEs with or without delays, where the technique of progressive contractions is a basic tool. Therefore, it deserves to study the uniqueness of solutions regarding the CFDE (1) with different variable time delays.

The primary goal of this work is to advance and contribute the theory of the existence and uniqueness of FDEs with and without delay, as well as to improve the results of Burton [5,7,8], Burton and Purnaras [6], Graef et al. [14], Ilea and Otrocol [9], and Tunç et al. [10,11] for CFDE (1). Indeed, we aim to acquire new insights in connection with the existence and uniqueness theory of FDEs. Therefore, this study aims to obtain a new result, i.e., Theorem 1, on the topic of CFDE (1) and to provide two new examples in relation to show the applications of this theorem by Examples 1 and 2. These data represent the primary contributions, novelty, and originality of this paper.

This paper's remaining sections are organized as follows: Section 2 allows some fundamental definitions and information in connection with the theory and application of fractional calculus. Section 3 presents the primary and novel outcome of the study concerning the existence of solutions of the nonlinear CFDE (1) through the application of the progressive contractions technique. Section 4 provides two examples of numerical applications. Section 5 summarizes the outcomes and contributions of the paper. Finally, the paper's conclusion is given in Section 6.

## 2. Preliminaries

Within this section, some common terminologies and background knowledge of fractional calculus are provided, as they are necessary for this study.

**Definition 1** (Kilbaset al. [19], Podlubny [21]). *The Riemann–Liouville fractional integral of order $q$ with the lower limit zero for a function $\vartheta$ is defined as*

$$ \left(I_{0^+}^q \vartheta\right)(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{\vartheta(s)}{(t-s)^{1-q}} ds, t > 0, \ \ q > 0, $$

*provided that the right side is point-wise defined on $\Re^+$, $\Re^+ = [0, \infty)$, where $\Gamma(.)$ is the gamma function.*

**Definition 2** (Kilbas et al. [19], Podlubny [21]). *The Caputo derivative of fractional-order $q$ for a function $\vartheta$ is defined as*

$$\left({}^{C}D_{0^+}^{q}\vartheta\right)(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{1}{(t-s)^{1+q-n}}\vartheta^{(n)}(s)ds, t > 0,\ \ q > 0,$$

*where $n = [q] + 1$, $[q]$ represents the integer part of $q$.*

## 3. Progressive Contractions and Existence of Solutions

We shall demonstrate the uniqueness of solutions of the CFDE (1), including different variable time delays in Theorem 1.

**Theorem 1.** *Let the following conditions hold:*

*(Hp1)* $0 \le t \le E$, $E > 0$, $E \in \Re$, $G_i \in C([0,E]\times\Re^2,\Re)$, $\theta_i \in C([0,E],\Re^+)$, $\theta_i(t) \le t$, $0 \le \theta_i(t) \le \omega_i$,

$$\omega_i \in \Re, \omega_i > 0, \omega = \max(\omega_i), i = 1,2,\ldots,N;$$

*(Hp2) there exist positive constants $L_{G_i}$ such that*

$$|G_i(t,\psi_1,\psi_1(\theta_i(t))) - G_i(t,\psi_2,\psi_2(\theta_i(t)))| \le L_{G_i}[|\psi_1-\psi_2| + |\psi_1(\theta_i(t)) - \psi_2(\theta_i(t))|]$$

*for all*

$$t \in [0,E], \psi_1,\ \psi_2 \in \Re, i = 1,2,\ldots,N;$$

*(Hp3)*

$$\frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)} \times \frac{(T-T_0)^q}{q} < \beta, \beta \in (0,1).$$

*Then, the CFDE (1) has a unique solution.*

**Remark 1.** *Here, $[T_0, E]$ is divided into equal portions, each having the endpoints $T_0,\ T_1,\ldots,\ T_n = E$ such that $E > T_0$ and the each part having also the length $S < T - T_0,\ T > 0$. Let $q \in (0,1)$ and $\beta \in (0,1)$. Therefore, it may be concluded that*

$$\frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)}\int_{T_0}^{T}(T-s)^{q-1}ds = \frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)} \times \frac{(T-T_0)^q}{q} < \beta, \beta \in (0,1). \tag{3}$$

*We will also need a form in connection with (3). Let $T_i - s = u$, $(i = 1,2,\ldots,n)$. Then, since $T_i - T_{i-1} = S$, we have*

$$\int_{T_{i-1}}^{T_i}(T_i-s)^{q-1}ds = \int_0^{T_i-T_{i-1}} u^{q-1}du = \frac{S^q}{q} < \frac{(T-T_0)^q}{q} < \frac{\Gamma(q)\beta}{2\sum_{i=1}^{N}\left(L_{G_i}\right)}. \tag{4}$$

**Remark 2.** *Since the functions $G_i(t,\vartheta(t),\vartheta(\theta_i(t)))$, $i = 1,2,\ldots,N$ are continuous and satisfy the Lipschitz condition of (Hp2), then, for each $q \in (0,1)$, there exists a $T_0 \in (0,E)$ such that Equation (5) has a unique continuous solution $\psi(t) = \zeta$ on $(0,T_0]$ with*

$$\lim_{t\to 0}\int_0^t\left[(t-s)^{q-1}\sum_{i=1}^{N}G_i(s,\vartheta(s),\vartheta(\theta_i(s)))\right]ds = 0 \text{ and } \lim_{t\to 0}\vartheta(t) = \psi(0)$$

*(see also Burton [8]).*

**Proof.** It is simple to observe that the CFDE (1) with (2) is equivalent to the singular integral system that follows:

$$\vartheta(t) = \begin{cases} \psi(t),\ t \in [-\omega, 0] \\ \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \vartheta(s), \vartheta(\theta_i(s))) \right] ds,\ t \in [0, E]. \end{cases} \tag{5}$$

**Step 1.** Let $(\aleph_1,\ \|.\|)$ be a complete metric space with the supremum norm such that it encompasses the functions $\psi \in C((0, T_1], \Re)$ with $\psi(t) = \zeta$ of Remark 2 on $(0, T_0]$.

We describe an operator

$$P_1 : \aleph_1 \to \aleph_1 \text{ with} \psi \in \aleph_1,$$

which allows that

$$(P_1\vartheta)(t) = \begin{cases} \psi(t),\ t \in [-\omega, 0] \\ \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \vartheta(s), \vartheta(\theta_i(s))) \right] ds,\ t \in (0, T_1]. \end{cases} \tag{6}$$

For $t \in [-\omega, 0]$, from Equation (6) we have

$$\|(P_1\psi_1)(t) - (P_1\psi_2)(t)\| = 0, \psi_1,\ \psi_2 \in \aleph_1.$$

Next, since $\zeta$ satisfies Equation (5) on $(0, T_0]$, i.e., for $0 < t \leq T_0$,then

$$(P_1\psi)(t) = \zeta(t) = \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \zeta(s), \zeta(\theta_i)) \right] ds \tag{7}$$

for $t \in [0, T_1]$. Therefore, $P_1$ of Equation (7) does the map $\aleph_1 \to \aleph_1$.

Let $\psi_1,\ \psi_2 \in \aleph_1$. Then, in the light of Equation (6), it follows that

$$\begin{aligned} &\|(\ P_1\psi_1)(t) - (P_1\psi_2)(t)\| \\ &= \left| \frac{1}{\Gamma(q)} \int_0^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} [G_i(s, \psi_1(s), \psi_1(\theta_i(s))) - G_i(s, \psi_2(s), \psi_2(\theta_i(s)))] \right\} ds \right| \end{aligned}$$

(since $\psi_1 = \psi_2 = \zeta$ on $(0, T_0]$, using (Hp2) and let $t > T_0$)

$$\begin{aligned} &\leq \left| \frac{1}{\Gamma(q)} \int_{T_0}^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} L_{G_i} [|\psi_1(s) - \psi_2(s)| + |\psi_1(\theta_i(s)) - \psi_2(\theta_i(s))|] \right\} ds \right| \\ &\leq \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} |\psi_1 - \psi_2|^{[T_0, T_1]} \int_{T_0}^{T_1} (T_1 - s)^{q-1} ds \\ &= \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} \times \frac{(T_1 - T_0)^q}{q} |\psi_1 - \psi_2|^{[T_0, T_1]} \\ &\leq \beta \|\psi_1 - \psi_2\|, \end{aligned}$$

where $|\psi_1 - \psi_2|^{[T_0, T_1]}$ denotes the supremum on the given interval. Therefore, $P_1$ is a contraction on the set $\aleph_1$ with the unique fixed point $\zeta_1$, where $\zeta_1$ agrees with $\zeta$ on $(0, T_0]$ since $\zeta$ and $\zeta_1$ are unique on the interval $(0, T_0]$.

**Step 2.** Let $(\aleph_2, \|.\|)$ be a complete metric space, which encompasses the functions $\psi \in C((0, T_2], \Re)$ such that $\psi(t) = \zeta_1(t)$ of Step 1 on the interval $(0, T_1]$ together with the supremum norm.

We describe a map

$$P_2 : \aleph_2 \to \aleph_2 \text{ with} \psi \in \aleph_2,$$

which allows that

$$(P_2\vartheta)(t) = \begin{cases} \psi(t), \ t \in [-\omega, 0] \\ \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \vartheta(s), \vartheta(\theta_i(s))) \right] ds, \ t \in (0, T_2]. \end{cases} \tag{8}$$

Hence, for $t \in [-\omega, 0]$, from Equation (8) we have

$$\|(P_2\psi_1)(t) - (P_2\psi_2)(t)\| = 0, \psi_1, \ \psi_2 \in \aleph_2.$$

Since $\zeta_1$ satisfies Equation (6) on the interval $(0, T_1]$, i.e., for $0 < t \leq T_1$ and in $\aleph_2$, each $\psi(t) = \zeta_1(t)$, then

$$(P_2\psi)(t) = \zeta_1(t) = \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \zeta_1(s), \zeta_1(\theta_i(s))) \right] ds, \ t \in (0, T_2]. \tag{9}$$

Therefore, $P_2$ of Equation (9) does the map $\aleph_2 \to \aleph_2$.

Let $\psi_1, \ \psi_2 \in \aleph_2$. Then, according to the conditions of Theorem 1, we derive that

$$\begin{aligned} &\|(P_2\psi_1)(t) - (P_2\psi_2)(t)\| \\ &= \left| \frac{1}{\Gamma(q)} \int_0^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} [G_i(s, \psi_1(s), \psi_1(\theta_i(s))) - G_i(s, \psi_2(s), \psi_2(\theta_i(s)))] \right\} ds \right| \end{aligned}$$

(since $\psi_1 = \psi_2 = \zeta_1$ on $(0, T_1]$, using (Hp2) and let $t > T_1$)

$$\leq \left| \frac{1}{\Gamma(q)} \int_{T_1}^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} L_{G_i} [|\psi_1(s) - \psi_2(s)| + |\psi_1(\theta_i(s)) - \psi_2(\theta_i(s))|] \right\} ds \right|$$

$$\leq \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} |\psi_1 - \psi_2|^{[T_1, T_2]} \int_{T_1}^{T_2} (T_2 - s)^{q-1} ds$$

(using (3), (4) and let $S = T_2 - T_1$)

$$\begin{aligned} &= \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} |\psi_1 - \psi_2|^{[T_1, T_2]} \int_{T_0}^{T_1} (T_1 - s)^{q-1} ds \\ &= \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} \times \frac{(T_1 - T_0)^q}{q} |\psi_1 - \psi_2|^{[T_0, T_1]} \\ &\leq \beta \|\psi_1 - \psi_2\|, \end{aligned}$$

where $|\psi_1 - \psi_2|^{[T_1, T_2]}$ denotes the supremum on the given interval. Therefore, $P_2$ is a contraction on the set $\aleph_2$ with the unique fixed point $\zeta_2$, where $\zeta_2$ agrees with $\zeta_1$ on $(0, T_1]$ since $\zeta_1$ and $\zeta_2$ are unique on the interval $(0, T_1]$.

**Step 3.** Let $(\aleph_3, \|.\|)$ be a complete metric space, which encompasses the functions $\psi \in C((0, T_3], \Re)$ such that $\psi(t) = \zeta_2(t)$ of Step 2 on the interval $(0, T_2]$ together with the supremum norm.

We describe a map

$$P_3 : \aleph_3 \to \aleph_3 \text{ with} \psi \in \aleph_3,$$

which allows that

$$(P_3\vartheta)(t) = \begin{cases} \psi(t), \ t \in [-\omega, 0] \\ \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \vartheta(s), \vartheta(\theta_i(s))) \right] ds, \ t \in (0, T_3]. \end{cases} \tag{10}$$

For $t \in [-\omega, 0]$, from Equation (10) we have

$$\|(P_3\psi_1)(t) - (P_3\psi_2)(t)\| = 0, \psi_1, \ \psi_2 \in \aleph_3.$$

Since $\zeta_2$ satisfies Equation (9) on the interval $(0, T_2]$, i.e., for $0 < t \leq T_2$ and in $\aleph_3$, each $\psi(t) = \zeta_2(t)$, then

$$(P_3\psi)(t) = \zeta_2(t) = \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \zeta_2(s), \zeta_2(\theta_i(s))) \right] ds, \ t \in (0, T_3]. \tag{11}$$

Therefore, $P_3$ of Equation (11) does the map $\aleph_3 \to \aleph_3$.

Let $\psi_1, \ \psi_2 \in \aleph_3$. Then, in the light of the conditions of Theorem 1, we derive that

$$\begin{aligned} &\|(P_3\psi_1)(t) - (P_3\psi_2)(t)\| \\ &= \left| \frac{1}{\Gamma(q)} \int_0^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} [G_i(s, \psi_1(s), \psi_1(\theta_i(s))) - G_i(s, \psi_2(s), \psi_2(\theta_i(s)))] \right\} ds \right| \end{aligned}$$

(since $\psi_1 = \psi_2 = \zeta_2$ on $(0, T_2]$, using (Hp3) and let $t > T_2$)

$$\begin{aligned} &\leq \left| \frac{1}{\Gamma(q)} \int_{T_2}^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} L_{G_i} [|\psi_1(s) - \psi_2(s)| + |\psi_1(\theta_i(s)) - \psi_2(\theta_i(s))|] \right\} ds \right| \\ &\leq \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} |\psi_1 - \psi_2|^{[T_2, T_3]} \int_{T_2}^{T_3} (T_3 - s)^{\alpha - 1} ds \end{aligned}$$

(using (3) and (4) also let $S = T_3 - T_2$)

$$\begin{aligned} &= \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} |\psi_1 - \psi_2|^{[T_2, T_3]} \int_{T_0}^{T_1} (T_1 - s)^{q-1} ds \\ &= \frac{2 \sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} \times \frac{(T_1 - T_0)^q}{q} |\psi_1 - \psi_2|^{[T_0, T_1]} \\ &\leq \beta \|\psi_1 - \psi_2\|, \end{aligned}$$

where $|\psi_1 - \psi_2|^{[T_2, T_3]}$ denotes the supremum on the given interval. Therefore, $P_3$ is a contraction on the set $\aleph_3$ with the unique fixed point $\zeta_3$, where $\zeta_3$ agrees with $\zeta_2$ on the interval $(0, T_2]$ since $\zeta_2$ and $\zeta_3$ are unique on the interval $(0, T_2]$.

**Step i.** Finally, for the $i-th$ step, by applying the mathematical induction method, we will be able to derive a unique solution on $[0, E]$. Showing this case is sufficient for full comprehension; however, the details on the induction are available in the following.

It is now clear that

$$\int_{T_{i-1}}^{T_i} (T_i - s)^{q-1} ds = \int_{T_0}^{T_1} (T_1 - u)^{q-1} du \leq \frac{\Gamma(q)\beta}{2\sum_{i=1}^{N} (L_{G_i})}.$$

Let $(\aleph_i, \|.\|)$ be a complete metric space, which encompasses the functions $\psi \in C((0, T_i], \Re)$ such that $\psi(t) = \zeta_{i-1}(t)$ of the previous step on the interval $(0, T_{i-1}]$ together with the supremum norm.

We describe the transformation

$$P_i : \aleph_i \to \aleph_i \text{with} \psi \in \aleph_i.$$

This transformation implies that

$$(P_i\vartheta)(t) = \begin{cases} \psi(t), \ t \in [-\omega, 0] \\ \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \vartheta(s), \vartheta(\theta_i(s))) \right] ds, \ t \in (0, T_i]. \end{cases} \tag{12}$$

For $t \in [-\omega, 0]$, from Equation (12) we have

$$\|(P_i\psi_1)(t) - (P_i\psi_2)(t)\| = 0, \psi_1, \ \psi_2 \in \aleph_i.$$

Similar as previous steps, $\zeta_{i-1}(t)$ also satisfies Equation (12) on the interval $(0, T_{i-1}]$, i.e., for $0 < t \leq T_{i-1}$ and in $\aleph_i$, $\psi(t) = \zeta_{i-1}(t)$, afterward we have

$$(P_i\psi)(t) = \zeta_{i-1}(t) = \psi(0) + \frac{1}{\Gamma(q)} \int_0^t \left[ (t-s)^{q-1} \sum_{i=1}^{N} G_i(s, \zeta_{i-1}(s), \zeta_{i-1}(\theta_i(s))) \right] ds, \ t \in (0, T_i]. \tag{13}$$

Therefore, $P_i$ of Equation (13) does a map $\aleph_i \to \aleph_i$.

Let $\psi_1, \ \psi_2 \in \aleph_i$. Consequently, according to Equation (12), it follows that

$$\begin{aligned} &\|(P_i\psi_1)(t) - (P_i\psi_2)(t)\| \\ &= \left| \frac{1}{\Gamma(q)} \int_0^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} [G_i(s, \psi_1(s), \psi_1(\theta_i(s))) - G_i(s, \psi_2(s), \psi_2(\theta_i(s)))] \right\} ds \right| \end{aligned}$$

(since $\psi_1 = \psi_2 = \zeta_{i-1}$ on $(0, T_{i-1}]$, using (Hp2) and let $t > T_{i-1}$)

$$\begin{aligned} &\leq \left| \frac{1}{\Gamma(q)} \int_{T_{i-1}}^t \left\{ (t-s)^{q-1} \sum_{i=1}^{N} L_{G_i} [|\psi_1(s) - \psi_2(s)| + |\psi_1(\theta_i(s)) - \psi_2(\theta_i(s))|] \right\} ds \right| \\ &\leq \frac{2\sum_{i=1}^{N} (L_{G_i})}{\Gamma(q)} |\psi_1 - \psi_2|^{[T_{i-1}, T_i]} \int_{T_{i-1}}^{T_i} (T_i - s)^{q-1} ds \end{aligned}$$

(using (3) and (4) also let $S = T_i - T_{i-1}$)

$$\begin{aligned}
&= \frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)}|\psi_1-\psi_2|^{[T_{i-1},T_i]}\int_{T_0}^{T_1}(T_1-s)^{q-1}ds\\
&= \frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)}\times\frac{(T_1-T_0)^q}{q}|\psi_1-\psi_2|^{[T_0,T_1]}\\
&\leq \beta\|\psi_1-\psi_2\|,
\end{aligned}$$

where $|\psi_1-\psi_2|^{[T_{i-1},T_i]}$ denotes the supremum on the given interval. Therefore, $P_i$ is a contraction on the set $\aleph_i$ with the unique fixed point $\zeta_i$, where $\zeta_i$ agrees with $\zeta_{i-1}$ on the interval $(0, T_{i-1}]$ since $\zeta_{i-1}$ and $\zeta_i$ are unique on the interval $(0, T_{i-1}]$. Therefore, after completing the $n$-steps, we obtain a solution $\zeta_n$ on the interval $(0, E]$. Additionally, to obtain a solution on the interval $(0, \infty)$, we obtain a sequence of solutions $\{\zeta_i\}$ on the intervals $(0, i]$ and construct the set of the functions $\{\zeta_i^*\}$, which are solutions from the earlier sequence, but continued for $t > i$ at the constant value $\zeta_i(i)$. This sequence converges uniformly on compacts sets to a continuous solution $\vartheta(t)$ on the interval $(0, \infty)$ since at every value of $t$ and for $t > i$, $\vartheta(t)$ aligns with $\zeta_i(t)$. This ends the proof of the theorem. □

## 4. Numerical Applications of Theorem 1

In the section on numerical applications, we present two examples that satisfy the conditions of Theorem 1 in connection with two specific cases of the CFDE (1).

**Example 1.** *Think about the following CpFrDE:*

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}}\vartheta(t) = \frac{1}{50}\sin(t) + \frac{1}{100}\vartheta(t) + \frac{1}{100}\frac{\vartheta^2(t)}{1+\vartheta^2(t)} + \frac{1}{100}\cos\left(\vartheta(t-5^{-1})\right),\ t\in[0,E] \\ \vartheta(t) = t,\ t\in[-5^{-1},0]. \end{cases} \tag{14}$$

*We note that the CFDE* (14) *has the same form as the CFDE* (1) *with the data as follows:*

$$q = \frac{1}{2}, [-\omega, 0] = [-5^{-1}, 0], 0 \leq t \leq E, E > 0,$$

$$0 < \theta_1(t) = 5^{-1} = \omega, \omega \in \Re, \omega > 0,$$

$$G_1(t, \vartheta, \vartheta(\theta_1(t))) = \frac{1}{50}\sin(t) + \frac{1}{100}\vartheta(t) + \frac{1}{100}\frac{\vartheta^2(t)}{1+\vartheta^2(t)} + \frac{1}{100}\cos\left(\vartheta(t-5^{-1})\right).$$

*We will now demonstrate that the conditions (Hp1), (Hp2) and (Hp3) of Theorem 1 hold. It is now evident that the function $G_1$ is continuous. Therefore, the condition (Hp1) holds.*

*Let $L_{G_1} = \frac{1}{100}$, $T_0 = 0$ and $\sqrt{T} < 25\sqrt{\pi}$. Then, we obtain*

$$
\begin{aligned}
&|G_1(t,\vartheta_1,\vartheta_1(\theta_1(t))) - G_1(t,\vartheta_2,\vartheta_2(\theta_1(t)))| \\
&\leq \frac{1}{100}|\vartheta_1-\vartheta_2| + \frac{1}{100}\left|\frac{\vartheta_1^2}{1+\vartheta_1^2} - \frac{\vartheta_2^2}{1+\vartheta_2^2}\right| \\
&\quad + \frac{1}{100}\left|\cos\left(\vartheta_1(t-5^{-1})\right) - \cos\left(\vartheta_2(t-5^{-1})\right)\right| \\
&= \frac{1}{100}|\vartheta_1-\vartheta_2| + \frac{1}{100}\frac{|\vartheta_1+\vartheta_2||\vartheta_1-\vartheta_2|}{(1+\vartheta_1^2)(1+\vartheta_2^2)} \\
&\quad + \frac{1}{50}\left|\sin\left(\frac{\vartheta_1(t-5^{-1})+\vartheta_2(t-5^{-1})}{2}\right)\sin\left(\frac{\vartheta_1(t-5^{-1})-\vartheta_2(t-5^{-1})}{2}\right)\right| \\
&= \frac{1}{100}|\vartheta_1-\vartheta_2| + \frac{1}{100}\frac{|\vartheta_1+\vartheta_2||\vartheta_1-\vartheta_2|}{(1+\vartheta_1^2)(1+\vartheta_2^2)} \\
&\quad + \frac{1}{50}\left|\sin\left(\frac{\vartheta_1(t-5^{-1})+\vartheta_2(t-5^{-1})}{2}\right)\right| \times \left|\sin\left(\frac{\vartheta_1(t-5^{-1})-\vartheta_2(t-5^{-1})}{2}\right)\right| \\
&\leq \frac{1}{100}|\vartheta_1-\vartheta_2| + \frac{1}{100}|\vartheta_1-\vartheta_2| + \frac{1}{100}\left|\vartheta_1(t-5^{-1})-\vartheta_2(t-5^{-1})\right| \\
&\leq \frac{1}{50}|\vartheta_1-\vartheta_2| + \frac{1}{100}\left|\vartheta_1(t-5^{-1})-\vartheta_2(t-5^{-1})\right|
\end{aligned}
$$

*and*

$$
\frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)} \times \frac{(T-T_0)^q}{q} = \frac{2L_{G_1}}{\Gamma(q)} \times 2\sqrt{T} = \frac{2\times\frac{1}{100}}{\Gamma\left(\frac{1}{2}\right)} \times 2\sqrt{T} = \frac{\sqrt{T}}{25\sqrt{\pi}} = \beta < 1.
$$

*Therefore, the conditions (Hp2) and (Hp3) of Theorem 1 are also met. Consequently, the CFDE (14) admits a unique solution.*

**Example 2.** *Let us consider the CFDE as follows:*

$$
\begin{cases}
{}_0^C D_t^{\frac{1}{3}}\vartheta(t) = \frac{1}{200\exp(t)}\left[\frac{1}{1+|\vartheta(t)|+\left|\vartheta\left(t-10^{-1}\right)\right|} + \frac{1}{2+|\vartheta(t)|+\left|\vartheta\left(t-20^{-1}|\sin(t)|\right)\right|}\right],\ t\in[0,E], \\
\vartheta(t) = t,\ t\in[-10^{-1},0].
\end{cases} \tag{15}
$$

*We see that the CFDE (15) and the CFDE (1) have the same form with the data as follows:*

$$
q = \frac{1}{3}, [-\omega,0] = [-10^{-1},0], 0\leq t\leq E, E>0,
$$

$$
0<\theta_1(t) = 10^{-1}, 0\leq\theta_2(t) = 20^{-1}|\sin(t)| \leq 20^{-1}, \omega = 10^{-1}, \omega\in\Re, \omega>0,
$$

$$
G_1(t,\vartheta,\vartheta(\theta_1(t))) = \frac{1}{200\exp(t)\left(1+|\vartheta|+\left|\vartheta\left(t-10^{-1}\right)\right|\right)},
$$

$$
G_2(t,\vartheta,\vartheta(\theta_2(t))) = \frac{1}{200\exp(t)\left(2+|\vartheta|+\left|\vartheta\left(t-20^{-1}|\sin(t)|\right)\right|\right)}.
$$

*We will now show that the conditions (Hp1), (Hp2) and (Hp3) of Theorem 1 hold. Subsequently, it is clear that the functions $G_1$ and $G_2$ are continuous. Therefore, the condition (Hp1) holds.*

*Let*

$$
L_{G_1} = \frac{1}{200}, L_{G_2} = \frac{1}{200}, T_0 = 0 \text{ and } \sqrt[3]{T} < 89.298.
$$

*Next, while considering the above data, we obtain*

$$
\begin{aligned}
&|G_1(t,\vartheta_1,\vartheta_1(\theta_1(t))) - G_1(t,\vartheta_2,\vartheta_2(\theta_1(t)))| \\
&\quad = \frac{1}{200\exp(t)}\left|\frac{1}{1+|\vartheta_1|+\left|\vartheta_1\left(t-10^{-1}\right)\right|} - \frac{1}{1+|\vartheta_2|+\left|\vartheta_2\left(t-10^{-1}\right)\right|}\right| \\
&\quad = \frac{1}{200\exp(t)}\left|\frac{\left(1+|\vartheta_2|+\left|\vartheta_2\left(t-10^{-1}\right)\right|\right) - \left(1+|\vartheta_1|+\left|\vartheta_1\left(t-10^{-1}\right)\right|\right)}{\left(1+|\vartheta_1|+\left|\vartheta_1\left(t-10^{-1}\right)\right|\right)\times\left(1+|\vartheta_2|+\left|\vartheta_2\left(t-10^{-1}\right)\right|\right)}\right| \\
&\quad \leq \frac{1}{200}\left|\left(1+|\vartheta_2|+\left|\vartheta_2\left(t-10^{-1}\right)\right|\right) - \left(1+|\vartheta_1|+\left|\vartheta_1\left(t-10^{-1}\right)\right|\right)\right| \\
&\quad \leq \frac{1}{200}|\vartheta_1-\vartheta_2| + \frac{1}{200}\left|\vartheta_1(t-10^{-1}) - \vartheta_2(t-10^{-1})\right|,
\end{aligned}
$$

$$
\begin{aligned}
&|G_2(t,\vartheta_1,\vartheta_1(\theta_1(t))) - G_2(t,\vartheta_2,\vartheta_2(\theta_1(t)))| \\
&\quad = \frac{1}{200\exp(t)}\left|\frac{1}{2+|\vartheta_1|+\left|\vartheta_1\left(t-20^{-1}|\sin(t)|\right)\right|} - \frac{1}{2+|\vartheta_2|+\left|\vartheta_2\left(t-20^{-1}|\sin(t)|\right)\right|}\right| \\
&\quad = \frac{1}{200\exp(t)}\left|\frac{\left(2+|\vartheta_2|+\left|\vartheta_2\left(t-20^{-1}|\sin(t)|\right)\right|\right) - \left(2+|\vartheta_1|+\left|\vartheta_1\left(t-20^{-1}|\sin(t)|\right)\right|\right)}{\left(2+|\vartheta_1|+\left|\vartheta_1\left(t-20^{-1}|\sin(t)|\right)\right|\right)\times\left(2+|\vartheta_2|+\left|\vartheta_2\left(t-20^{-1}|\sin(t)|\right)\right|\right)}\right| \\
&\quad \leq \frac{1}{200}\left|\left(2+|\vartheta_2|+\left|\vartheta_2\left(t-20^{-1}|\sin(t)|\right)\right|\right) - \left(2+|\vartheta_1|+\left|\vartheta_1\left(t-20^{-1}|\sin(t)|\right)\right|\right)\right| \\
&\quad \leq \frac{1}{200}\left[|\vartheta_1-\vartheta_2| + \left|\vartheta_1\left(t-20^{-1}|\sin(t)|\right) - \vartheta_2\left(t-20^{-1}|\sin(t)|\right)\right|\right]
\end{aligned}
$$

*and*

$$
\begin{aligned}
\frac{2\sum_{i=1}^{N}\left(L_{G_i}\right)}{\Gamma(q)}\times\frac{(T-T_0)^q}{q} &= \frac{2\left(L_{G_1}+L_{G_2}\right)}{\Gamma(q)}\times 3\sqrt[3]{T} \\
&= \frac{2\times\frac{1}{200}}{\Gamma\left(\frac{1}{3}\right)}\times 3\sqrt[3]{T} = \frac{3}{100\times 2.67894}\times\sqrt[3]{T} \\
&= \frac{3}{267.894}\times\sqrt[3]{T} = \beta < 1,
\end{aligned}
$$

*where* $\Gamma\left(\frac{1}{3}\right) \cong 2.67894$.

*Therefore, the conditions (Hp2) and (Hp3) of Theorem 1 hold. Consequently, the CFDE (15) admits a unique solution.*

## 5. The Outcomes and Contributions

We will now succinctly summarize the findings and contributions of this study.

- In view of the literature review that is provided in this study's introduction and available in the pertinent literature in order to investigate the existence and uniqueness of solutions for a variety of mathematical models, including IDEs, delay IEs, a scalar FDE of the Riemann–Liouville type, scalar nonlinear IEs, Hammerstein-type FIEs, nonlinear IEs including several variable time delays and nonlinear IDEs without delay on the intervals $[0,E]$ and $[0,\infty)$, the method of progressive contractions, which belongs to T.A. Burton, has effectively been applied and very interesting results have been obtained in the pertinent literature by this time. Nevertheless, to the best of our information from the relevant literature, there is no result on the same topic for CFDEs, including time delay(s) and without time delay(s), where the method of progressive contractions is used as a basic technique on the subject. Therefore, the aim of this study is to achieve the application of the progressive contractions technique to the existence

and uniqueness of solutions for CFDEs, including several variable time delays. This fact represents the paper's novelty and originality.

- Burton [8], Theorem 1.2 dealt with a scalar FDE of the Riemann–Liouville type without delay. To the best of our knowledge, the fractional differential equation considered in Burton [8] has a simple form and does not include any time delay. Moreover, Burton [8] provided no numerical application to validate the primary outcome of [8], Theorem 1.2. Despite this case, in this paper, we examine a distinct kind of fractional differential equation, namely CFDE (1) with time delays in multiple variables, and also offer two new two examples as numerical applications of our primary finding, i.e., Theorem 1.
- Regarding the benefits of the technique called progressive contractions, which was used in the proof of this paper, the existence and the uniqueness of solutions of different kinds of nonlinear mathematical models, as well as FDEs of various kinds, for example, FDEs of the Riemann–Liouville type with time delay and without time delay, CFDEs with time delay and without time delay, Hilfer type FDEs with time delay and without time delay, etc. can be discussed via the progressive contractions throughout very simple and short steps for the finite and infinite interval cases.

## 6. Conclusions

The uniqueness of solutions to a nonlinear CFDE with various variable time delays is the main problem of this paper. In the current work, we have deduced novel sufficient conditions for the uniqueness of solutions to the considered nonlinear CFDE with multiple variable time delays, which is called a qualitative characteristic. The fixed-point method, together with the supremum norm, is the basic tool in the proof of the main result of the paper, which is applied throughout the technique called progressive contractions.

To the best of our knowledge, Theorem 1 is the first result in the pertinent literature on the uniqueness of solutions for nonlinear delay CFDEs. The unique solution on the interval $[0, \infty)$ is obtained using the progressive contractions together supremum norm. The findings of this paper complement those that can be found in the pertinent literature and offer new insights into the qualitative theory of FDEs with and without time delays. In this study, it is a desirable case to provide Example 1 and Example 2 as numerical applications of Theorem 1.

Considering the present new result at hand, as future recommendations, the uniqueness of solutions to the following proper mathematical models can be investigated using the progressive contractions technique: CFIDEs with or without time delays, Riemann–Liouville FDEs with time delays, Riemann–Liouville IDEs with or without time delays, Hilfer FDEs with or without time delays, and Hilfer fractional IDEs with or without time delays.

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## Abbreviations

The following abbreviations will be used throughout this paper:

| | |
|---|---|
| DE | Differential equation |
| DEs | Differential equations |
| DDEs | Delay differential equations |
| CFDE | Caputo fractional-order differential equation |
| CFDEs | Caputo fractional-order differential equations |
| CFIDEs | Caputo fractional-order integro-differential equations |
| IE | Integral equation |
| IEs | Integral equations |
| IDE | Integro-differential equation |
| IDEs | Integro-differential equations |
| FDE | Fractional differential equation |
| FDEs | Fractional differential equations |
| FIE | Functional integral equation |
| FIEs | Functional integral equations |
| UHML stability | Ulam–Hyers–Mittag–Leffler stability |

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