mathematics

MDPI

Article

Ostrowski-Type Inequalities for Functions of Two Variables in Banach Spaces

Muhammad Amer Latif [1] and Ohud Bulayhan Almutairi [2,*]

[1] Department of Mathematics, Faculty of Sciences, King Faisal University, Hofuf 31982, Saudi Arabia; mlatif@kfu.edu.sa
[2] Department of Mathematics, University of Hafr Al-Batin, Hafr Al-Batin 31991, Saudi Arabia
* Correspondence: ohudbalmutairi@gmail.com or dr.ohudalmutairi@uhb.edu.sa

Abstract: In this paper, we offer Ostrowski-type inequalities that extend the findings that have been proven for functions of one variable with values in Banach spaces, conducted in a remarkable study by Dragomir, to functions of two variables containing values in the product Banach spaces. Our findings are also an extension of several previous findings that have been established for functions of two variable functions. Prior studies on Ostrowski-type inequalities incriminated functions that have values in Banach spaces or Hilbert spaces. This study is unique and significant in the field of mathematical inequalities, and specifically in the study of Ostrowski-type inequalities, because they have been established for functions having values in a product of two Banach spaces.

Keywords: Ostrowski's inequality; Hölder inequality; Banach spaces; Bochner integrable function; Hölder continuous function

MSC: Primary 26D15; 26A45; 26D10; 41A55; Secondary 46B20; 47A63; 47A99

check for updates

Citation: Latif, M.A.; Almutairi, O.B. Ostrowski-Type Inequalities for Functions of Two Variables in Banach Spaces. *Mathematics* **2024**, *12*, 2748. https://doi.org/10.3390/math12172748

Academic Editor: Paul Bracken

Received: 6 August 2024
Revised: 31 August 2024
Accepted: 3 September 2024
Published: 4 September 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Inequalities involving integrals that set up bounds on the physical quantities are significant, since they are applied in a variety of disciplines, such as biology, engineering, and physics, in addition to approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, and probability theory. Inequalities involving integrals that set up bounds on the physical quantities are significant, since they are applied in a variety of disciplines of biology, engineering, and physics in addition to approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, and probability theory (see [1,2]).

In 1938, A. Ostrowski proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x), x \in [a,b]$ (please see [3,4]):

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b). If f' is bounded on (a,b), i.e., $\|f'\| := \sup_{t\in(a,b)} |f(t)| < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty. \tag{1}$$

Furthermore, over a given interval, the deviation of a function can be estimated through its mean value by the Ostrowski inequality, i.e., Ostrowski's result gives us the bound of the values of a function from its mean value. This inequality has a great impact on different topics in mathematical analysis, including the estimation of error bounds

in Banach spaces, numerical integration, and approximation theory. Many papers about generalizations of Ostrowski's inequality have been written in the contemporary age; a few examples are mentioned in Anastassiou [4], who proved multivariate Ostrowski-type inequalities; Cheng gave some improvement to Ostrowski–Grüss-Type Inequalities in [5]; Irshad and Khan [6] obtained some quadrature rules for mappings on $L_p[u,v]$ space via Ostrowski-type inequality; and Liu [7] proved some generalized Ostrowski-type inequalities to obtain improvements of the results from the previous studies. It has been shown that Ostrowski's inequality is a useful tool for developing several mathematical science fields.

An improvement to the Ostrowski's inequality was investigated by Dragomir in [8].

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$, whose derivative $f'\in\mathcal{L}_\infty[a,b]$. Then,*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| &\le \frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]\\ &\le\begin{cases}\|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a);\\ \frac{1}{2}\left[\|f'\|^\alpha_{[a,x],\infty}+\|f'\|^\alpha_{[x,b],\infty}\right]^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{2\beta}+\left(\frac{b-t}{b-a}\right)^{2\beta}\right]^{\frac{1}{\beta}};\\ \text{where } q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}\left[\|f'\|_{[b,x],\infty}+\|f'\|_{[x,a],\infty}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a)\end{cases}\end{aligned} \tag{2}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[a,b],\infty}$ denotes the usual norm on $\mathcal{L}_\infty[a,b]$, i.e., $\|f\|_{[a,b],\infty}=\operatorname{ess}\sup_{t\in[a,b]}|f(t)|<\infty$.

In [9], Dragomir established a generalization of the weighted companion of Ostrowski integral inequality for mappings of bounded variation.

Theorem 3 ([9])**.** *Let $f:[a,b]\to\mathbb{R}$ be function of bounded of variation on $[a,b]$ and $\bigvee_a^b(f)$ its total variation on $[a,b]$. Then, the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\bigvee_a^b(f) \tag{3}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible constant in (3).

Definition 1 ([10])**.** *A mapping $B:[a,b]\to\mathcal{L}(X)$ is Hölder continuous on $[a,b]$, if the inequality*

$$\|B(t)-B(s)\|\le H|t-s|^\alpha,$$

holds for all $t,s\in[a,b]$, where $H>0$ and $\alpha\in(0,1]$.

In [10], Dragomir presented weighted Ostrowski-type inequalities for operators and vector-valued functions.

Theorem 4 ([10]). *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$\|B(t) - B(s)\| \leq H|t-s|^{\alpha}, \text{ for all } t,s \in [a,b],$$

where $H > 0$ and $\alpha \in (0,1]$. If $g : [a,b] \to X$ is Bochner integrable on $[a,b]$, then we have the inequality

$$\begin{aligned}\left|\int_a^b B(s)g(s)ds - B(t)\int_a^b g(s)ds\right| &\\ \leq H\int_a^b |t-s|^{\alpha}\|g(s)\|\,ds &\\ \leq H \times \begin{cases} \frac{(t-a)^{\alpha+1}+(b-t)^{\alpha+1}}{\alpha+1} \operatorname*{ess\,sup}_{t\in[a,b]}\|g(t)\|; \\ \left[\frac{(t-a)^{q\alpha+1}+(b-t)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}\left(\int_a^b \|g(t)\|^p dt\right)^{\frac{1}{p}}; \\ \quad \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]^{\alpha}\int_a^b \|g(t)\|dt \end{cases} \end{aligned} \tag{4}$$

for any $t \in [a,b]$, provided the integrals and $\operatorname*{ess\,sup}_{t\in[a,b]}\|g(t)\|$ from the right hand side are finite.

Anastassiou [3] also proved an Ostrowski-type inequality that we mention in the theorem below.

Theorem 5 ([3]). *Let $(X, \|\cdot\|)$ be a Banach space, and $f \in C^1([a,b], X)$, with $x \in [a,b]$, then we obtain the following inequality:*

$$\left\|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right\| \leq \left(\frac{(b-x)^2+(x-a)^2}{2(b-a)}\right)\|\|f'\|\|_{\infty}, \tag{5}$$

where $\|\|f'\|\|_{\infty} := \sup_{t\in[a,b]}\|f'(x)\| < \infty$. Inequality (5) is sharp. In particular, the optimal function is $f^*(x) := |x-t|^{\alpha}\cdot(b-a)\cdot t_0$, $\alpha > 1$, and t_0 is a fixed unit vector in X.

The weighted version of Ostrowski's inequality for two functions with values in Banach spaces was established by Dragomir in [11].

Theorem 6 ([11]). *Assume that $f : [a,b] \to \mathbb{C}$ and $g : [a,b] \to X$ are continuous, and g is strongly differentiable on (a,b), then for all $u \in [a,b]$, we obtain the inequality*

$$\left\|\int_a^b f(t)g(t)dt - \left(\int_a^b g(s)ds\right)f(u)\right\| \leq C(f,g,u),$$

where

$$C(f,g,u) \leq \int_u^b\left(\int_t^b |f(s)|ds\right)\|g'(t)\|dt + \int_a^u\left(\int_a^t |f(s)|ds\right)\|g'(t)\|dt. \tag{6}$$

We also have the bounds

$$C(f,g,u) \leq \begin{cases} \int_t^b |f(s)|ds \int_u^b \|g'(t)\|dt + \int_a^t |f(s)|ds \int_a^u \|g'(t)\|dt, \\ \left[\int_u^b \left(\int_t^b |f(s)|^p ds\right) dt\right]^{\frac{1}{p}} \left(\int_u^b \|g'(t)\|^q dt\right)^{\frac{1}{q}} \\ + \left[\int_a^u \left(\int_a^t |f(s)|^p ds\right) dt\right]^{\frac{1}{p}} \left(\int_a^u \|g'(t)\|^q dt\right)^{\frac{1}{q}}, \\ \sup\limits_{t\in[u,b]} \|g'(t)\| \int_u^b \left(\int_t^b |f(s)|ds\right) dt \\ + \sup\limits_{t\in[a,u]} \|g'(t)\| \int_a^u \left(\int_a^t |f(s)|ds\right) dt, \end{cases} \tag{7}$$

where $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Functional analysis, an active research area in mathematical analysis, has immensely contributed to the advancement of contemporary mathematics, along with their applications. Central to this field, Banach spaces, discovered by Stefan Banach, a Polish mathematician, analyze numerous linear spaces together with their continuous transformations. Serving as the basis of many theories in analysis, Banach spaces can be described as complete normed vector spaces. In addition, various applications of Banach spaces, such as Tikhonov regularization, least square approximation, and inverse problems, exist not only in applied mathematics, but also in their related fields.

Suppose that X is a Banach space for $-\infty < a < b < \infty$. Let $\mathcal{L}(X)$ be the Banach algebra of linear operators acting on X. We denote the norms of those operators by $\|\cdot\|$.

A function $f : [a,b] \to X$ is said to be measurable given that the sequence of simple functions $f_n : [a,b] \to X$ converges everywhere on the interval $[a,b]$ at f.

It is known that a measurable function $f : [a,b] \to X$ can only be Bochner integrable on the condition that the norm of this function $f(t)$ is Lebesgue integrable on the interval $[a,b]$.

One important area of interest emerging in the realm of Banach spaces is the theory of inequalities. This offers insights for understanding bounds, as well as approximating functional expressions. In Banach spaces, inequalities of Hermite–Hadamard, Ostrowski, and Lebesgue types play an important role in analysis. Banach space theory has been expanded to provide insights into complex mathematical issues.

In [12], Alomari establish generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation. In [10], Barnett et al. presented weighted Ostrowski-type inequalities for operators and vector-valued functions. Barnett and Dragomir [13] established a weighted version of Ostrowski's inequality for two functions with values in Banach spaces.

This paper deals with the Ostrowski-type inequalities for functions with values in the product of two Banach spaces. Such types of Ostrowski-type inequalities have not been discussed in any previous studies, i.e., our results extend the results of Dragomir from [11] to functions of two variables which, in turn, generalize many of those results that have been proved so far for functions of two variables, for instance [13–18] and the references cited in them. Many researchers have, so far, considered Ostrowski-type inequalities for real valued functions of two variables, for functions of one variable with values in Banach spaces, or for functions with values Hilbert spaces. The results of this study are important and different than those studies, since they have been established for functions with values in the product of two Banach spaces. The results of our study have applications in several areas of mathematical sciences, particularly, in the area of functional analysis, which is a branch of mathematical analysis that studies vector spaces with a limit structure (such as a norm or inner product), and functions or operators defined on these spaces. Functional analysis also provides a useful framework and abstract approach for some

applied problems in variety of disciplines.The study have limitations when we have sum of two or more functions with values in Banach spaces, since the distributive property does not hold for integrals in this case.

2. Main Results

We begin this section with the following definitions, followed by our results.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $V = X \times Y$ be a direct product of Banach spaces X and Y together with induced component-wise operations. Let $\|\cdot\|_{X\times Y}$ be the direct product norm of $V = X \times Y$. Then, V is a Banach space. Indeed, a number of equivalent norms are possible on $V = X \times Y$, but the most common norms on $V = X \times Y$ are the following:

$$\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y,$$

$$\|(x,y)\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\},$$

or, more generally,

$$\|(x,y)\|_{X\times Y} = \left(\|x\|_X^p + \|y\|_Y^p\right)^{\frac{1}{p}}, 1 \leq p < \infty,$$

for all $(x,y) \in V = X \times Y$.

We begin our first result assuming that the function has the values in the direct product to two Banach spaces X and Y in the norm topology $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.

Theorem 7. *Assume that $\alpha : [a,b] \times [c,d] \to \mathbb{C}$ and $Y : [a,b] \times [c,d] \to E_1 \times E_2$ are continuous, and $\frac{\partial Y}{\partial t}$, $\frac{\partial Y}{\partial w}$, $\frac{\partial^2 Y}{\partial t \partial w}$ exist in the norm topology $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ of $E_1 \times E_2$ on $(a,b) \times (c,d)$; then, for all $(u,v) \in [a,b] \times [c,d]$, we have the following inequalities:*

$$\begin{aligned}\left\| \left(\int_a^b \int_c^d \alpha(s,r) dr ds \right) Y(u,v) + \int_a^b \int_c^d \alpha(t,w) Y(t,w) dw dt \right. \\ \left. - \int_a^b \left(\int_c^d \alpha(t,r) dr \right) Y(t,v) dt - \int_c^d \left(\int_a^b \alpha(s,w) ds \right) Y(u,w) dw \right\|_{E_1\times E_2} \\ \leq B(\alpha, Y, u, v), \quad (8)\end{aligned}$$

where

$$\begin{aligned}B(\alpha, Y, u, v) := & \int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)| dr ds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dw dt \\ & + \int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)| dr ds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dw dt \\ & + \int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)| dr ds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dw dt \\ & + \int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)| dr ds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dw dt. \quad (9)\end{aligned}$$

We also have the following bounds for $B(\alpha, Y, u, v)$:

$B(\alpha, Y, u, v)$

$$
\leq \begin{cases}
\sup\limits_{(t,w)\in[u,b]\times[v,d]} \left(\int_t^b \int_w^d |\alpha(s,r)|drds\right) \int_u^b \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt \\
+ \sup\limits_{(t,w)\in[a,u]\times[v,d]} \left(\int_a^t \int_w^d |\alpha(s,r)|drds\right) \int_a^u \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt \\
+ \sup\limits_{(t,w)\in[u,b]\times[c,v]} \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right) \int_u^b \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt \\
+ \sup\limits_{(t,w)\in[a,u]\times[c,v]} \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right) \int_a^u \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt, \\
\left[\int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_u^b \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}} \\
+ \left[\int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_a^u \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}} \\
+ \left[\int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_u^b \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}} \\
+ \left[\int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_a^u \int_c^v \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}, \\
\int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[u,b]\times[v,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} \\
+ \int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[a,u]\times[v,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} \\
+ \int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[u,b]\times[c,v]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} \\
+ \int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[a,u]\times[c,v]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}.
\end{cases} \tag{10}
$$

Proof. Let $(u,v) \in [a,b]\times[c,d]$. Using the integration by parts formula for Bochner integral, we have

$$
\begin{aligned}
&\int_u^b \int_v^d \left(\int_t^b \int_w^d \alpha(s,r)drds\right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt \\
&\quad = \int_u^b \left[\left(\int_t^b \int_w^d \alpha(s,r)drds\right) \frac{\partial}{\partial t} Y(t,w)\Big|_v^d + \left(\int_t^b \int_w^d \alpha(s,r)drds\right) \frac{\partial}{\partial t} Y(t,v) \right] dt \\
&= -\int_u^b \left(\int_t^b \int_v^d \alpha(s,r)drds\right) \frac{\partial}{\partial t} Y(t,v)dt + \int_u^b \int_v^d \left(\int_t^b \alpha(s,w)ds\right) \frac{\partial}{\partial t} Y(t,w)dwdt.
\end{aligned} \tag{11}
$$

Using the integration by parts formula for Bochner integral again for the two integrals in the last part of (11), we have

$$
\begin{aligned}
&\int_u^b \left(\int_t^b \int_v^d \alpha(s,r)drds\right) \frac{\partial}{\partial t} Y(t,v)dt \\
&\qquad = \left(\int_t^b \int_v^d \alpha(s,r)drds\right) Y(t,v)\Big|_u^b + \int_u^b \left(\int_v^d \alpha(t,r)dr\right) Y(t,v)dt \\
&\qquad = -\left(\int_u^b \int_v^d \alpha(s,r)drds\right) Y(u,v) + \int_u^b \left(\int_v^d \alpha(t,r)dr\right) Y(t,v)dt
\end{aligned} \tag{12}
$$

and

$$
\int_u^b \int_v^d \left(\int_t^b \alpha(s,w)ds\right) \frac{\partial}{\partial t} Y(t,w)dwdt
$$

$$= \int_v^d \left(\int_t^b \alpha(s,w)ds \right) Y(t,w) \Big|_u^b + \int_u^b \int_v^d \alpha(t,w) Y(t,w) dwdt$$
$$= -\int_v^d \left(\int_t^b \alpha(s,w)ds \right) Y(u,w)dw + \int_u^b \int_v^d \alpha(t,w) Y(t,w) dwdt. \quad (13)$$

Using (12) and (13) in (11), we obtain

$$\int_u^b \int_v^d \left(\int_t^b \int_w^d \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt$$
$$= \left(\int_u^b \int_v^d \alpha(s,r)drds \right) Y(u,v) - \int_u^b \left(\int_v^d \alpha(t,r)dr \right) Y(t,v)dt$$
$$- \int_v^d \left(\int_u^b \alpha(s,w)ds \right) Y(u,w)dw + \int_u^b \int_v^d \alpha(t,w) Y(t,w) dwdt. \quad (14)$$

Using the integration by parts formula for Bochner integral, we have

$$\int_a^u \int_v^d \left(\int_a^t \int_w^d \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt$$
$$= \int_a^u \left[\left(\int_a^t \int_w^d \alpha(s,r)drds \right) \frac{\partial}{\partial t} Y(t,w) \Big|_v^d + \int_v^d \left(\int_a^t \alpha(s,w)ds \right) \frac{\partial}{\partial t} Y(t,w)dw \right] dt$$
$$= -\int_a^u \left(\int_a^t \int_v^d \alpha(s,r)drds \right) \frac{\partial}{\partial t} Y(t,v)dt + \int_a^u \int_v^d \left(\int_a^t \alpha(s,w)ds \right) \frac{\partial}{\partial t} Y(t,w)dwdt$$
$$= -\left(\int_a^t \int_v^d \alpha(s,r)drds \right) Y(t,v) \Big|_a^u + \int_a^u \left(\int_v^d \alpha(t,r)dr \right) Y(t,v)dt$$
$$+ \int_v^d \left(\int_a^t \alpha(s,w)ds \right) Y(t,w)dw \Big|_a^u - \int_a^u \int_v^d \alpha(t,w) Y(t,w) dwdt$$
$$= -\left(\int_a^u \int_v^d \alpha(s,r)drds \right) Y(u,v) + \int_a^u \left(\int_v^d \alpha(t,r)dr \right) Y(t,v)dt$$
$$+ \int_v^d \left(\int_a^u \alpha(s,w)ds \right) Y(u,w)dw - \int_a^u \int_v^d \alpha(t,w) Y(t,w) dwdt. \quad (15)$$

In a similar way, we have

$$\int_u^b \int_c^v \left(\int_t^b \int_c^w \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt$$
$$= -\left(\int_u^b \int_c^v \alpha(s,r)drds \right) Y(u,v) + \int_u^b \left(\int_c^v \alpha(t,r)dr \right) Y(t,v)dt$$
$$+ \int_u^b \left(\int_c^v \alpha(s,w)ds \right) Y(u,w)dw - \int_u^b \int_c^v \alpha(t,w) Y(t,w) dwdt \quad (16)$$

and

$$\int_a^u \int_c^v \left(\int_a^t \int_c^w \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt$$
$$= \left(\int_a^u \int_c^v \alpha(s,r)drds \right) Y(u,v) - \int_a^u \left(\int_c^v \alpha(t,r)dr \right) Y(t,v)dt$$
$$- \int_a^u \left(\int_c^v \alpha(s,w)ds \right) Y(u,w)dw + \int_a^u \int_c^v \alpha(t,w) Y(t,w) dwdt. \quad (17)$$

The equalities (14)–(17) give us

$$\begin{aligned}
&\int_u^b \int_v^d \left(\int_t^b \int_w^d \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt \\
&\qquad - \int_a^u \int_v^d \left(\int_a^t \int_w^d \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt \\
&\qquad - \int_u^b \int_c^v \left(\int_t^b \int_c^w \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt \\
&\qquad + \int_a^u \int_c^v \left(\int_a^t \int_c^w \alpha(s,r)drds \right) \frac{\partial^2}{\partial w \partial t} Y(t,w)dwdt \\
&= \left(\int_a^b \int_c^d \alpha(s,r)drds \right) Y(u,v) + \int_a^b \int_c^d \alpha(t,w) Y(t,w)dwdt \\
&\qquad - \int_a^b \left(\int_c^d \alpha(t,r)dr \right) Y(t,v)dt - \int_c^d \left(\int_a^b \alpha(s,w)ds \right) Y(u,w)dw \qquad (18)
\end{aligned}$$

$(u,v) \in [a,b] \times [c,d]$.

By taking the norm $\|\cdot\|_{E_1 \times E_2}$ on both sides of (18), we obtain

$$\begin{aligned}
&\left\| \left(\int_a^b \int_c^d \alpha(s,r)drds \right) Y(u,v) + \int_a^b \int_c^d \alpha(t,w) Y(t,w)dwdt \right. \\
&\qquad \left. - \int_a^b \left(\int_c^d \alpha(t,r)dr \right) Y(t,v)dt - \int_c^d \left(\int_a^b \alpha(s,w)ds \right) Y(u,w)dw \right\|_{E_1 \times E_2} \\
&\qquad \le \int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&\qquad + \int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&\qquad + \int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&\qquad + \int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&\qquad := B(\alpha, Y, u, v). \qquad (19)
\end{aligned}$$

Using Hölder's inequality, properties of supremum norm and the properties of supremum, we obtain, for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that

$$\begin{aligned}
&\int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt \\
&\le \begin{cases}
\sup\limits_{(t,w) \in [u,b] \times [v,d]} \left(\int_t^b \int_w^d |\alpha(s,r)|drds \right) \int_u^b \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt, \\
\left[\int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds \right)^p dwdt \right]^{\frac{1}{p}} \left[\int_u^b \int_v^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}^q dwdt \right]^{\frac{1}{q}}, \\
\int_u^b \int_v^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds \right) dwdt \sup\limits_{(t,w) \in [u,b] \times [v,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2},
\end{cases} \qquad (20)
\end{aligned}$$

$$\int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt$$

$$\leq \begin{cases} \sup\limits_{(t,w)\in[a,u]\times[v,d]} \left(\int_a^t \int_w^d |\alpha(s,r)|drds\right) \int_a^u \int_v^d \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt, \\ \left[\int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_a^u \int_v^d \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}, \\ \int_a^u \int_v^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[a,u]\times[v,d]} \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}, \end{cases} \tag{21}$$

$$\int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right) \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt$$

$$\leq \begin{cases} \sup\limits_{(t,w)\in[u,b]\times[c,v]} \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right) \int_u^b \int_c^v \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt, \\ \left[\int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_u^b \int_c^v \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}, \\ \int_u^b \int_c^v \left(\int_t^b \int_c^w |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[u,b]\times[c,v]} \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}, \end{cases} \tag{22}$$

and

$$\int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right) \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt$$

$$\leq \begin{cases} \sup\limits_{(t,w)\in[a,u]\times[c,v]} \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right) \int_a^u \int_c^v \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt, \\ \left[\int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}} \left[\int_a^u \int_c^v \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right]^{\frac{1}{q}}, \\ \int_a^u \int_c^v \left(\int_a^t \int_c^w |\alpha(s,r)|drds\right) dwdt \sup\limits_{(t,w)\in[a,u]\times[c,v]} \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}. \end{cases} \tag{23}$$

From (20)–(23), we obtain the bounds for $B(\alpha, Y, u, v)$. This complete the proof of the theorem. □

We can obtain the following inequalities as a consequence of Theorem 7:

Corollary 1. *In view of the assumptions of Theorem 7, we have*

$$\begin{aligned} &\left\| \left(\int_a^b \int_c^d \alpha(s,r)drds\right) Y(u,v) + \int_a^b \int_c^d \alpha(t,w) Y(t,w) dwdt \right. \\ &\quad \left. - \int_a^b \left(\int_c^d \alpha(t,r)dr\right) Y(t,v)dt - \int_c^d \left(\int_a^b \alpha(s,w)ds\right) Y(u,w)dw \right\|_{E_1\times E_2} \\ &\qquad \leq \left(\int_u^b \int_v^d |\alpha(s,r)|drds\right) \int_u^b \int_v^d \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt \\ &\qquad + \left(\int_a^u \int_v^d |\alpha(s,r)|drds\right) \int_a^u \int_v^d \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt \\ &\qquad + \left(\int_u^b \int_c^v |\alpha(s,r)|drds\right) \int_u^b \int_c^v \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2} dwdt \end{aligned}$$

$$
\begin{aligned}
&+\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\
\leq&\begin{cases}
\max\left\{\int_u^b\int_v^d|\alpha(s,r)|drds,\int_a^u\int_v^d|\alpha(s,r)|drds\right.\\
\left.+\int_u^b\int_c^v|\alpha(s,r)|drds,\int_a^u\int_c^v|\alpha(s,r)|drds\right\}\\
\times\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\\
\max\left\{\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\right.\\
\left.+\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\right\}\\
\times\int_a^b\int_c^d|\alpha(s,r)|drds.
\end{cases}\\
&\leq\int_a^b\int_c^d|\alpha(s,r)|drds\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt. \qquad (24)
\end{aligned}
$$

The following result can be obtained for the case when $m,n\in(a,b)\times(c,d)$.

Proposition 1. *With the assumptions of Theorem 7, and if $m,n\in(a,b)\times(c,d)$, then the following inequalities hold:*

$$
\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y(m,n)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\quad\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y(t,v)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y(u,w)dw\right\|_{E_1\times E_2}\\
&\qquad\leq\frac{1}{4}\int_a^b\int_c^d|\alpha(s,r)|drds\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt. \qquad (25)
\end{aligned}
$$

Proof. If $m,n\in(a,b)\times(c,d)$, then

$$
\begin{aligned}
\int_a^m\int_c^n|\alpha(s,r)|drds&=\int_m^b\int_c^n|\alpha(s,r)|drds\\
&=\int_a^m\int_n^d|\alpha(s,r)|drds=\int_m^b\int_n^d|\alpha(s,r)|drds=\frac{1}{4}\int_a^b\int_c^d|\alpha(s,r)|drds. \qquad (26)
\end{aligned}
$$

Using (26) in Corollary 1, we have (25). □

Another important result can be stated and proved as a consequence from Theorem 7 as follows.

Corollary 2. *Suppose that the assumptions of Theorem 7 are satisfied, then the following inequalities hold:*

$$
\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y(u,v)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\quad\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y(t,v)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y(u,w)dw\right\|_{E_1\times E_2}\\
&\leq\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[u,b]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[a,u]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}
\end{aligned}
$$

$$+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[u,b]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}$$
$$+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[a,u]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}$$
$$\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\int_a^b\int_c^d|t-u||w-v||\alpha(s,r)|dwdt \quad (27)$$

for all $(u,v)\in[a,b]\times[c,d]$.

Proof. By integration by parts, we have

$$\begin{aligned}
&\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)dwdt\\
&\quad=-v\int_u^b\left(\int_t^b\int_v^d|\alpha(s,r)|drds\right)dt+\int_u^b\int_v^d\left(w\int_t^b|\alpha(s,w)|ds\right)dwdt\\
&\quad=-v\left[-u\int_u^b\int_v^d|\alpha(s,r)|drds+\int_u^b\int_v^d t|\alpha(t,r)|drdt\right]\\
&\quad+\left[-u\int_v^d\int_u^b w|\alpha(s,w)|dsdw+\int_u^b\int_v^d wt|\alpha(t,w)|dwdt\right]\\
&\quad=vu\int_u^b\int_v^d|\alpha(s,r)|drds-v\int_u^b\int_v^d t|\alpha(t,r)|\,drdt\\
&\quad-u\int_v^d\int_u^b w|\alpha(s,w)|dsdw+\int_u^b\int_v^d tw|\alpha(t,w)|\,dwdt\\
&\quad=\int_u^b\int_v^d(vu-tv-wu+tw)|\alpha(s,r)|\,drds\\
&\quad=\int_u^b\int_v^d|t-u||w-v||\alpha(s,r)|drds. \quad (28)
\end{aligned}$$

Similarly, by suing integration by parts, we can prove that

$$\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)dwdt=\int_a^u\int_v^d|t-u||w-v||\alpha(s,r)|drds, \quad (29)$$

$$\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)dwdt=\int_u^b\int_c^v|t-u||w-v||\alpha(s,r)|drds, \quad (30)$$

and

$$\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)dwdt=\int_a^u\int_c^v|t-u||w-v||\alpha(s,r)|drds. \quad (31)$$

From the first part of Theorem 7, we have

$$\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y(u,v)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\quad\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y(t,v)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y(u,w)dw\right\|_{E_1\times E_2}\\
&\quad\leq\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[u,b]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&\quad+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[a,u]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}
\end{aligned}$$

$$
\begin{aligned}
&+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[u,b]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in[a,u]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\left\{\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)dwdt\right.\\
&+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)dwdt+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)dwdt\\
&\left.+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)dwdt\right\}\\
&\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\left\{\int_u^b\int_v^d|t-u||w-v||\alpha(s,r)|drds\right.\\
&+\int_a^u\int_v^d|t-u||w-v||\alpha(s,r)|drds+\int_u^b\int_c^v|t-u||w-v||\alpha(s,r)|drds\\
&\left.+\int_a^u\int_c^v|t-u||w-v||\alpha(s,r)|drds\right\}\\
&\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\int_a^b\int_c^d|t-u||w-v||\alpha(s,r)|drds. \qquad (32)
\end{aligned}
$$

Inequality (32) establishes the proof of (27). □

We can also have another important Ostrowski-type inequalities that can be deduced from the result of Corollary 2, given below.

Proposition 2. *With the assumptions of Theorem 7, we have the following inequalities for functions with values in Banach spaces:*

$$
\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y(u,v)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y(t,v)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y(u,w)dw\right\|_{E_1\times E_2}\\
&\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&\times\begin{cases}\left\{\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right\}\left\{\frac{1}{2}(d-c)+\left|v-\frac{c+d}{2}\right|\right\}\int_a^b\int_c^d|\alpha(s,r)|dwdt,\\ \left[\frac{[(u-a)^{q+1}+(b-u)^{q+1}][(v-c)^{q+1}+(d-v)^{q+1}]}{(q+1)^2}\right]^{\frac{1}{q}}\left(\int_a^b\int_c^d|\alpha(t,s)|^p dwdt\right)^{\frac{1}{p}},\\ \frac{1}{4}\left\{[(u-a)^2+(b-u)^2][(v-c)^2+(d-v)^2]\right\}\sup_{(t,w)\in[a,b]\times[c,d]}|\alpha(t,s)|\end{cases} \qquad (33)
\end{aligned}
$$

for all $u,v\in[a,b]\times[c,d]$.

Proof. Applying Hölder's integral inequality, we obtain

$$\int_a^b\int_c^d|t-u||w-v||\alpha(s,r)|\,dwdt$$

$$\leq \begin{cases} \sup\limits_{(t,w)\in[a,b]\times[c,d]} |t-u||w-v| \int_a^b \int_c^d |\alpha(s,r)|dwdt, \\ \left(\int_a^b \int_c^d |t-u|^q |w-v|^q dwdt\right)^{\frac{1}{q}} \left(\int_a^b \int_c^d |\alpha(s,r)|^p dwdt\right)^{\frac{1}{p}}; \\ \int_a^b \int_c^d |t-u||w-v|dwdt \sup\limits_{(t,w)\in[a,b]\times[c,d]} |\alpha(s,r)|, \end{cases}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.
Since

$$\sup_{(t,w)\in[a,b]\times[c,d]} |t-u||w-v| = \max\{u-a, b-u\} \max\{v-c, d-v\}$$
$$= \left\{\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right\}\left\{\frac{1}{2}(d-c) + \left|v - \frac{c+d}{2}\right|\right\},$$

$$\left(\int_a^b \int_c^d |t-u|^q |w-v|^q dwdt\right)^{\frac{1}{q}}$$
$$= \left\{\frac{[(u-a)^{q+1} + (b-u)^{q+1}][(v-c)^{q+1} + (d-v)^{q+1}]}{(q+1)^2}\right\}^{\frac{1}{q}},$$

and

$$\int_a^b \int_c^d |t-u||w-v|dwdt = \frac{1}{4}\Big\{[(u-a)^2 + (b-u)^2][(v-c)^2 + (d-v)^2]\Big\}.$$

Using (27), we obtain the inequalities (33). □

Different bounds of the results of Theorem 7 are proved in the following corollary and its subsequent results:

Corollary 3. *Suppose that the assumptions of Theorem 7 are satisfied, we have for all* $(u, v) \in [a, b] \times [c, d]$*:*

$$\left\| \left(\int_a^b \int_c^d \alpha(s,r)drds\right) Y(u,v) + \int_a^b \int_c^d \alpha(t,w)Y(t,w)dwdt \right.$$
$$\left. - \int_a^b \left(\int_c^d \alpha(t,r)dr\right) Y(t,v)dt - \int_c^d \left(\int_a^b \alpha(s,w)ds\right) Y(u,w)dw \right\|_{E_1\times E_2}$$
$$\leq \left\{ (b-u)(d-v)\left(\int_u^b \int_v^d |\alpha(s,r)|drds\right)^p + (u-a)(d-v) \right.$$
$$\times \left(\int_a^u \int_v^d |\alpha(s,r)|drds\right)^p + (b-u)(v-c)\left(\int_u^b \int_c^v |\alpha(s,r)|drds\right)^p$$
$$\left. + (u-a)(v-c)\left(\int_a^u \int_c^v |\alpha(s,r)|drds\right)^p \right\}^{\frac{1}{p}} \left(\int_a^b \int_c^d \left\|\frac{\partial^2}{\partial w \partial t} Y(t,w)\right\|_{E_1\times E_2}^q\right)^{\frac{1}{q}}$$
$$\leq (b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}} \left\{ \left(\int_u^b \int_v^d |\alpha(s,r)|drds\right)^p \right.$$
$$\left. + \left(\int_a^u \int_v^d |\alpha(s,r)|drds\right)^p + \left(\int_u^b \int_c^v |\alpha(s,r)|drds\right)^p \right.$$

$$+\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p\Bigg\}^{\frac{1}{p}}\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}\right)^{\frac{1}{q}}, \quad (34)$$

where $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the elementary inequality for $x_1,x_2,x_3,x_3,x_5,x_6,x_7,x_8$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$x_1x_2+x_3x_4+x_5x_6+x_7x_8\le\left(x_1^p+x_3^p+x_5^p+x_7^p\right)^{\frac{1}{p}}\left(x_2^q+x_4^q+x_6^q+x_8^q\right)^{\frac{1}{q}},$$

we have

$$\begin{aligned}
&\left[\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}}\\
&\quad+\left[\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}}\\
&\quad+\left[\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}}\\
&\quad+\left[\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)^p dwdt\right]^{\frac{1}{p}}\left[\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right]^{\frac{1}{q}}\\
&\quad\le\left\{\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)^p dwdt\right.\\
&\quad\left.+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)^p dwdt\right\}^{\frac{1}{p}}\\
&\qquad\times\left\{\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt+\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right.\\
&\qquad\left.+\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt+\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right\}^{\frac{1}{q}}\\
&\quad=\left\{\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)^p dwdt\right.\\
&\quad\left.+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)^p dwdt\right\}^{\frac{1}{p}}\\
&\qquad\times\left(\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2}dwdt\right)^{\frac{1}{q}}. \quad (35)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)^p dwdt\\
&\quad+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)^p dwdt\\
&\quad\le\int_u^b\int_v^d\left(\int_u^b\int_v^d|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_a^u\int_v^d|\alpha(s,r)|drds\right)^p dwdt
\end{aligned}$$

$$
\begin{aligned}
&+\int_u^b\int_c^v\left(\int_u^b\int_c^v|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p dwdt\\
&=\left(\int_u^b\int_v^d|\alpha(s,r)|drds\right)^p\int_u^b\int_v^d dwdt+\left(\int_a^u\int_v^d|\alpha(s,r)|drds\right)^p\int_a^u\int_v^d dwdt\\
&+\left(\int_u^b\int_c^v|\alpha(s,r)|drds\right)^p\int_u^b\int_c^v dwdt+\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p\int_a^u\int_c^v dwdt\\
&=(b-u)(d-v)\left(\int_u^b\int_v^d|\alpha(s,r)|drds\right)^p+(u-a)(d-v)\left(\int_a^u\int_v^d|\alpha(s,r)|drds\right)^p\\
&+(b-u)(v-c)\left(\int_u^b\int_c^v|\alpha(s,r)|drds\right)^p+(u-a)(v-c)\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p. \quad (36)
\end{aligned}
$$

Applying (36) in (35), we obtain the first inequality in (34).
Moreover, we also

$$
\begin{aligned}
&\int_u^b\int_v^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)^p dwdt\\
&+\int_u^b\int_c^v\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)^p dwdt+\int_a^u\int_c^v\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)^p dwdt\\
&\le(b-u)(d-v)\left(\int_u^b\int_v^d|\alpha(s,r)|drds\right)^p+(u-a)(d-v)\left(\int_a^u\int_v^d|\alpha(s,r)|drds\right)^p\\
&+(b-u)(v-c)\left(\int_u^b\int_c^v|\alpha(s,r)|drds\right)^p+(u-a)(v-c)\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p\\
&\le(b-a)(d-c)\left\{\left(\int_u^b\int_v^d|\alpha(s,r)|drds\right)^p+\left(\int_a^u\int_v^d|\alpha(s,r)|drds\right)^p\right.\\
&\left.+\left(\int_u^b\int_c^v|\alpha(s,r)|drds\right)^p+\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p\right\}. \quad (37)
\end{aligned}
$$

Applying (37) in (35), we obtain the second inequality in (34). □

Proposition 3. *Suppose that the assumptions of Theorem 7 are satisfied, and $m,n\in(a,b)\times(c,d)$, then the following inequalities hold: if (25) is valid, then, using (34), we have*

$$
\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y(m,n)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y(t,n)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y(m,w)dw\right\|_{E_1\times E_2}\\
&\le\left\{(b-u)(d-v)\left(\int_u^b\int_v^d|\alpha(s,r)|drds\right)^p+(u-a)(d-v)\right.\\
&\times\left(\int_a^u\int_v^d|\alpha(s,r)|drds\right)^p+(b-u)(v-c)\left(\int_u^b\int_c^v|\alpha(s,r)|drds\right)^p\\
&\left.+(u-a)(v-c)\left(\int_a^u\int_c^v|\alpha(s,r)|drds\right)^p\right\}^{\frac{1}{p}}\le\{(b-a)(d-c)\}^{\frac{1}{p}}\\
&\times\int_a^b\int_c^d|\alpha(s,r)|drds\left(\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^q dwdt\right)^{\frac{1}{q}}. \quad (38)
\end{aligned}
$$

Proof. Since $m,n\in(a,b)\times(c,d)$, we have

$$\int_a^m \int_c^n |\alpha(s,r)|drds = \int_m^b \int_c^n |\alpha(s,r)|drds$$
$$= \int_a^m \int_n^d |\alpha(s,r)|drds = \int_m^b \int_n^d |\alpha(s,r)|drds = \frac{1}{4}\int_a^b \int_c^d |\alpha(s,r)|drds. \quad (39)$$

From the first inequality in (34) and (39), we have

$$\left\| \left(\int_a^b \int_c^d \alpha(s,r)drds \right) Y(m,n) + \int_a^b \int_c^d \alpha(t,w)Y(t,w)dwdt \right.$$
$$- \int_a^b \left(\int_c^d \alpha(t,r)dr \right) Y(t,n)dt - \int_c^d \left(\int_a^b \alpha(s,w)ds \right) Y(m,w)dw \Big\|_{E_1\times E_2}$$
$$\leq \left\{ (b-m)(d-n) \left(\int_m^b \int_n^d |\alpha(s,r)|drds \right)^p + (m-a)(d-n) \right.$$
$$\times \left(\int_a^m \int_n^d |\alpha(s,r)|drds \right)^p + (b-m)(n-c) \left(\int_m^b \int_c^n |\alpha(s,r)|drds \right)^p$$
$$\left. + (m-a)(n-c) \left(\int_a^m \int_c^n |\alpha(s,r)|drds \right)^p \right\}^{\frac{1}{p}} \left(\int_a^b \int_c^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q \right)^{\frac{1}{q}}$$
$$\leq \left\{ \frac{(b-a)(d-c)}{4} \right\}^{\frac{1}{p}} \left\{ \left(\frac{1}{4}\int_a^b \int_c^d |\alpha(s,r)|drds \right)^p + \left(\frac{1}{4}\int_a^b \int_c^d |\alpha(s,r)|drds \right)^p \right.$$
$$\left. + \left(\frac{1}{4}\int_a^b \int_c^d |\alpha(s,r)|drds \right)^p + \left(\frac{1}{4}\int_a^b \int_c^d |\alpha(s,r)|drds \right)^p \right\}^{\frac{1}{p}}$$
$$\times \left(\int_a^b \int_c^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q \right)^{\frac{1}{q}} \leq \{(b-a)(d-c)\}^{\frac{1}{p}}$$
$$\times \int_a^b \int_c^d |\alpha(s,r)|drds \left(\int_a^b \int_c^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2}^q \right)^{\frac{1}{q}}. \quad (40)$$

□

If u and v are the midpoints of the the intervals $[a,b]$ and $[c,d]$, respectively, then we can obtain the followings Ostrowski-type inequalities from Theorem 7 and its subsequent corollaries.

Proposition 4. *Suppose that the assumptions of Theorem 7 are satisfied, then we have the following midpoint inequality:*

$$\left\| \left(\int_a^b \int_c^d \alpha(s,r)drds \right) Y\left(\frac{a+b}{2}, \frac{c+d}{2} \right) + \int_a^b \int_c^d \alpha(t,w)Y(t,w)dwdt \right.$$
$$\left. - \int_a^b \left(\int_c^d \alpha(t,r)dr \right) Y\left(t, \frac{c+d}{2} \right) dt - \int_c^d \left(\int_a^b \alpha(s,w)ds \right) Y\left(\frac{a+b}{2}, w \right) dw \right\|_{E_1\times E_2}$$
$$\leq M(\alpha, Y), \quad (41)$$

where

$$M(\alpha,Y) := \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d \left(\int_t^b \int_w^d |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt$$
$$+ \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d \left(\int_a^t \int_w^d |\alpha(s,r)|drds \right) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1\times E_2} dwdt$$

$$+\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left(\int_{t}^{b}\int_{c}^{w}|\alpha(s,r)|drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left(\int_{a}^{t}\int_{c}^{w}|\alpha(s,r)|drds\right)\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt. \quad (42)$$

We also have the following bounds:

$$M(\alpha,Y)$$
$$\leq\begin{cases}\sup\limits_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[\frac{c+d}{2},d\right]}\left(\int_{t}^{b}\int_{w}^{d}|\alpha(s,r)|drds\right)\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\ \quad+\sup\limits_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[\frac{c+d}{2},d\right]}\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\ +\sup\limits_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[c,\frac{c+d}{2}\right]}\left(\int_{t}^{b}\int_{c}^{w}|\alpha(s,r)|drds\right)\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\ +\sup\limits_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]}\left(\int_{a}^{t}\int_{c}^{w}|\alpha(s,r)|drds\right)\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\\ \\ \left[\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left(\int_{t}^{b}\int_{w}^{d}|\alpha(s,r)|drds\right)^{p}dwdt\right]^{\frac{1}{p}}\left[\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^{q}dwdt\right]^{\frac{1}{q}}\\ +\left[\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left(\int_{a}^{t}\int_{w}^{d}|\alpha(s,r)|drds\right)^{p}dwdt\right]^{\frac{1}{p}}\left[\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^{q}dwdt\right]^{\frac{1}{q}}\\ +\left[\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left(\int_{t}^{b}\int_{c}^{w}|\alpha(s,r)|drds\right)^{p}dwdt\right]^{\frac{1}{p}}\left[\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^{q}dwdt\right]^{\frac{1}{q}}\\ +\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left(\int_{a}^{t}\int_{c}^{w}|\alpha(s,r)|drds\right)^{p}dwdt\right]^{\frac{1}{p}}\left[\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}^{q}dwdt\right]^{\frac{1}{q}},\\ \\ \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left(\int_{t}^{b}\int_{w}^{d}|\alpha(s,r)|drds\right)dwdt\sup\limits_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[\frac{c+d}{2},d\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\ +\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left(\int_{a}^{t}\int_{w}^{d}|\alpha(s,r)|drds\right)dwdt\sup\limits_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[\frac{c+d}{2},d\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\ +\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left(\int_{t}^{b}\int_{c}^{w}|\alpha(s,r)|drds\right)dwdt\sup\limits_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[c,\frac{c+d}{2}\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\ +\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left(\int_{a}^{t}\int_{c}^{w}|\alpha(s,r)|drds\right)dwdt\sup\limits_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}.\end{cases} \quad (43)$$

Using inequality (24), we have

$$\left\|\left(\int_{a}^{b}\int_{c}^{d}\alpha(s,r)drds\right)Y(u,v)+\int_{a}^{b}\int_{c}^{d}\alpha(t,w)Y(t,w)dwdt\right.$$
$$\left.-\int_{a}^{b}\left(\int_{c}^{d}\alpha(t,r)dr\right)Y(t,v)dt-\int_{c}^{d}\left(\int_{a}^{b}\alpha(s,w)ds\right)Y(u,w)dw\right\|_{E_1\times E_2}$$
$$\leq\left(\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}|\alpha(s,r)|drds\right)\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\left(\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}|\alpha(s,r)|drds\right)\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$
$$+\left(\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}|\alpha(s,r)|drds\right)\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt$$

$$
\begin{aligned}
&+\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|\alpha(s,r)|drds\right)\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\\
&\leq\begin{cases}
\max\left\{\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}|\alpha(s,r)|drds,\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}|\alpha(s,r)|drds\right.\\
\left.+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}|\alpha(s,r)|drds,\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|\alpha(s,r)|drds\right\}\\
\times\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\\
\max\left\{\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\right.\\
\left.+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\right\}\\
\times\int_a^b\int_c^d|\alpha(s,r)|drds.
\end{cases}\\
&\leq\int_a^b\int_c^d|\alpha(s,r)|drds\int_a^b\int_c^d\frac{\partial^2}{\partial w\partial t}\|Y(t,w)\|_{E_1\times E_2}dwdt. \qquad (44)
\end{aligned}
$$

Applying (27), we obtain

$$
\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y\left(t,\frac{c+d}{2}\right)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y\left(\frac{a+b}{2},w\right)dw\right\|_{E_1\times E_2}\\
&\leq\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left(\int_t^b\int_w^d|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[\frac{c+d}{2},d\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&+\int_a^{\frac{a+b}{2}}\int_v^d\left(\int_a^t\int_w^d|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[\frac{c+d}{2},d\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\left(\int_t^b\int_c^w|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in\left[\frac{a+b}{2},b\right]\times\left[c,\frac{c+d}{2}\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(\int_a^t\int_c^w|\alpha(s,r)|drds\right)dwdt\sup_{(t,w)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\int_a^b\int_c^d\left|t-\frac{a+b}{2}\right|\left|w-\frac{c+d}{2}\right||\alpha(s,r)|dwdt. \qquad (45)
\end{aligned}
$$

Using (33), we derive the following inequality through functions in Banach spaces:

$$
\begin{aligned}
&\left\|\left(\int_a^b\int_c^d\alpha(s,r)drds\right)Y\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\int_a^b\int_c^d\alpha(t,w)Y(t,w)dwdt\right.\\
&\left.-\int_a^b\left(\int_c^d\alpha(t,r)dr\right)Y\left(t,\frac{c+d}{2}\right)dt-\int_c^d\left(\int_a^b\alpha(s,w)ds\right)Y\left(\frac{a+b}{2},w\right)dw\right\|_{E_1\times E_2}\\
&\leq\sup_{(t,w)\in[a,b]\times[c,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}\\
&\times\begin{cases}
\frac{1}{4}(b-a)(d-c)\int_a^b\int_c^d|\alpha(s,r)|dwdt,\\
\left[\frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{2(q+1)^{\frac{2}{q}}}\right]^{\frac{1}{q}}\left(\int_a^b\int_c^d|\alpha(t,s)|^pdwdt\right)^{\frac{1}{p}},\\
\frac{1}{4}[(b-a)^2(d-c)^2]\sup_{(t,w)\in[a,b]\times[c,d]}|\alpha(t,s)|.
\end{cases} \qquad (46)
\end{aligned}
$$

Using (34), we have

$$\left\| \left(\int_a^b \int_c^d \alpha(s,r)drds \right) Y\left(\frac{a+b}{2}, \frac{c+d}{2} \right) + \int_a^b \int_c^d \alpha(t,w)Y(t,w)dwdt \right.$$
$$\left. - \int_a^b \left(\int_c^d \alpha(t,r)dr \right) Y\left(t, \frac{c+d}{2} \right) dt - \int_c^d \left(\int_a^b \alpha(s,w)ds \right) Y\left(\frac{a+b}{2}, w \right) dw \right\|_{E_1 \times E_2}$$
$$\leq \frac{(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}}{2^{\frac{2}{p}}} \left\{ \left(\int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d |\alpha(s,r)|drds \right)^p \right.$$
$$+ \left(\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d |\alpha(s,r)|drds \right)^p + \left(\int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} |\alpha(s,r)|drds \right)^p$$
$$\left. + \left(\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} |\alpha(s,r)|drds \right)^p \right\}^{\frac{1}{p}} \left(\int_a^b \int_c^d \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2}^q \right)^{\frac{1}{q}}. \quad (47)$$

If we consider the case when $\alpha(t,w) = 1$, $(t,w) \in [a,b] \times [c,d]$ then, from (8), we obtain

$$\left\| (b-a)(d-c)Y(u,v) + \int_a^b \int_c^d Y(t,w)dwdt \right.$$
$$\left. - (d-c) \int_a^b Y(t,v)dt - (b-a) \int_c^d Y(u,w)dw \right\|_{E_1 \times E_2} \leq B(Y,u,v), \quad (48)$$

where

$$B(Y,u,v) := \int_u^b \int_v^d (b-t)(d-w) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt$$
$$+ \int_a^u \int_v^d (t-a)(d-w) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt$$
$$+ \int_u^b \int_c^v ((b-t)(w-c)) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt$$
$$+ \int_a^u \int_c^v (t-a)(w-c) \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} dwdt. \quad (49)$$

We have the following bounds for $B(\alpha, Y, u, v)$ from (10):

$B(Y,u,v)$

$$\leq \begin{cases} (b-u)(d-v)\int_u^b\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt \\ +(u-a)(d-v)\int_a^u\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt \\ +(b-u)(v-c)\int_u^b\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt \\ +(u-a)(v-c)\int_a^u\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} dwdt, \\ \\ \frac{[(b-u)(d-v)]^{\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_u^b\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}} \\ +\frac{[(u-a)(d-v)]^{\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_a^u\int_v^d \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}} \\ +\frac{[(b-u)(v-c)]^{\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_u^b\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}} \\ +\frac{[(u-a)(v-a)]^{\frac{1}{p}}}{(1+p)^{\frac{2}{p}}}\left[\int_a^u\int_c^v \left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|^q_{E_1\times E_2} dwdt\right]^{\frac{1}{q}}, \\ \\ \frac{(b-u)^2(d-v)^2}{4}\sup_{(t,w)\in[u,b]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\frac{(u-a)^2(d-v)^2}{4}\sup_{(t,w)\in[a,u]\times[v,d]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\frac{(b-u)^2(v-c)^2}{4}\sup_{(t,w)\in[u,b]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2} \\ +\frac{(u-a)^2(v-c)^2}{4}\sup_{(t,w)\in[a,u]\times[c,v]}\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}. \end{cases} \tag{50}$$

for all $(u,v)\in[a,b]\times[c,d]$.

From (24), we obtain

$$\begin{aligned} &\left\|(b-a)(d-c)Y(u,v)+\int_a^b\int_c^d Y(t,w)dwdt\right. \\ &\qquad\left.-\int_a^b(d-c)Y(t,v)dt-\int_c^d(b-a)Y(u,w)dw\right\|_{E_1\times E_2} \\ &\qquad\leq (b-u)(d-v)\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ &\qquad+(u-a)(d-v)\int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ &\qquad+(b-u)(v-c)\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ &\qquad+(u-a)(v-c)\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt \\ &\leq \begin{cases} \left\{\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right\}\left\{\frac{1}{2}(d-c)+\left|v-\frac{c+d}{2}\right|\right\} \\ \qquad\times\int_a^b\int_c^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt, \\ \\ (b-a)(d-c)\max\left\{\int_u^b\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\right. \\ \int_a^u\int_v^d\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt,\int_u^b\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt, \\ \left.\int_a^u\int_c^v\left\|\frac{\partial^2}{\partial w\partial t}Y(t,w)\right\|_{E_1\times E_2}dwdt\right\} \\ . \end{cases} \end{aligned} \tag{51}$$

for all $(u,v) \in [a,b] \times [c,d]$.

From (33), we have the following Ostrowski-type inequalities for functions with values in the product Banach spaces:

$$\begin{aligned}\Bigg\| (b-a)(d-c)Y(u,v) + \int_a^b \int_c^d Y(t,w)dwdt \\ , - \int_a^b (d-c)Y(t,v)dt - \int_c^d (b-a)Y(u,w)dw \Bigg\|_{E_1 \times E_2} \\ \leq \sup_{(t,w)\in[a,b]\times[c,d]} \left\| \frac{\partial^2}{\partial w \partial t} Y(t,w) \right\|_{E_1 \times E_2} \\ \times \left\{ \frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2 \right\} \left\{ \frac{1}{4}(d-c)^2 + \left(v - \frac{c+d}{2} \right)^2 \right\} \end{aligned} \quad (52)$$

for all $u, v \in [a,b] \times [c,d]$.

3. Conclusive Remarks

In the past four decades, there has been significant growth in the field of mathematical inequalities, which includes a variety of new results with numerous applications in different fields of mathematical sciences. Many researchers have published a plethora of articles using innovative approaches and applications. Within the extensive literature on mathematical inequalities, Ostrowski and Ostrowski-type inequalities have profound importance. These inequalities are utilized to estimate the absolute deviation of the average value of a function from its integral mean over an interval of real line. Mathematicians have proven various generalizations and diverse forms of the Ostrowski-type inequalities, such as those for functions of bounded variation, Lipschitzian mappings, absolutely continuous functions, and those involving two functions, of which one has values in Banach spaces and the other function has values in the field of complex numbers. One of the remarkable studies on the generalizations of Ostrowski-type inequalities are highlighted in [11]. In the present study, we proved a more general results of the Ostrowski-type for functions of two variables with values in the direct product of Banach spaces having the norm topology $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$, which generalizes the results from [11] to the functions of two variables, and extends the results from [10,12,13,15–18]. In order to obtain our results, we used the novel results from the theory of Banach spaces to provide various bounds of the average value of a function of two variables from its values at a specified point, in using the geometry of direct product of Banach spaces. We hope that the results of this study can be a good source to obtain more new results for the researchers working in the field of mathematical inequalities in the geometry of direct products of Banach spaces, and an inspiration to your researchers working to serve the mathematical community.

Author Contributions: Conceptualization, M.A.L. and O.B.A.; Validation, M.A.L. and O.B.A.; Formal analysis, M.A.L. and O.B.A.; Investigation, M.A.L. and O.B.A.; Writing—original draft, M.A.L. and O.B.A.; Writing—review & editing, M.A.L. and O.B.A.; Visualization, M.A.L. and O.B.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data has been used in the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Anastassiou, G.; Argyros, I.K. *Intelligent Numerical Methods: Applications to Fractional Calculus*; Studies in Computational Intelligence, 624; Kacprzyk, J., Ed.; Springer: London, UK, 2016.
2. Anastassiou, G.; Argyros, I.K. *Functional Numerical Methods: Applications to Abstract Fractional Calculus*; Studies in Systems, Decision and Control 130; Kacpryzyk, J., Ed.; Springer: Cham, Switzerland, 2018.

3. Anastassiou, G.A. Ostrowski and Landau inequalities for Banach space valued functions. *Math. Comput. Model.* **2012**, *55*, 312–329. [CrossRef]
4. Anastassiou, G.A. Multivariate Ostrowski type inequalities. *Acta Math. Hungar.* **1997**, *76*, 267–278. [CrossRef]
5. Cheng, X.-L. Improvement of Some Ostrowski-Grüss Type Inequalities. *Comput. Math. Appl.* **2001**, *42*, 109–114. [CrossRef]
6. Irshad, N.; Khan, A.R. Some applications of quadrature rules for mappings on $L_p[u, v]$ space via Ostrowski-type inequality. *J. Numer. Anal. Approx. Theory* **2017**, *46*, 141–149. [CrossRef]
7. Liu, Z. Some Ostrowski type inequalities. *Math. Comput. Model.* **2008**, *48*, 949–960. [CrossRef]
8. Dragomir, S.S. A Refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and Applications. *Libertas Math. I-XXXI* **2002**, *22*, 49–64.
9. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. *Inequalities Involving Functions and Their Integrals and Derivatives*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1994.
10. Barnett, N.S.; Buşe, C.; Cerone, P.; Dragomir, S.S. *On Weighted Ostrowski Type Inequalities for Operators and Vector-Valued Functions*; RGMIA Research Report Collection; RGMIA: Melbourne, Australia, 2001; Volume 4, 20p.
11. Dragomir, S.S. Ostrowski type inequalities for functions with values in Banach spaces. *RGMIA Res. Rep. Collect.* **2020**, *24*, 20–21.
12. Alomari, M.W. A generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation. *Int. J. Nonlinear Sci. Numer. Simul.* **2020**, *21*, 667–673. [CrossRef]
13. Barnett, N.S.; Dragomir, S.S. An Ostrowski type inequality for double integrals and applications for cubature formulae. *Soochow J. Math.* **2001**, *27*, 1–10.
14. Budak, H.; Sarikaya, M.Z. On Ostrowski type inequalities for functions of two variables with bounded variation. *Int. J. Anal. Appl.* **2016**, *12*, 142–156.
15. Özdemir, M.E.; Akdemir, A.O.; Set, E. A new Ostrowski-type inequality for double integrals. *J. Inequalities Spec. Funct.* **2011**, *2*, 27–34.
16. Pachpatte, B.G. On a new Ostrowski type inequality in two independent variables. *Tamkang J. Math.* **2001**, *32*, 45–49. [CrossRef]
17. Pachpatte, B.G. A new Ostrowski type inequality for double integrals. *Soochow J. Math.* **2006**, *32*, 317–322.
18. Sarikaya, M.Z. On the Ostrowski type integral inequality for double integrals. *Demonstratio Math.* **2012**, *XLV*, 533–540. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.