

Article

Generating Functions for Binomial Series Involving Harmonic-like Numbers

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Abstract: By employing the coefficient extraction method, a class of binomial series involving harmonic numbers will be reviewed through three hypergeometric ${}_2F_1(y^2)$ -series. Numerous closed-form generating functions for infinite series containing binomial coefficients and harmonic numbers will be established, including several conjectured ones.

Keywords: harmonic numbers; central binomial coefficient; hypergeometric series

MSC: 11B65; 33C20; 65B10

1. Introduction and Motivation

Harmonic numbers and their variants have been examined since Euler's era for more than three centuries and involved in many mathematical fields (cf. [1–6]), such as the analysis of algorithms (cf. [7]) in computer science, various topics of number theory (cf. [8–11]), and combinatorial analysis (cf. [12–14]). In this paper, we are going to review a particular class of binomial series (cf. [15–18]) with a free variable “ y ” and harmonic-like numbers. In order to proceed smoothly, it is necessary to go over some basic facts about harmonic numbers, the “coefficient extraction” method (cf. [19–21]), and classical hypergeometric series.

For the sake of brevity, we shall make use of the following notations throughout the paper for Catalan's constant and the Riemann zeta function (cf. [22]):

$$G = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2} \quad \text{and} \quad \zeta(m) = \sum_{k=1}^{\infty} \frac{1}{k^m} \quad \text{for } m > 1.$$

1.1. Harmonic Numbers

For $\lambda \in \mathbb{R}$ and $m, n \in \mathbb{N}_0$, the harmonic-like numbers are defined by

$$\begin{aligned} \mathbf{H}_n^{(m)}(\lambda) &= \sum_{k=0}^{n-1} \frac{1}{(\lambda+k)^m}, & \mathbf{O}_n^{(m)} &= \sum_{k=0}^{n-1} \frac{1}{(1+2k)^m}; \\ \bar{\mathbf{H}}_n^{(m)}(\lambda) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(\lambda+k)^m}, & \bar{\mathbf{O}}_n^{(m)} &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(1+2k)^m}. \end{aligned}$$

When $m = 1$ and/or $\lambda = 1$, they will be omitted from these notations. We also record the following simple but useful relations:

$$\begin{aligned} \mathbf{H}_n^{(m)}\left(\frac{1}{2}\right) &= 2^m \mathbf{O}_n^{(m)}, & \mathbf{H}_{2n}^{(m)} &= \mathbf{O}_n^{(m)} + 2^{-m} \mathbf{H}_n^{(m)}; \\ \bar{\mathbf{H}}_n^{(m)}\left(\frac{1}{2}\right) &= 2^m \bar{\mathbf{O}}_n^{(m)}, & \bar{\mathbf{H}}_{2n}^{(m)} &= \bar{\mathbf{O}}_n^{(m)} - 2^{-m} \mathbf{H}_n^{(m)}. \end{aligned}$$



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1.2. Coefficient Extraction

For $n \in \mathbb{N}_0$ and an indeterminate x , the shifted factorials are usually defined (cf. Bailey [23] §1.1) by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad \text{for } n \in \mathbb{N}.$$

Let $[x^m]\phi(x)$ stand for the coefficient of x^m in the formal power series $\phi(x)$. Then, the harmonic-like numbers can be expressed as coefficients extracted from factorial quotients:

$$\begin{aligned} [x] \frac{(1+x)_n}{n!} &= \mathbf{H}_n, & [x^2] \frac{(1+x)_n}{n!} &= \frac{\mathbf{H}_n^2 - \mathbf{H}_n^{(2)}}{2}, \\ [x] \frac{n!}{(1-x)_n} &= \mathbf{H}_n, & [x^2] \frac{n!}{(1-x)_n} &= \frac{\mathbf{H}_n^2 + \mathbf{H}_n^{(2)}}{2}, \\ [x] \frac{(\frac{1}{2}+x)_n}{(\frac{1}{2})_n} &= 2\mathbf{O}_n, & [x^2] \frac{(\frac{1}{2}+x)_n}{(\frac{1}{2})_n} &= 2(\mathbf{O}_n^2 - \mathbf{O}_n^{(2)}), \\ [x] \frac{(\frac{1}{2})_n}{(\frac{1}{2}-x)_n} &= 2\mathbf{O}_n, & [x^2] \frac{(\frac{1}{2})_n}{(\frac{1}{2}-x)_n} &= 2(\mathbf{O}_n^2 + \mathbf{O}_n^{(2)}). \end{aligned}$$

For a real number $\lambda \notin \mathbb{Z} \setminus \mathbb{N}$, it is not difficult to verify that

$$\mathbf{H}_n(\lambda) = [x] \frac{(\lambda+x)_n}{(\lambda)_n} = [x] \frac{(\lambda)_n}{(\lambda-x)_n}.$$

In general, the following formulae hold (cf. Chu [5,21]):

$$[x^m] \frac{(\lambda-x)_n}{(\lambda)_n} = \Omega_m \left\{ -\mathbf{H}_n^{(k)}(\lambda) \right\} \quad \text{and} \quad [x^m] \frac{(\lambda)_n}{(\lambda-x)_n} = \Omega_m \left\{ \mathbf{H}_n^{(k)}(\lambda) \right\}, \quad (1)$$

where the Bell polynomials (cf. [24] §3.3) are expressed by the multiple sum

$$\Omega_m \left\{ \pm \mathbf{H}_n^{(k)}(\lambda) \right\} = \sum_{\omega(m)} \prod_{k=1}^m \frac{\left\{ \pm \mathbf{H}_n^{(k)}(\lambda) \right\}^{i_k}}{i_k! k^{i_k}}, \quad (2)$$

with $\omega(m)$ standing for the set of m -partitions represented by $(i_1, i_2, \dots, i_m) \in \mathbb{N}_0^m$ subject to the linear condition $\sum_{k=1}^m k i_k = m$.

1.3. Three ${}_2F_1(y^2)$ -Series

By making use of the “coefficient extraction” method, we shall examine three closed formulae for ${}_2F_1$ -series (cf. Chu [25]) convergent with $|y| < 1$ and establish several infinite series identities.

$$\begin{aligned} \boxed{A(x, y)} \quad {}_2F_1 \left[x, -x \middle| \frac{1}{2} y^2 \right] &= \sum_{n=0}^{\infty} \frac{(x)_n (-x)_n}{n! (\frac{1}{2})_n} y^{2n} = \cos(2x \arcsin y), \\ \boxed{B(x, y)} \quad {}_2F_1 \left[x, 1-x \middle| \frac{1}{2} y^2 \right] &= \sum_{n=0}^{\infty} \frac{(x)_n (1-x)_n}{n! (\frac{1}{2})_n} y^{2n} = \frac{\cos((2x-1) \arcsin y)}{\sqrt{1-y^2}}, \\ \boxed{C(x, y)} \quad {}_2F_1 \left[x, 1-x \middle| \frac{3}{2} y^2 \right] &= \sum_{n=0}^{\infty} \frac{(x)_n (1-x)_n}{n! (\frac{3}{2})_n} y^{2n} = \frac{\sin((2x-1) \arcsin y)}{(2x-1)y}. \end{aligned}$$

The above three equalities are analytic in the neighborhood of $x = 0$ and can be expanded into Maclaurin series. For a real number λ , $m \in \mathbb{N}_0$, and $W \in \{A, B, C\}$, we denote by $W_m(\lambda+x, y) = [x^m]W(\lambda+x, y)$ the resulting equality from the coefficients of x^m extracted across the equality $W(\lambda+x, y)$. Then, numerous infinite series identities can be established

using the “coefficient extraction method”. The aim of this paper is to utilize the above three series to review the summation formulae involving both the binomial coefficient and harmonic-like numbers.

The rest of the paper will be divided into five sections in accordance with the binomial/multinomial structures. Each section consists of several algebraic formulae for infinite series involving harmonic-like numbers, which can be considered generating functions (cf. Wilf [26]) for binomial sequences, together with the harmonic-like numbers. A number of known results will be recorded for comparison, including several series conjectured by Sun [27–29]. In order to assure accuracy, all the displayed equalities are verified numerically through appropriately devised Mathematica commands.

2. Series with the Central Binomial Coefficient $\binom{2n}{n}$ in the Denominator

This section is devoted to harmonic series with the central binomial coefficient in the denominator.

2.1. Series from $A(x, y)$

Since $A(x, y)$ can be expanded into a power series of x , the initial coefficients give rise to the following three infinite series identities, where $A_4(x, y)$ can be found in Sun [29] (Equation 1.4).

$$\begin{aligned} A_2(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2 \binom{2n}{n}} = 2 \arcsin^2 y, \\ A_4(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2 \binom{2n}{n}} \mathbf{H}_{n-1}^{(2)} = \frac{2 \arcsin^4 y}{3}, \\ A_6(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2 \binom{2n}{n}} \left\{ (\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)} \right\} = \frac{8 \arcsin^6 y}{45}. \end{aligned}$$

We remark that these series are examples of the formula discovered by Borwein and Chamberland [30]. Some interesting numerical series are recorded as follows.

- Two numerical series from $A_2(x, y)$: $\varphi = \frac{1 + \sqrt{5}}{2}, \bar{\varphi} = \frac{1 - \sqrt{5}}{2}$.

$$\begin{aligned} A_2\left(x, \frac{\varphi}{2}\right) & \sum_{n=1}^{\infty} \frac{\varphi^{2n}}{n^2 \binom{2n}{n}} = 2 \arcsin^2\left(\frac{\varphi}{2}\right), \\ A_2\left(x, \frac{\bar{\varphi}}{2}\right) & \sum_{n=1}^{\infty} \frac{\bar{\varphi}^{2n}}{n^2 \binom{2n}{n}} = 2 \arcsin^2\left(\frac{\bar{\varphi}}{2}\right). \end{aligned}$$

Their linear combination leads to the following identity conjectured by Sun [27] (Page 7):

$$\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2 \binom{2n}{n}} = \frac{\pi^2}{5},$$

where the Lucas numbers are given by

$$L_{n+1} = L_n + L_{n-1} \quad \text{with} \quad L_0 = 2 \quad \text{and} \quad L_1 = 1$$

and can be expressed by Binet’s formula

$$L_n = \varphi^n + \bar{\varphi}^n.$$

- Two further numerical series from $A_2(x, y)$:

$$\boxed{A_2(x, \frac{\sqrt{5}\varphi}{2})} \quad \sum_{n=1}^{\infty} \frac{5^n \varphi^{2n}}{n^2 \binom{2n}{n}} = 2 \arcsin^2 \left(\frac{\sqrt{5}\varphi}{2} \right),$$

$$\boxed{A_2(x, \frac{\sqrt{5}\bar{\varphi}}{2})} \quad \sum_{n=1}^{\infty} \frac{5^n \bar{\varphi}^{2n}}{n^2 \binom{2n}{n}} = 2 \arcsin^2 \left(\frac{\sqrt{5}\bar{\varphi}}{2} \right).$$

These are refined identities of the following identity conjectured by Sun [27] (Page 7):

$$\sum_{n=1}^{\infty} \frac{V_n}{n^2 \binom{2n}{n}} = \frac{2\pi^2}{5},$$

where the V -sequence is defined by

$$V_{n+1} = 5(V_n - V_{n-1}) \quad \text{with} \quad V_0 = 2 \quad \text{and} \quad V_1 = 5$$

and can be expressed by Binet's formula

$$V_n = (\sqrt{5})^n \{ \varphi^n + (-\bar{\varphi})^n \}.$$

- Analogous evaluations detected by Sun [28] (Conjectures 10.62 and 10.63):

$$\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2 \binom{2n}{n}} \{ \mathbf{H}_{2n} - \mathbf{H}_{n-1} \} = \frac{41\zeta(3) + 4\pi^2 \ln \varphi}{25},$$

$$\sum_{n=1}^{\infty} \frac{V_n}{n^2 \binom{2n}{n}} \{ \mathbf{H}_{2n} - \mathbf{H}_{n-1} \} = \frac{124\zeta(3) + \pi^2 \ln(5^5 \varphi^6)}{50}.$$

- Numerical series from $A_4(x, y)$:

$$\boxed{A_4(x, \frac{\sqrt{2}}{2})} \quad \sum_{n=1}^{\infty} \frac{2^n \mathbf{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}} = \frac{\pi^4}{384},$$

$$\boxed{A_4(x, \frac{\sqrt{3}}{2})} \quad \sum_{n=1}^{\infty} \frac{3^n \mathbf{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}} = \frac{2\pi^4}{243},$$

$$\boxed{A_4(x, \frac{\varphi}{2})} \quad \sum_{n=1}^{\infty} \frac{\varphi^{2n} \mathbf{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}} = \frac{27\pi^4}{5000},$$

$$\boxed{A_4(x, \frac{\bar{\varphi}}{2})} \quad \sum_{n=1}^{\infty} \frac{\bar{\varphi}^{2n} \mathbf{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}} = \frac{\pi^4}{15000}.$$

- Numerical series from $A_6(x, y)$:

$$\boxed{A_6(x, \frac{\sqrt{2}}{2})} \quad \sum_{n=1}^{\infty} \frac{2^n \{ (\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)} \}}{n^2 \binom{2n}{n}} = \frac{21\zeta(6)}{512},$$

$$\boxed{A_6(x, \frac{\sqrt{3}}{2})} \quad \sum_{n=1}^{\infty} \frac{3^n \{ (\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)} \}}{n^2 \binom{2n}{n}} = \frac{56\zeta(6)}{243},$$

$$\boxed{A_6(x, \frac{\varphi}{2})} \quad \sum_{n=1}^{\infty} \frac{\varphi^{2n} \{ (\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)} \}}{n^2 \binom{2n}{n}} = \frac{15309\zeta(6)}{125000},$$

$$\boxed{A_6(x, \frac{\bar{\varphi}}{2})} \quad \sum_{n=1}^{\infty} \frac{\bar{\varphi}^{2n} \{ (\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)} \}}{n^2 \binom{2n}{n}} = \frac{21\zeta(6)}{125000}.$$

- Particularly for $y = \frac{1}{2}$, we deduce two numerical series

$$\boxed{A_4(x, \frac{1}{2})} \quad \sum_{n=1}^{\infty} \frac{\mathbf{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n}} = \frac{\pi^4}{1944},$$

$$\boxed{A_6(x, \frac{1}{2})} \quad \sum_{n=1}^{\infty} \frac{(\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)}}{n^2 \binom{2n}{n}} = \frac{\pi^6}{262440}.$$

The former is a combination of the following two formulae (cf. Comtet [24] (Page 89) and Chu [20] (Example 3.8)):

$$\sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = \frac{17\pi^4}{3240} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mathbf{H}_n^{(2)}}{n^2 \binom{2n}{n}} = \frac{7\pi^4}{1215},$$

while for the latter, three refined formulae exist (cf. Chu [20])

$$\sum_{n=1}^{\infty} \frac{7 - 3n^4 (\mathbf{H}_n^{(2)})^2}{n^6 \binom{2n}{n}} = \frac{65\pi^6}{34992},$$

$$\sum_{n=1}^{\infty} \frac{2 - n^2 \mathbf{H}_n^{(2)}}{n^6 \binom{2n}{n}} = \frac{313\pi^6}{612360},$$

$$\sum_{n=1}^{\infty} \frac{3n^4 \mathbf{H}_n^{(4)} - 1}{n^6 \binom{2n}{n}} = \frac{163\pi^6}{136080}.$$

It should be pointed out that the last two identities were first found by Bailey et al. [31], and amusingly, conjectured by Sun [32] (Equations 4.6 and 4.7) subsequently.

2.2. Series from $B(x, y)$

Considering that $B(x, y)$ is a power series of x , the initial coefficients yield the identities below.

$$\boxed{B_1(x, y)} \quad \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n \binom{2n}{n}} = \frac{2y \arcsin y}{\sqrt{1-y^2}},$$

$$\boxed{B_3(x, y)} \quad \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n \binom{2n}{n}} \mathbf{H}_{n-1}^{(2)} = \frac{4y \arcsin^3 y}{3\sqrt{1-y^2}},$$

$$\boxed{B_5(x, y)} \quad \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n \binom{2n}{n}} \{ (\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)} \} = \frac{8y \arcsin^5 y}{15\sqrt{1-y^2}}.$$

For comparison, we record five sample series conjectured experimentally by Sun [27], where the fourth and fifth series were evaluated by Chu [20].

$$\sum_{k=1}^{\infty} \frac{2^k \mathbf{H}_k}{k^2 \binom{2k}{k}} = \frac{7\zeta(3)}{16} + \frac{\pi^2}{8} \ln 2,$$

$$\sum_{k=1}^{\infty} \frac{2^k \mathbf{O}_k}{k^2 \binom{2k}{k}} = \frac{7\zeta(3)}{2} - \pi G,$$

$$\sum_{k=1}^{\infty} \frac{3^k (1 + 2k \mathbf{H}_k)}{k^3 \binom{2k}{k}} = \frac{2\pi^2}{3} \ln 3,$$

$$\sum_{k=1}^{\infty} \frac{\mathbf{H}_{2k} + 2\mathbf{H}_k}{k^2 \binom{2k}{k}} = \frac{5\zeta(3)}{3},$$

$$\sum_{k=1}^{\infty} \frac{\mathbf{H}_k^{(3)}}{k^2 \binom{2k}{k}} = \frac{\zeta(5) + 2\zeta(2)\zeta(3)}{9}.$$

2.3. Series from $C(x, y)$

By expanding $C(x, y)$ into a power series of x , we record the following infinite series identities from the initial coefficients.

$$\begin{aligned}
 C_1(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n \binom{2n}{n} (2n+1)} = 2 - \frac{2\sqrt{1-y^2} \arcsin y}{y}, \\
 C_2(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2 \binom{2n}{n} (2n+1)} = 2 \arcsin^2 y - 4 + \frac{4\sqrt{1-y^2} \arcsin y}{y}, \\
 C_3(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n} \mathbf{H}_{n-1}^{(2)}}{n \binom{2n}{n} (2n+1)} = 4 \arcsin^2 y - 8 - \frac{4\sqrt{1-y^2}}{3y} \left\{ \arcsin^3 y - 6 \arcsin y \right\}, \\
 C_4(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n} \mathbf{H}_{n-1}^{(2)}}{n^2 \binom{2n}{n} (2n+1)} = 16 - 8 \arcsin^2 y + \frac{2}{3} \arcsin^4 y + \frac{8\sqrt{1-y^2}}{3y} \left\{ \arcsin^3 y - 6 \arcsin y \right\}, \\
 C_5(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n \binom{2n}{n}} \frac{(\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)}}{(2n+1)} = \left\{ \begin{array}{l} \frac{8}{3} \arcsin^4 y - 32 \arcsin^2 y + 64 \\ - \frac{8\sqrt{1-y^2}}{15y} [120 \arcsin y - 20 \arcsin^3 y + \arcsin^5 y] \end{array} \right\}, \\
 C_6(x, y) & \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2 \binom{2n}{n}} \frac{(\mathbf{H}_{n-1}^{(2)})^2 - \mathbf{H}_{n-1}^{(4)}}{(2n+1)} = \left\{ \begin{array}{l} \frac{8}{45} \arcsin^6 y - \frac{16}{3} \arcsin^4 y + 64 \arcsin^2 y - 128 \\ + \frac{16\sqrt{1-y^2}}{15y} [120 \arcsin y - 20 \arcsin^3 y + \arcsin^5 y] \end{array} \right\}.
 \end{aligned}$$

3. Series with the Central Binomial Coefficient $\binom{2n}{n}$ in the Numerator

This section is devoted to harmonic series with the central binomial coefficient in the numerator.

3.1. Series from $A\left(\frac{1}{2} + x, y\right)$

Since $A\left(\frac{1}{2} + x, y\right)$ can be expanded into a power series of x , the initial coefficients give rise to the following six infinite series identities.

$$\begin{aligned}
 A_0\left(\frac{1}{2} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{1}{2n-1} = -\sqrt{1-y^2}, \\
 A_1\left(\frac{1}{2} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{2n}{(2n-1)^2} = y \arcsin y, \\
 A_2\left(\frac{1}{2} + x, y\right) & \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \left\{ \frac{\mathbf{O}_{n-1}^{(2)}}{2n-1} - \frac{1}{(2n-1)^2} \right\} = \frac{\sqrt{1-y^2} \arcsin^2 y}{-2}, \\
 A_3\left(\frac{1}{2} + x, y\right) & \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{n \mathbf{O}_{n-1}^{(2)}}{(2n-1)^2} = \frac{y \arcsin^3 y}{12}, \\
 A_4\left(\frac{1}{2} + x, y\right) & \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \left\{ \frac{(\mathbf{O}_{n-1}^{(2)})^2 - \mathbf{O}_{n-1}^{(4)}}{2n-1} - \frac{2 \mathbf{O}_{n-1}^{(2)}}{(2n-1)^2} \right\} = \frac{\sqrt{1-y^2} \arcsin^4 y}{-12}, \\
 A_5\left(\frac{1}{2} + x, y\right) & \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{n \{ (\mathbf{O}_{n-1}^{(2)})^2 - \mathbf{O}_{n-1}^{(4)} \}}{(2n-1)^2} = \frac{y \arcsin^5 y}{120}.
 \end{aligned}$$

3.2. Series from $B\left(\frac{1}{2} + x, y\right)$

Considering that $B\left(\frac{1}{2} + x, y\right)$ is a power series of x , the initial coefficients yield the identities below, where $B_2\left(\frac{1}{2} + x, y\right)$ is equivalent to the formula by Sun [29] (Equation 1.5).

$$\begin{aligned}
\boxed{B_0\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=0}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} = \frac{1}{\sqrt{1-y^2}}, \\
\boxed{B_2\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \mathbf{O}_n^{(2)} = \frac{\arcsin^2 y}{2\sqrt{1-y^2}}, \\
\boxed{B_4\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{\arcsin^4 y}{12\sqrt{1-y^2}}, \\
\boxed{B_6\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \{(\mathbf{O}_n^{(2)})^3 - 3\mathbf{O}_n^{(2)}\mathbf{O}_n^{(4)} + 2\mathbf{O}_n^{(6)}\} = \frac{\arcsin^6 y}{120\sqrt{1-y^2}}.
\end{aligned}$$

Unlike these series, there are two similar identities conjectured by Sun [29] (Equations 2.4 and 2.5):

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{8^k} \mathbf{O}_k^{(3)} = \frac{35\sqrt{2}}{64} \zeta(3) - \frac{\sqrt{2}}{8} \pi G \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} \mathbf{O}_k^{(3)} = \frac{2\zeta(3)}{3\sqrt{3}} - \frac{\pi K}{8}.$$

The former was confirmed by the authors [33] (Theorem 19), while the latter remains unproven.

3.3. Series from $C\left(\frac{1}{2} + x, y\right)$

By expanding $C\left(\frac{1}{2} + x, y\right)$ into a power series of x , we record the following infinite series identities from the initial coefficients, which are particular cases of a more general formula appearing in [30].

$$\begin{aligned}
\boxed{C_0\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=0}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{1}{2n+1} = \frac{\arcsin y}{y}, \\
\boxed{C_2\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{\mathbf{O}_n^{(2)}}{2n+1} = \frac{\arcsin^3 y}{6y}, \\
\boxed{C_4\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}}{2n+1} = \frac{\arcsin^5 y}{60y}, \\
\boxed{C_6\left(\frac{1}{2} + x, y\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{(\mathbf{O}_n^{(2)})^3 - 3\mathbf{O}_n^{(2)}\mathbf{O}_n^{(4)} + 2\mathbf{O}_n^{(6)}}{2n+1} = \frac{\arcsin^7 y}{840y}.
\end{aligned}$$

For these identities, many numerical series related to them exist in the literature.

- Numerical series from $C_2\left(\frac{1}{2} + x, y\right)$:

$$\begin{aligned}
\boxed{C_2\left(\frac{1}{2} + x, \frac{1}{2}\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n} \mathbf{O}_n^{(2)}}{2n+1} = \frac{\pi^3}{648}, \\
\boxed{C_2\left(\frac{1}{2} + x, \frac{\sqrt{2}}{2}\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n \frac{\binom{2n}{n} \mathbf{O}_n^{(2)}}{2n+1} = \frac{\pi^3}{192\sqrt{2}}, \\
\boxed{C_2\left(\frac{1}{2} + x, \frac{\sqrt{3}}{2}\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{3}{16}\right)^n \frac{\binom{2n}{n} \mathbf{O}_n^{(2)}}{2n+1} = \frac{\pi^3}{81\sqrt{3}}, \\
\boxed{C_2\left(\frac{1}{2} + x, \frac{\varphi}{2}\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{\varphi}{4}\right)^n \frac{\binom{2n}{n} \mathbf{O}_n^{(2)}}{2n+1} = \frac{9\pi^3}{1000\varphi}, \\
\boxed{C_2\left(\frac{1}{2} + x, \frac{\bar{\varphi}}{2}\right)} & \quad \sum_{n=1}^{\infty} \left(\frac{\bar{\varphi}}{4}\right)^n \frac{\binom{2n}{n} \mathbf{O}_n^{(2)}}{2n+1} = \frac{\pi^3 \varphi}{3000}.
\end{aligned}$$

- Numerical series from $C_4(\frac{1}{2} + x, y)$:

$$\boxed{C_4(\frac{1}{2} + x, \frac{1}{2})} \quad \sum_{n=1}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}}{2n+1} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{\pi^5}{233280},$$

$$\boxed{C_4(\frac{1}{2} + x, \frac{i}{2})} \quad \sum_{n=1}^{\infty} \left(\frac{-1}{16}\right)^n \frac{\binom{2n}{n}}{2n+1} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{\ln^5(\frac{1+\sqrt{5}}{2})}{30},$$

$$\boxed{C_4(\frac{1}{2} + x, \frac{\sqrt{2}}{2})} \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n \frac{\binom{2n}{n}}{2n+1} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{\pi^5}{30720\sqrt{2}},$$

$$\boxed{C_4(\frac{1}{2} + x, \frac{\sqrt{3}}{2})} \quad \sum_{n=1}^{\infty} \left(\frac{3}{16}\right)^n \frac{\binom{2n}{n}}{2n+1} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{\pi^5}{7290\sqrt{3}},$$

$$\boxed{C_4(\frac{1}{2} + x, \frac{\varphi}{2})} \quad \sum_{n=1}^{\infty} \left(\frac{\varphi}{4}\right)^{2n} \frac{\binom{2n}{n}}{2n+1} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{81\pi^5}{10^6\varphi},$$

$$\boxed{C_4(\frac{1}{2} + x, \frac{\bar{\varphi}}{2})} \quad \sum_{n=1}^{\infty} \left(\frac{\bar{\varphi}}{4}\right)^{2n} \frac{\binom{2n}{n}}{2n+1} \{(\mathbf{O}_n^{(2)})^2 - \mathbf{O}_n^{(4)}\} = \frac{\pi^5\varphi}{3 \times 10^6}.$$

- The identity corresponding to $\boxed{C_4(\frac{1}{2} + x, \frac{1}{2})}$ was refined by the following ones due to Li and Chu [21] (Equations 30 and 32):

$$\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}}{2n+1} \left\{3\mathbf{O}_n^{(4)} + \frac{2}{(2n+1)^4}\right\} = \frac{121\pi^5}{17280},$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \frac{\binom{2n}{n}}{2n+1} \left\{3(\mathbf{O}_n^{(2)})^2 + \frac{2}{(2n+1)^4}\right\} = \frac{1091\pi^5}{155520};$$

where the former was experimentally detected by Sun [27] (Equation 1.50).

- We remark that for the formula corresponding to $C_4(\frac{1}{2} + x, \frac{i}{2})$, the following counterpart exists:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2(-16)^k} \left\{5\mathbf{O}_n^{(4)} + \frac{1}{(2n+1)^4}\right\} = \frac{7\pi^6}{7200},$$

which was conjectured by Sun [28] (Conjecture 10.74) and confirmed in [5] (Equation 7).

- For comparison, we record below variant series involving skew harmonic numbers in odd order, conjectured by Sun [27,28] and evaluated by Li and Chu [5,21]:

★ Sun [27] (Equation 1.43) and Li–Chu [21] (Equation 25):

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}\mathbf{O}_{n+1}^{(3)}}{(2n+1)16^k} = \frac{5\pi}{18}\zeta(3).$$

★ Sun [28] (Conjecture 10.71) and Li–Chu [5] (Equation 6):

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^2(-16)^k} \left\{5\mathbf{O}_{n+1}^{(3)} + \frac{1}{(2n+1)^3}\right\} = \frac{\pi^2}{2}\zeta(3).$$

★ Sun [27] (Equation 1.53) and Li–Chu [21] (Equation 35):

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)16^k} \left\{33\mathbf{O}_{n+1}^{(5)} + \frac{4}{(2n+1)^5}\right\} = \frac{35\pi^3}{288}\zeta(3) + \frac{1003\pi}{96}\zeta(5).$$

★ Sun [27] (Equation 1.56) and Li–Chu [21] (Equation 34):

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)^3 16^k} \left\{ 33\mathbf{O}_{n+1}^{(3)} + \frac{8}{(2n+1)^3} \right\} = \frac{245\pi^3}{216} \zeta(3) - \frac{49\pi}{144} \zeta(5).$$

4. Series Containing the Binomial Coefficient $\binom{3n}{n}$

This section is devoted to harmonic series with the binomial coefficient $\binom{3n}{n}$. By integrating Lambert's series and manipulating the cubic transformations for the ${}_3F_2$ -series, the authors evaluated several similar series in a recent paper [34]. Adegoke–Frontczak–Goy [35] evaluated different series without harmonic numbers.

4.1. Series from $A\left(\frac{1}{3} + x, y\right)$

Since $A\left(\frac{1}{3} + x, y\right)$ can be expanded into a power series of x , the initial coefficients give rise to the following three infinite series identities.

$$\begin{aligned} A_0\left(\frac{1}{3} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \frac{1}{3n-1} = -\cos\left(\frac{2\arcsin y}{3}\right), \\ A_1\left(\frac{1}{3} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \left\{ \frac{\mathbf{H}_n\left(\frac{1}{3}\right) - \mathbf{H}_n\left(\frac{2}{3}\right)}{3n-1} + \frac{9n}{(3n-1)^2} \right\} = 2\arcsin y \sin\left(\frac{2\arcsin y}{3}\right), \\ A_2\left(\frac{1}{3} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \left\{ \frac{\mathbf{H}_n^{(2)}\left(\frac{1}{3}\right) + \mathbf{H}_n^{(2)}\left(\frac{2}{3}\right) - [\mathbf{H}_n\left(\frac{1}{3}\right) - \mathbf{H}_n\left(\frac{2}{3}\right)]^2}{2(3n-1)} \right. \\ & \quad \left. - \frac{9n[\mathbf{H}_n\left(\frac{1}{3}\right) - \mathbf{H}_n\left(\frac{2}{3}\right)]}{(3n-1)^2} - \frac{27n}{(3n-1)^3} \right\} = -2\arcsin^2 y \cos\left(\frac{2\arcsin y}{3}\right). \end{aligned}$$

4.2. Series from $B\left(\frac{1}{3} + x, y\right)$

Considering that $B\left(\frac{1}{3} + x, y\right)$ is a power series of x , the initial coefficients yield the identities below.

$$\begin{aligned} B_0\left(\frac{1}{3} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} = \frac{\cos\left(\frac{\arcsin y}{3}\right)}{\sqrt{1-y^2}}, \\ B_1\left(\frac{1}{3} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \left\{ \mathbf{H}_n\left(\frac{1}{3}\right) - \mathbf{H}_n\left(\frac{2}{3}\right) \right\} = \frac{2\arcsin y \sin\left(\frac{\arcsin y}{3}\right)}{\sqrt{1-y^2}}, \\ B_2\left(\frac{1}{3} + x, y\right) & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \left\{ \mathbf{H}_n^{(2)}\left(\frac{1}{3}\right) + \mathbf{H}_n^{(2)}\left(\frac{2}{3}\right) - [\mathbf{H}_n\left(\frac{1}{3}\right) - \mathbf{H}_n\left(\frac{2}{3}\right)]^2 \right\} = \frac{4\arcsin^2 y \cos\left(\frac{\arcsin y}{3}\right)}{\sqrt{1-y^2}}. \end{aligned}$$

4.3. Series from $C\left(\frac{1}{3} + x, y\right)$

By expanding $C\left(\frac{1}{3} + x, y\right)$ into a power series of x , we record the following infinite series identities from the initial coefficients.

$$\begin{aligned}
\boxed{C_0(\frac{1}{3} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \frac{1}{2n+1} = \frac{3 \sin(\frac{\arcsin y}{3})}{y}, \\
\boxed{C_1(\frac{1}{3} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \frac{\mathbf{H}_n(\frac{1}{3}) - \mathbf{H}_n(\frac{2}{3})}{2n+1} \\
& = \frac{18}{y} \sin\left(\frac{\arcsin y}{3}\right) - \frac{6}{y} \arcsin y \cos\left(\frac{\arcsin y}{3}\right), \\
\boxed{C_2(\frac{1}{3} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{4y^2}{27}\right)^n \binom{3n}{n} \frac{\mathbf{H}_n^{(2)}(\frac{1}{3}) + \mathbf{H}_n^{(2)}(\frac{2}{3}) - [\mathbf{H}_n(\frac{1}{3}) - \mathbf{H}_n(\frac{2}{3})]^2}{2n+1} \\
& = \frac{12}{y} (\arcsin^2 y - 18) \sin\left(\frac{\arcsin y}{3}\right) + \frac{72}{y} \arcsin y \cos\left(\frac{\arcsin y}{3}\right).
\end{aligned}$$

5. Series Containing the Central Binomial Coefficient $\binom{4n}{2n}$

This section is devoted to harmonic series with the central binomial coefficient $\binom{4n}{2n}$.

5.1. Series from $A\left(\frac{1}{4} + x, y\right)$

Since $A\left(\frac{1}{4} + x, y\right)$ can be expanded into a power series of x , the initial coefficients give rise to the following three infinite series identities.

$$\begin{aligned}
\boxed{A_0(\frac{1}{4} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \frac{1}{4n-1} = \frac{\sqrt{1 + \sqrt{1-y^2}}}{-\sqrt{2}}, \\
\boxed{A_1(\frac{1}{4} + x, y)} & \sum_{n=1}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \left\{ \frac{\bar{\mathbf{O}}_{2n-1}}{4n-1} + \frac{1}{4n-1} \right\} = \frac{y \arcsin y}{2\sqrt{2}\sqrt{1 + \sqrt{1-y^2}}}, \\
\boxed{A_2(\frac{1}{4} + x, y)} & \sum_{n=1}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \left\{ \frac{\mathbf{O}_{2n-1}^{(2)} - \bar{\mathbf{O}}_{2n-1}^2}{4n-1} - \frac{2\bar{\mathbf{O}}_{2n-1}}{4n-1} \right\} = \frac{\arcsin^2 y \sqrt{1 + \sqrt{1-y^2}}}{-4\sqrt{2}}.
\end{aligned}$$

5.2. Series from $B\left(\frac{1}{4} + x, y\right)$

Considering that $B\left(\frac{1}{4} + x, y\right)$ is a power series of x , the initial coefficients yield the identities below.

$$\begin{aligned}
\boxed{B_0(\frac{1}{4} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} = \frac{\sqrt{1 + \sqrt{1-y^2}}}{\sqrt{2}\sqrt{1-y^2}}, \\
\boxed{B_1(\frac{1}{4} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \bar{\mathbf{O}}_{2n} = \frac{y \arcsin y}{2\sqrt{2}\sqrt{1-y^2}\sqrt{1 + \sqrt{1-y^2}}}, \\
\boxed{B_2(\frac{1}{4} + x, y)} & \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \left\{ \mathbf{O}_{2n}^{(2)} - \bar{\mathbf{O}}_{2n}^2 \right\} = \frac{\arcsin^2 y \sqrt{1 + \sqrt{1-y^2}}}{4\sqrt{2}\sqrt{1-y^2}}.
\end{aligned}$$

Among these identities, Sun [29] (Remark 2.8) recorded an equivalent formula for $\boxed{B_0(\frac{1}{4} + x, y)}$ and conjectured the following counterpart for $\boxed{B_1(\frac{1}{4} + x, y)}$ (see [29] Equation 2.16):

$$\sum_{k=0}^{\infty} \left(\frac{x(1-x)}{4}\right)^k \binom{4k}{2k} \left\{ \mathbf{O}_{2k} - \mathbf{O}_k \right\} = \frac{\sqrt{1-x}}{2(2x-1)} \ln(1-x).$$

5.3. Series from $C\left(\frac{1}{4} + x, y\right)$

By expanding $C\left(\frac{1}{4} + x, y\right)$ into a power series of x , we record the following infinite series identities from the initial coefficients.

$$\begin{aligned} \boxed{C_0\left(\frac{1}{4} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \frac{1}{2n+1} &= \frac{\sqrt{2}}{\sqrt{1+\sqrt{1-y^2}}}, \\ \boxed{C_1\left(\frac{1}{4} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \frac{\bar{O}_{2n}}{2n+1} &= \frac{2y - (1 + \sqrt{1-y^2}) \arcsin y}{y\sqrt{2}\sqrt{1+\sqrt{1-y^2}}}, \\ \boxed{C_2\left(\frac{1}{4} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \frac{O_{2n}^{(2)} - \bar{O}_{2n}^2}{2n+1} &= \frac{8y - 4(1 + \sqrt{1-y^2}) \arcsin y - y \arcsin^2 y}{-2y\sqrt{2}\sqrt{1+\sqrt{1-y^2}}}. \end{aligned}$$

6. Series Involving Trinomial/Binomial Quotient $\binom{6n}{n, 2n, 3n} / \binom{2n}{n}$

This section is devoted to harmonic series with a trinomial/binomial quotient.

6.1. Series from $A\left(\frac{1}{6} + x, y\right)$

Since $A\left(\frac{1}{6} + x, y\right)$ can be expanded into a power series of x , the initial coefficients give rise to the following three infinite series identities.

$$\begin{aligned} \boxed{A_0\left(\frac{1}{6} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \frac{1}{6n-1} &= -\cos\left(\frac{\arcsin y}{3}\right), \\ \boxed{A_1\left(\frac{1}{6} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \left\{ \frac{H_n\left(\frac{1}{6}\right) - H_n\left(\frac{5}{6}\right)}{6n-1} + \frac{36n}{(6n-1)^2} \right\} &= 2 \arcsin y \sin\left(\frac{\arcsin y}{3}\right), \\ \boxed{A_2\left(\frac{1}{6} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \left\{ \frac{H_n^{(2)}\left(\frac{1}{6}\right) + H_n^{(2)}\left(\frac{5}{6}\right) - [H_n\left(\frac{1}{6}\right) - H_n\left(\frac{5}{6}\right)]^2}{\frac{2(6n-1)}{36n[H_n\left(\frac{1}{6}\right) - H_n\left(\frac{5}{6}\right)]} - \frac{216n}{(6n-1)^3}} \right\} &= -2 \arcsin^2 y \cos\left(\frac{\arcsin y}{3}\right). \end{aligned}$$

6.2. Series from $B\left(\frac{1}{6} + x, y\right)$

Considering that $B\left(\frac{1}{6} + x, y\right)$ is a power series of x , the initial coefficients yield the identities below.

$$\begin{aligned} \boxed{B_0\left(\frac{1}{6} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} &= \frac{\cos\left(\frac{2\arcsin y}{3}\right)}{\sqrt{1-y^2}}, \\ \boxed{B_1\left(\frac{1}{6} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \left\{ H_n\left(\frac{1}{6}\right) - H_n\left(\frac{5}{6}\right) \right\} &= \frac{2 \arcsin y \sin\left(\frac{2\arcsin y}{3}\right)}{\sqrt{1-y^2}}, \\ \boxed{B_2\left(\frac{1}{6} + x, y\right)} \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \left\{ \frac{H_n^{(2)}\left(\frac{1}{6}\right) + H_n^{(2)}\left(\frac{5}{6}\right)}{[H_n\left(\frac{1}{6}\right) - H_n\left(\frac{5}{6}\right)]^2} \right\} &= \frac{4 \arcsin^2 y \cos\left(\frac{2\arcsin y}{3}\right)}{\sqrt{1-y^2}}. \end{aligned}$$

6.3. Series from $C\left(\frac{1}{6} + x, y\right)$

By expanding $C\left(\frac{1}{6} + x, y\right)$ into a power series of x , we record the following infinite series identities from the initial coefficients.

$$\begin{aligned}
C_0\left(\frac{1}{6} + x, y\right) &= \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \frac{1}{2n+1} = \frac{3 \sin\left(\frac{2 \arcsin y}{3}\right)}{2y}, \\
C_1\left(\frac{1}{6} + x, y\right) &= \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \frac{\mathbf{H}_n\left(\frac{1}{6}\right) - \mathbf{H}_n\left(\frac{5}{6}\right)}{2n+1} \\
&= \frac{9}{2y} \sin\left(\frac{2 \arcsin y}{3}\right) - \frac{3}{y} \arcsin y \cos\left(\frac{2 \arcsin y}{3}\right), \\
C_2\left(\frac{1}{6} + x, y\right) &= \sum_{n=0}^{\infty} \left(\frac{y^2}{108}\right)^n \frac{\binom{6n}{n, 2n, 3n}}{\binom{2n}{n}} \frac{\mathbf{H}_n^{(2)}\left(\frac{1}{6}\right) + \mathbf{H}_n^{(2)}\left(\frac{5}{6}\right) - [\mathbf{H}_n\left(\frac{1}{6}\right) - \mathbf{H}_n\left(\frac{5}{6}\right)]^2}{2n+1} \\
&= \frac{18 \arcsin y \cos\left(\frac{2 \arcsin y}{3}\right) + 6 \arcsin^2 y \sin\left(\frac{2 \arcsin y}{3}\right) - 27 \sin\left(\frac{2 \arcsin y}{3}\right)}{y}.
\end{aligned}$$

7. Concluding Remarks

By means of the “coefficient extraction” method, we succeeded in deriving several algebraic formulae for harmonic series containing a free variable y . This fact is remarkable since almost all of the binomial series conjectured by Sun [27–29] are numerical series without variables.

7.1. General Forms of Power Series Expansions

For simplicity, throughout the paper, we examined Maclaurin polynomials only up to x^6 for three functions $A(x, y)$, $B(x, y)$, and $C(x, y)$. In fact, it is possible to expand these functions to higher degrees. Here, we record three examples as representatives. The remaining ones can be determined similarly.

- Coefficient of x^{2m+2} in $A(x, y)$ (general form of Section 2.1):

$$\sum_{n=1}^{\infty} \frac{(2y)^{2n}}{\binom{2n}{n} n^2} \Omega_m(-\mathbf{H}_{n-1}^{(2k)}) = (-1)^m \frac{(2 \arcsin y)^{2m+2}}{(2m+2)!}.$$

- Coefficient of x^{2m} in $B(\frac{1}{2} + x, y)$ (general form of Section 3.2):

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{y}{2}\right)^{2n} \Omega_m(-\mathbf{H}_n^{(2k)}(\frac{1}{2})) = (-1)^m \frac{(2 \arcsin y)^{2m}}{(2m)! \sqrt{1-y^2}}.$$

- Coefficient of x^m in $C(\frac{1}{3} + x, y)$ (general form of Section 4.3):

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\binom{3n}{n}}{2n+1} \left(\frac{4y^2}{27}\right)^n \sum_{\ell=0}^m (-1)^\ell \Omega_\ell(-\mathbf{H}_n^{(k)}(\frac{1}{3})) \Omega_{m-\ell}(-\mathbf{H}_n^{(k)}(\frac{2}{3})) \\
&= \frac{3}{y} \sin\left(\frac{\arcsin y}{3}\right) \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{(2k)!} (2 \arcsin y)^{2k} 6^{m-2k} \\
&+ \frac{18}{y} \cos\left(\frac{\arcsin y}{3}\right) \sum_{k=1}^{\lceil \frac{m}{2} \rceil} \frac{(-1)^k}{(2k-1)!} (2 \arcsin y)^{2k-1} 6^{m-2k}.
\end{aligned}$$

7.2. Linearly Combined Series

Some of our series can be combined further to produce closed formulae for simpler series.

- $A_2(\frac{1}{2} + x, y) + A_1(\frac{1}{2} + x, y) - A_0(\frac{1}{2} + x, y)$

$$\sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{\mathbf{O}_{n-1}^{(2)}}{2n-1} = y \arcsin y - 1 + \frac{\sqrt{1-y^2}}{2} \{2 - \arcsin^2 y\}.$$

- $$\boxed{A_1\left(\frac{1}{4} + x, y\right) - A_0\left(\frac{1}{4} + x, y\right)}$$

$$\sum_{n=1}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \frac{\bar{\mathbf{O}}_{2n-1}}{4n-1} = \frac{\sqrt{1+\sqrt{1-y^2}}}{\sqrt{2}} - 1 + \frac{y \arcsin y}{\sqrt{8}\sqrt{1+\sqrt{1-y^2}}}.$$
- $$\boxed{2A_3\left(\frac{1}{2} + x, y\right) - A_2\left(\frac{1}{2} + x, y\right) - A_1\left(\frac{1}{2} + x, y\right) + A_0\left(\frac{1}{2} + x, y\right)}$$

$$\sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{\mathbf{O}_{n-1}^{(2)}}{(2n-1)^2} = 1 - y \arcsin y + \frac{y \arcsin^3 y}{6} - \frac{\sqrt{1-y^2}}{2} \{2 - \arcsin^2 y\}.$$
- $$\boxed{A_2\left(\frac{1}{4} + x, y\right) + 2A_1\left(\frac{1}{4} + x, y\right) - 2A_0\left(\frac{1}{4} + x, y\right)}$$

$$\sum_{n=1}^{\infty} \left(\frac{y^2}{16}\right)^n \binom{4n}{2n} \frac{\mathbf{O}_{2n-1}^{(2)} - \bar{\mathbf{O}}_{2n-1}^2}{4n-1} = \sqrt{2}\sqrt{1+\sqrt{1-y^2}} + \frac{y \arcsin y}{\sqrt{2}\sqrt{1+\sqrt{1-y^2}}} - 2 - \frac{\arcsin^2 y \sqrt{1+\sqrt{1-y^2}}}{4\sqrt{2}}.$$
- $$\boxed{A_4\left(\frac{1}{2} + x, y\right) + 4A_3\left(\frac{1}{2} + x, y\right) - 2A_2\left(\frac{1}{2} + x, y\right) - 2A_1\left(\frac{1}{2} + x, y\right) + 2A_0\left(\frac{1}{2} + x, y\right)}$$

$$\sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{(\mathbf{O}_{n-1}^{(2)})^2 - \mathbf{O}_{n-1}^{(4)}}{2n-1} = 2 - 2y \arcsin y + \frac{y \arcsin^3 y}{3} - \frac{\sqrt{1-y^2}}{12} \{24 - 12 \arcsin^2 y + \arcsin^4 y\}.$$
- $$\boxed{2A_5\left(\frac{1}{2} + x, y\right) - A_4\left(\frac{1}{2} + x, y\right) - 4A_3\left(\frac{1}{2} + x, y\right) + 2A_2\left(\frac{1}{2} + x, y\right) + 2A_1\left(\frac{1}{2} + x, y\right) - 2A_0\left(\frac{1}{2} + x, y\right)}$$

$$\sum_{n=1}^{\infty} \left(\frac{y}{2}\right)^{2n} \binom{2n}{n} \frac{(\mathbf{O}_{n-1}^{(2)})^2 - \mathbf{O}_{n-1}^{(4)}}{(2n-1)^2} = 2y \arcsin y - \frac{y \arcsin^3 y}{3} + \frac{y \arcsin^5 y}{60} + \frac{\sqrt{1-y^2}}{12} \{24 - 12 \arcsin^2 y + \arcsin^4 y\}.$$

7.3. Further Contiguous Series

Besides the three series $A(x, y)$, $B(x, y)$ and $C(x, y)$ examined in this paper, several contiguous series were evaluated by Chu [25]. For instance, the formula displayed in [25] (Corollary 15) can be reproduced as

$$\begin{aligned} \Lambda(x, y) &= {}_2F_1 \left[\begin{matrix} 1+x, 1-x \\ \frac{5}{2} \end{matrix} \middle| y^2 \right] = \sum_{n=0}^{\infty} \frac{(1+x)_n (1-x)_n}{n! (\frac{5}{2})_n} y^{2n} \\ &= \frac{3 \cos(2x \arcsin y)}{y^2 (1-4x^2)} - \frac{3 \sqrt{1-y^2} \sin(2x \arcsin y)}{2xy^3 (1-4x^2)}. \end{aligned}$$

Three initial coefficients of the Maclaurin series in x are recorded as follows:

$$\begin{aligned}
\boxed{\Lambda_0(x, y)} \sum_{n=0}^{\infty} \frac{(2y)^{2n}}{\binom{2n}{n}(2n+1)(2n+3)} &= \frac{1}{y^2} - \frac{\sqrt{1-y^2}}{y^3} \arcsin y, \\
\boxed{\Lambda_2(x, y)} \sum_{n=0}^{\infty} \frac{(2y)^{2n} \mathbf{H}_n^{(2)}}{\binom{2n}{n}(2n+1)(2n+3)} &= \frac{2}{y^2} \left\{ \arcsin^2 y - 2 \right\} + \frac{2\sqrt{1-y^2}}{3y^3} \left\{ 6 \arcsin y - \arcsin^3 y \right\}, \\
\boxed{\Lambda_4(x, y)} \sum_{n=0}^{\infty} \frac{(2y)^{2n} \left\{ (\mathbf{H}_n^{(2)})^2 - \mathbf{H}_n^{(4)} \right\}}{\binom{2n}{n}(2n+1)(2n+3)} &= \frac{4(24 - 12 \arcsin^2 y + \arcsin^4 y)}{3y^2} \\
&\quad - \frac{4\sqrt{1-y^2}}{15y^3} \left\{ 120 \arcsin y - 20 \arcsin^3 y + \arcsin^5 y \right\}.
\end{aligned}$$

More identities of a similar nature can be derived. Interested readers may make further explorations.

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