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Review

On the Sub and Supersolution Method for Nonlinear Elliptic Equations with a Convective Term, in Orlicz Spaces

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Abstract: In this note we provide an overview of some existence (with sign information) and regularity results for differential equations, in which the method of sub and supersolutions plays an important role. We list some classical results and then we focus on the Dirichlet problem, for problems driven by a general differential operator, depending on $(x, u, \nabla u)$, and with a convective term f. Our framework is that of Orlicz–Sobolev spaces. We also present several examples.

Keywords: nonlinear elliptic equations; Orlicz–Sobolev spaces; gradient dependence; subsolution and supersolution

MSC: 35J25; 35J60; 46E30; 47J05

1. Introduction

This note is devoted to a report on some recent existence and regularity results for nonlinear elliptic equations, obtained via the sub and supersolutions method. Our aim is to present this technique and illustrate, especially to those who are not familiar with it, the situations in which it can be easily applied. This techique has been widely used for problems like

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0\,. \end{cases} \tag{1}$$

Here, $\Delta_p u = |\nabla u|^{p-2}\nabla u$ is the p-Laplacian (see, for instance, [1,2]). Equations like (1) are called convective, because of the dependence of f from the gradient of the unknown.

A natural generalization of (1) is the following quasilinear elliptic equation, involving the A-Laplacian operator

$$\begin{cases} -\Delta_A u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

Here, Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure $|\Omega|$, $A : [0, \infty) \to [0, \infty)$ is a continuously differentiable, strictly convex function, vanishing at 0, and such that $A'(0) = 0$. The A-Laplacian operator is defined by $\Delta_A u = \operatorname{div}\left(A'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$. In addition, $f : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a Carathéodory function.

Solutions to (1) belong in the classical Sobolev space $W_0^{1,p}(\Omega)$, whereas the natural framework for problem (2) is the Orlicz space $W_0^{1,A}(\Omega)$ (see Definition 16).

We are interested in a more general problem, which includes all of the above :

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, u, \nabla u)) = f(x, u, \nabla u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3}$$

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Citation: Barletta, G. On the Sub and Supersolution Method for Nonlinear Elliptic Equations with a Convective Term, in Orlicz Spaces. *Mathematics* **2024**, *12*, 2506. https://doi.org/10.3390/math12162506

Academic Editor: Luigi Rodino

Received: 10 July 2024
Revised: 6 August 2024
Accepted: 8 August 2024
Published: 14 August 2024

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Here, Ω and f are as in (2), and $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function.

Variational structure fails for all the problems above: this is due to the presence of u in the differential operator and of the convective term.

The interest in the study of such problems has increased in recent years, as indicated by the large number of papers on existence results for Dirichlet problems with the convection term. It is impossible to give an exhaustive list of them, so we limit ourselves to some of those where the method of sub and supersolutions (see Definitions 9 and 10) comes into play. We recall, for instance, [1–6] for problems with the p-Laplace operator ($\mathcal{A}(\nabla u) = |\nabla u|^{p-2}\nabla u$) or the ($p$,$q$)-Laplace operator ($\mathcal{A}(\nabla u) = |\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u$). There is also a good literature on problems that can be studied with the same techniques adopted for the $p-$Laplacian (that is in $W_0^{1,p}(\Omega)$): $\mathcal{A}(\nabla u)$ in [7], $\mathcal{A}(x, \nabla u)$ in [5,6], and, finally, $\mathcal{A}(x, u, \nabla u)$ in [3,8]. Existence (and regularity) results in Orlicz spaces, with an operator $\mathcal{A}(\nabla u) = a(|\nabla u|)\nabla u$, can be found in [9,10]. We also cite [11–13] for more general situations.

In most of the papers cited above, the differential operator can be handled like the p-Laplacian, so the framework is the classical Sobolev space $W_0^{1,p}(\Omega)$. This leads to polynomial growth conditions on $\mathcal{A}$ and for the function f, with respect to the unknown and to its gradient.

In Section 2, we present some recent existence results to (3) in the framework of Orlicz spaces, and Section 3 is devoted to the regularity results. Even if the aim of the paper is to highlight the versatility of this method in Orlicz spaces, for the sake of completeness, and to give an idea of the problems where the method of sub and supersolutions has a meaningful role, we present an overview of some results in the framework of classical Sobolev spaces. Of course, this list is far from exhaustive: we chose some different problems that can be treated with the technique above. To avoid too many notations, we only write hypotheses that are immediately readable. For those that are more technical or require a long introduction, we refer to the original papers. For more general knowledge, one can also refer to the references of the paper that we illustrate here. In all the results that we recall in this section, the set Ω is sufficiently smooth and bounded.

In [14], the authors study the following parametric problem:

$$\begin{cases} -\Delta u = \beta(u)|\nabla u|^2 + \lambda f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4}$$

The paper contains several results, and the authors use different techniques. The results in which the method we talk about comes into play are Corollary 3.3 and Corollary 3.11 (of [14]), which are a no existence and a bifurcation result, respectively, and Theorem 4.4 (of [14]),which guarantees the existence of infinitely many solutions. The reader may consult [14] for the conditions assumed time by time on β and f, as well as for a complete treatment of the topic.

In [1], the authors consider the problem

$$\begin{cases} -\Delta_p u = u^q + h(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{5}$$

where $1 < p < n$ and $p - 1 < q < p^* - 1$ with $p^* = \frac{np}{n-p}$. The convection term h is a continuous, nonnegative function with subcritical growth with respect to u and growth less then p with respect to ∇u. The main result of [1], namely, Theorem 1, guarantees the existence of a positive solution to problem (5).

The results in [15] concern problem (1), where $1 < p < n$, and f is a Carathéodory function, whose growth is at most $p - 1$ with respect to u and ∇u. Under two sets of readable and general conditions on f, the authors prove the existence of a positive and a negative solution to (1) (Theorems 1.1 and 1.2 of [15]).

In [2], the author establishes the existence and regularity of positive solutions for

problem (1) (Theorem 4.2 in [2]), without assuming a priori the existence of sub and supersolutions. The convection term is a continuous, nonnegative function, with subcritical growth with respect to u and growth less than p with respect to ∇u.

In [7] the authors study the problem

$$\begin{cases} -div(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{6}$$

The structure of a dictates that the space for studying (6) is $W_0^{1,q}(\Omega)$, with $p \le q < n$. In Theorem 1.2 of [7], the authors prove the existence of a solution to (6), under the assumption of the existence of a subsolution $\underline{u}$ and a supersolution, $\overline{u}$, as well as suitable growth condition on f. As often happens when we deal with these results, the condition on f, with respect to the s variable, is requested only for $s \in [\underline{u}(x), \overline{u}(x)]$.

In [8], the author studies problem (3) in a Sobolev space. The difference with other results is that now the sub and supersolutions are in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$: this means that the solution will automatically be in $L^\infty(\Omega)$.

Now we consider [3]. This paper contains several existence and regularity results for different choices of $\mathcal{A}$: the general one, namely, the case of $\mathcal{A}$ depending also on u, is studied as well. The differential operator satisfies a growth condition and a standard coercivity condition of polynomial type, as well as the monotonicity condition. These assumptions mean that the problem is always studied in a classical Sobolev space, even when its structure may suggest the use of Orlicz spaces. Just to give an idea, one of the results in [3] guarantees the existence of a solution to

$$\begin{cases} -div(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{7}$$

In [5], the authors establish some existence and regularity results, with precise sign information on the solutions for the problem

$$\begin{cases} -\Delta_p u - \mu(x)\Delta_q(u) = a|u|^{q-2}u - g(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{8}$$

in a bounded domain, with a C^2 boundary. Here, $1 < q < p < +\infty$, $a > 0$, and the weight function μ is essentially bounded, with $essinf_{x\in\Omega}\mu(x) > 0$. Under a mix of global growth conditions and local conditions, they construct pairs of sub and supersolutions, and, then, via suitable arguments, they achieve their result.

In addition, paper [4] contains existence and regularity results for (8), but with $\mu \equiv const > 0$.

Some interesting results, where the method that we are discussing plays a crucial role, can be found in [6]. The authors work as usual in regular domains and consider two different problems

$$\begin{cases} -div(\mathcal{A}(x,\nabla u)) = f(x,u,\nabla u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0\,, \end{cases} \tag{9}$$

and a problem with the p-Laplacian (see (1)), but with a function f that can be singular near 0. They adopt two different definitions of sub and supersolutions for the two problems and obtain some existence and regualrity results (Theorems 2.1 and 2.4 of [6] for problem (9), and Theorem 4.1 of [6] for problem (1)).

Remark 1. *In general, sub and supersolution method allow one to study problems with a singular convection term, provided the interval of sub and supersolutions does not contain the singular point.*

The use of the Orlicz space $W_0^{1,A}(\Omega)$ either allows one to take into account a larger class of operator or allows for a wider choice for f than the polynomial one, even when the problem concerns the p-Laplace operator. The growth condition on $\mathcal{A}$ and f are not necessarily of a power type. In particular, in some theorems in Sections 2 and 3, $|f|$ is assumed to be bounded from above with respect to the gradient in terms of a Young function E that grows essentially more slowly then the optimal Sobolev conjugate of A (see Definition 6), whereas in others, the conditions are more restrictive, but formulated always in terms of Young functions. Roughly speaking, in our context, we can consider problems driven by an operator having a power-times-logarithmic type growth (see Section 4):

$$\mathcal{A}(x,s,\xi) = a(x)|s|^{\beta}\lg^{\beta_1}(1+|s|)|\xi|^{p-2-\delta}\lg^{q\left(1-\frac{\delta}{p-1}\right)}(1+|\xi|)\xi + |\xi|^{p-2}\lg^{q}(1+|\xi|)\xi\,. \quad (10)$$

Moreover, also for a problem with the p-Laplacian, the function

$$f(x,s,\xi) = -c + \frac{|s|^{p^*-1}}{\lg(1+|s|)} + a(|s|)|\xi|^{p} \quad (11)$$

is allowed. Our conditions cover several situations appearing in most of the papers dealing with standard problems.

Let us say something on the sub and supersolutions method: once we know that our problem admits a pair of sub and supersolutions, $\underline{u}(x)$ and $\overline{u}(x)$, satisfying $\underline{u}(x) \le \overline{u}(x)$ a.e. in Ω, then we can find, via suitable methods, a solution $u(x)$ satisfying $u(x) \in [\underline{u}(x), \overline{u}(x)]$ for a.e. $x \in \Omega$. The difficulty in finding an ordered pair of sub and supersolutions is a weak point of this method. There are many results where the existence of such a pair is assumed a priori (see Theorems 1, 3, and 4). On the contrary, in other papers, the conditions imposed on f allow one to find such a pair (see, for instance, Theorems 2 and 5). Needless to say, those results are immediately applicable, and the most meaningful are those where the assumptions on f are quite natural and not a stretch. Nevertheless, sometimes the function f has such a structure that we can easily find the pair of sub and supersolutions. The precise knowledge of the couple is also useful from another point of view: it allows the original operator (that may not satisfy suitable growth conditions) to be replaced by an appropriate truncation of it, with a "good" growth. We will present some examples of this situation in Section 4, so to give the right account to our theorems.

As far as the regularity of Ω is concerned, we are able (see [9,11]) to weaken the regularity assumptions on Ω: some of the results in [9,11] hold in arbitrary open sets Ω with finite measure.

The paper is arranged as follows. In Section 2, we introduce the Orlicz–Sobolev spaces and state the main existence theorems. Section 3 is devoted to the regularity results: in particular, we present a new result, Theorem 5, that extends Theorem 4.4 of [11]. Finally, in Section 4, we present some applications of the abstract results, in which we highlight how the function f can have a behavior that is outside the classical settings, or for which it can be easy to find the pair of sub and supersolutions.

2. Existence Results in Orlicz Spaces

Let us briefly recall some basic definitions and introduce the abstract framework of our results. We limit ourselves only to the concepts necessary for reading the work. For a complete discussion on Young functions and for the theory of Orlicz spaces (properties, embedding theorems, etc.), we refer to [16–18].

Definition 1. *A function $A : [0,\infty) \to [0,\infty]$ is called a Young function if it is convex, vanishes at 0, and is neither identically equal to 0 nor to infinity.*

Definition 2. *The Young conjugate of a Young function A is the Young function $\widetilde{A}$, defined as*

$$\widetilde{A}(s) = \sup\{st - A(t) : t \ge 0\} \quad \text{for } s \ge 0.$$

Definition 3. *A Young function B is said to increase essentially more slowly than A near infinity (briefly, $B \ll A$), if B is finite valued and*

$$\lim_{t\to\infty} \frac{B(\lambda t)}{A(t)} = 0 \ \text{for all } \lambda > 0\,. \tag{12}$$

Definition 4. *A Young function A is said to satisfy the Δ_2-condition near infinity (briefly, $A \in \Delta_2$ near infinity) if it is finite valued and there exist two constants $K \geq 2$ and $M \geq 0$ such that*

$$A(2t) \leq KA(t) \quad \text{for } t \geq M\,. \tag{13}$$

Definition 5. *The function A is said to satisfy the ∇_2-condition near infinity (briefly, $A \in \nabla_2$ near infinity) if there exist two constants $K > 2$ and $M \geq 0$ such that*

$$A(2t) \geq KA(t) \quad \text{for } t \geq M\,. \tag{14}$$

Let Ω be a measurable set in $\mathbb{R}^n$, with $n \geq 1$. Given a Young function A, the Orlicz space $L^A(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\left(\tfrac{1}{\lambda}|u|\right) dx \leq 1\right\}$$

is finite. The functional $\|\cdot\|_{L^A(\Omega)}$ is a norm on $L^A(\Omega)$, and it is a Banach space.

The isotropic Orlicz–Sobolev spaces $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$ are defined as

$$W^{1,A}(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is weakly differentiable in } \Omega,\, u,\, |\nabla u| \in L^A(\Omega)\} \tag{15}$$

and

$$W_0^{1,A}(\Omega) = \{u : \Omega \to \mathbb{R} : \text{the continuation of } u \text{ by 0 outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n,\ \text{u},\ |\nabla u| \in L^A(\Omega)\}. \tag{16}$$

The spaces $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$ equipped with the norms

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega)}, \ \text{and} \ \|u\|_{W_0^{1,A}(\Omega)} = \|\nabla u\|_{L^A(\Omega)}, \tag{17}$$

are Banach spaces. The norm on $W_0^{1,A}(\Omega)$ is equivalent to the standard one

$$\|u\|_{W_0^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega)}\,.$$

Definition 6. *The optimal Sobolev conjugate of A is defined by $A_n : [0,\infty) \to [0,\infty]$*

$$A_n(t) = A(H^{-1}(t)) \quad \text{for } t \geq 0, \tag{18}$$

where $H : [0,\infty) \to [0,\infty)$ is given by

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}} \quad \text{for } t \geq 0,$$

provided that the integral is convergent. Here, H^{-1} denotes the generalized left-continuous inverse of H.

If

$$\int_0 \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau < \infty\,, \tag{19}$$

then (see Theorem 1 of [19])

$$W_0^{1,A}(\Omega) \to L^{A_n}(\Omega)\,. \tag{20}$$

Definition 7. *Let X be a real reflexive Banach space. A mapping $\mathcal{B} : X \to X^*$ is called*

(i) Coercive if $\lim_{\|u\|\to\infty} \frac{\langle \mathcal{B}u,u\rangle}{\|u\|} = +\infty$;

(ii) Bounded if it maps bounded sets into bounded sets;

(iii) Pseudomonotone if $u_n \rightharpoonup u$ *and* $\limsup_{k\to+\infty}\langle \mathcal{B}u_k, u_k - u\rangle \le 0$ *imply that* $\mathcal{B}u_k \rightharpoonup \mathcal{B}u$ *and* $\langle \mathcal{B}u_k, u_k\rangle \to \langle \mathcal{B}u, u\rangle$.

We can now give the fundamental definitions of weak solution, subsolution, and supersolution to (3).

Definition 8. *A function $u \in W_0^{1,A}(\Omega)$ is a weak solution to problem (3) if*

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla v dx = \int_\Omega f(x,u,\nabla u)v dx \text{ for all } v \in W_0^{1,A}(\Omega),$$

and $\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla v dx \in \mathbb{R}$ for all $v \in W_0^{1,A}(\Omega)$.

The classical definitions of sub and supersolution are by now well known: it is worth noting that there are different definitions, but they have in common that for them it is necessary to be able to define the trace of a function on the boundary of Ω. When the domain is not regular enough, then we need a new definition of sub and super solution, because we deal with spaces where the trace operator may be not defined. This new definition covers the classical one (see, for instance, [4]) for regular domains. For a real number r, we put $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$.

Definition 9. *We say that $\underline{u} \in W^{1,A}(\Omega)$ is a subsolution to (3) if $\underline{u}(x)^+ \in W_0^{1,A}(\Omega)$ and*

$$-\infty < \int_\Omega \mathcal{A}(x,\underline{u},\nabla\underline{u})\cdot\nabla v dx \le \int_\Omega f(x,\underline{u},\nabla\underline{u})v dx < +\infty$$

for all $v \in W_0^{1,A}(\Omega)$, $v \ge 0$ a.e. in Ω.

Definition 10. *We say that $\overline{u} \in W^{1,A}(\Omega)$ is a supersolution to (3) if $(\overline{u}(x))^- \in W_0^{1,A}(\Omega)$,*

$$+\infty > \int_\Omega \mathcal{A}(x,\overline{u},\nabla\overline{u})\cdot\nabla v dx \ge \int_\Omega f(x,\overline{u},\nabla\overline{u})v dx > -\infty$$

for all $v \in W_0^{1,A}(\Omega)$, $v \ge 0$ a.e. in Ω.

Remark 2. *For domains where it is possible to define a trace operator, the conditions $\underline{u}(x)^+ \in W_0^{1,A}(\Omega)$ and $(\overline{u}(x))^- \in W_0^{1,A}(\Omega)$ are replaced with $\underline{u}(x)_{|\partial\Omega} \le 0$ and $(\overline{u(x)})_{|\partial\Omega} \ge 0$, respectively. Our definitions coincide with the classical ones for regular domains.*

We can now introduce some existence results for (3). Let us start with some theorems of [11].

Let $\Omega \subset \mathbb{R}^n$ be a set with finite measure and let A be a Young function, $A \in \Delta_2 \cap \nabla_2$ near infinity. Consider the vector valued function $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A} = (a_1, \dots a_n)$, enjoying with the properties that each $a_i(x,s,\xi)$ is a Carathéodory function, and

$$|\mathcal{A}(x,s,\xi)| \le q(x) + b\Big[\widetilde{A}^{-1}(F(b|s|)) + \widetilde{A}^{-1}(A(|\xi|))\Big] \text{ for a.e. } x \in \Omega, \text{ all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^n\,. \tag{21}$$

In (2) $q \in L^{\widetilde{A}}(\Omega)$, $b > 0$, and F is a Young function, $F \ll A_n$ near infinity.

$$\sum_{i=1}^{n}\big(a_i(x,s,\xi) - (a_i(x,s,\xi')\big)\cdot(\xi_i - \xi_i') > 0 \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ all } \xi,\xi' \in \mathbb{R}^n,\ \xi \ne \xi'\,. \tag{22}$$

$$\sum_{i=1}^{n} a_i(x,s,\xi)\cdot\xi_i \ge cA(|\xi|) - dG(d|s|) - r(x) \qquad \text{for a. e. } x \in \Omega, \text{ all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^n\,. \tag{23}$$

In (23) $c, d > 0$, G is a Young function, $G \ll A_n$ near infinity, and $r \in L^1(\Omega)$.

Remark 3. *We can weaken* (21) *- - -* (23)*, requiring them to hold for* $s \in [\underline{u}(x), \overline{u}(x)]$*, with* $\underline{u}$ *and* $\overline{u}$ *being an ordered pair of sub and supersolutions. We write the hypotheses for* $s \in \mathbb{R}$ *just for reader convenience.*

Theorem 1 (see Theorem 3.4 of [11])**.** *Let* Ω *be an open set in* $\mathbb{R}^n$*, with* $n \ge 2$*, such that* $|\Omega| < \infty$*. Let* $A \in C^1([0,+\infty))$ *be a Young function,* $A \in \Delta_2 \cap \nabla_2$ *near infinity. Assume also that* A *satisfies* (19)*. Let* $\underline{u}$ *and* $\overline{u}$ *be a subsolution and a supersolution of problem* (3)*, respectively, with* $\underline{u} \le \overline{u}$ *a.e. in* Ω*, and* $\underline{u}, \overline{u} \in L^{A_n}(\Omega)$*. Assume that the function* $\mathcal{A}$ *satisfies* (21)*–*(23)*. Let* $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ *be a Carathéodory function fulfilling*

$$|f(x,s,\xi)| \le \sigma(x) + \overline{\gamma}\widetilde{E}^{-1}(A(|\xi|)) \text{ for a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x),\overline{u}(x)], \text{ all } \xi \in \mathbb{R}^n\,, \tag{24}$$

where $\sigma \in L^{\widetilde{A}_n}(\Omega)$*,* $\overline{\gamma} > 0$ *and* $E : [0,+\infty[\to [0,+\infty[$ *is a Young function,* $E \ll A_n$ *near infinity. Then, problem* (P) *has a solution* $u \in W_0^{1,A}(\Omega)$ *such that* $\underline{u} \le u \le \overline{u}$ *a.e. in* Ω*.*

Sketch of the Proof. To prove Theorem 1, we perturb problem (3). We define the truncation operator $T : W_0^{1,A}(\Omega) \to W_0^{1,A}(\Omega)$ by

$$T(u) = \begin{cases} \overline{u} & \text{if } u > \overline{u} \\ u & \text{if } \underline{u} \le u \le \overline{u} \\ \underline{u} & \text{if } u < \underline{u}\,. \end{cases} \tag{25}$$

Let $\mathcal{S}_T : W_0^{1,A}(\Omega) \to (W_0^{1,A}(\Omega))^*$, given by

$$\mathcal{S}_T(u)(v) = \int_\Omega \mathcal{A}(x,Tu,\nabla u)\cdot\nabla v\,dx\,, \text{ for } u,v \in W_0^{1,A}(\Omega), \tag{26}$$

Let $\Pi : W_0^{1,A}(\Omega) \to (W_0^{1,A}(\Omega))^*$, given by

$$\Pi(u)(v) = \int_\Omega \pi(x,u(x))v(x)dx\,, \text{ for } u,v \in W_0^{1,A}(\Omega), \tag{27}$$

where

$$\pi(x,s) = \begin{cases} \widetilde{E}^{-1}(E(s-\overline{u}(x)) & \text{if } s > \overline{u}(x) \\ 0 & \text{if } \underline{u}(x) \le s \le \overline{u}(x) \\ -\widetilde{E}^{-1}(E(\underline{u}(x)-s)) & \text{if } s < \underline{u}(x). \end{cases}$$

Let $\mathcal{N}_f \circ T : W_0^{1,A}(\Omega) \to (W_0^{1,A}(\Omega))^*$ be the operator defined as

$$\langle \mathcal{N}_f \circ T(u), v\rangle = \int_\Omega f(x,Tu,\nabla Tu)v(x)dx\,, \text{ for } u,v \in W_0^{1,A}(\Omega).$$

Given $\mu > 0$, we consider the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, Tu, \nabla u)) + \mu\Pi(u) = N_f(Tu) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

and the operator $\mathcal{A}_\mu : W_0^{1,A}(\Omega) \to (W_0^{1,A}(\Omega))^*$, defined by

$$\langle \mathcal{A}_\mu(u), v\rangle = \int_\Omega \mathcal{A}(x, Tu, \nabla u)\cdot\nabla v\, dx + \mu\int_\Omega \pi(x,u)v\, dx - \int_\Omega f(x, Tu, \nabla Tu)v\, dx = \langle \mathcal{S}_T u + \mu\Pi u - \mathcal{N}_f \circ T(u), v\rangle \qquad \text{for } u, v \in W_0^{1,A}(\Omega).$$

Let us prove that $\mathcal{A}_\mu$ is well defined, bounded, and pseudomonotone, and there is $\mu_0 > 0$ such that $\mathcal{A}_\mu$ is coercive for all $\mu > \mu_0$. Due to Corollary 2.18 of [11], Propositions 4.3 and 4.5 of [9], $\mathcal{A}_\mu$ is well defined, bounded, and continuous. To prove that it is pseudomonotone, we take $u \in W_0^{1,A}(\Omega)$, and a sequence $\{u_k\} \subset W_0^{1,A}(\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,A}(\Omega), \text{ and } \limsup_{k\to\infty} \langle \mathcal{A}_\mu(u_k), u_k - u\rangle \le 0.$$

Equations (4.6) and (4.17) of [9] allow us to write

$$\limsup_{k\to\infty} \langle \mathcal{S}_T(u_k), u_k - u\rangle \le 0.$$

Thus, $u_k \to u$ in $W_0^{1,A}(\Omega)$ (see Corollary 2.18 of [11]), and, consequently,

$$\lim_{k\to\infty} \|\mathcal{A}_\mu(u_k) - \mathcal{A}_\mu(u)\|_{(W_0^{1,A}(\Omega))^*} = 0,$$

therefore $\langle \mathcal{A}_\mu(u_k), u_k\rangle \to \langle \mathcal{A}_\mu(u), u\rangle$, $\langle \mathcal{A}_\mu(u_k), v\rangle \to \langle \mathcal{A}_\mu(u), v\rangle$ for all $v \in W_0^{1,A}(\Omega)$, and $\mathcal{A}_\mu$ is a pseudomonotone operator. The coercivity of $\mathcal{A}_\mu$, for μ big enough, follows from Equations (2.16), (3.9), and (3.10) of [11] and from Lemma 4.6 of [9]. Thus,

$$\lim_{\|u\|\to+\infty} \frac{\langle \mathcal{A}_\mu(u), u\rangle}{\|u\|_{W_0^{1,A}(\Omega)}} = +\infty.$$

Theorem 2.99 of [20] guarantees that there exists $u \in W_0^{1,A}(\Omega)$ such that $\mathcal{A}_\mu(u) \equiv 0$. Thus,

$$\int_\Omega \mathcal{A}(x, Tu, \nabla u)\cdot\nabla v dx + \mu\int_\Omega \pi(x, u(x))v(x)dx - \int_\Omega f(x, Tu, \nabla Tu)v dx = 0. \tag{29}$$

for all $v \in W_0^{1,A}(\Omega)$. Therefore, there exists a solution $u \in W_0^{1,A}(\Omega)$ of the truncated auxiliary problem (28) provided $\mu > 0$ is sufficiently large. Let us fix such a μ and u. Via the same comparison arguments of the proof of Theorem 3.6 of [9], we can prove that the solution of (28) has the enclosure property $u \in [\underline{u}, \overline{u}]$. Thus, it follows from (25) and (27) that $Tu = u$ and $\Pi(u) = 0$. Consequently, u is a solution of (3). □

Remark 4. *Theorem 1 extends Theorems 3.2 and 3.3 of [9], where the authors study (2), with $A \in \nabla_2$ globally.*

Remark 5. *If the function $u \equiv 0$ does not belong in the interval $[\underline{u}, \overline{u}]$, then the solution in Theorem 1 is nontrivial. This happens, for instance, when the subsolution (or the supersolution) changes sign, and the sets which have opposite sign have positive measure.*

Theorems 2 and 5, as well as Corollaries 1 and 2, guarantee the existence of a nontrivial solution.

If $\underline{u}$, $\overline{u} \in W_0^{1,A}(\Omega)$, then we have

Corollary 1 (see Corollary 3.8 of [11]). *Let Ω be an open set in $\mathbb{R}^n$, with $n \geq 2$, such that $|\Omega| < \infty$. Let $A \in C^1([0,+\infty))$ be a Young function, $A \in \Delta_2 \cap \nabla_2$ near infinity. Assume also that A satisfies (19). Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem (3), respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω, $\underline{u}, \overline{u} \in W_0^{1,A}(\Omega)$, and such that the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ fulfills*

$$|f(x,s,\xi)| \leq \rho(x) + g(|s|) + \overline{\gamma}\widetilde{E}^{-1}(A(|\xi|)) \text{ a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x), \overline{u}(x)],\ \xi \in \mathbb{R}^n, \quad (30)$$

where $\rho \in L^{\widetilde{A}_n}(\Omega)$, $\overline{\gamma}$, E, are as in Theorem 1, and $g : [0,+\infty[\to [0,+\infty[$ is a nondecreasing function such that $g(0) = 0$ and there exist s_0, $h > 0$ such that $g(|s|)|s| \leq A_n(h|s|)$ for all $|s| \geq s_0$. Then problem (P) possesses a nontrivial solution $u \in W_0^{1,A}(\Omega)$.

Remark 6. *The corollary above extends Corollary 5.2 in [9].*

In Theorem 1, we assumed a priori the existence of a pair of sub and supersolutions. The next result is formulated under a quite natural unified assumption on f that guarantees the existence of such a pair and allows one to prove also the existence of a solution. Even if this hypothesis covers a large class of problems, it is necessary to say that now the differential operator does not depend on the real variable s. It is still an open question to guarantee the existence of a subsolution and a supersolution for general $\mathcal{A}$, under suitable hypotheses on f.

Therefore, let $\Omega \subset \mathbb{R}^n$ be a set of finite measure and let A, B be two Young functions such that $A \in \Delta_2 \cap \nabla_2$ near infinity, $B \in \nabla_2$ near zero, and $A \circ B^{-1}$ is a Young function as well. We assume that $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A} = (a_1, \dots a_n)$, is such that each $a_i(x,\xi)$ is a Carathéodory function, and

$$\begin{aligned} &|\mathcal{A}(x,\xi)| \leq q(x)\widetilde{B}^{-1}(B(b|\xi|)) + b\widetilde{A}^{-1}(A(b|\xi|)) \\ &\text{for some } q \in L^{\widetilde{A \circ B^{-1}}}(\Omega), \text{ some } b > 0, \text{ for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^n, \end{aligned} \quad (31)$$

$$\sum_{i=1}^{n} \big(a_i(x,\xi) - a_i(x,\xi')\big) \cdot (\xi_i - \xi_i') > 0 \quad \text{for a. e. } x \in \Omega, \text{ all } \xi, \xi' \in \mathbb{R}^n,\ \xi \neq \xi', \quad (32)$$

$$\sum_{i=1}^{n} a_i(x,\xi) \cdot \xi_i \geq cA(c|\xi|) \quad \text{for some } c > 0, \text{ for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^n. \quad (33)$$

Furthermore, we assume that there exists a measurable function $\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$, even with respect to $\xi \in \mathbb{R}^n$ and such that

$$\Phi_\xi(x,\xi) = \mathcal{A}(x,\xi) \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}^n, \Phi(x,0) = 0 \text{ for all } x \in \Omega. \quad (34)$$

Condition (32) ensures that $\Phi(x,\cdot)$ is convex for every $x \in \Omega$. From (31) and (33), there exist $k_4, k_5 > 0$ such that

$$k_4 A(k_4|\xi|) \leq \Phi(x,\xi) \leq 2q(x)B(b|\xi|) + k_5 A(k_5|\xi|) \text{ for all } (x,\xi) \in \mathbb{R}^n. \quad (35)$$

Now, problem (3) reads as

$$\begin{cases} -\mathrm{div}(\Phi_\xi(x,\nabla u)) = f(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (36)$$

The Δ_2 and ∇_2 conditions, as well as some consequences of the new assumptions (see [11] for the details), have a key role for the existence of the sub and the supersolution for problem (36).

Theorem 2 (see Theorem 3.9 of [11]). *Let Ω be an open set in $\mathbb{R}^n$, with $n \geq 2$, such that $|\Omega| < \infty$. Let $A \in C^1([0,+\infty))$ be a Young function, $A \in \Delta_2 \cap \nabla_2$ at infinity. Assume also that A satisfies (19). Let $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be two Carathéodory functions satisfying (31)---(34). Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function fulfilling*

$$-\rho_1(x) - g_1(|s|) \leq f(x,s,\xi) \leq \rho_2(x) + g_2(|s|) + \overline{\gamma}\widetilde{E}^{-1}(A(|\xi|)) \text{ for a.e. } x \in \Omega, \text{ all } s \leq 0, \tag{37}$$
$$\text{all } \xi \in \mathbb{R}^n,\ f(x,0,0) \leq 0 \text{ in } \Omega \text{ and } f(x,0,0) < 0 \text{ on a set of positive measure},$$

or

$$-\rho_2(x) - g_2(|s|) - \overline{\gamma}\widetilde{E}^{-1}(A(|\xi|)) \leq f(x,s,\xi) \leq \rho_1(x) + g_1(|s|) \text{ for a.e. } x \in \Omega, \text{ all } s \geq 0, \tag{38}$$
$$\text{all } \xi \in \mathbb{R}^n,\ f(x,0,0) \geq 0 \text{ in } \Omega, \text{ and } f(x,0,0) > 0 \text{ on a set of positive measure},$$

where $\overline{\gamma} > 0$, E is a Young function, $E \ll A_n$ near infinity, $\rho_1, \rho_2 : \Omega \to [0,+\infty[$ are two measurable functions, $\rho_i \in L^{\widetilde{A}_n}(\Omega)$, $i = 1,2,$, $g_1, g_2 : [0,+\infty[\to [0,+\infty[$ are two nondecreasing functions such that $g_1(0) = g_2(0) = 0$ and there exist $s_0 > 0$, $h_0 \in]0, \tau\omega_n^{\frac{1}{n}}|\Omega|^{-\frac{1}{n}}[$, $h_1 > 0$ such that

$$g_1(|s|)|s| \leq A(h_0|s|) \text{ and } g_2(|s|)|s| \leq A_n(h_1|s|) \text{ for all } |s| \geq s_0\,. \tag{39}$$

In (39) ω_n is the measure of the unit ball in $\mathbb{R}^n$, $\tau = \min\{1, k_4^2\}$ where k_4 is that of (35). Then, problem (36) possesses a nontrivial constant sign solution $u \in W_0^{1,A}(\Omega)$.

Sketch of the Proof. Suppose that (37) is in force. We construct a subsolution $\underline{u} \leq 0$ a.e., $\underline{u} \not\equiv 0$, and show that $\overline{u} \equiv 0$ is a supersolution but not a solution to (36). Then, we show that f satisfies (30).

Put $G_1(t) = \int_0^t g_1(\tau)d\tau$, $t \geq 0$ and consider the functional $J : W_0^{1,A}(\Omega) \to \mathbb{R}$, defined as

$$J(u) = \int_\Omega (\Phi(x,\nabla u) + \rho_1(x)u - G_1(|u|))dx \quad \text{for } u \in W_0^{1,A}(\Omega).$$

We prove that J is well defined, weakly lower semicontinuous, coercive, and

$$J'(u)v = \int_\Omega \Phi_\xi(x,\nabla u)\nabla v dx + \int_\Omega \rho_1(x)v(x)dx - \int_\Omega g_1(|u|)sign\, u\, v(x)dx \tag{40}$$

for all $u, v \in W_0^{1,A}(\Omega)$. Due to (35), the fact that $A \in \Delta_2$ at infinity, and the convexity of $\Phi(x,\cdot)$, for all $x \in \Omega$, the functional $u \mapsto \int_\Omega \Phi(x,\nabla u)dx$ is well defined in $W_0^{1,A}(\Omega)$, convex. The proof of its regularity makes use of standard arguments like the Lebesgue Theorem and the properties of Young's functions. Thus, taking into account the properties of the other two integrals, the weak lower semicontinuity of J and Equation (40) follow.

To prove the coercivity of J, we choose $\varepsilon > 0$, such that $h_0\omega_n^{-\frac{1}{n}}|\Omega|^{\frac{1}{n}} < \tau - \varepsilon$. Making use of Proposition 3.2 of [21], (39), (35), Hölder inequality for Young functions, and Lemma 2.9 of [11], we obtain

$$\begin{aligned}\frac{J(u)}{\|u\|_{W_0^{1,A}(\Omega)}} &\geq \frac{\left(k_4 - \sqrt{\tau-\varepsilon}\right)\int_\Omega A(k_4|\nabla u|)dx - c_4\|u\|_{W_0^{1,A}(\Omega)} - G_1(s_0)|\Omega|}{\|u\|_{W_0^{1,A}(\Omega)}} \\ &\geq \left(k_4 - \sqrt{\tau-\varepsilon}\right)(k_4)^{i_A^\infty}\|u\|^{i_A^\infty - 1}_{W_0^{1,A}(\Omega)} - c_4 - \frac{k_3 + G_1(s_0)|\Omega|}{\|u\|_{W_0^{1,A}(\Omega)}},\end{aligned}$$

where i_A^∞ comes from Lemma 2.9 of [11]. This proves that J is coercive. Thus, it has a global minimum. Let $\underline{u}$ be a global minimum point for J. We prove that $\underline{u} \not\equiv 0$. To this end, consider a function $v \in C_0^1(\Omega)$, such that $b|\nabla v(x)| \leq \overline{t}$ and $k_5\overline{k}|\nabla v(x)| \leq \overline{t}$ for all

$x \in \Omega$. Also, $v \leq 0$ and $\rho_1(x)v(x) \not\equiv 0$ in Ω. The inequality $\frac{B(t_1)}{B(t_0)} > \left(\frac{t_1}{t_0}\right)^{k_B}$ holds for $0 < t_0 < t_1 < \bar{t}$, and some $k_B > 1$, by virtue of the ∇_2 condition near zero. Then, choosing once $t_1 = b|\nabla v|$, $t_0 = bt|\nabla v|$, and secondly $t_1 = k_5\bar{k}|\nabla v|$, $t_0 = tt_1$, with $t < 1$, and taking into account (35)

$$J(tv) \leq 2t^{k_B}\int_\Omega q(x)B(b|\nabla v|)dx + k_5 t^{k_B}\int_\Omega B(k_5\bar{k}|\nabla v|)dx + t\int_\Omega \rho_1(x)vdx < 0 \ \text{ for } t << 1\,,$$

and this proves that $J(\underline{u}) < 0$. Using $J(-|\underline{u}|) \geq J(\underline{u})$ and the fact that $\Phi(x,\cdot)$ is even, we obtain $\underline{u} \leq 0$ a.e. in Ω. Now, we prove that $\underline{u}$ is a subsolution and $u \equiv 0$ is a supersolution but not a solution to (3). Note that

$$J'(\underline{u})(v) = \int_\Omega \Phi_{\tilde{\xi}}(x,\nabla\underline{u})\nabla v dx + \int_\Omega (\rho_1(x) + g_1(|\underline{u}(x)|))vdx \equiv 0, \text{ for all } v \in W_0^{1,A}(\Omega)\,. \tag{41}$$

Acting with any $v \in W_0^{1,A}(\Omega)$, $v \geq 0$, in (41) and using (37)

$$\int_\Omega \Phi_{\tilde{\xi}}(x,\nabla\underline{u})\nabla v dx - \int_\Omega f(x,\underline{u},\nabla\underline{u})vdx \leq 0\,,$$

that is, $\underline{u}$ is a subsolution to (36). Using (37) and choosing $v \in W_0^{1,A}(\Omega)$, $v \geq 0$

$$0 - \int_\Omega f(x,0,0)vdx \geq 0\,,$$

thus, $u \equiv 0$ is a supersolution to (3) and the assumption on $f(x,0,0)$ guarantees that it is not a solution.

We put $\rho(x) = \max\{\rho_i(x),\ i = 1,2\}$, $g(|s|) = \max\{g_i(|s|),\ i = 1,2\}$ and use (37)

$$|f(x,s,\xi)| \leq \rho(x) + g(|s|) + \overline{\gamma}\widetilde{E}^{-1}(A(|\xi|)) \ \text{ for } x \in \Omega,\ s \in [\underline{u}(x),0],\ \xi \in \mathbb{R}^n\,.$$

Then, f satisfies (30) and, from Corollary 1, problem (36) has a nontrivial solution $u \in W_0^{1,A}(\Omega)$ and $u \in [\underline{u},0]$.

When (38) is in force, we consider $f_1(x,s,\xi) = -f(x,-s,-\xi)$. Then, by virtue of the proof above, problem

$$\begin{cases} -\mathrm{div}\left(\Phi_{\tilde{\xi}}(x,\nabla v)\right) = f_1(x,v,\nabla v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{42}$$

has a nontrivial solution $v \in W_0^{1,A}(\Omega)$, $v \leq 0$ a.e. in Ω. Then, the function $u = -v$ is a nontrivial solution to (3) and $u \geq 0$ a.e. in Ω. □

Remark 7. *The result above extends Theorems 3.2 and 3.2 of [9] in several directions. The results in [9] deal with an operator that does not depend on x, the growth condition on f are stronger than those in Theorem 2, and $A \in \nabla_2$ globally.*

Another existence result within Orlicz spaces can be found in [13], where the authors consider sub and supersolutions in $W_0^{1,A}(\Omega)$, rather than in $W^{1,A}(\Omega)$, as it is usually done: it is not a trivial matter, because it prevents us from considering constant sub and supersolutions. They do not require the ∇_2 condition on A and their operator $\mathcal{A}$ satisfies (22) and (23), with $G \equiv 0$, and a condition slightly different (and stronger, in general) than (21):

$$|\mathcal{A}(x,s,\xi)| \leq q(x) + b\left[\widetilde{P}^{-1}(A(|s|)) + \widetilde{A}^{-1}(A(|\xi|))\right] \text{ for a.e. } x \in \Omega, \text{ all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^n\,. \tag{43}$$

Here, $q \in L^{\widetilde{A}}(\Omega)$, $b > 0$, and P is a Young function, $P \ll A$ near infinity.

To formulate the result proved in [13], we must introduce the space $E_B(\Omega)$, namely the closure in $L^B(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$. Here, B is a Young function. For functions $B \in \Delta_2$ near infinity, one has $E_B(\Omega) \equiv L^B(\Omega)$.

Theorem 3 (Theorem 6 of [13]). *Let Ω be a bounded domain with a Lipschitz boundary. Let A and P be two N-functions, with $P \ll A$ and $A \in \Delta_2$ near infinity. Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem (3), respectively, such that $\underline{u} \leq \overline{u}$. Assume (22), (23), and (43), with $G \equiv 0$. Also,*

$$|f(x,s,\xi)| \leq k_1(x) + c\widetilde{P}^{-1}(A(|\xi|)) \text{ for a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x), \overline{u}(x)], \text{ all } \xi \in \mathbb{R}^n, \quad (44)$$

for some $k_1 \in E_{\tilde{P}}(\Omega)$, $k_1 \geq 0$, $c \geq 0$. Then, there exists at least one solution u of problem (3), with $u \in [\underline{u}, \overline{u}]$.

Finally, it is important to say that in [12], the results are formulated in separable Musielak Orlicz spaces: for their definitions and their properties, we refer to [12,16]. The authors consider the problem

$$-div(a_1(x, \nabla u)) + a_0(x,u) = f(x,u,\nabla u) \quad \text{in } \Omega \quad (45)$$

coupled with Dirichlet or Neumann boundary conditions. The principal part of the operator does not depend on u and satisfies a suitable growth condition, a standard coercivity condition, and the monotonicity condition. The function a_0 satisfies a suitable growth condition without any coercivity or monotonicity assumption. We recall here only the growth condition on f, for the others we refer to [12], assumptions Φ, Ψ, $(A1)$, (A_0):

$$|f(x,s,\xi)| \leq q(x) + b_3\widetilde{\Phi}^{-1}(x,\Phi(x,|s|)) + b_4\widetilde{\Psi}^{-1}(x,\Phi(x,|\xi|)) \quad (46)$$
$$\text{for a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x), \overline{u}(x)], \text{ all } \xi \in \mathbb{R}^n.$$

Here, Ψ (among the other properties) is such that the embedding $W_0^{1,\Phi}(\Omega) \to L^{\Psi}(\Omega)$ is compact, $q \in E^{\widetilde{\Psi}}(\Omega)$ (see page 5 of [12] for the definition), $b_3, b_4 > 0$ and $\underline{u}$, $\overline{u}$ are, respectively, the minimum and the maximum of a finite set of subsolutions and supersolutions to (45), respectively. Here, the sub and supersolutions belong in $W_0^{1,\Phi}(\Omega)$. Theorem 3.1 of [12] guarantees the existence of a solution to (45), coupled with homogeneous Dirichlet bouondary conditions.

3. Regularity Results

In this section, we state some existence and regularity results. The proof of the existence is based on sub and supersolution methods, while the main tool for the regularity is Theorem 1.7 of [22] (see also the remark after that result and [23]). The theorem below is a weak extension of [11] (Theorem 4.4), as well as a significant generalization of [10] (Theorem 3.5).

Theorem 4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a $C^{1,\alpha}$ boundary, for some $0 < \alpha \leq 1$. Let A be a Young function satisfying*

$$A \in C^2(]0,+\infty[), \text{ and there exist two positive constants } \delta,\ g_0 > 0, \quad (47)$$
$$\text{such that } \delta \leq \frac{tA''(t)}{A'(t)} \leq g_0 \quad \text{for } t > 0.$$

Let $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector valued function, with Carathéodory components, a_i, $i = 1, \ldots, n$ and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. Assume that problem (3) has a subsolution $\underline{u} \in W^{1,\infty}(\Omega)$, and a supersolution $\overline{u} \in W^{1,\infty}(\Omega)$, with $\underline{u}(x) < \overline{u}(x)$ a.e.

$x \in \Omega$. *Let* $M = \max\{\|\overline{u}\|_\infty, \|\underline{u}\|_\infty\} > 0$. *The functions* $\mathcal{A}$ *(here* $a_{ij}(x,s,\eta) = \frac{\partial a_i}{\partial \eta_j}$*) and* f *satisfy the structure conditions*

$$\sum_{i,j=1}^{n} a_{ij}(x,s,\eta)\xi_i\xi_j \geq \frac{A'(|\eta|)}{|\eta|}|\xi|^2\,, \tag{48}$$

$$\sum_{i,j=1}^{n} |a_{ij}(x,s,\xi)| \leq \Lambda \frac{A'(|\xi|)}{|\xi|}\,, \tag{49}$$

$$|\mathcal{A}(x,s,\xi) - \mathcal{A}(y,w,\xi)| \leq \Lambda_1(1 + A'(|\xi|)(|x-y|^\alpha + |s-w|^\alpha)\,, \tag{50}$$

$$\begin{aligned} |f(x,s,\xi)| \leq \sigma(x) + \overline{\gamma}(s)A'(|\xi|)|\xi| \quad & \textit{where } \sigma \in L^\infty(\Omega) \textit{ and} \\ \overline{\gamma} : [0,+\infty[\to [0,+\infty[\quad & \textit{is locally essentially bounded,} \end{aligned} \tag{51}$$

for some positive constants Λ, Λ_1, *for all* x *and* $y \in \Omega$, *for all* $s, w \in [-M, M]$, *and for all* $\xi \in \mathbb{R}^n$. *Then, problem* (3) *admits at least a solution* $u \in C_0^{1,\beta}(\overline{\Omega})$. *Moreover,* $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ *a.e in* Ω.

Under a local growth condition on f, near zero, we guarantee that the solution is positive. This result is proved for (2), but can be easily extended to (3).

Corollary 2 (see Corollary 3.7 of [10]). *Under the hypotheses of Theorem 4, assume that* f *satisfies*

$$f(x,s,\xi) \geq -aA'(|\xi|) - b(s) \textit{ for all } x \in \Omega,\ s > 0,\ \xi \in \mathbf{R}^n,\ |\xi| \leq 1, \tag{52}$$

where $a > 0$, $b : [0,+\infty[\to [0,+\infty[$ *is a function increasing in* $(0,\overline{\delta})$ *(for some* $\overline{\delta} > 0$*),* $b(0) = 0$, *and* $b(s) = \frac{A(ks)}{s}$ *for* $s \in (0,\overline{\delta})$ *and some* $k > 0$. *Then, any nonnegative, nontrivial solution to* (2) *is positive.*

When $\mathcal{A}$ does not depend on s, then we can formulate the following result, where the growth conditions on f and on $\mathcal{A}$ guarantee the existence of a subsolution and of a supersolution.

Theorem 5 (see Theorem 4.5 of [11]). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* $C^{1,\alpha}$ *boundary. Let* A *be a Young function satisfying* (47), *and let* $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ *be a vector valued function satisfying* (48) *and* (49) *for a.e.* $x \in \Omega$, *all* $\xi \in \mathbb{R}^n$, *and such that* $a_i(x,0) \equiv 0$ *for a.e.* $x \in \Omega$, *all* $i = 1,\dots,n$. *We further assume that* (34) *holds. Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *be a Carathéodory function fulfilling*

$$\begin{aligned} -\rho_2(x) - g_2(s) - \overline{\gamma}(s)A'(|\xi|)|\xi| \leq f(x,s,\xi) \leq \rho_1(x) + g_1(s) \textit{ for a.e. } x \in \Omega,\ \textit{all } s \geq 0, \\ \textit{all } \xi \in \mathbb{R}^n,\ f(x,0,0) \geq 0 \textit{ in } \Omega, \textit{ and } f(x,0,0) > 0 \textit{ on a set of positive measure,} \end{aligned} \tag{53}$$

or

$$\begin{aligned} -\rho_1(x) - g_1(|s|) \leq f(x,s,\xi) \leq \rho_2(x) + g_2(|s|) + \overline{\gamma}(s)A'(|\xi|)|\xi| \textit{ for a.e. } x \in \Omega,\ \textit{all } s \leq 0, \\ \textit{all } \xi \in \mathbb{R}^n,\ f(x,0,0) \leq 0 \textit{ in } \Omega, \textit{and } f(x,0,0) < 0 \textit{ on a set of positive measure.} \end{aligned} \tag{54}$$

Here, $\rho_1, \rho_2 : \Omega \to [0,+\infty[$ *are two measurable functions,* $\rho_1,\ \rho_2 \in L^\infty(\Omega)$; g_1 *is like in Theorem 2,* $g_2 : [0,+\infty[\to [0,+\infty[$ *is a nondecreasing function such that* $g_2(0) = 0$, *and* $\overline{\gamma}(s)$ *is a locally essentially bounded function. Then problem* (P) *has a nontrivial, solution* $u \in C_0^{1,\beta}(\overline{\Omega})$. *If* (53) *holds, then* $u \geq 0$ *in* Ω. *In the other case,* $u \leq 0$ *in* Ω.

Remark 8. *Regularity results, with sign information on the solutions, for problem (2), can be found in Theorems 3.2 and 3.3 of [10]. The following growth condition on g_2 near 0*

$$\text{there exist } \overline{\delta} > 0 \text{ and } k_3 > 0 \text{ such that } g_2(s)s \leq A(k_3 s) \text{ for every } s \in (0, \overline{\delta}), \tag{55}$$

guarantees that $u(x) \neq 0$ for all $x \in \Omega$.

Remark 9. *Our theorem extends Theorem 1 of [1] and Theorems 3.2 and 3.3 of [10].*

4. Examples

In this section, we present some examples introduced in [9,11]. The first of them involves Theorem 1 (Section 4.1) and two interesting and meaningful problems for which we can easily find constant sub and supersolutions. In the first of them, the growth of $\mathcal{A}$ with respect to s can be whatever we want.

In Section 4.2, we deal with an application of Theorem 2, and Section 4.3 is devoted to some applications of Theorem 5.

For more details, we refer to [11]. Other interesting examples can be found in [9,10].

4.1. Applications of Theorem 1

In this example, the structure of the convection term and the condition $\mathcal{A}(x, s, 0) \equiv 0_{\mathbb{R}^n}$ allow us to easily find a pair of constant sub and supersolutions to (3).

Let $0 < \delta < p - 1$, $p + q > 1$, $a : \Omega \to [0, +\infty[$ be a measurable function, $a \in L^{\frac{p}{\delta}}(\Omega)$, and let $b : [0, +\infty[\to [0, +\infty[$ be a continuous function. Consider the problem

$$\begin{cases} -\operatorname{div}\left(\left(a(x)b(|u|)|\nabla u|^{p-2-\delta} \lg^{q(1-\frac{\delta}{p-1})}(1 + |\nabla u|) + |\nabla u|^{p-2} \lg^{q}(1 + |\nabla u|)\right)\nabla u\right) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{56}$$

The Young function A governing the differential operator $\mathcal{A}$ obeys

$$A(t) \approx t^p \lg^{p_1}(t) \text{ near infinity}\,. \tag{57}$$

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be

$$f(x, s, \xi) = (h(x) + k(|\xi|))g(s) \quad \text{for } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n\,. \tag{58}$$

Here, $h : \Omega \to [0, +\infty[$ is a measurable function, $h \in L^{\widetilde{A}_n}(\Omega)$, $k : [0, +\infty] \to \mathbb{R}$ is a continuous function, $k(0) > 0$, and k has the following behavior near infinity:

$$\begin{cases} |k(s)| \approx s^{\frac{p}{(p^*)'}} \lg^r(s) & \text{for some } r < \frac{(n+1)q}{n}, \text{ when } p < n\,, \\ |k(s)| \approx s^n \lg^r(s) & \text{for some } r < q - 1 + \frac{q+1}{n}, \text{ when } p = n,\ q < n - 1\,, \\ |k(s)| \approx s^n \lg^{n-1}(s) \lg^{-\frac{1}{r}}(\lg s) & \text{for some } r < \frac{n-1}{n}, \text{ when } p = n,\ q = n - 1\,, \\ |k(s)| \approx s^p \lg^r(s) & \text{for some } r < q, \text{ when } p > n \text{ or } p = n, q > n - 1\,. \end{cases} \tag{59}$$

The function $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g(s) > 0$ for $s \in [0, \overline{s})$, and $g(\overline{s}) = 0$.

We note that $u_1 = 0$ and $u_2 = \overline{s}$ are a subsolution and a supersolution to (56), and $u \equiv 0$ is not a solution. The growth of the functions h and k, and of those appearing in $\mathcal{A}$, as well as the continuity of b, guarantee that we can apply Theorem 1. Thus, problem (56) has a nontrivial solution $u \in [u_1, u_2]$.

The same arguments work for different choices of h and g. We can take nonlinearities having two zeros s_1 and s_2, with $s_1 < 0 < s_2$, ($f(x, s_1, 0) = f(x, s_2, 0) = 0$ for all $x \in \Omega$) and such that $f(x, s, 0)$ has constant sign for $s \in]s_1, s_2[$.

We can also consider functions f or for which $f(x, 0, 0)$ has constant sign and there exists

$s \in \mathbb{R}$ such that $f(x,s,0) \equiv 0$ and $s \cdot f(x,0,0) > 0$.

We now examine the problem

$$\begin{cases} -\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{\lg^q(1+|\nabla u|)}\right) = f(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{60}$$

where $p > q+1 \geq 1$. Then (according to the convention to modify A near 0 when (19) does not hold),

$$A_n(t) \approx \begin{cases} \frac{t^{p*}}{\lg^{\frac{qn}{n-p}}(1+t)} \text{ near infinity} & if\ p < n \\ e^{t^{\frac{n}{q+n-1}}} \text{ near infinity} & if\ p = n \\ +\infty \text{ near infinity} & if\ p > n\,. \end{cases}$$

Let $g_1, g_2 : [0,+\infty[\to [0,+\infty[$ be two unbounded, nondecreasing functions, such that $g_1(0) = g_2(0) = 0$. Let $a_1, a_2, c_1, c_2 > 0$.

Assume $p < n$. Let $\rho \in L^{(p^*)'}(\Omega)$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function such that

$$-c_1 + g_1(|s|) - a_1\frac{|\xi|^{(p^*)'}}{\lg^r(1+|\xi|)} \leq f(x,s,\xi) \leq -c_2 + g_2(|s|)\rho(x) + a_2\frac{|\xi|^{(p^*)'}}{\lg^r(1+|\xi|)}$$
$$\text{for } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \text{ and } r \in \left]q\frac{n+1}{n}, \frac{q}{(p^*)'}+1\right[.$$

If $k := \inf\{s > 0 : g_1(s) \geq c_1\}$, then $\underline{u} \equiv -k$ is a subsolution to (60), and $\overline{u} \equiv 0$ is a supersolution but not a solution to (60). We can find $a, b, c > 0$ such that

$$|f(x,s,\xi)| \leq b|\rho(x)| + c + a\widetilde{E}^{-1}(A(|\xi|)) \text{ for } x \in \Omega,\ s \in [-k,0],\ \xi \in \mathbb{R}^n\,,$$

where $E(t) = \frac{t^{p^*}}{\lg^{\lambda p^*}(1+t)}$, $\lambda = r - \frac{q}{(p^*)'}$. Thus, f satisfies (24) with $\sigma(x) = b|\rho(x)| + c$. By Theorem 1, problem (60) has a nontrivial solution $u \in [-k,0]$.

Let now $p = n$, $\rho \in L^t(\Omega)$ for some $t > 1$, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function such that

$$-c_1 + g_1(|s|) - a_1\frac{|\xi|^p}{\lg^{q+1}(1+|\xi|)} \leq f(x,s,\xi) \leq -c_2 + g_2(|s|)\rho(x) + a_2\frac{|\xi|^p}{\lg^{q+1}(1+|\xi|)}$$
$$\text{for } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n\,.$$

Then, $\underline{u} \equiv -k$, $\overline{u} \equiv 0$ are a subsolution and a supersolution to (60), and $u \equiv 0$ is not a solution to (60). Taking $E(t) = e^t - 1$ and arguing as in Example 3.2 of [9], we can prove that f satisfies (24). By Theorem 1, problem (60) has a nontrivial solution $u \in [-k,0]$.

Under the same assumptions on the functions g_1, g_2 and on the constants a_1, a_2, c_1, c_2 listed above, the same arguments can be used to prove that problem (60) has a nontrivial solution $u \in [0,k]$, where $k := \inf\{s > 0 : g_2(s) \geq c_2\}$, provided that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function such that

$$c_1 - g_1(|s|)\rho(x) - a_1\frac{|\xi|^{(p^*)'}}{\lg^r(1+|\xi|)} \leq f(x,s,\xi) \leq c_2 - g_2(|s|) + a_2\frac{|\xi|^{(p^*)'}}{\lg^r(1+|\xi|)}$$
$$\text{for } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \text{ and } r \in \left]q\frac{n+1}{n}, \frac{q}{(p^*)'}+1\right[\text{ if } p < n\,,$$

or

$$c_1 - g_1(|s|)\rho(x) - a_1\frac{|\xi|^{p^*}}{\lg^{q+1}(1+|\xi|)} \le f(x,s,\xi) \le c_2 - g_2(|s|) + a_2\frac{|\xi|^{p^*}}{\lg^{q+1}(1+|\xi|)}$$
$$\text{for } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \text{ if } p = n\,.$$

4.2. Applications of Theorem 2

Let $p, q, r \in \mathbb{R}$ be such that $1 < r < p < n$, $1 < q < p$, and let $m < 0$. Let $a, \rho : \Omega \to [0, +\infty[$ be two measurable functions, $a \in L^{\frac{p}{p-r}}(\Omega)$ and $\rho > 0$ on a subset of Ω having positive measure.
We show that problem

$$\begin{cases} -\operatorname{div}\left((a(x)|\nabla u|^{r-2} + |\nabla u|^{p-2})\nabla u\right) = \frac{\rho(x)+|u|^{q-1}}{1+|\nabla u|} - |\nabla u|^{\frac{p}{p^{*r}}}|\lg(|\nabla u|)|^{\frac{m}{p}^*} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{61}$$

has a nontrivial solution $u \in W_0^{1,p}(\Omega)$, $u \ge 0$.

The functions

$$\Phi(x,\xi) = a(x)\frac{|\xi|^r}{r} + \frac{|\xi|^p}{p}, \quad \text{and} \quad \mathcal{A}(x,\xi) = \left(a(x)|\xi|^{r-2} + |\xi|^{p-2}\right)\xi$$

satisfy (31)–(34). Condition (38) holds as well, with

$$\rho_1(x) \equiv \rho(x), \quad g_1(|s|) = |s|^{q-1},$$
$$\rho_2(x) = -M^{\frac{p}{p^{*r}}}|\lg(|M|)|^{\frac{m}{p}^*}, \quad \text{for a suitable constant } M > 0,$$
$$g_2(|s|) \equiv 0 \quad \text{and } E(t) \approx t^{p^*}\lg^m(t), \text{for } t >> 1.$$

Thus, from Theorem 2, problem (61) has a nontrivial solution $u \in W_0^{1,p}(\Omega)$, $u \ge 0$.

4.3. Applications of Theorem 4

Let Ω be a bounded domain with a $C^{1,\alpha}$ boundary. Let $\gamma \ge \alpha$, $p > 1$ and $q \in \mathbb{R}$, satisfying $p + q - 1 > 0$. Let $h : \Omega \to [0, +\infty[$ be a measurable function, $h \in L^\infty(\Omega)$, $h \ne 0$ on a subset of Ω having finite measure, and let $g : \mathbf{R} \to \mathbb{R}$ be a continuous function such that $g(s) > 0$ for all $s \in [0, \bar{s}[$ and $g(\bar{s}) \equiv 0$. We consider the problem

$$\begin{cases} -\operatorname{div}\left(\|x\|^\gamma e^{|u|}|\nabla u|^{p-2}\lg^q(1+|u|)\nabla u\right) = (h(x) + |\nabla u|^p\lg^q(1+|\nabla u|))g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{62}$$

and prove that it has a nontrivial solution $u \in W_0^{1,A}(\Omega)$, $u \ge 0$. The Young function A is defined via

$$A'(t) = t^{p-1}\lg^q(1+t) \quad \text{for } t \ge 0\,. \tag{63}$$

The operator $\mathcal{A}$ and the function f, defined, respectively, as

$$\mathcal{A}(x,s,\xi) = \|x\|^\gamma e^{|s|}|\xi|^{p-2}\lg^q(1+|\xi|)\xi \quad \text{and}$$
$$f(x,s,\xi) = (h(x) + |\xi|^p\lg^q(1+|\xi|))g(s),$$

satisfy the hypotheses of Theorem 4. Furthermore, $\underline{u} \equiv 0$ and $\overline{u} = \bar{s}$ are a subsolution and a supersolution to (62), and $\underline{u} \equiv 0$ is not a solution. Thus, from Theorem 4, the problem that we are dealing with has a regular solution $u : \Omega \to [0, \bar{s}]$.

Funding: The author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This research is partially supported by the Ministry of Education, University and Research of Italy, Prin 2022 Nonlinear differential problems with applications to real phenomena (Project No. 2022ZXZTN2, CUP C53D23002600006).

Conflicts of Interest: The author declares no competing interests.

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