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On the Injectivity of Euler Integral Transforms with Hyperplanes and Quadric Hypersurfaces

Mattie Ji

Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912, USA; mattie_ji@brown.edu or mattie.hd.ji@gmail.com

Abstract: The Euler characteristic transform (ECT) is an integral transform used widely in topological data analysis. Previous efforts by Curry et al. and Ghrist et al. have independently shown that the ECT is injective on all compact definable sets. In this work, we first study the injectivity of the ECT on definable sets that are not necessarily compact and prove a complete classification of constructible functions that the Euler characteristic transform is not injective on. We then introduce the quadric Euler characteristic transform (QECT) as a natural generalization of the ECT by detecting definable shapes with quadric hypersurfaces rather than hyperplanes. We also discuss some criteria for the injectivity of QECT.

Keywords: Euler characteristic; Euler characteristic transform (ECT); topological data analysis (TDA); quadric hypersurfaces; Radon transform

MSC: 46M20; 52A22

1. Introduction

The Euler characteristic transform (ECT) is an integral transform in topological data analysis (TDA) introduced in by Turner and more authors in [1]. Since then, the ECT itself and its variants have been widely used in the applied sciences [2–8]. On a high level, the ECT takes in a shape S in $\mathbb{R}^n$, "scans" S through each direction $v \in \mathbb{S}^{n-1}$, and keeps track of the Euler characteristics of the sublevel sets of S. Concretely, the Euler characteristic transform of S may be formalized into a function $\mathrm{ECT}(S) : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{Z}$ defined by

$$(v, t) \mapsto \mathrm{ECT}(S)(v, t) = \chi(\{x \in S \mid x \cdot v \leq t\}),$$

where $\chi(\cdot)$ denotes the combinatorial Euler characteristic (see Definition 4). Note that $x \cdot v = t$ defines the equation of a hyperplane in $\mathbb{R}^n$.

In the conclusion of [9] by Curry and more authors, they posed the question of how the Euler characteristic transform (ECT) would behave on shapes cut out by quadraticequations rather than linear equations. Inspired by this question, we consider a "converse" of this question in this work—what if we replace hyperplanes in the ECT with quadric hypersurfaces?

The equation of a quadric hypersurface may be written as $x^T Ax + v \cdot x = t$, where A is a symmetric $n \times n$ real matrix and v is a vector in $\mathbb{R}^n$. Based on this notion, we can define the quadric Euler characteristic transform (QECT) of the shape S as a function given by $(A, v, t) \mapsto \mathrm{QECT}(S)(A, v, t) := \chi(\{x \in S \mid x^T Ax + x \cdot v \leq t\})$. This definition will be made more precise in Section 4. By extending the class of hyperplanes to quadric hypersurfaces, the hope is that the QECT would add an extra variable that takes into account of curvatures.

In this work, a central question we are interested in is the injectivity of the ECT and the QECT. Curry et al. [9] and Ghrist et al. [10] have independently shown that the ECT is injective on a "reasonable" class of compact shapes. Furthermore, the work in [10] showed

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that this injectivity result extends to finite sums of indicator functions on a collection of shapes (known as *constructible functions*) that are compactly supported.

We will first investigate the injectivity of the ECT on constructible functions that are not compactly supported. As we will see in Example 3 and Theorem 3 in Section 3, there are many pairs of constructible functions that the ECT is not injective on. We then extend the ECT to the QECT and discuss its injectivity in Theorems 4 and 5. Specifically, we will prove the following main results in this paper:

1. We completely classify all the pairs of constructible functions that the Euler characteristic transform is not injective on in Theorem 3.
2. We show that the transform $S \mapsto \{(A,t) \mapsto \mathrm{QECT}(S)(A,0,t)\}$ is injective up to sign in Theorem 4.
3. Suppose the classes of "reasonable" shapes (see Definition 1) we are considering are all contained in $B_R(0) := \{x \in \mathbb{R}^n \mid |x| \leq R\}$ for some $R \geq 0$. Let $||\cdot||_{op}$ denote the operator norm of a symmetric matrix. For a fixed A such that $||A||_{op} < \frac{1}{1+2R^2}$, we show in Theorem 5 that the transform $S \mapsto \{(v,t) \mapsto \mathrm{QECT}(S)(A,v,t)\}$ is injective. In particular, this serves as an interpolation between the injectivity of the ECT and Theorem 4 (see Remark 1). A similar statement holds for other definable norms with reasonable adjustments to the bounds.

These statements will be made more precise in their respective theorems.

Outline

This paper is organized as follows. In Section 2, we introduce the relevant backgrounds in o-minimal structures, Euler calculus, and the ECT. In Section 3, we discuss the injectivity of the ECT on all constructible functions, leading to a complete characterization of injectivity in Theorem 2. In Section 4, we extend the ECT to the QECT by considering quadric surfaces rather than hyperplanes in the sublevel sets of the integral transform and discuss several results on the injectivity of the QECT, leading to Theorems 4 and 5 in the end.

2. Background

2.1. O-Minimal Structures

O-minimal structures are widely used as the mathematical representation of a shape [9,11] in applied topology and topological data analysis. In the theoretical foundation of these applied areas, we often want to consider shapes that have some level of "tameness" to avoid pathological examples, and o-minimal structures offer one way to capture the idea of "tameness". We refer the reader to [12] for a comprehensive introduction to o-minimal structures.

Definition 1. *Let $\mathcal{O}_n$ be a collection of subsets of $\mathbb{R}^n$ and $\mathcal{O} = \{\mathcal{O}_n\}_{n\geq 1}$. We say that $\mathcal{O}$ is an o-minimal structure if it satisfies the following seven axioms:*

1. *The collection $\mathcal{O}_n$ is a Boolean algebra, meaning it is closed under finite unions and finite intersections.*
2. *If $A \in \mathcal{O}_n$, then $A \times \mathbb{R} \in \mathcal{O}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{O}_{n+1}$.*
3. *The subset $\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ belongs to $\mathcal{O}_n$ for $1 \leq i < j \leq n$.*
4. *The collection $\mathcal{O}$ is closed under all projections of the form*

$$\pi_n^i : \mathbb{R}^n \to \mathbb{R}^{n-1}, \pi_n^i(x_1,\ldots,x_n) := (x_1,\ldots,\hat{x_i},\ldots,x_n).$$

5. *The singleton set $\{r\}$ belongs to $\mathcal{O}_1$ for all $r \in \mathbb{R}$. The halfspace $\{(x,y) \in \mathbb{R}^2 \mid x < y\}$ belongs to $\mathcal{O}_2$.*
6. *The collection $\mathcal{O}_1$ is exactly the finite unions of points and open intervals.*
7. *The collection $\mathcal{O}$ contains all real algebraic sets.*

An element of $\mathcal{O}_n$ is called a definable set.

In this paper, we will fix an arbitrary *o*-minimal structure $\mathcal{O}_n$. Note that the collection of all real semi-algebraic sets is an example of an o-minimal structure. In fact, Definition 1 implies that any o-minimal structure has to contain all real semi-algebraic sets (see Remark 2.2 of [9]). We also want a notion of "definability" for functions between definable sets.

Definition 2. *Let $f : X \to Y$ be a function between definable sets.*

1. *The function f is called definable if its graph is a definable set.*
2. *If f is continuous and definable with a continuous and definable inverse, then f is called a definable homeomorphism, and X and Y are said to be definably homeomorphic.*
3. *If f is an integer valued function, then $f : X \to \mathbb{Z}$ is called a constructible function. Let $\mathrm{CF}(X)$ denote the space of constructible functions on X. Note that Definition 1(4) implies that the image of f is a discrete definable subset of $\mathcal{O}_1$ and is thus finite.*

When we later define the quadric Euler characteristic transform, we will want to consider a suitable norm on the space of symmetric $n \times n$ matrices. There are many choices of norms for matrices that are popular in machine learning, such as the Schatten norm, cut norms, and $L_{p,q}$ norms, or the operator norm (see [13]), so we need to consider norms that are compatible with our *o*-minimal structure.

Definition 3. *Let $(V, ||\cdot||)$ be a finite-dimensional normed real vector space. $||\cdot||$ is called a definable norm if the norm function $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ is definable. We denote $S^V := \{x \in V \mid ||x|| = 1\}$ as the unit sphere with respect to $||\cdot||$. Note that S^V is a compact definable set when $||\cdot||$ is a definable norm.*

In particular, we will use $|\cdot|$ to denote the usual ℓ_2 norm on $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ to denote the usual unit sphere in $\mathbb{R}^n$ with respect to the ℓ_2 norm.

Example 1. *Here are two examples of definable norms that will be relevant to our discussions in Section 4.*

Let $|\cdot|$ be the ℓ_2 norm on $\mathbb{R}^n$. The graph of $|\cdot| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the set

$$\{(x_1, \ldots, x_n, y) \in \mathbb{R}^n \times \mathbb{R} \mid x_1^2 + \ldots + x_n^2 = y^2 \text{ and } y \geq 0\},$$

which is semi-algebraic and hence definable.

Let V be the vector space of $n \times n$ symmetric real matrices and $||\cdot||_{\mathrm{op}}$ be the operator norm on V. The graph of $||\cdot||_{op} : V \to \mathbb{R}_{\geq 0}$ may be realized as an axis-aligned projection of the following definable set

$$\{(A, \lambda_1, \ldots, \lambda_n) \in V \times \mathbb{R}^n \mid \det(A - \lambda_i I) = 0, e_i(\lambda_1, \ldots, \lambda_n) = \frac{a_i(A)}{a_n(A)}$$

$$\text{for all } i = 1, \ldots, n \text{ and } \lambda_1^2 \leq \ldots \leq \lambda_n^2\},$$

where e_i denotes the i-th elementary symmetric polynomial, and $a_i(A)$ denotes the i-th coefficient of the characteristic polynomial $\det(A - xI)$. Note that a_i is a polynomial function on the components of the matrix A. Thus, the operator norm is definable.

We also state the following technical lemma on o-minimal structures that will be used later in the paper.

Lemma 1 (Rephrased from Proposition 2.10 of Chapter 4 of [12])**.** *Let $S \subseteq \mathbb{R}^{m+n}$ be a definable set. For any $a \in \mathbb{R}^m$, define $S_a := \{x \in \mathbb{R}^n | (a, x) \in S\}$. Then, $\chi(S_a)$ takes only finitely many values as a runs through $\mathbb{R}^m$, and for each integer e the set $\{a \in \mathbb{R}^m : \chi(S_a) = e\}$ is definable.*

2.2. Euler Calculus and the Euler Characteristic Transform

Let $S \subseteq \mathbb{R}^n$ be a definable set. The cell decomposition theorem ([12] Chapter 3, Theorem 2.11) asserts that there is a disjoint partition of S into open-cells $C_1, \ldots, C_N$ such that each C_i is definably homeomorphic to $\mathbb{R}^{a_i}$ for some a_i.

Definition 4. *Choose S as above; the Euler characteristic of S is $\chi(S) := \sum_{i=1}^{N}(-1)^{a_i}$. This quantity is independent of the cell partition and is preserved under definable homeomorphisms (see Chapter 4 of [12]). If S is a locally compact definale set, $\chi(S)$ would be the alternating sum of the ranks of cohomology with compact support (see Lemma 8.5 of [14]).*

Euler calculus is an integral calculus based on the observation that the *Euler characteristic* $\chi(\cdot)$ exhibits a finitely additive property similar to a signed measure,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B),$$

on definable sets A and B. The field seeks to develop a theory of integration for constructible functions, similar to how regular calculus developed a theory of integration for measurable functions. We refer the reader to [14,15] for more details on Euler calculus.

Definition 5. *Let X be a definable function and $f : X \to \mathbb{Z}$ be a constructible function. The Euler integral of f is*

$$\int_X f(x)d\chi(x) := \sum_{n=-\infty}^{\infty} n\chi(\{x \in X \mid f(x) = n\}).$$

Note that this quantity is well-defined by the discussions in Definition 2(3). The Euler characteristic transform of f is defined as

$$\mathrm{ECT}(f) : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{Z}, (v, t) \mapsto \mathrm{ECT}(f)(v, t) = \int_X f(x)\mathbb{1}_{X_t^v}(x)d\chi(x),$$

where X_t^v denotes the set $\{x \in X \mid v \cdot x \leq t\}$. For a definable subset $S \subseteq X$, we use $\mathrm{ECT}(S)$ to indicate the Euler characteristic transform of the indicator function on S.

We refer the reader to [16] for a general review of the Euler characteristic transform. Here is an example of computation with the Euler characteristic transform.

Example 2. *Take $B_1(0) = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ to be the closed unit ball with respect to the ℓ_2 norm. For any $v \in \mathbb{S}^{n-1}$, we have that $\mathrm{ECT}(B_1(0)(v, t)) = 1$ if $-1 \leq t$ and $\mathrm{ECT}(B_1(0)(v, t)) = 0$ if $t < -1$.*

Euler calculus also enjoys its version of Fubini's Theorem.

Theorem 1 (Fubini's Theorem for Euler integrals)**.** *Let $f : X \to Y$ be a definable function between definable sets and $h : X \to \mathbb{Z}$ be a constructible function. The following equality is true*

$$\int_X h(x)d\chi(x) = \int_Y \left(\int_{f^{-1}(y)} h(x)d\chi(x) \right) d\chi(y).$$

A proof of Theorem 1 may be found in Theorem 4.5 of [14]. Note that while the authors assumed h to be compactly supported, the condition is not strictly required in the proof (see Page 5 of [14]). Theorem 1 of [15] presents an explicit proof of Theorem 1 for the case of semi-algebraic sets without the assuming h to be compactly supported, and the case for a general o-minimal structure follows similarly.

For convenience, we will also briefly explain what a Radon transform is and how it relates to the Euler characteristic transform.

Definition 6. *Let (X, Y) be a pair of definable sets and $K \in \mathrm{CF}(X \times Y)$ (known as a kernel function). The Radon transform is a function $R_K : \mathrm{CF}(X) \to \mathrm{CF}(Y)$ defined by*

$$(R_K h)(y) = \int_X h(x)K(x,y)d\chi(x), h(x) \in \mathrm{CF}(X), \text{ for all } y \in Y.$$

In particular, when $Y = \mathbb{S}^{n-1} \times \mathbb{R}$ and K is the indicator function on $\{(x, v, t) \in X \times Y \mid v \cdot x \leq t\}$, then R_K is the ECT.

3. Euler Characteristic Transform with Hyperplanes

In [10], the authors proved the following result on the injectivity of the Euler characteristic transform based on the Schapira inversion formula in [17].

Theorem 2 (Theorem 1 of [10], modified)**.** *Let $X = \mathbb{R}^n$, $Y = \mathbb{S}^{n-1} \times \mathbb{R}$, $K \in \mathrm{CF}(X \times Y)$ be the indicator function on $\{(x, v, t) \in X \times Y \mid v \cdot x \leq t\}$ and $K' \in \mathrm{CF}(Y \times X)$ be the indicator function on $\{(v, t, x) \in Y \times X \mid v \cdot x \geq t\}$. For any $h \in \mathrm{CF}(X)$, the following formula holds*

$$(R_{K'} \circ R_K)h = (\mu - \lambda)h + \lambda(\int_X h d\chi)\mathbb{1}_X, \tag{1}$$

where $\mu = \chi(\mathbb{S}^{n-1})$ and $\lambda = 1$. Moreover, when restricted to the class of compactly supported functions on X, $R_K = \mathrm{ECT}$ is injective.

In the original proof by the authors of [10], this formula is only stated in the case where h is compactly supported. However, the formula still holds when h is not compactly supported. Please see Appendix A for a proof of Equation (1) without assuming that h is compactly supported. While the ECT is injective on compactly supported constructible functions, it is not injective on $\mathrm{CF}(\mathbb{R}^n)$. We illustrate this with the following counter-example.

Example 3. *Let $X = \mathbb{R}^n$, $S_1 = \mathbb{R}^n$, and $S_2 = \emptyset$, then $\mathrm{ECT}(S_1)(v, t) = \mathrm{ECT}(S_2)(v, t)$ for all $(v, t) \in \mathbb{S}^n \times \mathbb{R}$. Indeed, the set $\{x \in \mathbb{R}^n \mid x \cdot v = t\}$ is definably homeomorphic to $\mathbb{R}^{n-1}$, and the set $\{x \in \mathbb{R}^n \mid x \cdot v > t\}$ is definably homeomorphic to $\mathbb{R}^n$. Hence, the additivity of Euler characteristic implies that*

$$\mathrm{ECT}(S_1)(v, t) = \chi(\{x \in \mathbb{R}^n \mid x \cdot v \leq t\}) = \chi(\mathbb{R}^{n-1}) + \chi(\mathbb{R}^n) = 0.$$

Hence $\mathrm{ECT}(S_1)$ is the zero function. On the other hand, the Euler characteristic of the empty set is always zero, so $\mathrm{ECT}(S_2)$ is also the zero function.

Fortunately, we can classify how non-injective is the ECT with the following theorem. From the theorem, we will also obtain a corollary that shows Example 3 is the only such counter-example for the case of definable sets.

Theorem 3. *Let $f, g : \mathbb{R}^n \to \mathbb{Z}$ be constructible functions. $\mathrm{ECT}(f) = \mathrm{ECT}(g)$ if and only if there exists some $c \in \mathbb{Z}$ such that*

$$f(x) = \sum_{n=-\infty}^{+\infty} n\mathbb{1}_{\{f^{-1}(n)\}}(x) \text{ and } g(x) = \sum_{n=-\infty}^{+\infty} (n+c)\mathbb{1}_{\{f^{-1}(n)\}}(x).$$

In particular, suppose $f(x) = \mathbb{1}_{S_1}(x)$ and $g(x) = \mathbb{1}_{S_2}(x)$ for distinct definable sets $S_1, S_2 \subseteq \mathbb{R}^n$ and $\mathrm{ECT}(f) = \mathrm{ECT}(g)$, then $S_1 = \mathbb{R}^n$ and $S_2 = \emptyset$ up to renaming of variables.

Proof. Suppose $\mathrm{ECT}(f) = \mathrm{ECT}(g)$, then Equation (1) implies that there exists integers $\mu \neq \lambda$ such that

$$(\mu - \lambda)f(x) + \lambda \int_{\mathbb{R}^n} f(x)d\chi = (\mu - \lambda)g(x) + \lambda \int_{\mathbb{R}^n} g(x)d\chi,$$

for all $x \in \mathbb{R}^n$. Hence, the difference $f(x) - g(x)$ is a constant integer, say c, and may be expressed as

$$g(x) - f(x) = c \coloneqq \frac{\lambda}{\mu - \lambda}\left(\int_{\mathbb{R}^n} f(x)d\chi - \int_{\mathbb{R}^n} g(x)d\chi\right). \tag{2}$$

Since the images of constructible functions are finite, we can write $f(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x)$ such that $A_i = f^{-1}(a_i)$ and a_i ranges through the image of $f(x)$. Similarly, we can write $g(x) = \sum_{j=1}^{m} b_j \mathbb{1}_{B_j}(x)$ such that $B_j = g^{-1}(b_j)$ and b_j ranges through the image of $g(x)$.

Let $x \in A_i$. Equation (2) implies that $f(x) - g(x) = c$. On the other hand $f(x) = a_i$, so $g(x) = b_j = a_i + c$ for some b_j in the image of $g(x)$. Thus, the set function $\{a_1, \ldots, a_n\} \mapsto \{b_1, \ldots, b_m\}$ by $a_i \mapsto a_i + c$ is a well-defined injective set function. Similarly, the set function $\{b_1, \ldots, b_m\} \mapsto \{a_1, \ldots, a_n\}$ is also a well-defined inverse of the previous set function. Thus, we conclude that $n = m$ and $b_i = a_i + c$ for $i = 1, \ldots, n$ up to reordering.

Now for any $x \in A_i$, $g(x) = c + f(x) = c + a_i = b_i$, so $x \in B_i$. Similarly for any $x \in B_i$, $f(x) = g(x) - c = a_i$, thus $x \in A_i$. Hence $A_i = B_i$. Thus, we conclude that

$$f(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x) \text{ and } g(x) = \sum_{i=1}^{n} (a_i + c)\mathbb{1}_{A_i}(x).$$

This concludes the proof of the "only if" direction.

Conversely, suppose $f(x) = \sum_{n=-\infty}^{+\infty} n\mathbb{1}_{\{f^{-1}(n)\}}(x)$ and $g(x) = \sum_{n=-\infty}^{+\infty}(n+c)\mathbb{1}_{\{f^{-1}(n)\}}(x)$, then for any $(\nu, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$, we will compute the difference of their respective Euler characteristic transforms.

$$\begin{aligned}
\mathrm{ECT}(g)(\nu,t) - \mathrm{ECT}(f)(\nu,t) &= \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}^n} (n+c)\mathbb{1}_{\{f^{-1}(n)\}\cap\{x\cdot\nu\leq t\}}(x)d\chi \\
&\quad - \int_{\mathbb{R}^n} n\mathbb{1}_{\{f^{-1}(n)\}\cap\{x\cdot\nu\leq t\}}(x)d\chi \\
&= \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}^n} (n+c-n)\mathbb{1}_{\{f^{-1}(n)\}\cap\{x\cdot\nu\leq t\}}(x)d\chi \\
&= c \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}^n} \mathbb{1}_{\{f^{-1}(n)\}\cap\{x\cdot\nu\leq t\}}(x)d\chi \\
&= c \int_{\mathbb{R}^n} \mathbb{1}_{\{x\cdot\nu\leq t\}}(x)d\chi \\
&= c\,\mathrm{ECT}(\mathbb{R}^n)(\nu,t) \\
&= 0,
\end{aligned}$$

where the fourth line follows from the fact that the sets $\{f^{-1}(n)\}_{n\in\mathbb{Z}}$ form a finite (disregarding empty sets) partition of $\mathbb{R}^n$, and the sixth line follows from Example 3.

Finally, we will focus on the specific case that $f(x) = \mathbb{1}_{S_1}(x)$ and $g(x) = \mathbb{1}_{S_2}(x)$. Without loss of generality, we will assume that S_1 is non-empty. Since $\mathrm{ECT}(f) = \mathrm{ECT}(g)$, there exists some $c \in \mathbb{Z}$ such that $g(x) = 1 + c$ for all $x \in S_1$ and $g(x) = 0 + c$ for all $x \notin S_1$.

Since S_1 is not empty, then let y be any point in S_1. The equality $f(y) = 1$ implies that $g(y) = 1 + c$. If $g(y) = 1$, then $c = 0$ and $g(x)$ becomes the indicator function on S_1, which is a contradiction to the assumption that $f(x) \neq g(x)$. If $g(y) = 1 + c = 0$, then it follows that $c = -1$ and $g(x) = -1$ for all $x \notin S_1$. Since $g(x)$ takes values only between 0 and 1, this can occur only when $S_1 = \mathbb{R}^n$.

Thus, $S_1 = \mathbb{R}^n$ and $f(x)$ is the constant function with value 1. Since $c = -1$, $g(x)$ is the constant function with value 0, which implies that S_2 is the empty set. □

4. Euler Characteristic Transform with Quadric Hypersurfaces

Before going into the quadric Euler characteristic transform specifically, we will first discuss some results on Radon transforms in Section 4.1 that will be useful in Section 4.2 (and the Appendix A).

4.1. Generalized Kernel Spaces

Here is the general setup we will consider.

Definition 7. *Let $X \subseteq \mathbb{R}^n$ be a definable set, $P \subseteq \mathbb{R}^k$ be a compact definable set (called the "parameter space"), and $f : X \times P \to \mathbb{R}$ a definable function.*

1. *We define $K_f(x,(\xi,t)) \in \mathrm{CF}(X \times P \times \mathbb{R})$ as the indicator function on $\{(x,(\xi,t)) \in X \times P \times \mathbb{R} \mid f(x,\xi) \leq t\}$ (the kernel function).*
2. *We define $K'_f((\xi,t),x) \in \mathrm{CF}(P \times \mathbb{R} \times X)$ as the indicator function on $\{(\xi,t),x) \in P \times \mathbb{R} \times X \mid f(x,\xi) \geq t\}$ (the dual kernel function).*
3. *We also define the fiber $K_{x,f} = \{(\xi,t) \in P \times \mathbb{R} \mid f(x,\xi) \leq t\}$ and the dual fiber $K'_{x',f} = \{(\xi,t) \in P \times \mathbb{R} \mid f(x',\xi) \geq t\}$.*

Given a constructible function $h : X \to \mathbb{Z}$, we are interested in what the function $(R_{K'_f} \circ R_{K_f})h$ is, to prove injectivity results similar to that of Theorem 2. We first prove a technical lemma.

Lemma 2. *Let $x, x' \in X$.*

1. *We have the equality $\chi(K_{x,f} \cap K'_{x',f}) = \chi(\{\xi \in P \mid f(x',\xi) - f(x,\xi) \geq 0\})$.*
2. *If $x = x'$, then $\chi(K_{x,f} \cap K'_{x',f}) = \chi(P)$.*
3. *As (x,x') ranges through $X \times X$, the function $(x,x') \mapsto \chi(K_{x,f} \cap K'_{x',f})$ can only take on finitely many values $c_1, \ldots, c_n$. Furthermore, the preimage S_i of each c_i is a definable subset of $X \times X$.*

Proof. For Lemma 2(1), we first rewrite the set $K_{x,f} \cap K'_{x',f}$ as follows.

$$\begin{aligned} K_{x,f} \cap K'_{x',f} &= \{(\xi,t) \in P \times \mathbb{R} \mid t \geq f(x,\xi) \text{ and } f(x',\xi) \geq t\} \\ &= \{(\xi,t) \in P \times \mathbb{R} \mid f(x,\xi) \leq t \leq f(x',\xi)\} \\ &= \{(\xi,t) \in P \times \mathbb{R} \mid t \in [f(x,\xi), f(x',\xi)]\}. \end{aligned}$$

By considering the definable homeomorphism $\phi : P \times \mathbb{R} \to P \times \mathbb{R}$ by $\phi(\xi,t) = (\xi, t - f(x,\xi))$, we have that

$$\begin{aligned} \chi(K_{x,f} \cap K'_{x',f}) &= \chi(\phi(K_{x,f} \cap K_{x',f})) \\ &= \chi(\{(\xi,t) \in P \times \mathbb{R} \mid t \in [0, f(x',\xi) - f(x,\xi)]\}). \end{aligned}$$

Since P is compact and definable, the set $A \coloneqq \{(\xi,t) \in P \times \mathbb{R} \mid t \in [0, f(x',\xi) - f(x,\xi)]\}$ is compact and definable. Define the straight-line homotopy $H : A \times [0,1] \to A$ as $H((\xi,t),s) = (\xi,(1-s)t)$ for all $((\xi,t),s) \in A \times [0,1]$; this produces a deformation retract of A onto the set $\{\xi \in P \mid f(x',\xi) - f(x,\xi) \geq 0\}$, which preserves the Euler characteristic because both sets are compact and definable. The proof of Lemma 2(1) is thus completed.

For Lemma 2(2), $x = x'$ implies that $f(x',p) - f(x,p) = 0$ for any $p \in P$. It then follows from Lemma 2(1) that $\chi(K_{x,f} \cap K'_{x,f}) = \chi\{p \in P \mid 0 = 0\} = \chi(P)$.

For Lemma 2(3), we implement Lemma 1 as follows. We define S as the definable set

$$S := \{(x, x', \xi) \in X \times X \times P \subseteq \mathbb{R}^{2n+k} \mid f(x', \xi) - f(x, \xi) \geq 0\}.$$

In this case, $S_{(x,x')} = \{\xi \in \mathbb{R}^k \mid f(x', \xi) - f(x, \xi) \geq 0\}$. Thus, Lemma 2(3) follows directly from Lemma 1. □

It is not generally true that for $x \neq x' \in P$, the value of $\chi(K_{x,f} \cap K'_{x',f})$ remains constant, as will be shown in the proof of Theorem 4. However, we can still compute the function $(R_{K'_f} \circ R_{K_f})h$.

Lemma 3. *Following the context of Definition 7 and Lemma 2(3) and fix $c_1 = \chi(P)$, then for any $x' \in X$,*

$$(R_{K'_f} \circ R_{K_f})h(x') = \chi(P) \int_X h(x) \mathbb{1}_{S_1}(x, x') d\chi(x) + \sum_{i=2}^{n} c_i \int_X h(x) \mathbb{1}_{S_i}(x, x') d\chi(x).$$

In particular, if $S_1 = \Delta$ is the diagonal of $X \times X$, then

$$(R_{K'} \circ R_K)h(x') = \chi(P)h(x') + \sum_{i=2}^{n} c_i \int_X h(x) \mathbb{1}_{S_i}(x, x') d\chi(x).$$

Proof. By Lemma 2(3), we may write $\chi(K_{x,f} \cap K'_{x',f}) = \sum_{i=1}^{n} c_i \mathbb{1}_{S_i}(x, x')$ as a function of x and x'. By Lemma 2, the diagonal $\Delta \subseteq X \times X$ is contained in exactly one of the S_i, say S_1. Then,

$$\begin{aligned}
(R_{K'_f} \circ R_{K_f})h(x') &= \int_{P \times \mathbb{R}} K'(y, x') \left[\int_X h(x) K(x, y) d\chi(x) \right] d\chi(y) \\
&= \int_X h(x) \left[\int_{P \times \mathbb{R}} K'_f(y, x') K_f(x, y) d\chi(y) \right] d\chi(x) \\
&= \int_X h(x) \chi(K_{x,f} \cap K'_{x',f}) \\
&= \int_X h(x) \sum_{i=1}^{n} c_i \mathbb{1}_{S_i}(x, x') d\chi(x) \\
&= c_1 \int_X h(x) \mathbb{1}_{S_1}(x, x') d\chi(x) + \sum_{i=2}^{n} c_i \int_X h(x) \mathbb{1}_{S_i}(x, x') d\chi(x) \\
&= \chi(P) \int_X h(x) \mathbb{1}_{S_1}(x, x') d\chi(x) + \sum_{i=2}^{n} c_i \int_X h(x) \mathbb{1}_{S_i}(x, x') d\chi(x),
\end{aligned}$$

where the second line follows from Theorem 1. □

4.2. Quadric Euler Characteristic Transform

Let V be the space of real $n \times n$ symmetric matrices equipped with a definable norm $|| \cdot ||_V$ (recall the notations in Definition 3). Recall that a general quadric surface is given by

$$x^T Ax + v \cdot x = t$$

where $A \in V$, $v \in \mathbb{R}^n$, and $t \in \mathbb{R}$. Then it would seem that a natural definition (that turns out to be not desirable) of QECT on a constructible function $f : \mathbb{R}^n \to \mathbb{Z}$ would be

$$\mathrm{QECT}(f) : S^V \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{Z}, \mathrm{QECT}(f)(A, v, t) = \int_X f(x) \mathbb{1}_{X_t^{A,v}}(x) d\chi(x),$$

where $X_t^{A,v}$ denotes the set $\{x^T Ax + v \cdot x \leq t\}$.

There is a question of whether our domain of choice is the best choice of domain. On one hand, this seems like a natural thematic generalization of the ECT. However, there are no choices of $A \in S^V$ such that $||A||_V = 0$, so we cannot recover the ECT from this definition of the QECT.

The domain of $\mathrm{ECT}(f)$ is $\mathbb{S}^{n-1} \times \mathbb{R}$ to avoid the degenerate case of the zero vector. Thus, what our more general QECT wants is to consider the case where A and v are both not identically zero. Thus, we adjust our definition to the following.

Definition 8. *Let W denote the space $V \times \mathbb{R}^n$. We define the norm $||\cdot||_W$ on W as*

$$||(A, v)||_W = ||A||_V + |v| \text{ for all } (A, v) \in W = V \times \mathbb{R}^n.$$

Let $X \subseteq \mathbb{R}^n$ be a definable set and $f : X \to \mathbb{Z}$ be a constructible function; the quadric Euler characteristic transform of f is the function $\mathrm{QECT}(f) : S^W \times \mathbb{R} \to \mathbb{Z}$ defined by

$$\mathrm{QECT}(f)(A, v, t) = \int_X \left(f(x) \mathbb{1}_{X_t^{A,v}}(x) \right) d\chi(x).$$

Example 4. *Let $S_1 = \mathbb{R}^n$, $S_2 = \emptyset$, and I be the $n \times n$ identity matrix. We have that $\mathrm{QECT}(S_1)(I, 0, t) \neq \mathrm{QECT}(S_2)(I, 0, t)$. Hence, the QECT can tell the difference between S_1 and S_2 compared to Example 3.*

Now we will analyze a few properties of the QECT. First of all, when $v = 0$ is fixed to be the zero vector, we note that the function $f \mapsto \{(A, t) \mapsto \mathrm{QECT}(f)(A, 0, t)\}$, which we will refer to as $\mathrm{QECT}(-, 0, -) : \mathrm{CF}(\mathbb{R}^n) \to \mathrm{CF}(S^V \times \mathbb{R})$, is not injective.

Example 5. *Let $p \in \mathbb{R}^n$ be the vector whose components are all unity, $f(x) = \mathbb{1}_p(x)$, and $g(x) = \mathbb{1}_{-p}(x)$. We have that $\mathrm{QECT}(f)(A, 0, t) = \mathrm{QECT}(g)(A, 0, t)$ for all $(A, t) \in S^V \times \mathbb{R}$.*

However, we can see that the failure to detect signs in Example 5 is the only such locus of non-injectivity with the following theorem.

Theorem 4. *Let $v = 0$ be fixed. The transform $\mathrm{QECT}(-, 0, -) : \mathrm{CF}(\mathbb{R}^n) \to \mathrm{CF}(S^V \times \mathbb{R})$ is "injective up to sign". More precisely, let $h : X \to \mathbb{R}^n$. We obtain an inversion formula reminiscent of the Schapira inversion formula,*

$$(R_{K'_f} \circ R_{K_f} h)(x') = (\mu - \lambda) \sum_{z \in \{+x', -x'\}} h(z) + \lambda \left(\int_X h d\chi \right) \mathbb{1}_X,$$

where μ and λ are distinct integers.

Proof. We will use Lemma 3 to prove this theorem. Since $v = 0$, $||A||_V = 1 - |v| = 1$. Thus, the function $(A, t) \mapsto \mathrm{QECT}(f)(A, 0, t)$ has domain $S^V \times \mathbb{R}$. Following the setup of Definition 7, we choose $X = \mathbb{R}^n$, $P = S^V$, and $f : X \times P \to \mathbb{R}$ to be the function $(x, A) \mapsto x^T A x$. By Lemma 2 and the property that A is symmetric,

$$\begin{aligned} \chi(K_{x,f} \cap K'_{x',f}) &= \chi(\{A \in P \mid (x')^T A (x') - (x)^T A (x) \geq 0\}) \\ &= \chi(\{A \in P \mid (x' + x)^T A (x' - x) \geq 0\}). \end{aligned}$$

If $x = x'$ or $x = -x'$, then $(x' + x)^T A (x' - x) = 0$. Hence, $\chi(K_{x,f} \cap K'_{x',f}) = \chi(S^V)$.

Otherwise, suppose $x \notin \{\pm x'\}$, then consider the function $\phi : V \to \mathbb{R}$ given by $\phi(A) = (x' + x)^T A (x' - x)$. Since $(x + x')$ and $(x' - x)$ are both non-zero vectors, ϕ is a surjective linear transformation, and hence $\ker(\phi)$ is a codimension 1 linear subspace of V. $\ker(\phi)$ inherits a natural norm from V and hence $S^V \cap \ker(\phi) = S^{\ker(\phi)}$.

Since all norms on a finite-dimensional real vector space are equivalent, the map $\psi : S^V \to \mathbb{S}^{\dim V-1}$ by $\psi(x) = \frac{x}{|x|}$ is a definable homeomorphism whose restriction to $S^{\ker(\phi)}$ gives a homeomorphism between $S^{\ker(\phi)}$ and the ℓ_2 unit sphere of $\ker(\phi)$.

By Alexander duality, $\tilde{H}^0(S^V \setminus S^{\ker(\phi)}) \cong \tilde{H}^{\dim V-1-1}(S^{\ker(\phi)}) \cong \mathbb{Z}$, so $S^V \setminus S^{\ker(\phi)}$ has two connected components. Since ϕ is an odd function, we will denote the two connected components as S^V_+ and S^V_- corresponding to the locus where ϕ is positive and negative, respectively. Hence, the map $\psi : S^V \to \mathbb{S}^{\dim V-1}$ brings $\ker(f) \cup S^V_+$ homeomorphically to a closed hemisphere of $\mathbb{S}^{\dim V-1}$. Thus, the set $\ker(f) \cup S^V_+$ is compact and contractible, so we conclude that $\chi(K_{x,f} \cap K'_{x',f}) = 1$.

Let $\mu = \chi(P)$ and $\lambda = 1$. By Lemma 3, we can write

$$(R_{K'_f} \circ R_{K_f})h(x') = \mu \int_{\mathbb{R}^n} h(x)\mathbb{1}_{\pm\Delta}(x,x')d\chi(x) + \lambda \int_{\mathbb{R}^n} h(x)\mathbb{1}_{\{X\times X-\pm\Delta\}}(x,x')d\chi(x),$$

where $\mathbb{1}_{\pm\Delta}(x,x') = 1$ if $x \in \{\pm x'\}$ and is 0 otherwise. Finally, we can furthermore simplify the expression as

$$\begin{aligned}(R_{K'_f} \circ R_{K_f})h(x') &= \mu \int_{\{\pm x'\}} h(x)d\chi(x) + \lambda \int_{\mathbb{R}^n-\{\pm x'\}} h(x)d\chi(x)\\ &= (\mu-\lambda)\int_{\{\pm x'\}} h(x)d\chi(x) + \lambda \int_{\mathbb{R}^n} h(x)d\chi(x)\\ &= (\mu-\lambda)\sum_{z\in\{\pm x'\}} h(z) + \lambda \int_{\mathbb{R}^n} h(x)d\chi(x).\end{aligned}$$

This concludes the proof of Theorem 4. □

Theorem 4 examined what happens to the QECT when its vector component is fixed. We are also interested in what happens to the QECT when its matrix component is fixed.

Theorem 5. *Let $B_R(0)$ denote the closed ball of radius R for some $R \geq 0$ centered at the origin. Fix the norm on V to be the operator norm $||\cdot||_{op}$, and let $A \in V$ be fixed such that $||A||_{op} < \frac{1}{1+2R^2}$. Define the transform $\mathrm{QECT}(A,-,-) : \mathrm{CF}(B_R(0)) \to \mathrm{CF}(P\times\mathbb{R})$ by $f \mapsto \{(v,t) \mapsto \mathrm{QECT}(f)(A,v,t)\}$, where P is the sphere of radius $1-||A||_{op}$ in $\mathbb{R}^n$. Then, $\mathrm{QECT}(A,-,-)$ is injective.*

Remark 1. *Theorem 5 interpolates between Theorems 2 and 4 in the following sense. When $||A||_{op} = 1$, the inequality $1 < \frac{1}{1+2R^2}$ does not hold for any value of R. This reflects the fact that $(A,t) \mapsto \mathrm{QECT}(A,0,t)$ is only injective up to signs in Theorem 4 and Example 5. When $||A||_{op} = 0$, A is the zero matrix and the* QECT *becomes the usual* ECT*, which is injective by Theorem 2 no matter what R is. This reflects the fact that the inequality $0 < \frac{1}{1+2R^2}$ is satisfied for any value of R.*

The requirement for the norm on V to be the operator norm is not strictly necessary. The same statement holds for definable norm $||\cdot||_V$ on V that satisfies the property $||x^TAx||_V \leq ||A||_V|x|^2$. Common examples include the Frobenius norm and the nuclear norm. Furthermore, by adjusting the constant $\frac{1}{1+2R^2}$ appropriately, similar statements for any definable norm on V will hold.

Now we will prove Theorem 5.

Proof of Theorem 5. We will use Lemma 3 to prove this theorem. Following the setup of Definition 7, we choose $X = B_R(0)$, $P = \{v \in \mathbb{R}^n \mid |v| = 1-||A||_{op}\}$, and $f : X\times P \to \mathbb{R}$ to be the function $(x,v) \mapsto x^TAx + v\cdot x$. By Lemma 2 and the property that A is symmetric,

$$\begin{aligned}\chi(K_{x,f} \cap K'_{x',f}) &= \chi(\{v \in P \mid (x')^TA(x') + v\cdot(x') - (x)^TA(x) - v\cdot x \geq 0\})\\ &= \chi(\{v \in P \mid (x'+x)^TA(x'-x) + v\cdot(x'-x) \geq 0\}).\end{aligned}$$

If $x = x'$, we have that $\chi(K_{x,f} \cap K'_{x',f}) = \chi(P)$. Otherwise, if $x \neq x'$, we can consider the function $\phi : \mathbb{R}^n \to \mathbb{R}$ by $\phi(v) = (x' + x)^T A(x' - x) + v \cdot (x' - x)$. In this case, we have that $\chi(K_{x,f} \cap K'_{x',f}) = \chi(\{v \in P \mid \phi(v) \geq 0\})$.

It remains for us to determine $\chi(\{v \in P \mid \phi(v) \geq 0\})$. Since $x' - x$ is not the zero vector, ϕ is a surjective affine map and $\ker(\phi)$ is a hyperplane in $\mathbb{R}^n$. Furthermore, the vector $\frac{(x'-x)}{|x-x'|}$ is a unit normal vector to $\ker(\phi)$, and $\ker(\phi)$ may be written as the sum

$$\ker(\phi) = \ker\{v \mapsto v \cdot (x' - x)\} - \frac{(x' + x)^T A(x' - x)}{|x - x'|}(x' - x).$$

This is because for any $v - \frac{(x'+x)^T A(x'-x)}{|x-x'|}(x - x')$ in the right hand side above,

$$\begin{aligned}
\phi(v - \frac{(x' + x)^T A(x' - x)}{|x - x'|}(x' - x)) &= (x' + x)^T A(x' - x) \\
&\quad + \left[v - \frac{(x' + x)^T A(x' - x)}{|x - x'|}(x - x')\right] \cdot (x' - x) \\
&= (x' + x)^T A(x' - x) + 0 - \frac{(x' + x)^T A(x' - x)}{|x - x'|}|x' - x| \\
&= (x' + x)^T A(x' - x) - (x' + x)^T A(x' - x) \\
&= 0.
\end{aligned}$$

Now we observe that

$$\begin{aligned}
|((x' + x)^T A(x' - x)| &= |(x')^T A(x') - (x)^T A(x)| \\
&\leq |x^T Ax| + |(x')^T A(x')| \\
&\leq ||A||_{op}|x|^2 + ||A||_{op}|x'|^2 \\
&\leq ||A||_{op}(2R^2) \\
&< \frac{2R^2}{1 + 2R^2} \\
&= 1 - \frac{1}{1 + 2R^2} \\
&< 1 - ||A||_{op},
\end{aligned}$$

where the fifth and last line both follow from the assumption $||A||_{op} < \frac{1}{1+2R^2}$. The space $\{v \in P \mid \phi(v) \geq 0\}$ is one of the two sides of the sphere P divided by the hyperplane $\ker(\phi)$. Since the radius of P is $1 - ||A||_{op}$ and $|(x + x')^T A(x - x')| < 1 - ||A||_{op}$, the normal vector $(x + x')^T A(x - x')$ supporting the hyperplane $\ker(\phi)$ has length strictly less than the radius of P. This means that the space $\{v \in P \mid \phi(v) \geq 0\}$ is a compact, contractible, and definable subset of P. It then follows that

$$\chi(K_{x,f} \cap K'_{x',f}) = \chi(\{v \in P \mid \phi(v) \geq 0\}) = 1.$$

Thus, by Lemma 3, for all $x' \in \mathbb{R}^n$,

$$\begin{aligned}
R_{K'_f} \circ R_{K_f} h &= [1 + (-1)^{n-1}]h(x') + \int_{\mathbb{R}^n} h(x) \mathbb{1}_{\{(x,x') \notin \Delta\}} d\chi(x) \\
&= [1 + (-1)^{n-1}]h(x') + \int_{\mathbb{R}^n - \{x'\}} h(x) d\chi(x) \\
&= [1 + (-1)^{n-1}]h(x') + \int_{\mathbb{R}^n} h(x) d\chi(x) - \int_{\{x'\}} h(x) d\chi(x) \\
&= [(-1)^{n-1}]h(x') + \int_{\mathbb{R}^n} h(x) d\chi(x).
\end{aligned}$$

Since $h(x) \in \mathrm{CF}(B_R(0))$ has compact support, QECT would determine the value of $\int_{\mathbb{R}^n} h(x)d\chi(x)$. Thus, the transform $\mathrm{QECT}(A,-,-) : \mathrm{CF}(B_R(0)) \to \mathrm{CF}(P \times \mathbb{R})$ is injective. □

5. Conclusions

In this paper, we completely classified the pairs of constructible functions that the Euler characteristic transform (ECT) is not injective on (Theorem 2). We then proposed the quadric Euler characteristic transform (QECT) as an algebraic generalization of the ECT. From here, we derived two injectivity results for special cases of the QECT in Theorems 4 and 5.

While we only considered quadratic polynomials in this paper, it would be interesting to investigate similar transforms using higher-degree polynomials. It also would be interesting to see a computational implementation of the quadric Euler characteristic transform and numerical experiments around this integral transform.

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Abbreviations

The following abbreviations are used in this manuscript:

ECT	Euler characteristic transform
QECT	Quadric Euler characteristic transform
TDA	Topological data analysis
CF	Constructible functions
$\|\|\cdot\|\|_{op}$	Operator norm
$\|\cdot\|$	ℓ_2 norm
V	The space of real $n \times n$ symmetric matrices
W	$V \times \mathbb{R}^n$

Appendix A. Inversion Formula of ECT without Compact Support

Here, we reprove the inversion formula of ECT in [10] without the assumption that $h : \mathbb{R}^n \to \mathbb{Z}$ is compactly supported. The proof will be similar to that of [10].

Proof of Equation (1). Following the setup of Definition 7, we choose $X = \mathbb{R}^n$, $P = \mathbb{S}^{n-1}$, and $f : X \times P \to \mathbb{R}$ to be the function $(x, v) \mapsto x \cdot v$. By Lemma 2, we can compute that

$$\begin{aligned}
\chi(K_{x,f} \cap K'_{x'}, f) &= \chi(\{v \in \mathbb{S}^{n-1} \mid f(x', v) - f(x, v) \geq 0\}) \\
&= \chi(\{v \in \mathbb{S}^{n-1} \mid (x' - x) \cdot v \geq 0\}) \\
&= \begin{cases} \chi(\mathbb{S}^{n-1}), x = x' \\ \chi(\mathbb{S}^{n-1}_+), x \neq x' \end{cases} \\
&= \begin{cases} 1 + (-1)^{n-1}, x = x' \\ 1, x \neq x' \end{cases},
\end{aligned}$$

where $\mathbb{S}_+^{n-1}$ denotes the closed upper hemisphere of $\mathbb{S}_+^{n-1}$. By Lemma 3, for all $x' \in \mathbb{R}^n$,

$$\begin{aligned}(R_{K'} \circ R_K)h &= [1+(-1)^{n-1}]h(x') + \int_{\mathbb{R}^n} h(x)\mathbb{1}_{\{(x,x')\notin\Delta\}}d\chi(x)\\ &= [1+(-1)^{n-1}]h(x') + \int_{\mathbb{R}^n} h(x)d\chi(x) - \int_{\{x'\}} h(x)d\chi(x)\\ &= [(-1)^{n-1}]h(x') + \int_{\mathbb{R}^n} h(x)d\chi(x).\end{aligned}$$

The proof of Equation (1) is completed. □

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