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On a Discrete Version of the Hardy–Littlewood–Polya Inequality Involving Multiple Parameters in the Whole Plane

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Abstract: In this paper, by introducing multiple parameters, we establish a discrete version of the Hardy–Littlewood–Polya inequality in the whole plane. For the obtained inequality, we give the equivalent statements of the best possible constant factor linked to the parameters and deal with the equivalent inequalities. Our main result provided a new generalization of Hardy–Littlewood–Polya inequality, and as a consequence, we show that some new inequalities of the Hardy–Littlewood–Polya type can be derived from the current results by taking the special values of parameters.

Keywords: Hardy–Littlewood–Polya inequality; multiple parameters; equivalent inequalities; best possible constant factor

MSC: 26D15; 26D10; 26A42

1. Introduction

The famous Hardy–Hilbert inequality states the following [1]:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m \geq 0, b_n \geq 0, 0<\sum_{m=1}^{\infty}a_m^p<\infty,\ 0<\sum_{n=1}^{\infty}b_n^q<\infty$, $\frac{\pi}{\sin(\pi/p)}$ is the best possible constant factor.

Under the same assumption condition as above, an analogous form of inequality (1), called the Hardy–Littlewood–Polya inequality, was transcribed in [1], i.e.,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor pq is the best possible.

In [2], Yang, Wu and Chen established an extended Hardy–Littlewood–Polya inequality, as follows:

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$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\})^{\lambda}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q\}^{\frac{1}{q}}, \quad (3)$$

where $p>1 (q>1)$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in(0,3]$, $\lambda_i\in(0,\frac{11}{8}]\cap(0,\lambda)$ $(i=1,2)$, $a_m, b_n\geq 0$ $(m,n\in \mathrm{N}=\{1,2,\cdots\})$.

For the continuous case, the following integral version of the Hardy–Hilbert inequality is well known (see [1]).

If $f(x)\geq 0, g(y)\geq 0, 0<\int_0^{\infty} f^p(x)dx<\infty$ and $0<\int_0^{\infty} g^q(y)dy<\infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}(\int_0^{\infty} f^p(x)dx)^{\frac{1}{p}}(\int_0^{\infty} g^q(y)dy)^{\frac{1}{q}}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is the best possible.

Recently, You [3] considered the integral version of Hardy–Littlewood–Polya inequality; he extended the integral interval to the whole plane and obtained the following result:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{\max\{|x|,|y|\}}dxdy < 2pq(\int_{-\infty}^{\infty} f^p(x)dx)^{\frac{1}{p}}(\int_{-\infty}^{\infty} g^q(y)dy)^{\frac{1}{q}}. \quad (4)$$

Some relevant results about Hardy–Hilbert-type inequalities in the whole plane were provided in [4–7].

As is known to us, Hardy–Littlewood–Polya inequality and Hardy–Hilbert-type inequalities play important roles in mathematical analysis. These inequalities provide a lot of practical application cases in the theories of double series, double integrals and special functions.

Motivated by the above-mentioned works [1,2], we devote to establish a new generalization of Hardy–Littlewood–Polya inequality (2). To achieve this goal, we construct the following weighted coefficients for the series of the right-hand-side inequality (2):

$$(\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-(\lambda-\lambda_1-\lambda_2)-1}a_m^p)(\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-(\lambda-\lambda_1-\lambda_2)-1}b_n^q).$$

Also, we extend the range of parameters by $1\leq m,n<\infty$ to $1\leq|m|,|n|<\infty$ in the newly established inequality.

Based on the above idea, in this article, by introducing multiple parameters and constructing weight coefficients, with the help of the techniques of real analysis, we acquire a discrete version of the Hardy–Littlewood–Polya inequality in the whole plane. Furthermore, we give the equivalent forms of the obtained inequality and characterize the equivalent conditions for the best possible constant factor linked to the parameters. Finally, we illustrate how the main results obtained can generate some new Hardy–Littlewood–Polya type inequalities.

2. Preliminaries and Lemmas

Definition 1. *We define the homogeneous function of degree $-\lambda$ as follows*:

$$k_{\lambda}^{(\eta)}(x,y):=\frac{x^{\eta}+y^{\eta}}{(\max\{x,y\})^{\lambda+\eta}}\ (x,y>0)$$

which satisfies $k_\lambda^{(\eta)}(ux,uy)=u^{-\lambda}k_\lambda^{(\eta)}(x,y)$ $(u,x,y>0)$. *It follows that* $k_\lambda^{(\eta)}(x,y)$ *is a positive and continuous function with respect to* $x,y>0$. *It is easy to observe that*

$$k_\lambda^{(\eta)}(1,u)u^{\lambda_i-1}=\frac{(1+u^\eta)u^{\lambda_i-1}}{(\max\{1,u\})^{\lambda+\eta}}=\begin{cases}u^{\lambda_i-1}+u^{\eta+\lambda_i-1},0<u<1\\ u^{\lambda_i-\lambda-\eta-1}+u^{\lambda_i-\lambda-1},u\geq 1\end{cases}. \tag{5}$$

Claim 1. *(i) For* $\lambda+\eta>0,\lambda_i\in(\max\{0,-\eta\},\min\{\lambda,\lambda+\eta\})$ $(i=1,2)$, *by (5), we obtain*

$$\begin{aligned}k_{\lambda,\eta}(\lambda_i)&:=\textstyle\int_0^\infty k_\lambda^{(\eta)}(1,u)u^{\lambda_i-1}du\\ &=\textstyle\int_0^1(u^{\lambda_i-1}+u^{\eta+\lambda_i-1})du+\int_1^\infty(u^{-\lambda-\eta+\lambda_i-1}+u^{-\lambda+\lambda_i-1})du\\ &=\frac{1}{\lambda_i}+\frac{1}{\eta+\lambda_i}+\frac{1}{\lambda+\eta-\lambda_i}+\frac{1}{\lambda-\lambda_i}=\frac{(\lambda+\eta)(\lambda\eta+2\lambda\lambda_i-2\lambda_i^2)}{\lambda_i(\lambda+\eta-\lambda_i)(\eta+\lambda_i)(\lambda-\lambda_i)}\in \mathrm{R}_+=(0,\infty).\end{aligned}$$

(ii) For $\lambda+\eta>0,\lambda_i\leq\min\{1,1-\eta\},\lambda_i<\min\{\lambda+\eta,\lambda\}$ $(i=1,2)$, *which implies that* $\lambda_i\leq 1<\lambda+\eta+1$, *hence* $u^{\lambda_i-\lambda-\eta-1}(u\geq 1)$ *is strictly decreasing. We deduce that* $k_\lambda^{(\eta)}(1,u)u^{\lambda_i-1}$ *is decreasing with respect to* $u\in\mathrm{R}_+$ *and strictly decreasing in* $u\in[1,\infty)$.

For convenience, let us state the conditions (C1) below, which will be used in what follows.

(C1) $p>1,\frac{1}{p}+\frac{1}{q}=1,-1<\alpha,\beta<1,\lambda+\eta>0$,

$$\lambda_i\in(\max_{j=0,1}\{-j\eta\},\min_{j=0,1}\{\lambda+j\eta\})\cap(\max_{j=0,1}\{-j\eta\},\min_{j=0,1}\{1-j\eta\}]\ (i=1,2). \tag{6}$$

$a_m,b_n\geq 0$ $(|m|,|n|\in\mathrm{N}=\{1,2,\cdots\})$, such that for $c:=\lambda-\lambda_1-\lambda_2$,

$$0<\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-c-1}a_m^p<\infty \text{ and } 0<\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-c-1}b_n^q<\infty,$$

where $\sum_{|j|=1}^{\infty}(\cdots)=\sum_{j=-1}^{-\infty}(\cdots)+\sum_{j=1}^{\infty}(\cdots)$ $(j=m,n)$.

Claim 2. *For* $-1<\eta<0$ $(\lambda>-2\eta>0)$, *by (6), we have* $\lambda_i\in(-\eta,\lambda+\eta)\cap(-\eta,1]$ $(i=1,2)$; *for* $\eta\geq 0$ $(\lambda>0)$, *we have* $\lambda_i\in(0,\lambda)\cap(0,1]$ $(i=1,2)$.

Lemma 1. *For* $\gamma>0$, *we have the following inequalities:*

$$\frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]<I_\alpha:=\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1}<\frac{\gamma+1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]. \tag{7}$$

Proof. By using the decreasing property of this series, we find

$$\begin{aligned}I_\alpha&=\sum_{m=-1}^{-\infty}[(1-\alpha)(-m)]^{-\gamma-1}+\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}=\sum_{m=1}^{\infty}[(1-\alpha)m]^{-\gamma-1}+\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}\\ &=[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](1+\sum_{m=2}^{\infty}m^{-\gamma-1})<[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](1+\textstyle\int_1^\infty x^{-\gamma-1}dx)\\ &=\frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](\gamma+1),\end{aligned}$$

$$\begin{aligned}I_\alpha&=\sum_{m=1}^{\infty}[(1-\alpha)m]^{-\gamma-1}+\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}=[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\sum_{m=1}^{\infty}m^{-\gamma-1}\\ &>[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\textstyle\int_1^\infty x^{-\gamma-1}dx=\frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}].\end{aligned}$$

Hence, the two-sided inequalities in (7) hold true. This proves Lemma 1. □

Definition 2. *Set*

$$k(m,n) := k_{\lambda}^{(\eta)}(|m|+\alpha m, |n|+\beta n) = \frac{(|m|+\alpha m)^{\eta}+(|n|+\beta n)^{\eta}}{(\max\{|m|+\alpha m, |n|+\beta n\})^{\lambda+\eta}} \ (|m|,|n| \in \mathrm{N})$$

We define the following weight functions:

$$\omega(\lambda_2, m) := (|m|+\alpha m)^{\lambda-\lambda_2} \sum_{|n|=1}^{\infty} k(m,n)(|n|+\beta n)^{\lambda_2-1} \ (|m| \in \mathrm{N}) \tag{8}$$

$$\varpi(\lambda_1, n) := (|n|+\beta n)^{\lambda-\lambda_1} \sum_{|m|=1}^{\infty} k(m,n)(|m|+\alpha m)^{\lambda_1-1} \ \ (|n| \in \mathrm{N}) \tag{9}$$

Lemma 2. *The following inequalities are valid:*

$$0 < \frac{2}{1-\beta^2} k_{\lambda,\eta}(\lambda_2)(1-\theta(\lambda_2,m)) < \omega(\lambda_2,m) < \frac{2}{1-\beta^2} k_{\lambda,\eta}(\lambda_2) \ (|m| \in \mathrm{N}), \tag{10}$$

$$0 < \frac{2}{1-\alpha^2} k_{\lambda,\eta}(\lambda_1)(1-\vartheta(\lambda_1,n)) < \varpi(\lambda_1,n) < \frac{2}{1-\alpha^2} k_{\lambda,\eta}(\lambda_1) \ (|n| \in \mathrm{N}), \tag{11}$$

where

$$\theta(\lambda_2,m) := \frac{1}{k_{\lambda,\eta}(\lambda_2)} \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} \frac{1+u^{\eta}}{\max\{1,u\}^{\lambda+\eta}} u^{\lambda_2-1} du = O(\frac{1}{(|m|+\alpha m)^{\widetilde{\lambda}_2}}) > 0 \ (\widetilde{\lambda}_2 = \min_{j=0,1}\{j\eta+\lambda_2\} > 0)$$

$$\vartheta(\lambda_1,n) := \frac{1}{k_{\lambda,\eta}(\lambda_1)} \int_0^{\frac{1+|\alpha|}{|n|+\beta n}} \frac{1+u^{\eta}}{\max\{1,u\}^{\lambda+\eta}} u^{\lambda_1-1} du = O(\frac{1}{(|n|+\beta n)^{\widetilde{\lambda}_1}}) > 0 \ (\widetilde{\lambda}_1 = \min_{j=0,1}\{j\eta+\lambda_1\} > 0).$$

Proof. For $|m| \in \mathrm{N}$, we define the following functions:

$$k^{(1)}(m,y) := k_{\lambda}^{(\eta)}(|m|+\alpha m, (1-\beta)(-y)), y < 0,$$
$$k^{(2)}(m,y) := k_{\lambda}^{(\eta)}(|m|+\alpha m, (1+\beta)y), y > 0,$$

where for $y > 0$, $k^{(1)}(m,-y) = k_{\lambda}^{(\eta)}(|m|+\alpha m, (1-\beta)y)$. It is easy to verify that

$$\begin{aligned}
\omega(\lambda_2,m) &= (|m|+\alpha m)^{\lambda-\lambda_2}\{\sum_{n=-1}^{-\infty} k^{(1)}(m,n)[(1-\beta)(-n)]^{\lambda_2-1} + \sum_{n=1}^{\infty} k^{(2)}(m,n)[(1+\beta)n]^{\lambda_2-1}\} \\
&= (|m|+\alpha m)^{\lambda-\lambda_2}\{\sum_{n=1}^{\infty} k^{(1)}(m,-n)[(1-\beta)n]^{\lambda_2-1} + \sum_{n=1}^{\infty} k^{(2)}(m,n)[(1+\beta)n]^{\lambda_2-1}\} \\
&= (|m|+\alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1} \sum_{n=1}^{\infty} k^{(1)}(m,-n)n^{\lambda_2-1} + (1+\beta)^{\lambda_2-1} \sum_{n=1}^{\infty} k^{(2)}(m,n)n^{\lambda_2-1}].
\end{aligned}$$

For fixed $|m| \in \mathrm{N}$, by the aid of Claim 1 (ii), we conclude that both $k^{(1)}(m,-y)y^{\lambda_2-1}$ and $k^{(2)}(m,y)y^{\lambda_2-1}$ are decreasing with respect to $y \in (0,\infty)$ and strictly decreasing in $y \in [1,\infty)$. Using the decreasing property of the series, it follows that

$$\begin{aligned}
&\omega(\lambda_2,m) < (|m|+\alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1} \textstyle\int_0^{\infty} k^{(1)}(m,-y)y^{\lambda_2-1}dy \\
&\quad + (1+\beta)^{\lambda_2-1} \textstyle\int_0^{\infty} k^{(2)}(m,y)y^{\lambda_2-1}dy], \\
&\omega(\lambda_2,m) > (|m|+\alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1} \textstyle\int_1^{\infty} k^{(1)}(m,-y)y^{\lambda_2-1}dy \\
&\quad + (1+\beta)^{\lambda_2-1} \textstyle\int_1^{\infty} k^{(2)}(m,y)y^{\lambda_2-1}dy].
\end{aligned}$$

Setting $u = \frac{(1-\beta)y}{|m|+\alpha m}$ in the above first integral, and setting $u = \frac{(1+\beta)y}{|m|+\alpha m}$ in the above second integral, respectively, we obtain

$$\omega(\lambda_2, m) < [(1-\beta)^{-1} + (1+\beta)^{-1}] \int_0^\infty k_\lambda^{(\eta)}(1,u) u^{\lambda_2 - 1} du = \frac{2k_{\lambda,\eta}(\lambda_2)}{1-\beta^2},$$

$$\begin{aligned}\omega(\lambda_2, m) &> \frac{1}{1-\beta} \int_{\frac{1-\beta}{|m|+\alpha m}}^\infty k_\lambda^{(\eta)}(1,u) u^{\lambda_2-1} du + \frac{1}{1+\beta} \int_{\frac{1+\beta}{|m|+\alpha m}}^\infty k_\lambda^{(\eta)}(1,u) u^{\lambda_2-1} du \\ &> \frac{2}{1-\beta^2} \int_{\frac{1+|\beta|}{|m|+\alpha m}}^\infty k_\lambda^{(\eta)}(1,u) u^{\lambda_2-1} du = \frac{2k_{\lambda,\eta}(\lambda_2)}{1-\beta^2}(1-\theta(\lambda_2, m)) > 0,\end{aligned}$$

where $\theta(\lambda_2, m) = \frac{1}{k_{\lambda,\eta}(\lambda_2)} \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} k_\lambda^{(\eta)}(1,u) u^{\lambda_2-1} du$, which satisfies, for $\frac{1+|\beta|}{|m|+\alpha m} < 1$, $\widetilde{\lambda}_2 = \min_{j=0,1}\{\eta + \lambda_2\} > 0$,

$$\begin{aligned}0 &< \int_0^{\frac{1+\beta}{|m|+\alpha m}} k_\lambda^{(\eta)}(1,u) u^{\lambda_2-1} du = \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} (u^{\lambda_2-1} + u^{\eta+\lambda_2-1}) du \\ &\le 2 \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} u^{\widetilde{\lambda}_2 - 1} du = \frac{2}{\widetilde{\lambda}_2} \left(\frac{1+|\beta|}{|m|+\alpha m}\right)^{\widetilde{\lambda}_2}.\end{aligned}$$

Hence, we derive the inequality (10). Moreover, inequality (11) can be deduced in the same way as above. The proof of Lemma 2 is complete. □

3. Main Results

Theorem 1. *Under the assumption condition (C1), we have the following parameterized Hardy–Littlewood–Polya inequality:*

$$\begin{aligned}H &:= \sum_{|n|=1}^\infty \sum_{|m|=1}^\infty k(m,n) a_m b_n \\ &< \frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \Big[\sum_{|m|=1}^\infty (|m|+\alpha m)^{p(1-\lambda_1)-c-1} a_m^p\Big]^{\frac{1}{p}} \Big[\sum_{|n|=1}^\infty (|n|+\beta n)^{q(1-\lambda_2)-c-1} b_n^q\Big]^{\frac{1}{q}}.\end{aligned} \tag{12}$$

Proof. By utilizing Hölder's inequality [8], we obtain

$$\begin{aligned}H &= \sum_{|n|=1}^\infty \sum_{|m|=1}^\infty k(m,n) \Big[\frac{(|n|+\beta n)^{(\lambda_2-1)/p}}{(|m|+\alpha m)^{(\lambda_1-1)/q}} a_m\Big] \Big[\frac{(|m|+\alpha m)^{(\lambda_1-1)/q}}{(|n|+\beta n)^{(\lambda_2-1)/p}} b_n\Big] \\ &\le \Big[\sum_{|m|=1}^\infty \sum_{|n|=1}^\infty k(m,n) \frac{(|n|+\beta n)^{\lambda_2-1}}{(|m|+\alpha m)^{(\lambda_1-1)(p-1)}} a_m^p\Big]^{\frac{1}{p}} \Big[\sum_{|n|=1}^\infty \sum_{|m|=1}^\infty k(m,n) \frac{(|m|+\alpha m)^{\lambda_1-1}}{(|n|+\beta n)^{(\lambda_2-1)(q-1)}} b_n^q\Big]^{\frac{1}{q}} \\ &= \Big[\sum_{|m|=1}^\infty \omega(\lambda_2, m)(|m|+\alpha m)^{p(1-\lambda_1)-c-1} a_m^p\Big]^{\frac{1}{p}} \Big[\sum_{|n|=1}^\infty \varpi(\lambda_1, n)(|n|+\beta n)^{q(1-\lambda_2)-c-1} b_n^q\Big]^{\frac{1}{q}}.\end{aligned}$$

By virtue of inequalities (10) and (11), we deduce inequality (12). This completes the proof of Theorem 1. □

Remark 1. *In (12), for* $\lambda_1+\lambda_2=\lambda$, *we derive from Claim 1 that* $k_{\lambda,\eta}(\lambda_1)=k_{\lambda,\eta}(\lambda_2)=\frac{(\lambda+\eta)(2\lambda_1\lambda_2+\lambda\eta)}{\lambda_1\lambda_2(\lambda_1+\eta)(\lambda_2+\eta)}$,

$$0<\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p<\infty,\ 0<\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-1}b_n^q<\infty,$$

and we have the following inequality:

$$H<\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}} \tag{13}$$

Theorem 2. *Theconstant factor* $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in inequality (13) is the best possible.*

Proof. For any $0<\varepsilon<p\min_{j=0,1}\{j\eta+\lambda_1\}$, we set

$$\widetilde{a}_m:=(|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1},\widetilde{b}_n:=(|n|+\beta n)^{(\lambda_2-\frac{\varepsilon}{q})-1}\ \ (|m|,|n|\in\mathrm{N})$$

By the way of contradiction, if there exists a constant M ($M<\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$) such that inequality (13) is valid, when we replace $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ by M, it follows that

$$\widetilde{H}:=\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)\widetilde{a}_m\widetilde{b}_n<M[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}\widetilde{a}_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-1}\widetilde{b}_n^q]^{\frac{1}{q}}$$

By Lemma 1, we obtain

$$\begin{aligned}\widetilde{H}&<M[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\varepsilon-1}]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{-\varepsilon-1}]^{\frac{1}{q}}\\&<\frac{M}{\varepsilon}(\varepsilon+1)[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}]^{\frac{1}{p}}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}]^{\frac{1}{q}}.\end{aligned}$$

In view of (8), $(\lambda_1-\frac{\varepsilon}{p})+(\lambda_2+\frac{\varepsilon}{p})=\lambda$ and

$$\lambda_1-\frac{\varepsilon}{p}>\max_{j=0,1}\{-j\eta\},\lambda_1-\frac{\varepsilon}{p}<\lambda_1\le\min_{j=0,1}\{1-j\eta,j\eta+\lambda\}.$$

Using inequality (11) and Lemma 1, we have

$$\begin{aligned}\widetilde{H}&=\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)(|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1}(|n|+\beta n)^{(\lambda_2-\frac{\varepsilon}{q})-1}\\&=\sum_{|n|=1}^{\infty}\varpi(\lambda_1-\frac{\varepsilon}{p},n)(|n|+\beta n)^{-\varepsilon-1}>\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{1-\alpha^2}\sum_{|n|=1}^{\infty}[1-O(\frac{1}{(|n|+\beta n)^{\widetilde{\lambda}_1-\varepsilon/p}})](|n|+\beta n)^{-\varepsilon-1}\\&=\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{1-\alpha^2}[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{-\varepsilon-1}-\sum_{|n|=1}^{\infty}O(\frac{1}{(|n|+\beta n)^{(\widetilde{\lambda}_1+\varepsilon/q)+1}})]\\&>\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{\varepsilon(1-\alpha^2)}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}-\varepsilon O(1)].\end{aligned}$$

Combining the above results, we have

$$\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{1-\alpha^2}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}-\varepsilon O(1)]<\varepsilon\widetilde{H}$$

$$<M(\varepsilon+1)[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}]^{\frac{1}{p}}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}]^{\frac{1}{q}}.$$

Taking a limit as $\varepsilon\to 0^+$, we acquire that $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\alpha^2)(1-\beta^2)}\le\frac{M}{(1-\alpha^2)^{1/p}(1-\beta^2)^{1/q}}$, namely $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\le M$, which means that the constant factor $M=\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ is the best possible in (13). The proof of Theorem 2 is complete. □

Remark 2. *Letting* $\hat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\lambda_1+\frac{c}{p}$, $\hat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda_2+\frac{c}{q}$, *we can rewrite (12) as follows:*

$$H<\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}. \tag{14}$$

Note that $\hat{\lambda}_1+\hat{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda$. *From* $0<\lambda_1<\lambda$, $0<\lambda-\lambda_2<\lambda$, *it follows that* $0<\hat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}<\lambda, 0<\hat{\lambda}_2=\lambda-\hat{\lambda}_1<\lambda$. *Moreover, from* $-\eta<\lambda_i<\lambda+\eta$ $(i=1,2)$, *we conclude that* $-\eta<\hat{\lambda}_i<\lambda+\eta$ $(i=1,2)$.

Thus, for

$$c\le\min\{p(1-\lambda_1),q(1-\lambda_2),p(1-\eta-\lambda_1),q(1-\eta-\lambda_2)\}$$

We can still find that $\hat{\lambda}_1,\hat{\lambda}_2\le\min\{1,1-\eta\}$. *Therefore, inequality (13) can be rewritten as*

$$H<\frac{2k_{\lambda,\eta}(\hat{\lambda}_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\hat{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\hat{\lambda}_2)-1}b_n^q]^{\frac{1}{q}} \tag{15}$$

Theorem 3. *Suppose that* $c\le\min\{p(1-\lambda_1),q(1-\lambda_2),p(1-\eta-\lambda_1),q(1-\eta-\lambda_2)\}$; *if the constant factor* $\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in (12) (or (14)) is the best possible, then* $\lambda_1+\lambda_2=\lambda$.

Proof. By employing Hölder's inequality [8], we obtain

$$\begin{aligned}
k_{\lambda,\eta}(\hat{\lambda}_2)&=k_{\lambda,\eta}(\hat{\lambda}_1)=k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})\\
&=\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du=\int_0^\infty k_\lambda^{(\eta)}(u,1)(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du\\
&\le(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda-\lambda_2-1}du)^{\frac{1}{p}}(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du)^{\frac{1}{q}}\\
&=(\int_0^\infty k_\lambda^{(\eta)}(1,v)v^{\lambda_2-1}dv)^{\frac{1}{p}}(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du)^{\frac{1}{q}}\\
&=k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)
\end{aligned} \tag{16}$$

If the constant factor $\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (12) (or (14)) is the best possible, then, by comparing with the constant factors in (14) and (15), we have the following inequality:

$$\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \le \frac{2k_{\lambda,\eta}(\hat{\lambda}_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} (\in \mathrm{R}_+),$$

namely $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) \le k_{\lambda,\eta}(\hat{\lambda}_1)$; it follows that (16) keeps the form of equality.

In view of the condition of equality for Hölder's inequality [8], we observe that (16) keeps the form of equality if and only if there exist constants C and D such that they are both not zero and $Cu^{\lambda-\lambda_2-1} = Du^{\lambda_1-1}$ *a.e.* in R_+. We may assume $C \neq 0$, which leads to $u^{\lambda-\lambda_2-\lambda_1} = D/C$ *a.e.* in R_+, hence $\lambda - \lambda_2 - \lambda_1 = 0$, namely $\lambda_1 + \lambda_2 = \lambda$. This completes the proof of Theorem 3. □

4. Equivalent Statements of the Main Results

Theorem 2. *We have the following inequality which is equivalent to inequality (14):*

$$\begin{aligned} L &:= \Big[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{p\hat{\lambda}_2-1}\Big(\sum_{|m|=1}^{\infty}k(m,n)a_m\Big)^p\Big]^{\frac{1}{p}} \\ &< \frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\Big[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\hat{\lambda}_1)-1}a_m^p\Big]^{\frac{1}{p}} \end{aligned} \tag{17}$$

Proof. (i) Suppose that (17) is valid. By using Hölder's inequality [8], we obtain

$$\begin{aligned} H &= \sum_{|n|=1}^{\infty}\Big[(|n|+\beta n)^{\frac{-1}{p}+\hat{\lambda}_2}\sum_{|m|=1}^{\infty}k(m,n)a_m\Big]\Big[(|n|+\beta n)^{\frac{1}{p}-\hat{\lambda}_2}b_n\Big] \\ &\le L\cdot\Big[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\hat{\lambda}_2)-1}b_n^q\Big]^{\frac{1}{q}} \end{aligned} \tag{18}$$

Applying inequality (17), we acquire inequality (14).

(ii) On the other hand, if inequality (14) is valid, taking

$$b_n := (|n|+\beta n)^{p\hat{\lambda}_2-1}\Big(\sum_{|m|=1}^{\infty}k(m,n)a_m\Big)^{p-1}, |n| \in \mathrm{N}$$

then we have $L^p = \sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-c-1}b_n^q = H$. If $L = 0$, (17) is naturally valid. Also, we find that $L = \infty$ is impossible to make (17) valid, namely $L < \infty$.

For $0 < L < \infty$, by (14), we deduce that

$$0 < \sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-c-1}b_n^q = L^p = H$$

$$< \frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\Big[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-c-1}a_m^p\Big]^{\frac{1}{p}}L^{p-1} < \infty$$

this yields

$$L = [\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-c-1}b_n^q]^{\frac{1}{p}}$$
$$< \frac{2k_{\lambda.\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-c-1}a_m^p]^{\frac{1}{p}},$$

which is the desired inequality (17). Therefore, inequality (17) is equivalent to inequality (14). Theorem 4 is proved. □

Theorem 5. *The following statements (i), (ii), (iii), (iv) and (v) are equivalent:*

(i) *Both* $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ *and* $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ *are independent of* p, q*;*

(ii) $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})(\in \mathrm{R}_+)$*;*

(iii) $\lambda_1+\lambda_2=\lambda$*;*

(iv) *Under the condition that* $c \le \min\{p(1-\lambda_1), q(1-\lambda_2), p(1-\eta-\lambda_1), q(1-\eta-\lambda_2)\}$, $\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of (14);*

(v) *Under the condition that* $c \le \min\{p(1-\lambda_1), q(1-\lambda_2), p(1-\eta-\lambda_1), q(1-\eta-\lambda_2)\}$, $\frac{2k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *is the best possible constant factor of (17).*

Remark 3. *Similarly to Theorem 4, if* $\lambda_1+\lambda_2=\lambda$*, then we have the following inequality with the best possible constant factor* $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$*, which is equivalent to (13).*

$$[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{p\lambda_2-1}(\sum_{|m|=1}^{\infty}k(m,n)a_m)^p]^{\frac{1}{p}} < \frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}} \tag{19}$$

Proof. (i)⇒(ii). Since both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p, q, it follows that

$$k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty}\lim_{q\to 1^+}k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}(\lambda_1)$$

$$k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}) = \lim_{p\to\infty}\lim_{q\to 1^+}k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}) = k_{\lambda,\eta}(\lambda_1) = k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$$

(ii)⇒(iii). If $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ $(\in \mathrm{R}_+)$, then (16) keeps the form of equality. From the proof of Theorem 3, we deduce that $\lambda_1+\lambda_2=\lambda$.

(iii)⇒(i). If $\lambda_1+\lambda_2=\lambda$, then we have

$$k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}) = k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}(\lambda_1).$$

We conclude that both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p, q. Hence, (i)⇔(ii)⇔(iii).

(iii)⇔(iv). From the results of Theorem 2 and Theorem 3, we can directly deduce the conclusions.

(iv)⇔(v). If the constant factor in (14) is the best possible, then so is the best possible constant factor in (17). Otherwise, by (18) (for $\hat{\lambda}_i = \lambda_i \ \ (i = 1, 2)$), we would reach a contradiction that the constant factor in (13) is not the best possible. On the other hand, if the constant factor in (17) is the best possible, then so is the best possible constant factor in (14). Otherwise, in view of the equivalency between (14) and (17), we would reach a contradiction that the constant in (14) is not the best possible.

Therefore, the statements (i), (ii), (iii), (iv) and (v) are equivalent. The proof of Theorem 5 is complete. □

5. Special Cases of Parameterized Inequalities

As applications of the main results, below we will deduce some new inequalities of the Hardy–Littlewood–Polya type from the special values of parameters.

Remark 4. *(i) Putting $\alpha = \beta = 0$ in (13) and (19), we obtain the following inequalities:*

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{|m|^{\eta}+|n|^{\eta}}{(\max\{|m|,|n|\})^{\lambda+\eta}}a_m b_n < 2\frac{(\lambda+\eta)(2\lambda_1\lambda_2+\lambda\eta)}{\lambda_1\lambda_2(\lambda_1+\eta)(\lambda_2+\eta)} \times[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}|n|^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}} \tag{20}$$

$$[\sum_{|n|=1}^{\infty}|n|^{p\lambda_2-1}(\sum_{|m|=1}^{\infty}\frac{|m|^{\eta}+|n|^{\eta}}{(\max\{|m|,|n|\})^{\lambda+\eta}}a_m)^p]^{\frac{1}{p}} < 2\frac{(\lambda+\eta)(2\lambda_1\lambda_2+\lambda\eta)}{\lambda_1\lambda_2(\lambda_1+\eta)(\lambda_2+\eta)}[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}} \tag{21}$$

(ii) Taking $\eta = 0$ in (13) and (19), we derive the following inequalities:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(\max\{|m|+\alpha m,|n|+\beta n\})^{\lambda}} < \frac{2\lambda}{\lambda_1\lambda_2}\frac{1}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \times[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}(|n|+\beta)^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}, \tag{22}$$

$$\{\sum_{|n|=1}^{\infty}(|n|+\beta n)^{p\lambda_2-1}[\sum_{|m|=1}^{\infty}\frac{a_m}{(\max\{|m|+\alpha m,|n|+\beta n\})^{\lambda}}]^p\}^{\frac{1}{p}} < \frac{2\lambda}{\lambda_1\lambda_2}\frac{1}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}} \tag{23}$$

(iii) Taking $\alpha = \beta = 0$ in (22) and (23), we acquire the inequalities

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{1}{(\max\{|m|,|n|\})^{\lambda}}a_m b_n < \frac{2\lambda}{\lambda_1\lambda_2}[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{|n|=1}^{\infty}|n|^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}, \tag{24}$$

$$[\sum_{|n|=1}^{\infty}|n|^{p\lambda_2-1}(\sum_{|m|=1}^{\infty}\frac{1}{(\max\{|m|,|n|\})^{\lambda}}a_m)^p]^{\frac{1}{p}} < \frac{2\lambda}{\lambda_1\lambda_2}[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}} \tag{25}$$

(iv) Choosing $a_{-m} = a_m, b_{-n} = b_n \ \ (m, n \in \mathrm{N})$ in (20) and (21), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\eta}+n^{\eta}}{(\max\{m,n\})^{\lambda+\eta}}a_m b_n < \frac{(\lambda+\eta)(2\lambda_1\lambda_2+\lambda\eta)}{\lambda_1\lambda_2(\lambda_1+\eta)(\lambda_2+\eta)}[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}, \tag{26}$$

$$[\sum_{n=1}^{\infty} n^{p\lambda_2-1}(\sum_{m=1}^{\infty} \frac{m^{\eta}+n^{\eta}}{(\max\{m,n\})^{\lambda+\eta}} a_m)^{p}]^{\frac{1}{p}} < \frac{(\lambda+\eta)(2\lambda_1\lambda_2+\lambda\eta)}{\lambda_1\lambda_2(\lambda_1+\eta)(\lambda_2+\eta)}[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}} \quad (27)$$

(v) *Choosing* $\eta = 0, \lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ *in (26), we obtain the classical Hardy–Littlewood–Polya inequality (2).*

6. Conclusions

Compared to the Hardy–Hilbert inequality, the Hardy–Littlewood–Polya inequality is relatively less investigated. In this work, we focus on the extension of the Hardy–Littlewood–Polya inequality in the whole plane. The main idea is to construct weight coefficients and introduce more parameters. In Theorem 1, we establish a discrete version of the Hardy–Littlewood–Polya inequality, involving multiple parameters in the whole plane. We give the equivalent forms of the parameterized Hardy–Littlewood–Polya inequality in Theorem 4. The characterization of the equivalent conditions via the parameters linked to the best possible constant factor are considered in Theorems 2, 3 and 5. At the end of the paper, in Remark 4, we show that some new inequalities of Hardy–Littlewood–Polya type can be derived from the special values of parameters. The main result provided a new generalization of the Hardy–Littlewood–Polya inequality. The limitation of the presented work is that we are unable to obtain a corresponding integral version of the generalized Hardy–Littlewood–Polya inequality. In the future, the Hardy–Littlewood–Polya will be studied in a wider range of areas, such as refined forms, reversed forms, integral forms and half-discrete forms.

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