mathematics

MDPI

Article

Optimal Solutions for a Class of Impulsive Differential Problems with Feedback Controls and Volterra-Type Distributed Delay: A Topological Approach

Paola Rubbioni

Department of Mathematics and Computer Science, University of Perugia, via L. Vanvitelli 1, 06123 Perugia, Italy; paola.rubbioni@unipg.it

Abstract: In this paper, the existence of optimal solutions for problems governed by differential equations involving feedback controls is established for when the problem must account for a Volterra-type distributed delay and is subject to the action of impulsive external forces. The problem is reformulated within the class of impulsive semilinear integro-differential inclusions in Banach spaces and is studied by using topological methods and multivalued analysis. The paper concludes with an application to a population dynamics model.

Keywords: optimal solutions; feedback controls; distributed delay; impulses; integro-differential inclusions; mild solutions; population dynamics

MSC: 34G20; 34G25; 34A37; 93B24; 93B52; 92D25

check for updates

Citation: Rubbioni, P. Optimal Solutions for a Class of Impulsive Differential Problems with Feedback Controls and Volterra-Type Distributed Delay: A Topological Approach. *Mathematics* **2024**, *12*, 2293. https://doi.org/10.3390/math12142293

Academic Editor: Rodica Luca

Received: 3 June 2024
Revised: 11 July 2024
Accepted: 15 July 2024
Published: 22 July 2024

Correction Statement: This article has been republished with a minor change. The change does not affect the scientific content of the article and further details are available within the backmatter of the website version of this article.

Copyright: © 2024 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In this article, we aim to provide sufficient conditions for the existence of optimal solutions for differential problems that describe models of real phenomena where the state of the system depends on its past evolution according to a fading memory process. This means that the further back in time an event occurs, the less it influences the current state of the system. This fading memory process is well described by the type of distributed delay induced by the exponential probability distribution $k(r) = \frac{e^{-r/\tau}}{\tau}$ used as a kernel in a Volterra-type integral involved in the system's equation. The formalization of the problem will thus be governed by an integro-differential equation. Examples of such problems include population dynamics models, where only fertile individuals are considered in the phenomenon under study. Often, in such cases, the time between an individual's birth and the moment it begins participating in the reproductive process is not negligible. Consequently, it is necessary to consider in the population development equation a term representing the delay with which an individual becomes an active part in the evolution of the considered process, which is linked to its maturation time. Other examples can be found in the context of beams fixed at one end with a mass at the free end, which were recently used to describe the robotic arms of flexible robots. In these cases, depending on the material used, the deflections of the beam (or arm) can significantly affect the system's current state, and thus, in these cases as well, the problem is better defined if a delay term is included in the equation. To explore these topics further, we refer to the papers [1–6] and books [7,8], acknowledging that they do not represent an exhaustive bibliography on the problem. In this article, by way of example, we will demonstrate the application of our results to a population dynamics model.

The differential problems we address in this article are subject to feedback controls, so the optimal solutions will actually be trajectory–control pairs where the trajectory minimizes or maximizes the cost functional (depending on whether it is lower semicontinuous

or upper semicontinuous) for a particular control. In fact, our study takes into consideration the semilinear integro-differential equation with fading memory

$$y'(t) = A(t)y(t) + f\Big(t, y(t), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} y(s)ds, \eta(t)\Big),\ t \in [t_0, T],$$

coupled with the feedback control condition

$$\eta(t) \in H\Big(t, y(t), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} y(s)ds\Big).$$

Here, $\{A(t)\}_{t\in[0,T]}$ is a family of linear operators in a Banach space E, $f : [t_0, T] \times E \times E \times E_1 \to E$ and $H : [t_0, T] \times E \times E \multimap E_1$ are given functions, and E_1 is separable Banach space.

From our perspective, this allows us to consider an integro-differential inclusion structure, as by f and H we can define the evolution multifunction

$$F(t, v, w) := f(t, v, w, H(t, v, w)),$$

leading to the semilinear integro-differential inclusion

$$y'(t) \in A(t)y(t) + F\Big(t, y(t), \int_{t_0}^{t} k(t, s)y(s)ds\Big),\ t \in [t_0, T]. \tag{1}$$

Here, k is a continuous kernel, generalizing the exponential one.

Furthermore, we allow external forces to act instantaneously on the system at predetermined times, abruptly modifying the trajectories that describe the evolutionary process of the problem. These instantaneous forces are represented by what are called "impulse functions". As a direct consequence of their action, the solutions of our problem will not be continuous but rather piecewise continuous with jump discontinuities. It is therefore clear that our study enables us to take into consideration a much wider range of situations than that usually reported in the literature, which is described by continuous trajectories. This type of phenomenon is quite common in the real world (think, for example, but not only, about the application of pesticides in plantations or the administration of antibiotics to patients with bacterial diseases, or the electrical stimulation of a limb), and it has been and remains a subject of study by the scientific community. In this regard, we refer to the works [9–12] and texts [13–17].

In our investigation, we make use of topological methods and multivalued analysis tools. Indeed, our results are based on the fixed-point theory for condensing multimaps in Banach spaces and on some isomorphisms and continuous embeddings of spaces. Our approach offers a twofold advantage. On one hand, it provides a novel contribution to the theory of integro-differential inclusions in abstract spaces, thanks to the new compactness theorems for the set of solutions, both in the case without impulses and in the impulsive case (cf. Theorems 1 and 2). On the other hand, it simultaneously establishes the existence results of optimal mild trajectory–control pairs for a wide range of real-world phenomena models as a consequence of those for the abstract case (cf. Theorem 4).

The article is structured as follows. In Section 2, we provide the definitions and preliminary results necessary for an easy understanding of the work. In Section 3, we position the problem in a Banach space E, introducing the space to which the mild trajectories belong, the assumptions on the problem data, and the definition of mild solution for the integro-differential inclusion (1) under conditions

$$y(t_0) = y_0 \in E; \quad y(t_j^+) = y(t_j) + I_j(y(t_j)),\ j = 1, \ldots, m. \tag{2}$$

The functions $I_j : E \to E$, $j = 1, \ldots, m$ provide the impulses.

Section 4 is dedicated to the compactness of the set of mild solutions of (1) in the non-impulsive case. As far as we know, indeed, our results are new in this case as well, and

therefore, we have decided to isolate them so that they can be used in the future separately from the rest of the article.

Section 5 contains the main results of the work, namely the compactness of the set of mild solutions of (1) under the action of impulses and the existence of optimal solutions for a cost functional.

In Section 6, we apply the theorem on optimal solutions from the preceding section to the cost functional of the impulsive feedback control system

$$y'(t) = A(t)y(t) + f\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds, \eta(t)\Big),\ t \in [t_0, T], \tag{3}$$

$$\eta(t) \in H\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\Big), \tag{4}$$

with conditions (2) in abstract spaces.

Finally, in Section 7, to illustrate by way of example how our abstract results can be applied to concrete models, we have examined the population dynamics system described by the differential equation

$$u_t(t,x) = -b(t,x)u(t,x) + g\Big(t, u(t,x), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau}\, u(s,x)\, ds\Big) + \omega(t,x),$$

subject to feedback controls

$$\omega(t,\cdot) \in \Omega(u(t,\cdot)),$$

and conditions

$$u(t_0,x) = u_0(x),\ \text{and}\ u\Big(t_j^+, x\Big) = u\big(t_j, x\big) + \mathcal{I}_j(u\big(t_j, x\big)\big)\ ,\ j = 1, \ldots, m.$$

Thanks to appropriate settings, we show that it can be formalized as the system (3)–(4) with conditions (2) in the function space $E = L^2([0,1])$. The optimization of a cost functional related to the model concludes the paper.

2. Notations and Reference Results

We recall some definitions and properties which will be used throughout the paper.

Let $\mathcal{X}, \mathcal{Y}$ be two topological spaces and $\mathcal{F} : \mathcal{X} \multimap \mathcal{Y}$ be a multifunction.

Definition 1. *The multifunction $\mathcal{F}$ is said to be closed if the set graph $\mathcal{F} := \{(x,y) \in \mathcal{X} \times \mathcal{Y} : y \in \mathcal{F}(x)\}$ is a closed subset of $\mathcal{X} \times \mathcal{Y}$.*

Definition 2. *The multifunction $\mathcal{F}$ is said to be upper semicontinuous at a point $x \in \mathcal{X}$ if for every open $W \subset \mathcal{Y}$ such that $\mathcal{F}(x) \subset W$ there exists a neighborhood $V(x)$ of x such that $\mathcal{F}(V(x)) \subset W$.*

If (Y,d) is a metric space and the multimap $\mathcal{F} : \mathcal{X} \multimap Y$ takes compact values, then $\mathcal{F}$ is upper semicontinuous at a point $x \in \mathcal{X}$ if and only if for every $\varepsilon > 0$ there exists a neighborhood $V(x)$ such that $\mathcal{F}(z) \subset W_\varepsilon(\mathcal{F}(x))$ for every $z \in V(x)$, where $W_\varepsilon(\mathcal{F}(x)) := \{y \in Y : d(y, \mathcal{F}(x)) < \varepsilon\}$ and $d(y, \mathcal{F}(x)) := \inf_{w \in \mathcal{F}(x)} d(y,w)$ (cf. [18], Theorem 1.1.8).

In $\mathbb{R}^n$, let 0_n be the zero-element and $\preccurlyeq$ the partial ordering defined by

$$x := (x_1, \ldots, x_n) \preccurlyeq y := (y_1, \ldots, y_n) \text{ if and only if } y_k - x_k \geq 0 \text{ for every } k = 1, \ldots, n.$$

Of course, $x \prec y$ stands for $x \preccurlyeq y \wedge x \neq y$.

Let $\mathcal{E}$ be a Banach space and $\mathcal{P}_b(\mathcal{E})$ be the family of all the bounded subsets of $\mathcal{E}$.

Definition 3. *A function $\beta : \mathcal{P}_b(\mathcal{E}) \to \mathbb{R}^n_{0,+}$ is said to be a measure of noncompactness in $\mathcal{E}$ if*

(β_1) $\beta(\Omega) = 0_n$ if and only if $\overline{\Omega}$ is compact, $\Omega \in \mathcal{P}_b(\mathcal{E})$;
(β_2) $\beta(\overline{co}(\Omega)) = \beta(\Omega)$, $\Omega \in \mathcal{P}_b(\mathcal{E})$.

Moreover, β is said to be

monotone if $\Omega_1 \subset \Omega_2$ *implies* $\beta(\Omega_1) \preccurlyeq \beta(\Omega_2)$, $\Omega_1, \Omega_2 \in \mathcal{P}_b(\mathcal{E})$;
nonsingular if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, *for every* $x \in \mathcal{E}$, $\Omega \in \mathcal{P}_b(\mathcal{E})$;
invariant under closure if $\beta(\overline{\Omega}) = \beta(\Omega)$, $\Omega \in \mathcal{P}_b(\mathcal{E})$;
invariant with respect to the union with compact sets if $\beta(\Omega \cup C) = \beta(\Omega)$, *for every relatively compact set* $C \subset \mathcal{E}$, $\Omega \in \mathcal{P}_b(\mathcal{E})$.

An example of a measure of noncompactness satisfying all the above properties is the Hausdorff measures of noncompactness in $\mathcal{E}$,

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ can be covered by finitely many balls with radius } \varepsilon\}.$$

Throughout the paper, we will use also the next monotone vectorial measure of noncompactness which was introduced in [19] (cf. also [18]). When $L > 0$ is fixed, let $\nu_L : \mathcal{P}_b(C([a,b];E)) \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$ be given by

$$\nu_L(\Omega) := \max_{\{w_n\}_n \subset \Omega} (\gamma(\{w_n\}_n), \eta(\{w_n\}_n)), \tag{5}$$

where

$$\gamma(\{w_n\}_n) := \sup_{t \in [a,b]} e^{-Lt} \left[\chi(\{w_n(t)\}_n) + \chi\left(\left\{ \int_a^t k(t,s) w_n(s) ds \right\}_n \right) \right], \tag{6}$$

and

$$\eta(\{w_n\}_n) := mod_C(\{w_n\}_n) + mod_C\left(\left\{ \int_a^{(\cdot)} k(\cdot, s) w_n(s) ds \right\}_n \right), \tag{7}$$

being mod_C the modulus of continuity in $C([a,b];E)$.

Definition 4. *If* D *is a nonempty subset of* $\mathcal{E}$, *a multifunction* $\mathcal{G} : D \multimap \mathcal{E}$ *is said to be* condensing *with respect to a measure of noncompactness* $\beta : \mathcal{P}_b(\mathcal{E}) \to \mathbb{R}^n_{0,+}$ *(*β*-condensing, for short) if* $\mathcal{G}(D)$ *is bounded and for every* $\Omega \in \mathcal{P}_b(D)$, *the inequality* $\beta(\Omega) \preccurlyeq \beta(\mathcal{G}(\Omega))$ *implies that* $\beta(\Omega) = 0_n$, *where* $\preccurlyeq$ *denotes the partial ordering induced in* $\mathbb{R}^n$ *by its normal cone* $\mathbb{R}^n_{0,+}$.

Equivalently, $\mathcal{G}$ is β-condensing if $\mathcal{G}(D)$ is bounded and for every $\Omega \in \mathcal{P}_b(D)$, the inequality $0_n \prec \beta(\Omega)$ implies that $\beta(\Omega) \not\preccurlyeq \beta(\mathcal{G}(\Omega))$, i.e., $\beta(\mathcal{G}(\Omega)) \prec \beta(\Omega)$ or $\beta(\mathcal{G}(\Omega))$, $\beta(\Omega)$ are not comparable.

Proposition 1 ([18], Proposition 3.5.1)**.** *Let* X *be a closed subset of* $\mathcal{E}$ *and* $\mathcal{G} : X \multimap \mathcal{E}$ *be a closed multimap with compact values and* β*-condensing on every bounded subset of* X, *where* β *is a monotone measure of noncompactness in* $\mathcal{E}$. *If the set* $Fix\, \mathcal{G} := \{x \in X : x \in \mathcal{G}(x)\}$ *is bounded, then it is compact.*

Let E be a real Banach space endowed with the norm $\|\cdot\|$.

For $[a,b] \subset \mathbb{R}$, we denote by $C([a,b];E)$ the space of all the E-valued continuous functions defined on $[a,b]$ endowed with the supremum norm $\|\cdot\|_{C([a,b];E)}$, and by $L^p([a,b];E)$, the space of all functions $v : [a,b] \to E$ with v^p is Bochner integrable equipped with the norm $\|v\|_{L^p([a,b];E)} = \left(\int_{[a,b]} \|v(z)\|^p \, dz \right)^{\frac{1}{p}}$ (if $E = \mathbb{R}$, $L^p([a,b])$ and $\|v\|_{L^p}$, respectively), $p \geq 1$.

Definition 5. *A countable set* $\mathcal{M} \subset L^1([a,b];E)$ *is said to be* integrably bounded *if there exists* $\omega \in L^1_+([a,b])$ *such that for every* $\mu \in \mathcal{M}$, *it is*

$$\|\mu(t)\| \leq \omega(t), \text{ a.e. } t \in [a,b].$$

Proposition 2 ([19], Proposition 3.1). *If $\mathcal{M} \subset L^1([a,b],E)$ is a countable and integrably bounded set, then the function $\chi(\mathcal{M}(\cdot))$ belongs to $L^1_+([a,b])$ and satisfies the inequality*

$$\chi\left(\int_a^b \mathcal{M}(s)\,ds\right) \leq 4\int_a^b \chi(\mathcal{M}(s))\,ds\,.$$

Definition 6. *A countable set $\{f_n\}_n \subset L^1([a,b];E)$ is said to be semicompact if*

(i) *It is integrably bounded;*
(ii) *The set $\{f_n(t)\}_n$ is relatively compact for a.e., $t \in [a,b]$.*

Proposition 3 (cf. [18], Proposition 4.2.1). *If $\{f_n\}_n \subset L^1([a,b];E)$ is a semicompact sequence, then it is weakly compact in $L^1([a,b];E)$.*

Let $T > 0$ and define $\Delta_0 := \{(t,s) : 0 \leq s \leq t \leq T\}$. Following [20], we report the next definitions.

Definition 7. *A family $\{U(t,s)\}_{(t,s)\in\Delta_0}$ of bounded linear operators on E is a strongly continuous evolution system (evolution system, for short) if*

$U(s,s) = I, \quad U(t,r)U(r,s) = U(t,s)$ *for* $0 \leq s \leq r \leq t \leq T$;
for every $x \in E$, the map $\xi_x : (t,s) \in \Delta_0 \mapsto U(t,s)x$ is continuous.

Definition 8. *A family of linear operators $\{A(t)\}_{t\in[0,T]}$, $A(t) : D(A) \subset E \to E$, $D(A)$ dense subset of E not depending on t, generates an evolution system $\{U(t,s)\}_{(t,s)\in\Delta_0}$ if (see, e.g., [21])*

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s) \quad \text{and} \quad \frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s), \quad (t,s) \in \Delta_0.$$

Moreover, if $\mathcal{L}(E)$ is the space of all bounded linear operators from E to E furnished with the strong operator topology, then for an evolution system, the next condition holds:

$$\exists D \geq 1 \ : \ \|U(t,s)\|_{\mathcal{L}(E)} \leq D, \ (t,s) \in \Delta_0. \tag{8}$$

Let us consider the generalized Cauchy operator $G : L^1([a,b];E) \to C([a,b];E)$,

$$Gf(t) \ = \ \int_a^t U(t,s)f(s)\,ds, \ t \in [a,b]. \tag{9}$$

By ([22], Theorem 2) joined with, respectively, ([18], Theorem 5.1.1) and ([18], Theorem 4.2.2), we can claim that the following results hold:

Proposition 4. *For every semicompact set $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b];E)$, the set $\{Gf_n\}_{n=1}^{\infty}$ is relatively compact in $C([a,b];E)$ and, moreover, if $f_n \rightharpoonup \bar{f}$ then $Gf_n \to G\bar{f}$.*

Proposition 5. *If $\{f_n\}_n \subset L^1([a,b];E)$ is integrably bounded and there exists $q \in L^1([a,b])$ such that*

$$\chi(\{f_n(t)\}_n) \leq q(t), \ a.e., \ t \in [a,b],$$

then

$$\chi(\{Gf_n(t)\}_n) \leq 2D\int_a^t q(s)ds,$$

where D is from (8).

3. The Impulsive Integro-Differential Problem in Banach Spaces

Let E be a real Banach space, $T > 0$, and $\{t_0,\ldots,t_{m+1}\}$ with $m > 0$ be a set of fixed real numbers such that $0 \leq t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$. By the symbol $\mathcal{PC}([t_0,T];E)$, we denote the Banach space

$$\mathcal{PC}([t_0,T];E) := \left\{ \begin{array}{ll} y:[t_0,T]\to E\ : & y_{|\,[t_0,t_1]} \text{ continuous;} \\ & y_{|\,]t_{j-1},t_j]} \text{ continuous, for all } j=2,\dots,m+1; \\ & \exists\, y\left(t_j^+\right) = \lim\limits_{h\to 0^+} y(t_j+h)\in E, \textit{all}\text{for all } j=1,\dots,m \end{array} \right\}$$

endowed with the norm $\|y\|_\infty = \sup_{t\in[t_0,T]}\|y(t)\|$.

Let $y_0 \in E$ be fixed and consider the corresponding initial value problem driven by a semilinear integro-differential inclusion subject to impulses $I_j : E \to E$, $j = 1,\dots,m$ at the given times $\{t_j : j = 1,\dots,m\}$

$$(P)\ \left\{ \begin{array}{l} y'(t) \in A(t)y(t) + F\left(t, y(t), \int_{t_0}^t k(t,s)y(s)ds\right),\ t\in[t_0,T],\ t\neq t_j,\ j=1,\dots,m, \\ \\ y(t_0) = y_0, \\ \\ y(t_j^+) = y(t_j) + I_j(y(t_j)),\ j=1,\dots,m, \end{array} \right.$$

where $\{A(t)\}_{t\in[0,T]}$ is a family of linear operators, $A(t) : D(A) \subset E \to E$, $D(A)$ is a dense subset of E not depending on t; $F : [t_0,T]\times E\times E \multimap E$ is a given multimap; $k : \Delta \to \mathbb{R}^+$, with $\Delta := \{(t,s) : t_0 \le s \le t \le T\}$, is a given kernel.

For the functions involved in the problem, we assume the following:

(A) The family $\{A(t)\}_{t\in[0,T]}$ of densely defined linear operators generates an evolution system $\{U(t,s)\}_{(t,s)\in\Delta_0}$;

(k) The kernel k is continuous, and we put

$$M := \max_{(t,s)\in\Delta} k(t,s); \tag{10}$$

(I) The impulse functions $I_1,\dots,I_m$ are continuous.

Further, on the nonlinear multifunction F, we will suppose that it satisfies the following properties:

(F1) F takes compact and convex values;
(F2) For every $v, w \in E$, the multimap $F(\cdot, v, w)$ admits a strongly measurable selection;
(F3) For a.e., $t \in [t_0,T]$, the multimap $F(t,\cdot,\cdot)$ is upper semicontinuous;
(F4) There exists a nonnegative function $\alpha \in L^1([t_0,T])$ such that

$$\|F(t,v,w)\| \le \alpha(t)(1+\|v\|+\|w\|), \tag{11}$$

for a.e., $t\in[t_0,T]$ and all $v, w \in E$;
(F5) There exists a nonnegative function $h \in L^1([t_0,T])$ such that

$$\chi(F(t,\Omega_1,\Omega_2)) \le h(t)[\chi(\Omega_1)+\chi(\Omega_2)], \tag{12}$$

for a.e., $t\in[t_0,T]$ and every bounded $\Omega_1,\Omega_2 \subset E$.

Definition 9. *A function $y \in \mathcal{PC}([t_0,T];E)$ is said to be a mild solution to (P) if*

$$y(t) = U(t,t_0)y_0 + \sum_{t_0<t_j<t} U(t,t_j)I_j(y(t_j)) + \int_{t_0}^t U(t,s)f(s)\,ds,\ t\in[t_0,T], \tag{13}$$

where $f : [t_0,T]\to E$ is a L^1-function on $[t_0,T]$ such that

$$f(s) \in F\left(s, y(s), \int_{t_0}^s k(s,r)y(r)dr\right) \textit{ for a.e., } s\in[t_0,T],$$

with the agreement that $\sum_{t_0<t_j<t} U(t,t_j)I_j(y(t_j)) = 0$ if $t\in[t_0,t_1]$.

Note that every mild solution also satisfies the conditions

$y(t_0) = y_0$;
$y(t_j^+) = y(t_j) + I_j(y(t_j))$, $j = 1, \dots, m$.

4. Compactness of the Mild Solutions Set in the Non-Impulsive Case

To obtain optimal solutions for problem (P), we aim to demonstrate the compactness of the set of its mild solutions. In the proof, we will use the compactness of the set of mild solutions of non-impulsive problems. This result for integro-differential problems like the one we are studying is not already present in the literature, at least as far as we know. Clearly, it potentially has relevance in itself, which is why we dedicate this paragraph to it.

Let us consider the following non-impulsive Cauchy problem

$$(P_v) \begin{cases} y'(t) \in A(t)y(t) + F\Big(t, y(t), \int_a^t k(t,s)y(s)ds\Big), & t \in [a,b], \\ y(a) = v, \end{cases}$$

where $[a,b]$ is a given subinterval of $[t_0, T]$, and $v \in E$.

A mild solution of (P_v) is a function $y \in C([a,b]; E)$ such that

$$y(t) \ = \ U(t,a)v + \int_a^t U(t,s)f(s)\,ds, \ t \in [a,b], \tag{14}$$

with $f \in L^1([a,b]; E)$ and such that $f(s) \in F\big(s, y(s), \int_a^s k(s,r)y(r)dr\big)$ for a.e., $s \in [a,b]$.

We put

$$\mathcal{S}_v := \{y \in C([a,b]; E) : y \text{ mild solution of } (P_v)\}. \tag{15}$$

We recall the following result on the weak closeness of the superposition operator for multifunctions involving the Volterra operator.

Lemma 1 ([19], Lemma 5.1). *Assume that k satisfies (k) and that for F, properties (F1)–(F5) hold. Then, the operator* $\mathcal{N}_F : C([a,b], E) \multimap L^1([a,b], E)$,

$$\mathcal{N}_F(y) := \{f \in L^1([a,b], E) : f(t) \in F(t, y(t), \textstyle\int_a^t k(t,s)y(s)\,ds), \ a.e.\ t \in [a,b]\}, \tag{16}$$

is correctly defined.

Moreover, if we consider sequences $(y_n)_n$, $y_n \in C([a,b], E)$, $(f_n)_n$, $f_n \in \mathcal{N}_F(y_n)$, $n \in \mathbb{N}$, *such that* $y_n \to \bar{y}$, *and* $f_n \rightharpoonup \bar{f}$, *then* $\bar{f} \in \mathcal{N}_F(\bar{y})$.

We also need the next technical result inspired by ([23], Lemma 2.1).

Lemma 2. *For every* $H, K > 0$ *and* $v, w \in L^1_+([a,b])$, *there exists* $\bar{n} = \bar{n}(H, K, v, w) \in \mathbb{N}$ *such that*

$$p_{\bar{n}} := \max_{t \in [a,b]} e^{-\bar{n}t} \left[\int_a^t He^{\bar{n}s} v(s)ds + \int_a^t \int_a^s Ke^{\bar{n}r} w(r)drds \right] < 1.$$

Proof. For every $n \in \mathbb{N}$, we consider

$$p_n := \max_{t \in [a,b]} e^{-nt} \left[\int_a^t He^{ns} v(s)ds + \int_a^t \int_a^s Ke^{nr} w(r)drds \right].$$

By the properties of the supremum, there exists $t_n \in [a,b]$ such that

$$\begin{aligned} p_n - \frac{1}{n} &< e^{-nt_n} \left[\int_a^{t_n} He^{ns} v(s)ds + \int_a^{t_n} \int_a^s Ke^{nr} w(r)drds \right] \\ &= \int_a^b \kappa_{[a,t_n]}(s) He^{-n(t_n - s)} v(s)ds + \int_a^b \kappa_{[a,t_n]}(s) \left(\int_a^s Ke^{-n(t_n - r)} w(r)dr \right) ds, \end{aligned} \tag{17}$$

where $\kappa_{[a,t_n]}$ is the characteristic function of interval $[a, t_n]$.

Now, let us put

$$\phi_n(s) := \kappa_{[a,t_n]}(s) H e^{-n(t_n - s)} v(s), \tag{18}$$

$$\psi_n(s) := \kappa_{[a,t_n]}(s) \int_a^s K e^{-n(t_n - r)} w(r) dr, \tag{19}$$

a.e., $s \in [a, b]$.

Notice that both $(\phi_n)_n$ and $(\psi_n)_n$ a.e. pointwise converge to zero. Indeed, the sequence $\{t_n\}_n \subset [a, b]$, eventually passing to a subsequence, converges to an element $t^* \in [a, b]$.

Clearly, if $s > t^*$, then definively $s > t_n$ as well, so $\kappa_{[a,t_n]}(s) = 0$, and it holds that

$$\lim_{n \to +\infty} \phi_n(s) = 0 \text{ and } \lim_{n \to +\infty} \psi_n(s) = 0.$$

On the other hand, if $s < t^*$, then definitively $s < t_n$. In this case,

$$\lim_{n \to +\infty} \phi_n(s) = \lim_{n \to +\infty} H e^{-n(t_n - s)} v(s) = 0$$

and, by the Lebesgue dominated convergence theorem,

$$\lim_{n \to +\infty} \psi_n(s) = \lim_{n \to +\infty} \int_a^s K e^{-n(t_n - r)} w(r) dr = 0.$$

Now, by (18) and (19), we have, respectively,

$$\|\phi_n(s)\| \leq H v(s) \text{ and } \|\psi_n(s)\| \leq \int_a^s K w(r) dr, \text{ a.e., } s \in [a, b].$$

Hence, we can use the Lebesgue dominated convergence theorem and pass the limit under the integral sign in (17), so that

$$0 \leq p_n \leq \frac{1}{n} + \int_a^b \phi_n(s) ds + \int_a^b \psi_n(s) ds \to 0.$$

Thus, $\lim_{n \to +\infty} p_n = 0$, from which the existence of $\bar{n} \in \mathbb{N}$ such that $p_{\bar{n}} < 1$. □

We can now state and prove the compactness result for the solutions set in the non-impulsive case. The proof is based on the use of Proposition 1.

Theorem 1. *Suppose that $\{A(t)\}_{t \in [0,T]}$, F and k, respectively, satisfy (A), (F1)–(F5), and (k). Then, the set of all mild solutions of (P_v) is a nonempty compact subset of $C([a, b]; E)$.*

Proof. Firstly, we notice that the solutions set $\mathcal{S}_v$ is nonempty. Indeed, the existence of mild solutions to (P_v) can be deduced, albeit for different reasons, both from ([6], Corollary 1) and from ([19], Theorem 5.1).

Let us show that $\mathcal{S}_v$ is bounded in the Banach space $C([a, b]; E)$.

For any fixed $y \in \mathcal{S}_v$, let $f \in L^1([a, b]; E)$ be an a.e., selector of $F(\cdot, y(\cdot), \int_a^{(\cdot)} k(\cdot, r) y(r) dr)$ for which y has the representation (14). Then, for every $t \in [a, b]$, by (8) and (F4), we obtain

$$\|y(t)\| \leq D\|v\| + \int_a^t D\alpha(s) \left(1 + \|y(s)\| + \int_a^s k(s, r) \|y(r)\| dr\right) ds. \tag{20}$$

Let us define the real positive function $m : [a, b] \to \mathbb{R}$,

$$m(t) := \sup_{a \leq s \leq t} \|y(s)\|, \ t \in [a, b]. \tag{21}$$

Thus, by (20), (10), and (21), for every $s \in [a,t]$, we have the estimate

$$\begin{aligned} \|y(s)\| &\leq D(\|v\| + \|\alpha\|_{L^1}) + \int_a^s D\alpha(r)\big(\|y(r)\| + M\textstyle\int_a^r \|y(\tau)\| d\tau\big)\, dr \\ &\leq D(\|v\| + \|\alpha\|_{L^1}) + \int_a^s D\alpha(r)[1 + M(b-a)]m(r)\, dr. \end{aligned}$$

By the monotonicity of the supremum and using again (21), for every $t \in [a,b]$, we have

$$\begin{aligned} m(t) &\leq \sup_{a\leq s\leq t}\left[D(\|v\| + \|\alpha\|_{L^1}) + \int_a^s D\alpha(r)[1 + M(b-a)]m(r)\, dr\right] \\ &\leq D(\|v\| + \|\alpha\|_{L^1}) + \int_a^t D\alpha(r)[1 + M(b-a)]m(r)\, dr. \end{aligned}$$

We can therefore apply the Gronwall inequality and obtain

$$m(t) \leq D(\|v\| + \|\alpha\|_{L^1})e^{D[1+M(b-a)]\|\alpha\|_{L^1}} := H. \tag{22}$$

It implies that $\|y(t)\| \leq H$ for every $t \in [a,b]$; hence,

$$\|y\|_{C([a,b];E)} \leq H.$$

From the arbitrariness of y, we obtain the boundedness of $\mathcal{S}_v$.

Now, in the Banach space $C([a,b];E)$, we consider the closed set

$$X := \{z \in C([a,b];E) : \|z\|_{C([a,b];E)} \leq H\}, \tag{23}$$

and define the multioperator $\Gamma : X \multimap C([a,b];E)$ as

$$\Gamma(y) = \left\{ z \in C([a,b];E) : \begin{array}{l} z(t) = U(t,a)v + \int_a^t U(t,s)f(s)ds,\ t \in [a,b], \\ \text{all } f \in L^1([a,b];E),\ f(s) \in F(s,y(s),\ \int_a^s k(s,r)y(r)dr) \text{ a.e. } s \in [a,b] \end{array} \right\}. \tag{24}$$

The multimap Γ is actually the solution multioperator to (P_v), because $Fix\,\Gamma = \mathcal{S}_v$.

This identity and what is shown above yield that $Fix\,\Gamma$ is a nonempty and bounded subset of $C([a,b];E)$.

We prove now that Γ takes compact values.

Let y be arbitrarily fixed in X and let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $C([a,b];E)$ such that $z_n \in \Gamma(y)$ for all $n \in \mathbb{N}$. Then, consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1([a,b];E)$ such that

$$z_n(t) = U(t,a)v + \int_a^t U(t,s)f_n(s)ds,\ t \in [a,b], \tag{25}$$

with

$$f_n(s) \in F(s,y(s),\textstyle\int_a^s k(s,r)y(r)dr),\ a.e.,\ s \in [a,b].$$

The set $\{f_n\}_{n\in\mathbb{N}}$ is integrably bounded. Indeed, by (F4) and (10), and recalling that $y \in X$ (see (23)), for a.e., $t \in [a,b]$, we obtain

$$\begin{aligned} \|f_n(t)\| &\leq \alpha(t)(1 + \|y(t)\| + \textstyle\int_a^t M\|y(s)\|ds) \\ &\leq \alpha(t)(1 + H + MH(b-a)), \end{aligned} \tag{26}$$

from which the integrably boundedness of $\{f_n\}_{n\in\mathbb{N}}$. Further, $\{f_n(t)\}_{n\in\mathbb{N}}$ is relatively compact for a.e., $t \in [a,b]$, since by the monotonicity of the Hausdorff measure of noncompactness and (F5), it is

$$\chi(\{f_n(t)\}_{n\in\mathbb{N}}) \leq h(t)[\chi(\{y(t)\}) + \chi(\{\textstyle\int_a^t k(t,s)y(s)ds\})] = 0,\ a.e.\ t \in [a,b].$$

Hence, we can apply Proposition 3, so that $\{f_n\}_{n\in\mathbb{N}}$ is weakly compact in $L^1([a,b];E)$. Without loss of generality, we can say that there exists $\tilde{f} \in L^1([a,b];E)$ such that $f_n \rightharpoonup \tilde{f}$ in $L^1([a,b];E)$.

Therefore, by Proposition 4, it follows that $Gf_n \to G\tilde{f}$ in $C([a,b];E)$ (see (9)). This implies that the sequence $(z_n)_{n\in\mathbb{N}}$ converges in $C([a,b];E)$ to the function

$$\tilde{z}(t) := U(t,a)v + \int_a^t U(t,s)\tilde{f}(s)ds,\ t \in [a,b].$$

By applying Lemma 1 to the sequence $\{f_n\}_{n\in\mathbb{N}}$, we have that

$$\tilde{f}(s) \in F(s, y(s), \textstyle\int_a^s k(s,r)y(r)dr),\ a.e.\ s \in [a,b],$$

so that $\tilde{z} \in \Gamma(y)$ (see (24)). Thus, $\Gamma(y)$ is compact.

Now, we prove that Γ is a closed multimap.

Let us consider the sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ with $y_n \in X$ and $z_n \in \Gamma(y_n)$ for all $n \in \mathbb{N}$ such that $y_n \to \bar{y}$ and $z_n \to \bar{z}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1([a,b];E)$ such that for each z_n, the representation (25) holds, where this time

$$f_n(s) \in F(s, y_n(s), \textstyle\int_a^s k(s,r)y_n(r)dr),\ a.e.,\ s \in [a,b].$$

The estimate (26) holds also for this sequence $\{f_n\}_n$, so its integrably boundedness follows.

Further, by (F5), we have

$$\chi(\{f_n(t)\}_{n\in\mathbb{N}}) \le h(t)[\chi(\{y_n(t)\}_{n\in\mathbb{N}}) + \chi(\{\textstyle\int_a^t k(t,s)y_n(s)ds\}_{n\in\mathbb{N}})],\ a.e.\ t \in [a,b]. \quad (27)$$

The convergence $y_n \to \bar{y}$ allows us to say that

$$y_n(t) \to \bar{y}(t) \quad \text{and} \quad \textstyle\int_a^t k(t,s)y_n(s)ds \to \int_a^t k(t,s)\bar{y}(s)ds, \quad \text{for all } t \in [a,b].$$

So both

$$\chi(\{y_n(t)\}_{n\in\mathbb{N}}) = 0 \quad \text{and} \quad \chi(\{\textstyle\int_a^t k(t,s)y_n(s)ds\}_{n\in\mathbb{N}}) = 0, \quad \text{for all } t \in [a,b],$$

and then by (27), it is $\chi(\{f_n(t)\}_{n\in\mathbb{N}}) = 0$ for a.e., $t \in [a,b]$, i.e., the relative compactness of the sets $\{f_n(t)\}_{n\in\mathbb{N}}$ for a.e., $t \in [a,b]$.

With the same reasoning as above, by Proposition 3, there exists $\bar{f} \in L^1([a,b];E)$ such that $f_n \rightharpoonup \bar{f}$ (eventually passing to a subsequence), and by Proposition 4, it holds that $Gf_n \to G\bar{f}$ in $C([a,b];E)$, from which

$$z_n(t) \to U(t,a)v + \int_a^t U(t,s)\bar{f}(s)ds,\ t \in [a,b].$$

Invoking the uniqueness of the limit, we have

$$\bar{z}(t) = U(t,a)v + \int_a^t U(t,s)\bar{f}(s)ds,\ t \in [a,b].$$

Also, in this case, we can use Lemma 1 and then deduce that $\bar{z} \in \Gamma(\bar{y})$.

Let us put

$$p_L := \max_{t\in[a,b]} e^{-Lt}\left[2D\int_a^t e^{Ls}h(s)ds + 8MD\int_a^t\int_a^s e^{Lr}h(r)drds\right],\ L > 0 \quad (28)$$

where D, h, M are from (8), (F5), and (10), respectively. By Lemma 2, there exists $L > 0$ large enough to have $p_L < 1$. For such an L, we consider the corresponding monotone measure of noncompactness ν_L on $C([a,b];E)$ (cf. (5)).

We are going to show that Γ is ν_L condensing. First, we fixed an arbitrary bounded set $\Omega \subset X$ such that

$$\nu_L(\Omega) \preccurlyeq \nu_L(\Gamma(\Omega)), \tag{29}$$

we have to show that $\nu_L(\Omega) = (0,0)$.

To this aim, let $\{z_n\}_{n\in\mathbb{N}} \subset \Gamma(\Omega)$ be a countable set where the maximum $\nu_L(\Gamma(\Omega))$ is achieved, and let $\{y_n\}_{n\in\mathbb{N}} \subset \Omega$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^1([a,b];E)$ with

$$f_n(s) \in F(s, y_n(s), \textstyle\int_a^s k(s,r)y_n(r)dr),\ a.e.,\ s \in [a,b],$$

be such that the representation (25) of z_n holds for every $n \in \mathbb{N}$. Thus, bearing in mind the definition of ν_L, it is immediate from (29) that

$$(\gamma(\{y_n\}_n), \eta(\{y_n\}_n)) \preccurlyeq \nu_L(\Omega) \preccurlyeq \nu_L(\gamma(\Omega)) = (\gamma(\{z_n\}_n), \eta(\{z_n\}_n)). \tag{30}$$

Let us show that $\gamma(\{z_n\}_n) = 0$ (cf. (6) for the definition of γ).

From (30), we can immediately say that

$$\gamma(\{y_n\}_n) \leq \gamma(\{z_n\}_n). \tag{31}$$

Of course, the estimate (26) holds for the sequence $\{f_n\}_n$, and hence it is integrably bounded. Further,

$$\begin{aligned} \chi(\{f_n(s)\}_{n\in\mathbb{N}}) &\leq h(s)[\chi(\{y_n(s)\}_n) + \chi(\{\textstyle\int_a^s k(s,r)y_n(r)dr\}_n)] \\ &\leq e^{Ls}h(s)\gamma(\{y_n\}_n),\ a.e.\ s \in [a,b], \end{aligned} \tag{32}$$

hence, by Proposition 5, we can write

$$\chi\left(\left\{\int_a^t U(t,s)f_n(s)ds\right\}_n\right) \leq 2D\int_a^t e^{Ls}h(s)\gamma(\{y_n\}_n)ds,\ t \in [a,b],$$

implying the estimate

$$\chi(\{z_n(t)\}_n) \leq 2D\gamma(\{y_n\}_n)\int_a^t e^{Ls}h(s)ds,\ t \in [a,b].$$

On the other hand, in our setting, we can apply Proposition 2 and obtain

$$\chi(\{\textstyle\int_a^t k(t,s)z_n(s)ds\}_n) \leq 8MD\gamma(\{y_n\}_n)\int_a^t\int_a^s e^{Lr}h(r)drds,\ t \in [a,b].$$

Therefore, recalling the defintions of function γ and number p_L (see (6) and (28), respectively), we have

$$\begin{aligned} \gamma(\{z_n\}_n) &\leq \gamma(\{y_n\}_n)\sup_{t\in[a,b]} e^{-Lt}\left[2D\int_a^t e^{Ls}h(s)ds + 8MD\int_a^t\int_a^s e^{Lr}h(r)drds\right] \\ &= \gamma(\{y_n\}_n)p_L. \end{aligned}$$

From this and by (31), we obtain $\gamma(\{y_n\}_n) \leq \gamma(\{z_n\}_n) \leq \gamma(\{y_n\}_n)p_L$.

Since $p_L < 1$, it follows that

$$\gamma(\{y_n\}_n) = 0 \tag{33}$$

and, as a consequence of the same inequality, also

$$\gamma(\{z_n\}_n) = 0. \tag{34}$$

We prove now that $\eta(\{z_n\}_n) = 0$ (cf. (7) for the definition of η). First of all, from (33), we deduce that $\chi(\{y_n(t)\}_n) = 0$, for every $t \in [a,b]$. Moreover, we know that the set $\{f_n\}_n$

is integrably bounded (see above), and by (32) and (33), we have $\chi(\{f_n(t)\}_n) = 0$ for a.e., $t \in [a,b]$. So $\{f_n\}_n$ is semicompact. By Propositions 3 and 4, we have the convergence of the sequence $(Gf_n)_n$ (see (9)). Hence, the set $\{Gf_n\}_n$ is relatively compact in $C([a,b],E)$ and hence, it is equicontinuous, so that

$$mod_C(\{z_n\}_n) = mod_C(\{Gf_n\}_n) = 0.$$

As for the other term of η, from the continuity of the Volterra operator, we have that

$$mod_C(\{\int_a^{(\cdot)} k(\cdot,s)z_n(s)ds\}_n) \ = \ 0.$$

We hence achieve

$$\eta(\{z_n\}_n) = 0. \tag{35}$$

Therefore, by (34), (35), and (30), we obtain $\nu_L(\Omega) = (0,0)$ as desired.

We have shown that all the hypotheses of Proposition 1 are satisfied, allowing the compactness of the set $\mathcal{S}_v = Fix\,\Gamma$. □

5. Existence of Optimal Solutions for Impulsive Integro-Differential Problems

We are here interested in the minimization or maximization of a cost functional of problem (P), say

$$\mathcal{J} : \mathcal{PC}([t_0,T];E) \to \mathbb{R}.$$

To this aim, in this section, we state and prove the compactness of the set of all mild solutions of problem (P). We preface the following lemma, which can be immediately deduced by ([6], Lemma 1).

Lemma 3. *Assume that F and k, respectively, satisfy (F1)–(F5) and (k).*

Then, for every $j = 1,\dots,m$ and every set of functions $\{y_i \in C([t_i,t_{i+1}],E) : i = 0,\dots, j-1\}$, the multimap $F_j : [t_j,t_{j+1}] \times E \times E \multimap E$ defined by

$$F_j(t,v,w) := F\left(t,v,w + \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} k(t,s)y_i(s)ds\right),\ t \in [t_j,t_{j+1}], v,w \in E \tag{36}$$

satisfies (F1)–(F5).

Theorem 2 (Compactness of the Mild Solutions Set under the Impulses' Effect)**.** *Suppose that $\{A(t)\}_{t\in[0,T]}$, F, k, and $I_1,\dots,I_m$, respectively, satisfy (A), (F1)–(F5), (k), and (I).*

Then, the set of all mild solutions of problem (P) is a nonempty compact subset of $\mathcal{PC}([t_0,T];E)$.

Proof. Let us denote the set of all mild solutions of (P) as $\mathcal{S}$. It is a nonempty set (cf. [6], Theorem 1).

In order to prove the compactness of the set of solutions $\mathcal{S}$, we suppose that the number of impulse times is $m = 1$. Clearly, if $m > 1$, we will only have to iterate the procedure a finite number of times to achieve the same conclusions.

We proceed by the following steps.

Step 1.

Let us consider the non-impulsive Cauchy problem

$$(P_1)\begin{cases} y'(t) \in A(t)y(t) + F\left(t,y(t),\int_{t_0}^t k(t,s)y(s)ds\right), & t \in [t_0,t_1], \\ y(t_0) = y_0. \end{cases}$$

We can apply Theorem 1 to (P_1) with $[a,b] = [t_0,t_1]$ and $v = y_0$, so that the set

$$\mathcal{S}_1 := \{y \in C([t_0,t_1];E) : y \text{ is a mild solution to } (P_1)\}$$

is a nonempty compact subset of $C([t_0, t_1]; E)$.

Step 2.

Let us fix any $y_1 \in \mathcal{S}_1$ and consider the problem

$$(P_2)_{y_1} \begin{cases} y'(t) \in A(t)y(t) + F_1\Big(t, y(t), \int_{t_1}^{t} k(t,s)y(s)ds\Big), & t \in [t_1, t_2], \\ y(t_1) = y_1(t_1) + I_1(y_1(t_1)), \end{cases}$$

where $F_1 : [t_1, t_2] \times E \times E \multimap E$ is defined by

$$F_1(t, v, w) := F\Big(t, v, w + \int_{t_0}^{t_1} k(t,s)y_1(s)ds\Big),\ t \in [t_1, t_2],\ v, w \in E. \tag{37}$$

Clearly, problem $(P_2)_{y_1}$ is of the type (P_v), just taking $[a, b] = [t_1, t_2]$, $v = y_1(t_1) + I_1(y_1(t_1))$, and $F = F_1$. Since by Lemma 3, the multimap F_1 satisfies properties (F1)–(F5), we can apply Theorem 1 and claim that the set

$$\mathcal{S}_{2,y_1} := \{y \in C([t_1, t_2]; E) : y \text{ is a mild solution to } (P_2)_{y_1}\}$$

is a nonempty compact subset of $C([t_1, t_2]; E)$.

Step 3.

Put

$$\mathcal{S}_2 := \cup_{y_1 \in \mathcal{S}_1} \mathcal{S}_{2,y_1},$$

we are going to show that it is a compact subset of $C([t_1, t_2]; E)$ as well.

To this aim, let us consider the multimap $\Phi_1 : \mathcal{S}_1 \multimap C([t_1, t_2]; E)$ given by

$$\Phi_1(y) = \mathcal{S}_{2,y},\ y \in \mathcal{S}_1. \tag{38}$$

From what was shown before, we know that this multimap takes nonempty compact values.

Let us prove that it is an upper semicontinuous multifunction. Suppose, on the contrary, that there exist $\bar{y} \in \mathcal{S}_1$, $\varepsilon_0 > 0$, and sequences $\{y_n\}_n \subset \mathcal{S}_1$ with $y_n \to \bar{y}$, $\{w_n\}_n \subset C([t_1, t_2], E)$ with $w_n \in \Phi_1(y_n)$ for every $n \in \mathbb{N}$, such that

$$w_n \notin V_{\varepsilon_0}(\Phi_1(\bar{y})),\ n \in \mathbb{N}, \tag{39}$$

where V_{ε_0} is a ε_0-neighborhood of $\Phi_1(\bar{y})$.

Clearly, since w_n belongs to $\Phi_1(y_n) = \mathcal{S}_{2,y_n}$ for every $n \in \mathbb{N}$, then there exists a sequence $\{f_n\}_n \subset L^1([t_1, t_2], E)$ with (cf. (37))

$$f_n(s) \in F_1(s, w_n(s), \textstyle\int_{t_1}^{s} k(s,r)w_n(r)\,dr),\ \text{a.e., } s \in [t_1, t_2],$$

and such that

$$w_n(t) = U(t, t_1)T_1(y_n) + \int_{t_1}^{t} U(t,s)f_n(s)\,ds,\ t \in [t_1, t_2],\ n \in \mathbb{N}, \tag{40}$$

where the function $T_1 : \mathcal{S}_1 \to E$ is defined by

$$T_1(y) = y(t_1) + I_1(y(t_1)),\ y \in \mathcal{S}_1.$$

It is easy to see that according to the continuity of I_1, the mapping T_1 is continuous.

The set $\{w_n\}_n$ is bounded in $C([t_1, t_2]; E)$. In fact, if we fixed $n \in \mathbb{N}$ and put

$$m_n(t) := \sup_{t_1 \le s \le t} \|w_n(s)\|,\ t \in [t_1, t_2],$$

by using the same arguments as in the proof of Theorem 1, we have

$$m_n(t) \leq D(\|T_1(y_n)\| + \|\alpha\|_{L^1}) + \int_{t_1}^{t} D\alpha(s)[1 + M(t_2 - t_1)]m_n(s)\,ds.$$

By the continuity of T_1 and the convergence of $\{y_n\}_n$, the set $\{T_1(y_n)\}_n$ is bounded. Thus, by the Gronwall inequality, the boundedness of $\{w_n\}_n$ follows.

Moreover, by (F4) of F_1 and by (10), we have

$$\begin{aligned}\|f_n(t)\| &\leq \|F_1(t, w_n(t), \textstyle\int_{t_1}^{t} k(t,s)w_n(s)\,ds)\| \\ &\leq \alpha(t)\Big(1 + \|w_n(t)\| + \textstyle\int_{t_1}^{t_2} M\|w_n(s)\|\,ds\Big),\ a.e.\ t \in [t_1, t_2].\end{aligned}$$

So, recalling the boundedness of $\{w_n\}_n$, we obtain that the integrably boundedness of the sequence $\{f_n\}_n$.

Further, the set $\{f_n(t)\}_n$ is relatively compact for a.e., $t \in [t_1, t_2]$. Indeed, by the properties of the Hausdorff measure of noncompactness, (F5) of F_1 and recalling Definition (6), we have the estimate

$$\begin{aligned}\chi(\{f_n(t)\}_n) &\leq \chi\Big(\{F_1(t, w_n(t), \textstyle\int_{t_1}^{t} k(t,s)w_n(s)\,ds)\}_n\Big) \\ &\leq h(t)\Big[\chi(\{w_n(t)\}_n) + \chi\Big(\{\textstyle\int_{t_1}^{t} k(t,s)w_n(s)\,ds\}_n\Big)\Big] \\ &\leq e^{Lt}h(t)\gamma(\{w_n\}_n),\ \text{a.e., } t \in [t_1, t_2],\end{aligned}$$

which is analogous to (32). So, with the same reasonings as in the proof of Theorem 1, we can claim that $\gamma(\{w_n\}_n) = 0$ (cf. (33)). Thus, $\chi(\{f_n(t)\}_n) = 0$ for a.e., $t \in [t_1, t_2]$.

Therefore, we have that the set $\{f_n\}_n$ is semicompact.

Now, considering the generalized Cauchy operator on $[t_1, t_2]$, i.e., $G_1 : L^1([t_1, t_2]; E) \to C([t_1, t_2]; E)$,

$$G_1 f(t) = \int_{t_1}^{t} U(t,s)f(s)\,ds,\ t \in [t_1, t_2], \tag{41}$$

by Proposition 4, we have that the set $\{G_1 f_n\}_n$ is relatively compact. Further, since T_1 is continuous, we have both

$$y_n \to \bar{y} \ \text{ and } \ T_1(y_n) \to T_1(\bar{y}).$$

We can hence say that the set $\{w_n\}_n$ is relatively compact in $C([t_1, t_2]; E)$. Therefore, there exists $\bar{w} \in C([t_1, t_2]; E)$ such that, eventually passing to a subsequence,

$$w_n \to \bar{w}. \tag{42}$$

Now, we prove that $\bar{w} \in \Phi_1(\bar{y})$. In fact, we can use Proposition 3 which yields that there exists $\bar{f} \in L^1([t_1, t_2], E)$ such that $f_n \rightharpoonup \bar{f}$. Then, we apply Lemma 1 to the operator

$$\mathcal{N}_{F_1}(y) := \{f \in L^1([t_1, t_2], E) : f(t) \in F_1(t, w(t), \textstyle\int_{t_1}^{t} k(t,s)w(s)\,ds),\ a.e.\ t \in [t_1, t_2]\},$$

so that

$$\bar{f}(t) \in F_1(t, \bar{w}(t), \textstyle\int_{t_1}^{t} k(t,s)\bar{w}(s)\,ds),\ a.e.\ t \in [t_1, t_2]. \tag{43}$$

Moreover, by Proposition 4, we obtain

$$G_1 f_n \to G_1 \bar{f}.$$

Therefore, for every $t \in [t_1, t_2]$, on the one hand, it is (see (40))

$$w_n(t) \to U(t, t_1)T_1(\bar{y}) + \int_{t_1}^{t} U(t,s)\bar{f}(s)\,ds,$$

on the other hand, we had $w_n(t) \to \bar{w}(t)$ (see (42)). Therefore, by the uniqueness of the limit algorithm, we obtain

$$\bar{w}(t) = U(t,t_1)T_1(\bar{y}) + \int_{t_1}^{t} U(t,s)\bar{f}(s)\,ds$$

with (see (43) and (37))

$$\begin{aligned}\bar{f}(s) &\in F_1(s, w_n(s), \textstyle\int_{t_1}^{s} k(s,r)w_n(r)\,dr)\\ &= F(s, w_n(s), \textstyle\int_{t_1}^{s} k(s,r)w_n(r)\,dr + \int_{t_0}^{t_1} k(s,r)y_n(r)\,dr), \text{ a.e., } s \in [t_1,t_2].\end{aligned}$$

Hence, $\bar{w}$ is a mild solution to $(P_2)_{\bar{y}}$, i.e., (see (38))

$$\bar{w} \in \mathcal{S}_{2,\bar{y}} = \Phi_1(\bar{y}).$$

Thus, $w_n \in V_{\varepsilon_0}(\Phi_1(\bar{y}))$ definitively, leading to a contradiction to (39).

So far, we have proven that Φ_1 takes compact values and is upper semicontinuous. Therefore, it maps compact sets into compact sets (see, e.g., [18], Theorem 1.1.7), and hence, we can conclude that the set $\Phi_1(\mathcal{S}_1)$ is compact. From the equality

$$\mathcal{S}_2 = \Phi_1(\mathcal{S}_1),$$

we ensure the compactness of $\mathcal{S}_2$.

Step 4.

Let us observe that the Banach space $\mathcal{PC}([t_0,T];E)$ is isomorphic to the space $C([t_0,t_1];E) \times C([t_1,t_2];E)$ endowed with the Chebyshev norm

$$\|(y_1,y_2)\|_C = \max\left\{\sup_{t\in[t_0,t_1]}\|y_1(t)\|, \sup_{t\in[t_1,t_2]}\|y_2(t)\|\right\}.$$

Indeed, first of all, recall that $m = 1$, so $T = t_2$.

Then, let us define the mapping $J : \mathcal{PC}([t_0,T];E) \to C([t_0,t_1];E) \times C([t_1,t_2];E)$ as

$$J(y) = (y_1,y_2),\ y \in \mathcal{PC}([t_0,T];E),$$

where

$$y_1(t) := y(t),\ t \in [t_0,t_1], \text{ and } y_2(t) := \begin{cases} y(t), & t \in]t_1,t_2], \\ y(t_1^+), & t = t_1. \end{cases} \tag{44}$$

It is easy to check that it is injective and continuous.

Moreover, also, its inverse function $J^{-1} : C([t_0,t_1];E) \times C([t_1,t_2];E) \to \mathcal{PC}([t_0,T];E)$,

$$J^{-1}(y_1,y_2) = y(t) := \begin{cases} y_1(t), & t \in [t_0,t_1], \\ y_2(t), & t \in]t_1,t_2], \end{cases}$$

is continuous.

Step 5.

We show that $J(\mathcal{S}) \subset \mathcal{S}_1 \times \mathcal{S}_2$. To this aim, let $y \in \mathcal{S}$ be arbitrarily fixed. By (13), there exists $f \in L^1([t_0,T];E)$ with $f(s) \in F\left(s, y(s), \int_{t_0}^{s} k(s,r)y(r)dr\right)$ for a.e., $s \in [t_0,T]$, and such that

$$y(t) = U(t,t_0)y_0 + \sum_{t_0<t_j<t} U(t,t_j)I_j(y(t_j)) + \int_{t_0}^{t} U(t,s)f(s)\,ds,\ t \in [t_0,T].$$

We put

$$f_1(s) = f(s),\ s \in [t_0, t_1], \text{ and } f_2(s) = f(s),\ s \in [t_1, t_2].$$

Consider now $J(y) = (y_1, y_2)$ where y_1 and y_2 are defind by (44). It is immediate to verify that

$$y_1(t) = U(t, t_0)y_0 + \int_{t_0}^{t} U(t, s)f_1(s)ds,\ t \in [t_0, t_1],$$

and $f_1(s) \in F\left(s, y_1(s), \int_{t_0}^{s} k(s, r)y_1(r)dr\right)$ a.e., $s \in [t_0, t_1]$. Therefore, y_1 is a mild solution of (P_1), i.e., $y_1 \in \mathcal{S}_1$.

For y_2, we have

$$\begin{aligned} y_2(t) &= U(t, t_0)y_0 + U(t, t_1)I_1(y(t_1)) + \int_{t_0}^{t} U(t, s)f(s)ds \\ &= U(t, t_1)\left[U(t_1, t_0)y_0 + I_1(y_1(t_1)) + \int_{t_0}^{t_1} U(t_1, s)f_1(s)ds\right] + \int_{t_1}^{t} U(t, s)f_2(s)ds \\ &= U(t, t_1)[y_1(t_1) + I_1(y_1(t_1))] + \int_{t_1}^{t} U(t, s)f_2(s)ds,\ t \in [t_0, t_1]. \end{aligned}$$

Further, recalling (36), we have

$$\begin{aligned} f_2(s) &\in F\left(s, y(s), \textstyle\int_{t_0}^{s} k(s, r)y(r)dr\right) \\ &= F\left(s, y_2(s), \textstyle\int_{t_0}^{t_1} k(s, r)y_1(r)dr + \int_{t_1}^{s} k(s, r)y_2(r)dr\right) \\ &= F_1\left(s, y_2(s), \textstyle\int_{t_1}^{s} k(s, r)y_2(r)dr\right), \text{ a.e., } s \in [t_1, t_2]. \end{aligned}$$

Hence, y_2 is a mild solution of $(P_2)_{y_1}$ and then belongs to $\mathcal{S}_{2,y_1}$.

So, we achieve

$$J(y) \in \{y_1\} \times \mathcal{S}_{2,y_1} \subset \mathcal{S}_1 \times \mathcal{S}_2.$$

Step 6.

Now, we prove that the nonempty set $\mathcal{S}$ is closed.

To this aim, let us fix any sequence $\{y_n\}_n$ in $\mathcal{S}$ converging to a function $\bar{y} \in \mathcal{PC}([t_0, T]; E)$. Since each $y_n \in \mathcal{S}$, then there exists a sequence $\{f_n\}_n$ in $L^1([t_0, T]; E)$ such that for y_n the representation in (13) holds, i.e., there exists a sequence $\{f_n\}_n$ in $L^1([t_0, T]; E)$ with $f_n(s) \in F\left(s, y_n(s), \int_{t_0}^{s} k(s, r)y_n(r)dr\right)$ a.e., $s \in [t_0, T]$, such that

$$y_n(t) = U(t, t_0)y_0 + U(t, t_1)I_1(y_n(t_1)) + \int_{t_0}^{t} U(t, s)f_n(s)\,ds,\ t \in [t_0, T].$$

Notice that $\mathcal{S}$ isomorphic to the set $J(\mathcal{S})$, which is a subset of the compact set $\mathcal{S}_1 \times \mathcal{S}_2$. Therefore, $\mathcal{S}$ is bounded in $\mathcal{PC}([t_0, T]; E)$. We can therefore use the same arguments as before to say that $\{f_n\}_n$ is semicompact. Then, by Proposition 3 on $[t_0, T]$, there exists $\bar{f} \in L^1([t_0, T]; E)$ such that without loss of generality, $f_n \rightharpoonup \bar{f}$. Hence, $\bar{f} \in \mathcal{N}_F(\bar{y})$.

Moreover, by the continuity of I_1 and by Proposition 4 applied to $G : L^1([t_0, T]; E) \to C([t_0, T]; E)$,

$$Gf(t) = \int_{t_0}^{t} U(t, s)f(s)ds,\ t \in [t_0, T],$$

we obtain

$$y_n(t) \to U(t, t_0)y_0 + U(t, t_1)I_1(\bar{y}(t_1)) + \int_{t_0}^{t} U(t, s)\bar{f}(s)\,ds,\ t \in [t_0, T].$$

Thus, by the uniqueness of the limit, we have

$$\bar{y}(t) = U(t,t_0)y_0 + U(t,t_1)I_1(\bar{y}(t_1)) + \int_{t_0}^{t} U(t,s)\bar{f}(s)\,ds,\ t \in [t_0,T],$$

and then $\bar{y} \in \mathcal{S}$.

Step 7.

In conclusion, we have

$$\mathcal{S} \cong J(\mathcal{S}) \subset \mathcal{S}_1 \times \mathcal{S}_2$$

with $\mathcal{S}$ closed and $\mathcal{S}_1 \times \mathcal{S}_2$ compact, from which we obtain the compactness of $\mathcal{S}$. □

Remark 1. *Let us note that unlike the case without delay, in this case, we do not have $J(\mathcal{S}) = \mathcal{S}_1 \times \mathcal{S}_2$, but only $J(\mathcal{S}) \subset \mathcal{S}_1 \times \mathcal{S}_2$. In fact, considering a pair $(y_1, y_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, there are no reasons why y_2 should belong exactly to $\mathcal{S}_{2,y_1}$. In other words, an element of $\mathcal{S}_2$ has a past that may not be y_1.*

Remark 2. *A different approach to the problem of the compactness of the solutions for the problem (P) could be to avoid the extensio-with-memory process by addressing the solutions globally over the entire interval at a single time. In this case, the proof of Theorem 1 should be retraced but using the Gronwall–Bellman inequality of ([24], Lemma 1) established in the impulsive case. To be able to apply it, it would be necessary to strengthen the hypothesis of continuity on the impulse functions, assuming $\|I_j(v)\| \leq \beta_j\|v\|$ for every $v \in E$ and some $\beta_j \in E$, $j = 1, \cdot, m$, thus making the result for all intents and purposes a mere corollary of the previous Theorem 2.*

We can finally provide the existence of optimal solutions to our problem (P), whose proof is immediate according to the compactness of the solutions set.

Theorem 3 (Existence of optimal solutions)**.** *Assume the same hypotheses as Theorem 2 and let $\mathcal{J} : \mathcal{PC}([t_0,T];E) \to \mathbb{R}$ be a cost functional for (P).*

If $\mathcal{J}$ is lower semicontinuous, then there exists a mild solution y_* of (P) such that

$$\mathcal{J}(y_*) = \min_{y \in \mathcal{S}} \mathcal{J}(y);$$

if $\mathcal{J}$ is upper semicontinuous, then there exists a mild solution y^* of (P) such that

$$\mathcal{J}(y^*) = \max_{y \in \mathcal{S}} \mathcal{J}(y),$$

where $\mathcal{S}$ is the set of all the mild solutions of (P).

6. Existence of Optimal Solutions for Feedback Control Systems under Impulses' Effects

We now deal with applying the theory up to now developed to the following feedback control system

$$y'(t) = A(t)y(t) + f\left(t, y(t), \textstyle\int_{t_0}^{t} k(t,s)y(s)ds, \eta(t)\right),\ t \in [t_0,T]\,,\ t \neq t_j\,,\ j = 1,\dots,m, \quad (45)$$

$$\eta(t) \in H(t, y(t), \textstyle\int_{t_0}^{t} k(t,s)y(s)ds)\,,\ t \in [t_0,T], \quad (46)$$

satisfying the initial condition and subject to the impulses' action

$$y(t_0) = y_0; \quad y(t_j^+) = y(t_j) + I_j(y(t_j))\,,\ j = 1,\dots,m. \quad (47)$$

Here, $f : [t_0,T] \times E \times E \times E_1 \to E$, E_1 is a separable Banach space and a function with the following properties:

(f1) $f(\cdot, v, w, \eta) : [t_0,T] \to E$ is measurable for every $(v, w, \eta) \in E \times E \times E_1$;
(f2) $f(t, \cdot, \cdot, \cdot) : E \times E \times E_1 \to E$ is continuous for a.e., $t \in [t_0,T]$;

(f3) $\|f(t,v_1,w_1,\eta) - f(t,v_2,w_2,\eta)\| \leq q(t)(\|v_1 - v_2\| + \|w_1 - w_2\|)$, for every $v_1, v_2, w_1, w_2 \in E$, $\eta \in E_1$, where $q \in L^1_+([t_0,T])$.

Moreover, $H : [t_0,T] \times E \times E \multimap E_1$ is a multifunction such that the following conditions hold:

(H1) H takes compact values;
(H2) $H(\cdot,v,w) : [t_0,T] \multimap E_1$ is measurable for every $v, w \in E$;
(H3) $H(t,\cdot,\cdot) : E \times E \multimap E_1$ is upper semicontinuous for a.e., $t \in [t_0,T]$;
(H4) H is superpositionally measurable, i.e., for every measurable multifunction $Q : [t_0,T] \multimap E \times E$ with compact values, the multifunction $\mathcal{Q} : [t_0,T] \multimap E$, $\mathcal{Q}(t) = H(t,Q(t))$, is measurable;
(H5) The set

$$F(t,v,w) := f(t,v,w,H(t,v,w)) \tag{48}$$

is convex for all $(t,v,w) \in [t_0,T] \times E \times E$;
(H6) The multimap F satisfies the sublinear growth (F4);
(H7) The set $f(t,v,w,H(t,D_1,D_2))$ is relatively compact for every $(t,v,w) \in [t_0,T] \times E \times E$ and D_1, D_2 bounded subsets of E.

A pair (y,η), where $y \in \mathcal{PC}([t_0,T];E)$ and $\eta : [t_0,T] \to E_1$ is measurable, is said to be a *mild solution of the control system* (45)–(47) if

$$\begin{aligned} y(t) \;=\; & U(t,t_0)y_0 + \sum_{t_0<t_j<t} U(t,t_j)I_j(y(t_j)) \\ & + \int_{t_0}^{t} U(t,s) f\Big(s, y(s), \textstyle\int_{t_0}^{s} k(s,r)y(r)dr, \eta(s)\Big)\, ds,\ t \in [t_0,T], \end{aligned}$$

with $\eta(s) \in H(s, y(s), \int_{t_0}^{s} k(s,r)y(r)dr)$, $s \in [t_0,T]$.

The piecewise continuous function y is the *mild trajectory* and the measurable function η is the *control*.

Theorem 4. *Let $\mathcal{J} : \mathcal{PC}([t_0,T];E) \to \mathbb{R}$ be a cost functional for the control system* (45) *and* (46). *Suppose that $\{A(t)\}_{t\in[0,T]}$, f, k, H, and $I_1, \ldots, I_m$, respectively, satisfy (A), (f1)–(f3), (k), (H1)–(H7), and (I).*

If $\mathcal{J}$ is lower semicontinuous, then there exists a mild solution (y_*,η_*) of the control system (45)–(47) *such that*

$$\mathcal{J}(y_*) = \min_{y\in\mathcal{S}} \mathcal{J}(y);$$

if $\mathcal{J}$ is upper semicontinuous, then there exists a mild solution (y^*,η^*) of the control system (45)–(47) *such that*

$$\mathcal{J}(y^*) = \max_{y\in\mathcal{S}} \mathcal{J}(y);$$

where $\mathcal{S}$ is the set of all mild trajectories of the control system with initial datum $y(t_0) = y_0$ and subject to the impulses' action $y(t_j^+) = y(t_j) + I_j(y(t_j))$, $j = 1, \ldots, m$.

Proof. Assume that $\mathcal{J}$ is lower semicontinuous. In the case of upper semicontinuity, the proof will be analogous.

From the system (45)–(46), by using the function F defined in (48), we obtain the associated integro-differential inclusion

$$y'(t) \in A(t)y(t) + F(t, y(t), \textstyle\int_{t_0}^{t} k(t,s)y(s)ds). \tag{49}$$

Notice that the multifunction F defined in (48) satisfies properties (F1)–(F5). This is a consequence of the hypotheses (f1)–(f3), (k), and (H1)–(H7), as well as the basic properties of multifunctions. The detailed proof can be immediately deduced by the one of ([18], Theorem 5.2.3), so we refer the interested reader to that.

We are hence in position to apply our Theorem 3 to the problem (49) and (47), so that there exists a function $y_* \in \mathcal{PC}([t_0, T]; E)$ minimizing the cost functional $\mathcal{J}$ over the nonempty compact set of all the mild solution problems (49) and (47).

Thus, the function y_* has the representation (13), i.e.,

$$y_*(t) = U(t, t_0)y_0 + \sum_{t_0 < t_j < t} U(t, t_j) I_j(y_*(t_j)) + \int_{t_0}^{t} U(t, s) g(s)\, ds,\ t \in [t_0, T],$$

where $g : [t_0, T] \to E$ is a L^1-function on $[t_0, T]$ such that (see (48))

$$\begin{aligned} g(s) &\in F\Big(s, y_*(s), \textstyle\int_{t_0}^{s} k(s, r) y_*(r) dr\Big) \\ &= f\Big(s, y_*(s), \textstyle\int_{t_0}^{s} k(s, r) y_*(r) dr, H(s, y_*(s), \textstyle\int_{t_0}^{s} k(s, r) y_*(r) dr)\Big), \end{aligned}$$

for a.e., $s \in [t_0, T]$.

By (H4) and the Filippov Implicit Function Lemma (see, e.g., [18], Theorem 1.3.3), there exists a measurable selection η_* of $H(\cdot, y_*(\cdot), \int_{t_0}^{(\cdot)} k(\cdot, r) y_*(r) dr)$ such that

$$g(s) = f\Big(s, y_*(s), \textstyle\int_{t_0}^{s} k(s, r) y_*(r) dr, \eta_*(s)\Big),\ a.e.,\ t \in [t_0, T].$$

The function η_* is the control which realizes the mild solution y_* to be a mild trajectory of the control system (45)–(47).

The pair (y_*, η_*) is therefore an optimal solution of the control system (45)–(47). □

7. Optimal Solutions for a Feedback Control Population Dynamics Model with Impulses and Fading Memory

In this section, we apply our optimality result to a feedback control population dynamics model subject to the action of instantaneous external forces and with fading memory.

The differential equation describing the population dynamics we deal with is

$$\begin{aligned} u_t(t, x) = -b(t, x)u(t, x) + g\Big(t, u(t, x), \textstyle\int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} u(s, x)\, ds\Big) + \omega(t, x), \\ t \in [t_0, T]\,,\ t \neq t_j\,,\ j = 1, \ldots, m,\ a.e.\, x \in [0, 1]. \end{aligned} \tag{50}$$

Here, $u(t, x)$ represents the local and instantaneous population density (in the normalized spatial interval $[0, 1]$); $b(t, x)$ is the removal rate coefficient due to death and displacement; g is the nonlinear law of population development; the Volterra integral $\int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} u(s, x)\, ds$ describes the distributed delay which affects the evolution of the population, where the positive number τ establishes the width of the action of a fading memory kernel; the control $\omega(t, x)$ belongs to a set of feedback controls

$$\omega(t, \cdot) \in \Omega(u(t, \cdot)),\ t \in [t_0, T]. \tag{51}$$

The system must satisfy the initial datum $u_0 \in L^2([0, 1])$ and the effects induced by the impulses $\mathcal{I}_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, m$, i.e.,

$$u(t_0, x) = u_0(x)\ \text{ and } u\Big(t_j^+, x\Big) = u\big(t_j, x\big) + \mathcal{I}_j(u\big(t_j, x\big))\,,\ j = 1, \ldots, m,\ a.e.\, x \in [0, 1]. \tag{52}$$

The controllability of the system (50)–(52) has already been demonstrated in our earlier work [6], even on the half-line $[t_0, +\infty[$. By reducing the therein assumptions to $[t_0, T]$, we can follow that paper and state the next propositions.

Proposition 6 (cf. [6], Proposition 4.1)**.** *Assume that the function $b : [0, T] \times [0, 1] \to \mathbb{R}^+$ satisfies properties*

(b1) b is measurable;

(b2) There exists $s \in L^1([0,T])$ *such that*

$$0 < b(t,x) \leq s(t)\,, \text{ for every } t \in [0,T], \text{ a.e., } x \in [0,1];$$

(b3) For every $x \in [0,1]$, *the function* $b(\cdot,x) : [0,T] \to \mathbb{R}^+$ *is continuous.*

Then, the family $\{A(t)\}_{t\in[0,T]}$, $A(t) : L^2([0,1]) \to L^2([0,1])$, $t \in [0,T]$, *defined by*

$$A(t)v(x) = -b(t,x)v(x),\ v \in L^2([0,1]),\ x \in [0,1],$$

satisfies property (A).

Moreover, the (noncompact) evolution system generated by $\{A(t)\}_{t\in[0,T]}$ *is given by*

$$[U(t,s)v](x) = e^{\int_s^t -b(\sigma,x)d\sigma}v(x),$$

for every $0 \leq s \leq t \leq T$, $v \in L^2([0,1])$, $x \in [0,1]$.

Proposition 7 (cf. [6], Theorem 4.1)**.** *Assume that the function* $b : [0,T] \times [0,1] \to \mathbb{R}^+$ *satisfies properties (b1)–(b3). Suppose also that the function* $g : [t_0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies*

(g1) For every $t \in [t_0,T]$, *the map* $x \mapsto g(t,v(x),w(x))$ *belongs to* $L^2([0,1])$, *for every* $v,w \in L^2([0,1])$;

(g2) For every $p,q \in \mathbb{R}$, *the function* $g(\cdot,p,q)$ *is (strongly) measurable;*

(g3) For a.e., $t \in [t_0,T]$, *the function* $g(t,\cdot,\cdot)$ *is continuous;*

(g4) There exists $\varphi \in L^1_+([t_0,T])$ *such that* $|g(t,p,q)| \leq \varphi(t)$, *for a.e.,* $t \in [t_0,T]$ *and every* $p,q \in \mathbb{R}$;

(g5) There exists $m \in L^1_+([t_0,T])$ *such that*

$$\chi_{L^2}(g(t,D_1(\cdot),D_2(\cdot))) \leq m(t)[\chi_{L^2}(D_1) + \chi_{L^2}(D_2)]\,,$$

for a.e., $t \in [t_0,T]$ *and every bounded* $D_1, D_2 \subset L^2([0,1])$.

Moreover, assume that for the multifunction $\Omega : L^2([0,1]) \multimap L^2([0,1])$, *the next properties hold:*

(Ω1) Ω *takes compact convex values;*

(Ω2) Ω *is upper semicontinuous;*

(Ω3) There exists $Q > 0$ *such that* $\chi_{L^2}(\Omega(D)) \leq Q\chi_{L^2}(D)$ *for every bounded* $D \subset L^2([0,1])$;

(Ω4) There exists $R > 0$ *such that* $\|\Omega(v)\|_{L^2} \leq R(1 + \|v\|_{L^2})$ *for every* $v \in L^2([0,1])$.

Then, the problem (50)–(52) *is controllable, i.e., there exists a mild trajectory-control pair* (u,ω), *where* $u,\omega : [t_0,T] \times [0,1] \to \mathbb{R}$ *with* $u(t,\cdot) \in L^2([0,1])$ *for every* $t \in [t_0,T]$, $u(\cdot,x) \in \mathcal{PC}([t_0,T],\mathbb{R})$ *for all* $x \in [0,1]$, *and* ω *measurable, such that*

$$\begin{aligned} u(t,x) \;=\; & e^{\int_{t_0}^t -b(\sigma,x)d\sigma}u_0(x) + \sum_{t_0<t_j<t} e^{\int_{t_j}^t -b(\sigma,x)d\sigma}\mathcal{I}_j(u(t_j,x)) + \\ & + \int_{t_0}^t e^{\int_s^t -b(\sigma,x)d\sigma}\left[g\left(s,u(s,x),\int_{t_0}^s \tfrac{e^{-(s-\tau)/\tau}}{\tau}\,u(\tau,x)\,d\tau\right) + \omega(s,x)\right]ds, \end{aligned}$$

for every $t \in [t_0,T]$, $x \in [0,1]$, *where* $\omega(s,\cdot) \in \Omega(u(s,\cdot))$, *a.e.,* $s \in [t_0,T]$.

We show now that the control system (50)–(52) admits optimal solutions. To this aim, we take the following positions:

- $E = E_1 = L^2([0,1])$;
- $y : [t_0,T] \to L^2([0,1])$,

$$y(t)(x) = u(t,x),\ t \in [t_0,T],\ x \in [0,1];$$

- $\eta : [t_0,T] \to L^2([0,1])$,

$$\eta(t)(x) = \omega(t,x),\ t \in [t_0,T],\ x \in [0,1];$$

- $H : [t_0,T] \times L^2([0,1]) \times L^2([0,1]) \multimap L^2([0,1])$,

$$H(t,v,w) = \Omega(v)\,,\ t \in [t_0,T],\ v,w \in L^2([0,1]); \tag{53}$$

- $f : [t_0,T] \times L^2([0,1]) \times L^2([0,1]) \times L^2([0,1]) \to L^2([0,1])$,

$$f(t,v,w,\eta)(x) = g(t,v(x),w(x)) + \eta(x), t \in [t_0,T],\ v,w,\eta \in L^2([0,1]),\ x \in [0,1]; \tag{54}$$

- $k : \Delta = \{(t,s) : t_0, \leq s \leq t \leq T\} \to \mathbb{R}^+$,

$$k(t,s) = \frac{e^{-(t-s)/\tau}}{\tau},\ (t,s) \in \Delta; \tag{55}$$

- $y_0 = u_0$;
- $I_j : L^2([0,1]) \to L^2([0,1]),\ j = 1\ldots,m$,

$$I_j(v)(x) = \mathcal{I}_j(v(x)), v \in L^2([0,1]),\ x \in [0,1]. \tag{56}$$

In this way, the feedback control population dynamics system (50)–(52) assumes the form of a control system of type (45)–(47).

To prove our goal, it is therefore sufficient to show that the maps f, H, and $I_1,\ldots,I_m$ above defined satisfy the hypotheses of Theorem 4.

Proposition 8. *Suppose that g, Ω, and $\mathcal{I}_1,\ldots,\mathcal{I}_m$ satisfy properties (g1)–(g4), (Ω1), (Ω2), (Ω4), and*
(g6) There exists $q \in L^1_+([t_0,T])$ such that for a.e., $t \geq 0$

$$\|g(t,v_1(\cdot),w_1(\cdot)) - g(t,v_2(\cdot),w_2(\cdot))\|_{L^2} \leq q(t)(\|v_1 - v_2\|_{L^2} + \|w_1 - w_2\|_{L^2}),$$

for all $v_1,v_2,w_1,w_2 \in L^2([0,1])$ and $\eta \in L^2([0,1])$;
(g7) The map $t \mapsto \|g(t,0_{L^2},0_{L^2})\|_{L^2}$ belongs to $L^1_+([t_0,T])$;
(Ω5) Ω is compact, i.e., maps bounded sets into relatively compact sets;
($\mathcal{I}$) The functions $\mathcal{I}_1,\ldots,\mathcal{I}_m$ are bounded and continuous.

Then, the functions f, H, and $I_1,\ldots,I_m$ defined, respectively, in (54), (53), and (56) satisfy (f1)–(f3), (H1)–(H7), and (I).

Proof. First of all, f is well defined by (g1). Now, we show that it satisfies (f1)–(f3).

Property (f1) easily follows from (g2), while (f2) comes from (g3) and the Lebesgue dominated converge theorem (see (g4)); indeed, for any $t \in [t_0,T]$, $(v_0,w_0,\eta_0) \in (L^2([0,1]))^3$, and $(v_n,w_n,\eta_n)_n \to (v_0,w_0,\eta_0)$ in $(L^2([0,1]))^3$, we obtain

$$\begin{aligned}&\|f(t,v_n,w_n,\eta_n) - f(t,v_0,w_0,\eta_0)\|_{L^2}\\ &\quad\leq \left[\int_0^1 |g(t,v_n(x),w_n(x)) - g(t,v_0(x),w_0(x))|^2 dx\right]^{1/2} + \|\eta_n - \eta_0\|_{L^2} \to 0.\end{aligned}$$

About (f3), fixed $v_1,v_2,w_1,w_2 \in L^2([0,1])$ and $\eta \in L^2([0,1])$, for a.e., $t \in [t_0,T]$ by (g6) we have

$$\begin{aligned}\|f(t,v_1,w_1,\eta) - f(t,v_2,w_2,\eta)\|_{L^2} &= \|g(t,v_1(\cdot),w_1(\cdot)) - g(t,v_2(\cdot),w_2(\cdot))\|_{L^2}\\ &\leq q(t)(\|v_1 - v_2\|_{L^2} + \|w_1 - w_2\|_{L^2}).\end{aligned}$$

On the other hand, H satisfies (H1)–(H7), as we are going to prove.

First of all, conditions (Ω1) and (Ω3) imply, respectively, (H1) and (H3). Moreover, (H2) is immediate since H with respect to t is constant.

Let us prove that (H4) holds. To this aim, let us consider any measurable multifunction $Q = (Q_1, Q_2) : [t_0, T] \multimap L^2([0,1]) \times L^2([0,1])$ with compact values and consider the multifunction $\mathcal{Q} : [t_0, T] \multimap L^2([0,1])$ defined by $\mathcal{Q}(t) = H(t, Q_1(t), Q_2(t))$.

By the definition of H, we have

$$\mathcal{Q}(t) = \Omega(Q_1(t)).$$

Since Ω is an upper semicontinuous multifunction with compact values (cf. ($\Omega 2$) and ($\Omega 1$)), then we can use ([18], Proposition 1.3.1) and claim that it is superpositionally measurable, so $\mathcal{Q}$ is measurable.

Property (H5) is satisfied, since (cf. (48), (53) and (54))

$$F(t, v, w) = f(t, v, w, H(t, v, w)) = f(t, v, w, \Omega(v)) = g(t, v(\cdot), w(\cdot)) + \Omega(v)$$

for every $t \in [t_0, T]$, $v, w \in L^2([0,1])$, and ($\Omega 1$) holds.

Now, let us check property (H6). For every $t \in [t_0, T]$, $v, w \in L^2([0,1])$, we have

$$\|F(t, v, w)\|_{L^2} \leq \|g(t, v(\cdot), w(\cdot))\|_{L^2} + \|\Omega(v)\|_{L^2}.$$

By using ($\Omega 4$) and (g6), we obtain

$$\begin{aligned} \|F(t, v, w)\|_{L^2} &\leq \|g(t, v(\cdot), w(\cdot))\|_{L^2} + R(1 + \|v\|_{L^2}) \\ &\leq q(t)(\|v\|_{L^2} + \|w\|_{L^2}) + \|g(t, 0(\cdot), 0(\cdot))\|_{L^2} + R(1 + \|v\|_{L^2}) \\ &\leq (q(t) + \|g(t, 0(\cdot), 0(\cdot))\|_{L^2} + R)(1 + \|v\|_{L^2} + \|w\|_{L^2}). \end{aligned}$$

Recalling the hypothesis (g7), property (H6) is true, taking

$$h(t) = q(t) + \|g(t, 0(\cdot), 0(\cdot))\|_{L^2} \; + \; R$$

To check that (H7) holds, it is enough to use ($\Omega 5$); in fact, for every $t \in [t_0, T]$, $v, w \in L^2([0,1])$, and D_1, D_2 bounded subsets of $L^2([0,1])$, we achieve

$$\chi_{L^2}(f(t, v, w, H(t, D_1, D_2))) \; = \; \chi_{L^2}(g(t, v(\cdot), w(\cdot)) + \Omega(D_1)) \leq \chi_{L^2}(\Omega(D_1)) = 0,$$

thus, the relative compactness of $f(t, v, w, H(t, D_1, D_2))$.

Finally, by ($\mathcal{I}$) and ($\mathcal{I}2$), we can apply the Lebesgue-dominated convergence theorem; hence, for every $j = 1, \ldots, m$ and every $v_0 \in L^2([0,1])$, $v_n \to v_0$ in $L^2([0,1])$, we obtain

$$\|\mathcal{I}_j(v_n) - \mathcal{I}_j(v_0)\|_{L^2}^2 \to 0.$$

Therefore, the functions $I_1, \ldots, I_m$ satisfy (I). □

Remark 3. *Notice that (g6) implies (g5) and ($\Omega 5$) implies ($\Omega 3$), so Proposition 7 still holds.*

Conclusion. In the end, by Propositions 6 and 8 and the continuity of k (cf. (55)), we can apply Theorem 4. Theorefore, the feedback control population dynamics system (50)–(52) admits optimal solutions, that is a mild trajectory-control pair (u, ω),

$$\begin{aligned} u(t,x) = e^{\int_{t_0}^{t} -b(\sigma,x)d\sigma} u_0(x) + \sum_{t_0 < t_j < t} e^{\int_{t_j}^{t} -b(\sigma,x)d\sigma} \mathcal{I}_j(u(t_j, x)) + \\ + \int_{t_0}^{t} e^{\int_{s}^{t} -b(\sigma,x)d\sigma} \left[g\left(s, u(s,x), \int_{t_0}^{s} \frac{e^{-(s-\tau)/\tau}}{\tau} u(\tau, x)\, d\tau\right) + \omega(s,x) \right] ds, \\ t \in [t_0, T],\ x \in [0,1], \end{aligned}$$

$$\omega(s, \cdot) \in \Omega(u(s, \cdot)),\ \textit{a.e. } s \in [t_0, T],$$

minimizing or maximizing a cost functional to the system, depending on whether it is lower or upper semicontinuous.

Funding: This study was partly funded by Research project of the MUR (Ministry of University and Research, Italy) PRIN 2022 "Nonlinear differential problems with applications to real phenomena"—Grant Number: 2022ZXZTN2.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The author is a member of the national group GNAMPA of INdAM (National Institute of Higher Mathematics, Italy) and of the group TAA of UMI (Italian Mathematical Union).

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Araújo, R.O.; Marinho, S.S.; Prates Filho, J.S. Uniform stability of a non-autonomous semilinear Bresse system with memory. *Appl. Math. Comput.* **2020**, *387*, 124418. [CrossRef]
2. Cardinali, T.; Matucci, S.; Rubbioni, P. Uniform asymptotic stability of a PDE's system arising from a flexible robotics model, submitted. *arXiv* **2024**, arXiv:2405.20949.
3. Chen, B.; Timoshin, S.A. Optimal control of a population dynamics model with hysteresis. *Acta Math. Sci. Ser. B* **2022**, *42*, 283–298. [CrossRef]
4. Metwali, M.M.A.; Cichoń, K. On solutions of some delay Volterra integral problems on a half-line. *Nonlinear Anal. Model. Control* **2021**, *26*, 661–677. [CrossRef]
5. Refaai, D.A.; El-Sheikh, M.M.A.; Ismail, G.A.F.; Bahaaeldin, A.; Thabet, A. Hyers-Ulam stability of impulsive Volterra delay integro-differential equations. *Adv. Difference Equ.* **2021**, *477*, 1–13. [CrossRef]
6. Rubbioni, P. Solvability for a Class of Integro-Differential Inclusions Subject to Impulses on the Half-Line. *Mathematics* **2022**, *10*, 224. [CrossRef]
7. Lakshmikantham, V.; Rama Mohana Rao, M. *Theory of Integro-Differential Equations*; Stability and Control: Theory, Methods and Applications, 1; Gordon and Breach Science Publishers: Lausanne, Switzerland, 1995.
8. Smith, H. *An Introduction to Delay Differential Equations with Applications to the Life Sciences*; Texts in Applied Mathematics, 57; Springer: New York, NY, USA, 2011.
9. Hua, L.; Zhu, H.; Zhong, S.; Zhang, Y.; Shi, K.; Kwon, O.-M. Fixed-time stability of nonlinear impulsive systems and its application to inertial neural networks. *IEEE Trans. Neural Netw. Learn. Syst.* **2024**, *35*, 1872–1883. [CrossRef] [PubMed]
10. Leiva, H.; Garcia, K.; Lucena, E. Existence of solutions for semilinear retarded equations with non-instantaneous impulses, non-local conditions, and infinite delay. *Open Math.* **2023**, *21*, 20230106. [CrossRef]
11. Li, Y.; Zhao, H.; Cheng, Y. Modeling mosquito control by an impulsive reaction-diffusion mosquito model with periodic evolution domain. Commun. *Nonlinear Sci. Numer. Simul.* **2024**, *130*, 107738. [CrossRef]
12. Ma, W.; Yang, B.; Zheng, Y. Practical exponential stability of impulsive stochastic functional differential systems with distributed-delay dependent impulses. *Nonlinear Anal. Hybrid Syst.* **2024**, *51*, 101424. [CrossRef]
13. Benchohra, M.; Henderson, J.; Ntouyas, S. *Impulsive Differential Equations and Inclusions*; Hindawi Publishing Corporation: New York, NY, USA, 2006.
14. Cardinali, T.; Rubbioni, P. Impulsive mild solutions for semilinear differential inclusions with nonlocal conditions in Banach spaces. *Nonlinear Anal.* **2012**, *75*, 871–879. [CrossRef]
15. Djebali, S.; Górniewicz, L.; Ouahab, A. *Existence and Structure of Solution Sets for Impulsive Differential Inclusions: A Survey*; Lecture Notes in Nonlinear Analysis, 13; Juliusz Schauder Center for Nonlinear Studies: Toruń, Poland, 2012; 148p.
16. Graef, J.R.; Henderson, J.; Ouahab, A. *Impulsive Differential Inclusions: A fixed Point Approach*; De Gruyter Series in Nonlinear Analysis and Applications, 20; De Gruyter: Berlin, Germany, 2013.
17. Perestyuk, N.A.; Plotnikov, V.A.; Samoilenko, A.M.; Skripnik, N.V. *Differential Equations with Impulse Effects: Multivalued Right-Hand Sides with Discontinuities*; De Gruyter Studies in Mathematics, 40; Walter de Gruyter & Co.: Berlin, Germany, 2011.
18. Kamenskii, M.; Obukhovskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; De Gruyter Series in Nonlinear Analysis and Applications 7; Walter de Gruyter: Berlin, Germany; New York, NY, USA, 2001.
19. Bungardi, S.; Cardinali, T.; Rubbioni, P. Nonlocal semilinear integro-differential inclusions via vectorial measures of noncompactness. *Appl. Anal.* **2017**, *96*, 2526–2544. [CrossRef]
20. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Applied Mathematical Sciences, 44; Springer: New York, NY, USA, 1983.
21. Kreĭn, S.G. *Linear Differential Equations in Banach Space*; Translated from the Russian by J. M. Danskin; Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 1971; Volume 29.
22. Cardinali, T.; Rubbioni, P. On the existence of mild solutions of semilinear evolution differential inclusions. *J. Math. Anal. Appl.* **2005**, *308*, 620–635. [CrossRef]

23. Cardinali, T.; Rubbioni, P. Hereditary evolution processes under impulsive effect. *Mediterr. J. Math.* **2021**, *18*, 91. [CrossRef]
24. Baĭnov, D.; Covachev, V.; Stamova, I. Estimates of the solutions of impulsive quasilinear functional-differential equations. *Ann. Fac. Sci. Toulouse Math.* **1991**, *12*, 149–161. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.