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A Maslov-Type Index in Dimension 2

Qiyu Zhong * and Hai-Long Her

Department of Mathematics, Jinan University, Guangzhou 510632, China; hailongher@jnu.edu.cn
* Correspondence: qiyuzhong@stu2021.jnu.edu.cn

Abstract: In this article, we define an index of the Maslov type for paths of 2×2 orthogonal symplectic matrices. The starting point is an arbitrary 2×2 orthogonal symplectic matrix rather than the identity matrix. We use this index to explain the geometric intersection number of a pair of Lagrangian paths and compare it with the Cappell–Lee–Miller index.

Keywords: Maslov index; symplectic path; Lagrangian path

MSC: 53D12

1. Introduction

The Maslov indices for paths of symplectic matrices and paths of Lagrangian subspaces have been important topics in the fields of Hamiltonian dynamical systems, symplectic geometry and mathematical physics for a long time [1–13]. For example, the Conley–Zehnder index is a special Maslov-type index for paths of symplectic matrices (symplectic paths). It was constructed by Conley and Zehnder [4] for non-degenerate periodic solutions of Hamiltonian systems and has been an important tool for the construction of Floer homology [14,15] and research on the Arnold conjecture [4,12,14–21]. The set of $2n \times 2n$ symplectic matrices is denoted by $Sp(2n, \mathbb{R})$. Then, the Conley–Zehnder index is defined for a continuous path $\Phi : [0,1] \to Sp(2n, \mathbb{R})$ such that $\Phi(0) = I_{2n}$ and $\Phi(1)$ has no eigenvalues equal to 1. Long [7,8] and Viterbo [13] independently proposed methods to deal with the degenerate symplectic paths. In particular, Long [7,8] generalized the Conley–Zehnder index and defined a Maslov-type index for the symplectic paths that satisfies the conditions that $\Phi(0) = I_{2n}$ and $\Phi(1)$ is arbitrary. This Maslov-type index is extremely useful for studying various problems related to the periodic solutions, closed characteristics, brake orbits and closed geodesics arising from celestial mechanics, contact geometry, Riemannian and Finsler geometry, etc. [22–27]. Furthermore, in order to study symplectic paths with Lagrangian boundary conditions, Liu [5] defined a Maslov-type index via a fixed Lagrangian subspace L_0, called the L_0-index.

In the study of Maslov-type indexes for the segments of symplectic paths (i.e., $\Phi(0)$ and $\Phi(1)$ are arbitrary), Long [7,8] and Liu [5,6] made significant contributions, respectively. Their constructions employ the point of view that the general symplectic paths can be viewed as the segment of symplectic paths that correspond to two symplectic paths starting from I_{2n}. Consequently, the index for the segment of symplectic paths can be defined as the difference between the indices of these two symplectic paths starting at I_{2n}. However, it should be noted that the methods of constructing Maslov-type indices for the segments of symplectic paths and those special paths starting from I_{2n} are not consistent.

Conversely, Arnold [2] elucidated Maslov's original index as the intersection number of a path of Lagrangian subspaces with the so-called Maslov cycle and defined an index of a pair of Lagrangian loops. Robbin and Salamon [10,11] generalized the work of Arnold and considered arbitrary Lagrangian paths instead of loops. They defined a kind of Maslov index for a pair of Lagrangian paths, which can induce an index for symplectic paths. However, this kind of index by Robbin and Salamon is merely a half-integer rather than an

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integer. To unify different definitions, Cappell, Lee and Miller [3], using somewhat different methods, gave a system of axioms for the pair of Lagrangian paths and introduced four definitions of Maslov indices. They showed that these definitions satisfy this system of axioms; and thereby, their equivalences to one another were established.

In this article, we consider a simple case in dimension 2 and give a consistent method for constructing a Maslov-type index for orthogonal symplectic paths. We compare this index with the Cappell–Lee–Miller index μ_{CLM} (see Definition 6), which is the index for a pair of Lagrangian paths. If we denote the set of general paths of 2×2 orthogonal symplectic matrices by

$$\mathcal{OP}(2,\mathbb{R}) := \{\Phi : [0,1] \to Sp(2,\mathbb{R}) \cap O(2) \text{ is continuous}\},$$

where $O(2)$ is the set of 2×2 orthogonal matrices, we obtain the following main results.

Theorem 1. *For any $\Phi \in \mathcal{OP}(2,\mathbb{R})$, there exists a well-defined integer $\mu(\Phi)$, called a Maslov-type index (see (22)), such that the following properties hold:*
(1) For $\forall\ a \in (0,1)$, $\mu(\Phi) = \mu(\Phi([0,a])) + \mu(\Phi([a,1]))$;
(2) For the orthogonal symplectic paths, as

$$\Phi_{k,b}(t) = \begin{pmatrix} \cos(k\pi + bt) & -\sin(k\pi + bt) \\ \sin(k\pi + bt) & \cos(k\pi + bt) \end{pmatrix},\ 0 \le t \le 1,\ 0 \le |b| < \delta,\ k \in \mathbb{Z}, \tag{1}$$

where $\delta > 0$ is small enough, we have

$$\mu(\Phi_{k,b}) = 0, \text{ for } b \le 0, \tag{2}$$
$$\mu(\Phi_{k,b}) = 1, \text{ for } b > 0. \tag{3}$$

More intuitively, if we do not take the endpoints into account, the Maslov-type index $\mu(\Phi)$ is equal to the intersection number Φ crossing the singular set $\mathscr{C}(2,\mathbb{R})$ (see (16)).

Theorem 2. *Let $f(t) = (L_1, L_2(t))$ be a pair of Lagrangian paths, where $L_1 = \mathbb{R} \times \{0\}$ and*

$$L_2(t) = \{(m(t)x, n(t)x)^T \mid x \in \mathbb{R},\ m^2(t) + n^2(t) \neq 0,\ 0 \le t \le 1\}. \tag{4}$$

Then, the path $f(t)$ corresponds to a unique orthogonal symplectic path

$$\Phi(t) = \begin{pmatrix} \frac{m(t)}{\sqrt{m^2(t)+n^2(t)}} & -\frac{n(t)}{\sqrt{m^2(t)+n^2(t)}} \\ \frac{n(t)}{\sqrt{m^2(t)+n^2(t)}} & \frac{m(t)}{\sqrt{m^2(t)+n^2(t)}} \end{pmatrix},\ 0 \le t \le 1. \tag{5}$$

If $e^{\varepsilon J_2} L_2(t)$ crosses the Maslov cycle $\Sigma(L_1)$ (see (13)) transversely at each intersection time t, where $\varepsilon > 0$ is small enough and J_2 is given by (6), then

$$\mu_{CLM}(f) = \mu(\Phi).$$

We remark that, at least for this special case of dimension 2, the intersection number of the Lagrangian pair can be described by the intersection number of the orthogonal symplectic path crossing the singular set. We will roughly discuss the idea of higher dimensions in Section 5 and study those more general cases as well as related applications in another paper.

2. Preliminaries

In this section, we introduce some definitions and results that we use in the article.

2.1. Symplectic Path and Lagrangian Path

Definition 1. *Let ω_0 be a bilinear form in $\mathbb{R}^{2n}$ given by*

$$\omega_0(\xi, \eta) = \xi^T J_{2n} \eta, \ \forall \ \xi, \eta \in \mathbb{R}^{2n},$$

where

$$J_{2n} = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}. \tag{6}$$

Then, the space $(\mathbb{R}^{2n}, \omega_0)$ is called the standard symplectic space. This bilinear form ω_0 is called the standard symplectic form.

Definition 2. *Let L be a n-dimensional subspace of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and*

$$L^{\perp} = \{\xi \in \mathbb{R}^{2n} | \ \omega_0(\xi, \eta) = 0, \ \forall \eta \in L\}, \tag{7}$$

where $L^{\perp}$ is called the skew-orthogonal complement [17] of L. If $L = L^{\perp}$, then L is called a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$.

The set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$ is called the Lagrangian Grassmannian, denoted by $\mathscr{L}(n)$. Let T be an isomorphism of $(\mathbb{R}^{2n}, \omega_0)$; if T keeps ω_0 invariant, then T is called a symplectic isomorphism. The set of all symplectic isomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ is called the symplectic group [9,28] and is denoted by $Sp(2n, \mathbb{R})$. Then, $M \in Sp(2n, \mathbb{R})$ is a $2n \times 2n$ real matrix and it satisfies $M^T J_{2n} M = J_{2n}$. Thus, we have the following definitions:

Definition 3. *Let $M \in \mathbb{R}^{2n \times 2n}$; M is called a symplectic matrix if it satisfies*

$$M^T J_{2n} M = J_{2n}.$$

Definition 4. *A continuous map*

$$\Phi : [0, 1] \to Sp(2n, \mathbb{R})$$

is called a symplectic path in $Sp(2n, \mathbb{R})$. Similarly, a continuous map

$$\Phi : [0, 1] \to \mathscr{L}(n)$$

is called a Lagrangian path in $\mathscr{L}(n)$.

According to the above definitions, we have some results for $n = 1$.

Proposition 1. *When $n = 1$, the Lagrangian Grassmannian*

$$\mathscr{L}(n) = \{L = (mx, nx)^T, x \in \mathbb{R} \mid m^2 + n^2 \neq 0\}. \tag{8}$$

Proof. The set $\{L = (mx, nx)^T, x \in \mathbb{R} \mid m^2 + n^2 \neq 0\}$ contains all the 1-dimensional subspaces of $(\mathbb{R}^2, \omega_0)$. We only need to prove that L is a Lagrangian subspace. By (7), we obtain the skew-orthogonal complement

$$L^{\perp} = \{(u, v)^T \mid u, v \in \mathbb{R}, mv = nu\},$$

then, $L^{\perp} = L$ and hence, L is a Lagrangian subspace. □

If $M \in Sp(2n, \mathbb{R})$ is an orthogonal matrix, then M is called an orthogonal symplectic matrix. The set of all orthogonal symplectic matrices is denoted by $OSp(2n, \mathbb{R})$. We have the following result when $n = 1$.

Proposition 2. *If $M \in OSp(2, \mathbb{R})$, then M has the form*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ \theta \in \mathbb{R}.$$

Proof. Set

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.$$

Since $M \in OSp(2, \mathbb{R})$, then

$$M^T J_2 M = J_2,\ M^T M = M M^T = I_2,$$

and we can obtain

$$m_1 m_4 - m_2 m_3 = 1,\ m_1^2 + m_3^2 = m_2^2 + m_4^2 = 1.$$

Then,

$$(m_1 - m_4)^2 + (m_2 + m_3)^2 = m_1^2 + m_3^2 + m_2^2 + m_4^2 - 2(m_1 m_4 - m_2 m_3) = 0,$$

and hence, $m_1 = m_4, m_2 + m_3 = 0$. Let $m_1 = \cos\theta, m_3 = \sin\theta$ and this completes the proof. □

2.2. The Rotation Number for Orthogonal Symplectic Paths

In this part, we will introduce the rotation number for orthogonal symplectic paths, which is used for the construction of the Maslov-type index. Let

$$\Phi : [0, 1] \to Osp(2n, \mathbb{R})$$

be an orthogonal symplectic path. If $n = 1$, then Φ has the form as

$$\begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix},\ 0 \le t \le 1, \tag{9}$$

where the angle mapping $\theta(t)$ is continuous.

According to [2], $Osp(2n, \mathbb{R}) = Sp(2n, \mathbb{R}) \cap O(2n)) \cong U(n)$, where $O(2n)$ is the orthogonal group and $U(n)$ is the unitary group. If $n = 1$, then $Osp(2n, \mathbb{R}) \cong \mathbb{S}^1$ (i.e., the unit circle). Thus, an orthogonal symplectic path can correspond to a path on $\mathbb{S}^1$. Denote the set of all orthogonal symplectic paths by

$$\mathcal{OP}(2n, \mathbb{R}) = \{\Phi : [0, 1] \to Osp(2n, \mathbb{R}) \text{ is a continuous map}\}. \tag{10}$$

For any $\Phi \in \mathcal{OP}(2, \mathbb{R})$ with the form (9), we construct the maps

$$\rho_\Phi(t) := \cos\theta(t) + \sqrt{-1}\sin\theta(t) = e^{\sqrt{-1}\theta(t)} \tag{11}$$

and

$$\Delta_\Phi(t) := \frac{\theta(t) - \theta(0)}{\pi},$$

then, we can define the rotation number:

Definition 5. *For any $\Phi \in \mathcal{OP}(2,\mathbb{R})$, the rotation number of Φ is defined by*

$$\Delta(\Phi) := \Delta_\Phi(1) = \frac{\theta(1)-\theta(0)}{\pi}.$$

Proposition 3. *If $\Phi, \Psi \in \mathcal{OP}(2,\mathbb{R})$ and they satisfy $\Phi(1) = \Psi(0)$, we define the catenation of Φ and Ψ by*

$$\Phi\#\Psi(t) := \begin{cases} \Phi(2t) & 0 \le t < \frac{1}{2} \\ \Psi(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}, \tag{12}$$

then, we have

$$\Delta(\Phi\#\Psi) = \Delta(\Phi) + \Delta(\Psi).$$

Proof. By (9), let $\theta(t)$ and $\theta'(t)$ be the angle mappings of Φ and Ψ, then they satisfy $\theta(1) = \theta'(0)$. Hence

$$\begin{aligned} \Delta(\Phi) + \Delta(\Psi) &= \frac{\theta(1)-\theta(0)}{\pi} + \frac{\theta'(1)-\theta'(0)}{\pi} \\ &= \frac{\theta'(1)-\theta(0)}{\pi} \\ &= \Delta(\Phi\#\Psi). \end{aligned}$$

This completes the proof. □

2.3. Cappell–Lee–Miller Index

In this part, we introduce the Cappell–Lee–Miller index, which is a geometrical definition of [3] by Cappell, Lee and Miller. It is the geometric intersection number of a Lagrangian path and the 1-codimensional cycle (i.e., the Maslov cycle) in the Lagrangian Grassmannian. Let

$$f(t) = (L_1, L_2(t)),\ 0 \le t \le 1$$

be the pair of two Lagrangian paths, where $L_1 = \mathbb{R}^n \times \{0\}$ is a constant path. $f(t)$ is called a proper path if

$$L_1 \cap L_2(t) = \{0\},\ t = 0, 1.$$

According to [3], for any path $f(t)$ above, there exists a sufficiently small $\varepsilon > 0$ such that

$$f_\varepsilon(t) = (L_1, e^{\varepsilon J_{2n}} L_2(t)),\ 0 \le t \le 1$$

is a proper path. Define the Maslov cycle of L as

$$\Sigma(L) = \{L' \in \mathscr{L}(n) | \dim(L' \cap L) \ge 1\}, \tag{13}$$

then, we can count the geometric intersection number with signs of $\Sigma(L_1)$ and $e^{\varepsilon J_{2n}} L_2(t)$, the Cappell–Lee–Miller index is given by

Definition 6. *For a Lagrangian pair $f(t) = (L_1, L_2(t))$, $L_1 = \mathbb{R}^n \times \{0\}$, if $e^{\varepsilon J_{2n}} L_2(t)$ crosses $\Sigma(L_1)$ transversely at each intersection time t, then the Cappell–Lee–Miller index is defined by*

$$\mu_{CLM}(f) = \mu_{proper}(f_\varepsilon),$$

which is the geometric intersection number counted with signs.

The signs depend on the orientation of $e^{\varepsilon J_{2n}}L_2(t)$ crossing $\Sigma(L_1)$. We consider the simple case of $n = 1$. For example, let

$$f(t) = (L_1, L_2(t)),\ L_2(t) = \{(x, (t - \frac{1}{2})x)\ | x \in \mathbb{R},\ 0 \le t \le 1\}.$$

Since the Maslov cycle $\Sigma(L_1) = \{L_1\}$, then, there is only one intersection time $t = \frac{1}{2} + \varepsilon$. Thus, we have

$$\begin{aligned}\mu_{CLM}(f([0, \frac{1}{2}])) &= 0,\\ \mu_{CLM}(f([\frac{1}{2}, 1])) &= 1,\\ \mu_{CLM}(f([0, 1])) &= 1.\end{aligned}$$

We can see that $e^{\varepsilon J_{2n}}L_2(t)$ crossing $\Sigma(L_1)$ anti-clockwise in $\mathbb{R}^2$, it will be counted with the positive sign, and if the crossing is in the opposite direction, it will be counted with the negative sign. For any $f(t)$, $f_\varepsilon(t)$ is a proper path. At each intersection time t_j, $f_\varepsilon(t)$ is locally isomorphic to one of the following two cases:

$$(\mathbb{R}, e^{\sqrt{-1}(t-t_j)}\mathbb{R}) \text{ or } (\mathbb{R}, e^{-\sqrt{-1}(t-t_j)}\mathbb{R}),\ |t - t_j| < \delta.$$

Suppose that there are p intersection points and q intersection points of these two cases, by Definition 6, we have

$$\mu_{CLM}(f) = p - q. \tag{14}$$

3. The Maslov-Type Index for Orthogonal Symplectic Paths

In this section, we will construct the Maslov-type index for orthogonal symplectic paths and use the methods of perturbation and extension. Recall that

$$\mathcal{OP}(2, \mathbb{R}) = \{\text{all paths in } Osp(2, \mathbb{R})\}, \tag{15}$$

and define the singular set

$$\mathscr{C}(2, \mathbb{R}) := \{I_2,\ -I_2\}. \tag{16}$$

To deal with the orthogonal symplectic paths whose endpoints are on the singular set, we need the following lemma:

Lemma 1. *For any $\Phi \in \mathcal{OP}(2, \mathbb{R})$, there exists a sufficiently small $\varepsilon \ge 0$ such that the end points of Φ under the perturbation are not on the singular set $\mathscr{C}(2, \mathbb{R})$, i.e., $e^{\varepsilon J_2}\Phi(0)$, $e^{\varepsilon J_2}\Phi(1) \notin \mathscr{C}(2, \mathbb{R})$. Moreover, this perturbation will not change the rotation number, and hence*

$$\Delta(e^{\varepsilon J_2}\Phi) = \Delta(\Phi).$$

Proof. Set

$$\Phi(t) = \begin{pmatrix}\cos\theta(t) & -\sin\theta(t)\\ \sin\theta(t) & \cos\theta(t)\end{pmatrix},\ 0 \le t \le 1,$$

If both $\theta(0)$ and $\theta(1)$ are not equal to $k\pi (k \in \mathbb{Z})$, then there exists $\varepsilon = 0$ such that this lemma holds. Otherwise, we choose $\varepsilon > 0$, and then

$$e^{\varepsilon J_2}\Phi(t) = \begin{pmatrix}\cos\varepsilon & \sin\varepsilon\\ -\sin\varepsilon & \cos\varepsilon\end{pmatrix}\begin{pmatrix}\cos\theta(t) & -\sin\theta(t)\\ \sin\theta(t) & \cos\theta(t)\end{pmatrix} = \begin{pmatrix}\cos(\theta(t) - \varepsilon) & -\sin(\theta(t) - \varepsilon)\\ \sin(\theta(t) - \varepsilon) & \cos(\theta(t) - \varepsilon)\end{pmatrix},$$

If ε is small enough, then $e^{\varepsilon J_2}\Phi(0)$ and $e^{\varepsilon J_2}\Phi(1)$ are not equal to $\pm I_2$, i.e., they are not on the singular set $\mathscr{C}(2,\mathbb{R})$. The rotation number

$$\Delta(e^{\varepsilon J_2}\Phi) = \frac{(\theta(1)-\varepsilon)-(\theta(0)-\varepsilon)}{\pi} = \frac{\theta(1)-\theta(0)}{\pi} = \Delta(\Phi).$$

This completes the proof of this lemma. □

In addition, the rotation number is not always an integer. To define an integer-valued index, we consider the extension of the paths to construct an integer. For any $M \in Osp(2,\mathbb{R})$ with the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

since $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R}) \cong \mathbb{S}^1\backslash\{\pm 1\}$, then $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$ has two connected components

$$Osp^+(2,\mathbb{R}) = \{M \in Osp(2,\mathbb{R}) \mid 2k\pi < \theta < (2k+1)\pi,\ k \in \mathbb{Z}\}, \tag{17}$$
$$Osp^-(2,\mathbb{R}) = \{M \in Osp(2,\mathbb{R}) \mid (2k-1)\pi < \theta < 2k\pi,\ k \in \mathbb{Z}\}. \tag{18}$$

We will construct the extension in these two connected components, which depends on the endpoints of the paths. To show that the index is independent of the choices of the extensions, we need the following lemma:

Lemma 2. *If Φ is a loop in $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$, i.e., $\Phi(0) = \Phi(1)$, then the rotation number of Φ is equal to zero, i.e., $\Delta(\Phi) = 0$.*

Proof. Choose any loop $\Phi : [0,1] \to Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$, which has the form

$$\begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix},\ 0 \le t \le 1,$$

where $\theta(t)$ is continuous and satisfies $\theta(0) = \theta(1) + 2k\pi (k \in \mathbb{Z})$. Since $\Phi(0) = \Phi(1)$ and they lie in $Osp^+(2,\mathbb{R})$ or $Osp^-(2,\mathbb{R})$, then $|\theta(0)-\theta(1)| < \pi$. Thus, we can obtain $k = 0$ and hence $\theta(0) = \theta(1)$, then

$$\Delta(\Phi) = \frac{\theta(1)-\theta(0)}{\pi} = 0.$$

This completes the proof. □

Now, we define the extension for orthogonal symplectic paths. For any $\Phi \in \mathcal{OP}(2,\mathbb{R})$, by Lemma 1, there exists a small enough $\varepsilon \ge 0$ such that $e^{\varepsilon J_2}\Phi(0), e^{\varepsilon J_2}\Phi(1) \notin \mathscr{C}(2,\mathbb{R})$. Let

$$e^{\varepsilon J_2}\Phi(0) = A, \quad e^{\varepsilon J_2}\Phi(1) = B. \tag{19}$$

We consider whether A and B lie in the same components of $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$. Denote the endpoint of the extension by $W_{A,B}$, then the rule to choose $W_{A,B}$ is given by

$$W_{A,B} = \begin{cases} A, & A \text{ and } B \text{ are in the same components,} \\ -A, & A \text{ and } B \text{ are in the different components.} \end{cases} \tag{20}$$

We define the extension for $e^{\varepsilon J_2}\Phi$ as

$$\beta : [0,1] \to Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R}),\ \beta(0) = B,\ \beta(1) = W_{A,B}, \tag{21}$$

then, the definition of the Maslov-type index is

Definition 7. *For any $\Phi \in \mathcal{OP}(2,\mathbb{R})$, the Maslov-type index is defined by*

$$\mu(\Phi) = \Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta). \tag{22}$$

4. Proof of the Main Results

In this section, we will complete the proofs of the main results.

Proof of Theorem 1. Let

$$\Phi'(t) = \begin{cases} e^{\varepsilon J_2}\Phi(2t), & 0 \le t < \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1, \end{cases} \tag{23}$$

then $\Phi'(0) = A, \Phi'(1) = W_{A,B}$. According to Definition 5 and (20), we have

$$\Delta(\Phi') = \begin{cases} 2k_1, & W_{A,B} = A,\ k_1 \in \mathbb{Z} \\ 2k_2+1, & W_{A,B} = -A,\ k_2 \in \mathbb{Z} \end{cases}. \tag{24}$$

By Definition 7 and Proposition 3, we can obtain

$$\mu(\Phi) = \Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta) = \Delta(\Phi'). \tag{25}$$

This proves that $\mu(\Phi)$ is an integer.

We continue to prove that Definition 7 is well-defined, which needs to show that μ is independent of the choices of the perturbations and the extensions. By Lemma 1, for sufficiently small ε and ε', the different perturbations will not change the rotation number. By Lemma 2, the rotation number of paths in $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$ only depends on the two endpoints. Let β be an extension of $e^{\varepsilon J_2}\Phi$, then $\Delta(\beta)$ is independent of the choices of β. For another perturbation $e^{\varepsilon' J_2}$, denote the endpoint of the extension of $e^{\varepsilon' J_2}\Phi$ by $W_{A',B'}$, we can choose this extension as the catenation

$$\beta' = (\beta_1' \# \beta) \# \beta_2', \tag{26}$$

where β_1' and β_2' are the path in $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$ and they satisfy

$$\beta_1'(0) = e^{\varepsilon' J_2}\Phi(1),\ \beta_1'(1) = e^{\varepsilon J_2}\Phi(1),\ \beta_2'(0) = W_{A,B}, \beta_2'(1) = W_{A',B'}. \tag{27}$$

By Proposition 3 and the continuity of $e^{\varepsilon J_2}$, we can see $|\Delta(\beta) - \Delta(\beta')| = |\Delta(\beta'_1) + \Delta(\beta'_2)|$ is small enough and then

$$|(\Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta)) - (\Delta(e^{\varepsilon' J_2}\Phi) + \Delta(\beta'))| = |\Delta(\beta) - \Delta(\beta')|$$

is small enough. Thus, we obtain

$$\Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta) = \Delta(e^{\varepsilon' J_2}\Phi) + \Delta(\beta'))$$

because they are integers, this shows that $\mu(\Phi)$ is independent of the small perturbations and the extensions. Hence, Definition 7 is well-defined.

Then, we prove the two properties. For $0 < a < 1$, we can choose a suitable ε such that $e^{\varepsilon J_2}\Phi(a) \notin \mathscr{C}(2,\mathbb{R})$ and set $C = e^{\varepsilon J_2}\Phi(a)$. The endpoints for extensions of $e^{\varepsilon J_2}\Phi$, $e^{\varepsilon J_2}\Phi([0,a])$ and $e^{\varepsilon J_2}\Phi([a,1])$ are $W_{A,B}, W_{A,C}$ and $W_{C,B}$. Denote these extensions by β, β_1 and β_2, by Proposition 3 and (22), we have

$$\mu(\Phi([0,a])) + \mu(\Phi([a,1])) = \Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta_1) + \Delta(\beta_2). \tag{28}$$

We only need to show that

$$\Delta(\beta_1) + \Delta(\beta_2) = \Delta(\beta). \tag{29}$$

Suppose A, B lie in the same connected component, then, if C is also in this connected component, we define $\beta^-(t) = \beta(1-t)$, then the catenation $(\beta_1\#\beta^-)\#\beta_2$ is a loop in $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$. By Lemma 2 and Proposition 3, $\Delta(\beta^-) = -\Delta(\beta)$, then (29) holds. If C is in another connected component, then $(-\beta_1\#\beta^-)\#\beta_2$ is a loop in $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$. Since $\Delta(-\beta) = \Delta(\beta)$, then (29) also holds. If A, B are in the different connected components, the proof is similar. By (28), (29) and (22), we have

$$\mu(\Phi([0,a])) + \mu(\Phi([a,1])) = \Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta) = \mu(\Phi). \tag{30}$$

This completes the proof of Theorem 1 (1). For

$$\Phi_{k,b}(t) = \begin{pmatrix} \cos(k\pi + bt) & -\sin(k\pi + bt) \\ \sin(k\pi + bt) & \cos(k\pi + bt) \end{pmatrix},\ 0 \le t \le 1,\ 0 \le |b| < \delta,\ k \in \mathbb{Z}, \tag{31}$$

where $\delta > 0$ is small enough. If $b \le 0$, $e^{\varepsilon J_2}\Phi_{k,b}$ is a path in $Osp^-(2,\mathbb{R})$, then the catenation $(e^{\varepsilon J_2}\Phi_{k,b})\#\beta$ is a loop in $Osp^-(2,\mathbb{R})$, where $0 < \varepsilon < \delta$ and β is the extension. Thus, by Proposition 3 and Lemma 2, the Maslov-type index

$$\mu(\Phi_{k,b}) = \Delta((e^{\varepsilon J_2}\Phi_{k,b})\#\beta) = 0,\ \text{for } b \le 0. \tag{32}$$

If $b > 0$, then $A = e^{\varepsilon J_2}\Phi_{k,b}(0) \in Osp^-(2,\mathbb{R})$ and $B = e^{\varepsilon J_2}\Phi_{k,b}(1) \in Osp^+(2,\mathbb{R})$. By (20), the endpoint of the extension is $-A$. According to (9), Definition 5 and Proposition 3, we have

$$\mu(\Phi_{k,b}) = \Delta((e^{\varepsilon J_2}\Phi_{k,b})\#\beta) = 1,\ \text{for } b > 0. \tag{33}$$

This completes the proof of Theorem 1 (2). For any $\Phi \in \mathcal{OP}(2,\mathbb{R})$, by Theorem 1 (1), it can be regarded as the catenation of some paths, then

$$\mu(\Phi) = \mu(\Phi([0,t])) + \mu(\Phi([t,s])) + \mu(\Phi([s,1])),\ \text{for } 0 \le t < s \le 1. \tag{34}$$

If $\Phi([t,s])$ is a path in $Osp(2,\mathbb{R})\backslash\mathscr{C}(2,\mathbb{R})$, then its Maslov-type index is equal to zero. If $\Phi([t,s]$ crosses the singular set $\mathscr{C}(2,\mathbb{R})$, then it is equivalent to $\Phi^-_{k,-b}\#\Phi_{k,b}(b \ne 0)$, where $\Phi^-_{k,-b} = \Phi_{k,-b}(1-t)$. Since

$$\mu(\Phi^-_{k,-b}\#\Phi_{k,b}) = \mu(\Phi_{k,b}) - \mu(\Phi(\Phi_{k,-b}) = 0,\ \text{for } b = 0, \tag{35}$$

$$\mu(\Phi^-_{k,-b}\#\Phi_{k,b}) = \mu(\Phi_{k,b}) - \mu(\Phi(\Phi_{k,-b}) = 1,\ \text{for } b > 0, \tag{36}$$

$$\mu(\Phi^-_{k,-b}\#\Phi_{k,b}) = \mu(\Phi_{k,b}) - \mu(\Phi(\Phi_{k,-b}) = -1,\ \text{for } b < 0. \tag{37}$$

Thus, the Maslov-type index of Φ is equal to the intersection number of Φ crossing the singular set $\mathscr{C}(2,\mathbb{R})$. This intersection number will be counted with signs of b. This completes the proof of Theorem 1. □

Proof of Theorem 2. Finally, we intend to prove Theorem 2. Let $f(t) = (L_1, L_2(t))$ be a pair of Lagrangian paths, where and $L_1 = \mathbb{R} \times \{0\}$ and

$$L_2(t) = \{(m(t)x, n(t)x)^T \mid x \in \mathbb{R},\ m^2(t) + n^2(t) \ne 0,\ 0 \le t \le 1\} \tag{38}$$

By Proposition 2, the path $f(t)$ can correspond to a unique orthogonal symplectic path

$$\Phi(t) = \begin{pmatrix} \frac{m(t)}{\sqrt{m^2(t)+n^2(t)}} & -\frac{n(t)}{\sqrt{m^2(t)+n^2(t)}} \\ \frac{n(t)}{\sqrt{m^2(t)+n^2(t)}} & \frac{m(t)}{\sqrt{m^2(t)+n^2(t)}} \end{pmatrix},\ 0 \le t \le 1. \tag{39}$$

We apply this result to the proper path $f_\varepsilon(t) = (L_1, e^{\varepsilon J_2}L_2(t))$. If

$$e^{\varepsilon J_2}L_2(t) = \{(m_\varepsilon(t)x, n_\varepsilon(t)x)^T \mid x \in \mathbb{R},\ m^2_\varepsilon(t) + n^2_\varepsilon(t) \ne 0,\ 0 \le t \le 1\}, \tag{40}$$

then $f_\varepsilon(t)$ corresponds to the orthogonal symplectic path

$$e^{\varepsilon J_2}\Phi(t) = \begin{pmatrix} \frac{m_\varepsilon(t)}{\sqrt{m_\varepsilon^2(t)+n_\varepsilon^2(t)}} & -\frac{n_\varepsilon(t)}{\sqrt{m_\varepsilon^2(t)+n_\varepsilon^2(t)}} \\ \frac{n_\varepsilon(t)}{\sqrt{m_\varepsilon^2(t)+n_\varepsilon^2(t)}} & \frac{m_\varepsilon(t)}{\sqrt{m_\varepsilon^2(t)+n_\varepsilon^2(t)}} \end{pmatrix}, \ 0 \le t \le 1. \tag{41}$$

Since $e^{\varepsilon J_2}L_2(t) \in \Sigma(L_1)$ if and only if $n_\varepsilon(t) = 0$, which corresponds to $e^{\varepsilon J_2}\Phi(t) \in \mathscr{C}(2,\mathbb{R})$, then $e^{\varepsilon J_2}L_2(t)$ crosses $\Sigma(L_1)$ if and only if $e^{\varepsilon J_2}\Phi(t)$ crosses $\mathscr{C}(2,\mathbb{R})$. After the perturbation, $e^{\varepsilon J_2}\Phi(t)$ has no intersections at the endpoints, then $\mu(e^{\varepsilon J_2}\Phi)$ is equal to intersection number of $e^{\varepsilon J_2}\Phi(t)$ crossing $\mathscr{C}(2,\mathbb{R})$. According to Definition 6, we obtain

$$\mu_{CLM}(f) = \mu_{proper}(f_\varepsilon) = \mu(e^{\varepsilon J_2}\Phi).$$

By Lemma 1, the perturbation $e^{\varepsilon J_2}$ will not change the rotation number. Since μ is defined by the rotation number, then $\mu(e^{\varepsilon J_2}\Phi) = \mu(\Phi)$ and hence

$$\mu_{CLM}(f) = \mu(\Phi).$$

This means that the intersection number of the Lagrangian pair can correspond to the intersection number of the orthogonal symplectic path crossing the singular set $\mathscr{C}(2,\mathbb{R})$, which can be calculated by the rotation number. This completes the proof of Theorem 2. □

5. Discussion

In this section, we present more details of the index constructed by Cappell, Lee and Miller [3]. Furthermore, we will consider the more general cases of higher dimensions and show the idea behind the construction of the Maslov-type index.

5.1. Comparison with μ_{CLM}

In [3], Cappell, Lee and Miller introduced four definitions for the index of a Lagrangian pair. These definitions satisfy the same system of axioms and are equivalent to one another. We have presented the special case of a geometrical definition in Section 2.3. More generally, μ_{CLM} can be defined for

$$f(t) = (L_1(t), L_2(t)), \ 0 \le t \le 1,$$

where $L_1(t), L_2(t)$ are two arbitrary Lagrangian paths. According to the system of axioms in [3], μ_{CLM} is a symplectic invariance. Thus, for any symplectic path Φ, we have

$$\mu_{CLM}(f) = \mu_{CLM}(\Phi f).$$

We can choose a suitable path Φ such that $\Phi(t)L_1(t) = \mathbb{R}^n \times \{0\}$, then, the general cases can correspond to the special case of $L_1 = \mathbb{R}^n \times \{0\}$. Therefore, Theorem 2 implies the relationship between the Maslov-type index and those four definitions in dimension 2.

5.2. The Idea Behind the Construction of the Maslov-Type Index in Higher Dimensions

In dimension 2, we use the methods of perturbation and extension to construct the Maslov-type index

$$\mu(\Phi) = \Delta(e^{\varepsilon J_2}\Phi) + \Delta(\beta).$$

In order to generalize this definition in dimension $2n$, it is necessary to address the following issues:

(1) How to define the rotation number Δ. The rotation number is defined by the paths on the unit circle; thus, we only need to map the orthogonal symplectic paths to the unit circle. For any orthogonal symplectic path Φ, it has the form

$$\Phi(t) = \begin{pmatrix} X(t) & -Y(t) \\ Y(t) & X(t) \end{pmatrix},\ 0 \le t \le 1,$$

where $X(t), Y(t)$ are two paths of $n \times n$ matrices and they satisfy $X^T(t)Y(t) = Y^T(t)X(t)$ and $X^T(t)X(t) + Y^T(t)Y(t) = I_n$. Thus, $X(t) + \sqrt{-1}Y(t)$ is a path of unitary matrices. Define

$$\rho_\Phi(t) := \det(X(t) + \sqrt{-1}Y(t)) = e^{\sqrt{-1}\theta(t)}$$

as the generalization of (11), then, we can also define the rotation number by $\theta(t)$ (see Definition 5).

(2) How to define the singular set $\mathscr{C}(2n, \mathbb{R})$. In [4,7], the authors considered the set of the symplectic matrices that have eigenvalues equal to 1. In our definition, $\mathscr{C}(2, \mathbb{R}) = \{I_2, -I_2\}$. Thus, we can consider the set of orthogonal symplectic matrices that have eigenvalues equal to 1 or -1. By perturbation $e^{\varepsilon J_{2n}}$, the two endpoints will lie in $Osp(2n, \mathbb{R})\backslash\mathscr{C}(2n, \mathbb{R})$. The rotation number

$$\Delta(e^{\varepsilon J_{2n}}\Phi) = \frac{(\theta(1) - n\varepsilon) - (\theta(0) - n\varepsilon)}{\pi} = \frac{\theta(1) - \theta(0)}{\pi} = \Delta(\Phi),$$

This means Lemma 1 holds in the general case. In fact, Lemma 2 also holds in the general case. As the state of reference [4,12], the rotation number can be viewed as the contributions of some eigenvalues on the unit circle. For any loops in $Osp(2n, \mathbb{R})\backslash\mathscr{C}(2n, \mathbb{R})$, all eigenvalues on the unit circle do not cross ± 1. Referring to the proof of Lemma 2, we see that the contributions of all eigenvalues are equal to zero. According to the discussion above, we can see that the methods of perturbation and extension can be generalized to the more general case.

(3) How to choose the endpoint of the extension β. The last issue is that one should choose a suitable endpoint of the extension such that the Maslov-type index is an integer. However, the topological structure of $Osp(2n, \mathbb{R})\backslash\mathscr{C}(2n, \mathbb{R})$ is complicated in the general case. We have to find all connected components of $Osp(2n, \mathbb{R})\backslash\mathscr{C}(2n, \mathbb{R})$ and hence, we can define the extension in one of these connected components. If we can choose a suitable endpoint of the extension in each connected component, the Maslov-type index can be generalized in dimension $2n$. This is the work that we need to study further.

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