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# Minimum Principles for Sturm–Liouville Inequalities and Applications

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**Abstract:** A minimum principle for a Sturm–Liouville (S-L) inequality is obtained, which shows that the minimum value of a nonconstant solution of a S-L inequality never occurs in the interior of the domain (a closed interval) of the solution. The minimum principle is then applied to prove that any nonconstant solutions of S-L inequalities subject to separated inequality boundary conditions (IBCs) must be strictly positive in the interiors of their domains and are increasing or decreasing for some of these IBCs. These positivity results are used to prove the uniqueness of the solutions of linear S-L equations with separated BCs. All of these results hold for the corresponding second-order differential inequalities (or equations), which are special cases of S-L inequalities (or equations). These results are applied to two models arising from the source distribution of the human head and chemical reactor theory. The first model is governed by a nonlinear S-L equation, while the second one is governed by a nonlinear second-order differential equation. For the first model, the explicit solutions are not available, and there are no results on the existence of solutions of the first model. Our results show that all the nonconstant solutions are increasing and are strictly positive solutions. For the second model, many results on the uniqueness of the solutions and the existence of multiple solutions have been obtained before. Our results are applied to prove that all the nonconstant solutions are decreasing and strictly positive.

**Keywords:** Sturm–Liouville inequalities; minimum principles; second-order differential inequalities; boundary value problems; strictly positive solutions

**MSC:** 34B24; 26D10; 34A12; 34A30; 34A34; 34B05

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## 1. Introduction

We study the properties of solutions of a Sturm–Liouville (S-L) inequality of the form

$$-(p(x)u'(x))' \geq 0 \quad \text{for each } x \in [a,b], \tag{1}$$

where $p : [a,b] \to \mathbb{R}_+$ is a function satisfying $p(x) > 0$ for each $x \in (a,b)$, and $u'$ denotes the first-order derivative of a function $u$.

A function $u : [a,b] \to \mathbb{R}$ is said to be a solution of (1) if $u \in C[a,b]$, $u'(x)$ and $(p(x)u'(x))'$ exist for each $x \in [a,b]$ and $u$ satisfies (1). A solution of (1) is said to be strictly positive if $u(x) > 0$ for each $x \in (a,b)$.

Note that $p$ may be zero at either $a$ or $b$, and $p$ is not required to be continuous or differentiable on $[a,b]$. We do not require $pu'$ to be in $L^1[a,b]$, so $pu'$ may not be an absolutely continuous function on $[a,b]$. Hence, if $u$ is a solution of (1), then the following assertion may not be true:

$$\int_a^x (p(x)u'(x))'\,dx = p(x)u'(x) - p(a)u'(a) \quad \text{for each } x \in [a,b].$$

This shows that we cannot obtain any results by taking the integral from $a$ to $x$ on both sides of (1).

However, in this paper, we use monotonicity of the function $pu'$ to derive a new minimum principle for nonconstant solutions of (1). More precisely, we prove that, if $u$ is a nonconstant solution of (1), then $u$ cannot reach its minimum in $(a, b)$, that is,

$$\min\{u(x) : x \in [a, b]\} = \min\{u(a), u(b)\} < u(x) \quad \text{for each } x \in (a, b). \tag{2}$$

It is well known that, if a function $u : [a, b] \to \mathbb{R}$ is twice differentiable on $[a, b]$ and satisfies that $u''(x) \leq 0$ for each $x \in [a, b]$, then $u$ is concave down on $[a, b]$, that is, $u$ satisfies

$$u(ta + (1-t)b) \geq tu(a) + (1-t)u(b) \quad \text{for each } t \in [0, 1].$$

This implies that

$$u(x) \geq \min\{u(a), u(b)\} \quad \text{for each } x \in [a, b]. \tag{3}$$

Hence, the new minimum principle (2) with $p \equiv 1$ enhances (3) by replacing the inequality sign with the strict inequality on $(a, b)$.

The minimum principle (2) holds for (1) without any boundary conditions (BCs). However, if we consider suitable BCs, then new properties of solutions for the boundary value problems can be obtained. Here, we apply the minimum principle to obtain new results on the positivity of the solutions of the S-L inequality (2) subject to the separated inequality boundary conditions (IBCs):

$$\alpha u(a) - \beta u'(a) \geq 0 \text{ and } \gamma u(b) + \delta u'(b) \geq 0, \tag{4}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ satisfy $(\alpha + \beta)(\gamma + \delta) > 0$. We refer to [1,2] for the study on a minimum principle (or strong minimum principle) and Hopf's boundary minimum principle for S-L inequality (1), which holds a.e on $(a, b)$ with the IBCs $u(a) \geq 0$ and $u(b) \geq 0$.

The separated IBCs contain Dirichlet ($\beta = \delta = 0$), Robin ($\alpha = \gamma$ and $\beta = \delta$) and Neumann ($\alpha = \gamma = 0$) IBCs. We show that (1) with the Neumann IBCs $u'(a) \leq 0$ and $u'(b) \geq 0$ only has constant solutions. By the minimum principle, we prove that all the nonconstant solutions of (1) with the other IBCs of (4) are strictly positive in $(a, b)$ and are increasing if $\alpha > 0$ and $\gamma = 0$ or decreasing if $\alpha = 0$ and $\gamma > 0$. We apply these positivity results to obtain the uniqueness of the solutions of linear S-L equations with separated BCs

$$\alpha u(a) - \beta u'(a) = 0 \text{ and } \gamma u(b) + \delta u'(b) = 0 \tag{5}$$

with $\alpha > 0$ or $\gamma > 0$.

An important special case of S-L inequality (1) is the following second-order differential inequalities:

$$-u''(x) - r(x)u'(x) \geq 0 \quad \text{for each } x \in [a, b], \tag{6}$$

where $r : [a, b] \to \mathbb{R}$ is a continuous function. We prove that, if $u$ is a solution of (6), then $u$ is a solution of (1). Hence, the minimum principle, positivity result and uniqueness for (6) without or with the separated IBCs can be obtained via the results on S-L inequalities or equations.

The minimum principle and positivity results can be used to study solutions of nonlinear S-L equations such as

$$-(p(x)u'(x))' = f(x, u(x)) \quad \text{for each } x \in [a, b] \tag{7}$$

when $f(x, u(x)) \geq 0$ for each $x \in [a, b]$. There are many results on the existence of nonnegative solutions of (7) with suitable boundary conditions, for example, in [3–7], and on the eigenvalues of the following S-L equations (see [8–10]).

$$-(p(x)u'(x))' = \lambda u(x) \quad \text{for each } x \in [a, b], \tag{8}$$

The minimum principle and positivity results can be used to obtain the minimum principle and positivity on nonconstant solutions of (7) and eigenfunctions of the eigenvalue problem (8).

As illustrations, we consider two models arising in the heat conduction of the human head and chemical reactor theory. The first model is governed by a nonlinear S-L equation (see [8,11–16] for computations of solutions). The second one is governed by a nonlinear second-order differential equation (see [17,18] for the existence of solutions). For the first model, the explicit solutions are not available, and there are no results on the existence of solutions of the first model. There is little study on the existence of solutions, possibly because of the lack of Green's functions. Our results show that, if the solutions exist, then all the solutions are increasing and are strictly positive. For the second models from the chemical reactor theory, there have been many results on the uniqueness of the solutions and the existence of multiple solutions (see [17,18] and the references therein). We prove that all the solutions are decreasing and are strictly positive.

The structure of this paper is as follows: In Section 2 of this paper, we study the minimum principle, positivity and uniqueness of solutions for the S-L inequalities and linear S-L equations. In Section 3, we apply these results on the S-L inequalities or linear S-L equations to deal with some second-order differential equations. In Section 4, we consider the two models governed by a nonlinear S-L equation and a nonlinear second-order differential equation, respectively, and obtain the minimum principles and monotonicity of their solutions.

## 2. Sturm–Liouville Inequalities

We study the properties of solutions for the Sturm–Liouville (S-L) inequality of the form

$$-(p(x)u'(x))' \geq 0 \quad \text{for each } x \in [a,b], \tag{9}$$

where $p : [a,b] \to \mathbb{R}_+$ is a function satisfying $p(x) > 0$ for each $x \in (a,b)$, and $u'$ denotes the first-order derivative of a function $u$. We allow $p$ to be zero at $a$ or $b$.

We denote by $C[a,b]$, $C^1[a,b]$ and $AC[a,b]$ the Banach space of continuous functions on $[a,b]$ with the maximum norm, the space of continuously differentiable functions on $[a,b]$ and the space of absolutely continuous functions on $[a,b]$, respectively. It is well known that

$$C^1[a,b] \subset AC[a,b] \subset C[a,b].$$

**Definition 1.** *A function $u : [a,b] \to \mathbb{R}$ is said to be a solution of (9) if $u \in C[a,b]$, $u'(x)$ and $(p(x)u'(x))'$ exist for each $x \in [a,b]$ and $u$ satisfies (9). A solution $u$ of (9) is said to be nonnegative if $u(x) \geq 0$ for each $x \in [a,b]$ and to be strictly positive on $(a,b)$ if $u(x) > 0$ for each $x \in (a,b)$.*

In Definition 1, we only require a solution $u$ to satisfy that $(p(x)u'(x))'$ exists for each $x \in [a,b]$. We do not require $(pu')' \in L^1[a,b]$, so $pu'$ may not be in $AC[a,b]$.

**Notation:** For $c \in \mathbb{R}$, we define a constant function $\hat{c} : [a,b] \to \mathbb{R}$ by

$$\hat{c}(x) = c \quad \text{for each } x \in [a,b]. \tag{10}$$

It is trivial that the constant function $\hat{c}$ is a solution of (9) for each $c \in \mathbb{R}$. Hence, we concentrate on the nonconstant solutions of (9), that is, the solution $u$ satisfies that there exist two different points $x_1, x_2 \in [a,b]$ such that $u(x_1) \neq u(x_2)$.

We first prove the following minimum principle for nonconstant solutions of (9), which shows that the minimum values of nonconstant solutions of (9) never occur at the interior points of $[a,b]$.

**Theorem 1.** *If $u$ is a nonconstant solution of (9), then*

$$\min\{u(a), u(b)\} < u(x) \quad \textit{for each } x \in (a,b). \tag{11}$$

**Proof.** Let $u$ be a nonconstant solution of (9). By Definition 1, $u$ is a continuous function on $[a,b]$. Let $m$ be the minimum value of $u$ on $[a,b]$, that is,

$$m = \min\{u(x) : x \in [a,b]\}.$$

We prove that

$$m < u(x) \quad \text{for each } x \in (a,b). \tag{12}$$

If (12) is false, then there exists $x^* \in (a,b)$ such that

$$u(x^*) = \min\{u(x) : x \in [a,b]\}. \tag{13}$$

Since $u'(x)$ exists for each $x \in (a,b)$, it follows from Fermat's Theorem that $u'(x^*) = 0$. By (9), $pu'$ is decreasing on $[a,b]$. Hence,

$$p(x)u'(x) \geq p(x^*)u'(x^*) = 0 \quad \text{for each } x \in [a,x^*] \tag{14}$$

and

$$0 = p(x^*)u'(x^*) \geq p(x)u'(x) \quad \text{for each } x \in [x^*,b]. \tag{15}$$

Since $p(x) > 0$ for each $x \in (a,b)$, by (14) and (15), we obtain $u'(x) \geq 0$ for each $x \in (a,x^*]$ and $u'(x) \leq 0$ for each $x \in [x^*,b)$. Hence, $u$ is increasing on $(a,x^*]$ and decreasing on $[x^*,b)$. It follows that

$$u(x) \leq u(x^*) \quad \text{for each } x \in (a,b).$$

This with (13) implies that

$$u(x) = u(x^*) = \min\{u(x) : x \in [a,b]\} \quad \text{for each } x \in (a,b). \tag{16}$$

Since $u \in C[a,b]$, taking limits on (16) as $x \to a^+$ and $x \to b^-$ implies that $u(a) = u(x^*)$ and $u(b) = u(x^*)$. This with (16) shows that $u$ is a constant function, which contradicts the hypothesis that $u$ is a nonconstant function. Hence, (12) holds. Since $u \in C[a,b]$, there exists $x_0 \in [a,b]$ such that $u(x_0) = m$. It follows from (12) that $x_0 = a$ or $x_0 = b$, and the result holds. □

As an illustration of Theorem 1, we consider the S-L inequality

$$-(x^2u'(x))' \geq 0 \quad \text{for each } x \in [0,1]. \tag{17}$$

**Example 1.** *Let*

$$u(x) = \cos\frac{\pi}{2}x \quad \textit{for } x \in [0,1]. \tag{18}$$

*Then, the following assertions hold:*
(1) *$u$ is a solution of* (17);
(2) $\min\{u(0),u(1)\} < u(x)$ *for each* $x \in (0,1)$.

**Proof.** (1) Differentiating both sides of (18), we have

$$u'(x) = -\frac{\pi}{2}\sin\frac{\pi}{2}x \quad \text{for each } x \in [0,1]. \tag{19}$$

By (19), we have

$$x^2u'(x) = -\frac{\pi}{2}x^2\sin\frac{\pi}{2}x \quad \text{for each } x \in [0,1]. \tag{20}$$

Taking derivatives on both sides of the above equation implies that

$$-\Big(x^2u'(x)\Big)' = \frac{\pi}{2}\Big[2x\sin\frac{\pi}{2}x + \frac{\pi}{2}x^2\cos\frac{\pi}{2}x\Big] \geq 0 \quad \text{for each } x \in [0,1]. \tag{21}$$

By (19), (20), (21) and Definition 1, $u$ is a solution of (17).

(2) By (18) and the result (1), $u$ is a nonconstant solution of (17). The result follows from Theorem 1. □

As an application of Theorem 1, we provide another new result which provides sufficient boundary value conditions ensuring that the first-order derivative of the nonconstant solutions of (9) at $a$ (or at $b$) is greater than 0 (or less than 0). The new result will be used to derive a Hopf's boundary minimum principle for the S-L inequalities with a Dirichlet-type inequality BC (see Theorem 6).

To do this, we first prove the following lemma, which shows that the signs of the first-order derivative of solutions of (9) at $a$ (or at $b$) determine the monotonicity that is decreasing or increasing on $[a, b]$ of solutions of (9).

**Lemma 1.** *Assume that $u : [a, b] \to \mathbb{R}$ is a solution of (9). Then, the following assertions hold:*
*(i) If $u'(a) \leq 0$, then $u$ is decreasing on $[a, b]$;*
*(ii) If $u'(b) \geq 0$, then $u$ is increasing on $[a, b]$.*

**Proof.** (*i*) By (9), $pu'$ is decreasing on $[a, b]$. This with $u'(a) \leq 0$ implies that

$$p(x)u'(x) \leq p(a)u'(a) \leq 0 \quad \text{for each } x \in [a, b].$$

Since $p(x) > 0$ for each $x \in (a, b)$, we have $u'(x) \leq 0$ for each $x \in (a, b)$, and $u$ is decreasing on $(a, b)$. Since $u \in C[a, b]$, we have

$$u(b) \leq u(x) \leq u(a) \quad \text{for each } x \in [a, b]$$

and $u$ is decreasing on $[a, b]$.

(*ii*) By (9), $pu'$ is decreasing on $[a, b]$. This with $u'(b) \geq 0$ implies that

$$0 \leq p(b)u'(b) \leq p(x)u'(x) \quad \text{for each } x \in [a, b].$$

Since $p(x) > 0$ for each $x \in (a, b)$, we have $u'(x) \geq 0$ for each $x \in (a, b)$, and $u$ is increasing on $(a, b)$. Since $u \in C[a, b]$, $u$ is increasing on $[a, b]$. □

By Theorem 1 and Lemma 1, we prove the new result, which is a key for obtaining the Hopf's boundary minimum principle (Theorem 6) for the S-L inequalities.

**Theorem 2.** *Assume that $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (9). Then, the following assertions hold:*
*(1) If $u(a) \leq u(b)$, then $u'(a) > 0$;*
*(2) If $u(a) \geq u(b)$, then $u'(b) < 0$.*

**Proof.** (1) If the result (1) is false, then $u'(a) \leq 0$. By Lemma 1 (1), $u$ is decreasing on $[a, b]$, and $u(x) \leq u(a)$ for each $x \in [a, b]$. This with Theorem 1 implies that

$$u(b) < u(x) \leq u(a) \quad \text{for each } x \in (a, b),$$

which contradicts the hypothesis that $u(a) \leq u(b)$.

(2) If the result (2) is false, then $u'(b) \geq 0$. By Lemma 1 (2), $u$ is increasing on $[a, b]$, and $u(x) \leq u(b)$ for each $x \in [a, b]$. This with Theorem 1 implies that

$$u(a) < u(x) \leq u(b) \quad \text{for each } x \in (a, b),$$

which contradicts the hypothesis that $u(a) \geq u(b)$. □

Theorem 1 applies to any nonconstant solutions of (9) in $C[a, b]$ and does not involve any inequality boundary conditions (IBCs) at $a$ or at $b$.

Below, we consider nonconstant solutions of (9) subject to the separated IBCs of the form

$$\alpha u(a) - \beta u'(a) \geq 0 \text{ and } \gamma u(b) + \delta u'(b) \geq 0. \tag{22}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ satisfy

$$(\alpha + \beta)(\gamma + \delta) > 0. \tag{23}$$

The separated IBCs (22) contain Dirichlet ($\beta = \delta = 0$), Robin ($\alpha = \gamma$ and $\beta = \delta$) and Neumann ($\alpha = \gamma = 0$) IBCs.

Under the assumption (23), it is easy to verify that (22) is equivalent to the following four IBCs:

$(B_1)$ $u(a) - \beta u'(a) \geq 0$ and $u(b) + \delta u'(b) \geq 0$ for $\beta, \delta \in \mathbb{R}_+$. ($\alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0$);
$(B_2)$ $u(a) - \beta u'(a) \geq 0$ and $u'(b) \geq 0$ for $\beta \in \mathbb{R}_+$. ($\alpha > 0, \beta \geq 0, \gamma = 0, \delta > 0$);
$(B_3)$ $u'(a) \leq 0$ and $u(b) + \delta u'(b) \geq 0$ for $\delta \in \mathbb{R}_+$. ($\alpha = 0, \beta > 0, \gamma > 0, \delta \geq 0$);
$(B_4)$ $u'(a) \leq 0$ and $u'(b) \geq 0$. ($\alpha = 0, \beta > 0, \gamma = 0, \delta > 0$).

It is clear that if $\alpha > 0$ or $\gamma > 0$, then the BC (22) is equivalent to the three BCs $(B_1)$, $(B_2)$ and $(B_3)$, and, if $\alpha = \gamma = 0$, then the BC (22) is $(B_4)$.

**Definition 2.** *A function $u : [a, b] \to \mathbb{R}$ is said to be a solution (nonnegative solution or strictly positive solution) of (9) with (22) if $u \in C[a, b]$ is a solution (nonnegative solution or strictly positive solution) of (9) and satisfies (22).*

We state the following simple result of constant solutions of (9) with (22).

**Theorem 3.** *Let $c \in \mathbb{R}$. Then, the following assertions hold:*
*$(i)$ $\hat{c}$ is a solution of (9) with $(B_i)$ if and only if $c \geq 0$ for each $i \in \{1, 2, 3\}$;*
*$(ii)$ $\hat{c}$ is a solution of (9) with $(B_4)$.*

**Theorem 4.** *Assume that $u : [a, b] \to \mathbb{R}$ is a solution of (9) with $(B_4)$. Then, $u$ is a constant solution.*

**Proof.** By $(B_4)$ and Lemma 1, $u$ is decreasing and increasing on $[a, b]$. It follows that

$$u(b) \leq u(x) \leq u(a) \leq u(x) \leq u(b) \quad \text{for each } x \in [a, b].$$

This implies that $u(x) = u(b)$ for each $x \in [a, b]$, and the result holds. □

By Theorem 3 $(ii)$ and Theorem 4, (9) with $(B_4)$ has no nonconstant solutions. Therefore, we only discuss nonconstant solutions of (9) with $(B_i)$ for each $i \in \{1, 2, 3\}$. Hence, from now on, we always assume that $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ satisfy

$$(\alpha + \beta)(\gamma + \delta) > 0 \quad \text{and either } \alpha > 0 \text{ or } \gamma > 0, \tag{24}$$

which excludes the BC $(B_4)$.

**Lemma 2.** *Assume that $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (9). Then, the following assertions hold:*
*$(i)$ If $u(a) \leq u(b)$ and $u(a) - \beta u'(a) \geq 0$, then $u(a) \geq 0$;*
*$(ii)$ If $u(a) \geq u(b)$ and $u(b) + \delta u'(b) \geq 0$, then $u(b) \geq 0$.*

**Proof.** Let $u : [a, b] \to \mathbb{R}$ be a nonconstant solution of (9) with (22). By Theorem 1, we have

$$\min\{u(a), u(b)\} < u(x) \quad \text{for each } x \in (a, b). \tag{25}$$

$(i)$ Since $u(a) \leq u(b)$, by (25), we have

$$u(a) = \min\{u(a), u(b)\} < u(x) \quad \text{for each } x \in (a, b). \tag{26}$$

If the result were false, then $u(a) < 0$. Since $u(a) - \beta u'(a) \geq 0$, we have

$$0 > u(a) \geq \beta u'(a).$$

This implies that $\beta > 0$, and $u'(a) < 0$. Hence, there exists $\varepsilon_0 \in (0, b-a)$ such that

$$u(x) \le u(a) \quad \text{for each } x \in [a, a+\varepsilon_0).$$

This with (26) implies that

$$u(a) < u(x) \le u(a) \quad \text{for each } x \in [a, a+\varepsilon_0),$$

which is a contradiction.

$(ii)$ Since $u(a) \ge u(b)$, by (25), we have

$$u(b) = \min\{u(a), u(b)\} < u(x) \quad \text{for each } x \in (a, b). \tag{27}$$

If the result is false, then $u(b) < 0$. Since $u(b) + \delta u'(b) \ge 0$, we have

$$0 > u(b) \ge -\delta u'(b).$$

This implies that $\delta > 0$, and $u'(b) > 0$. Hence, there exists $\varepsilon_0 \in (0, b-a)$ such that

$$u(x) \le u(b) \quad \text{for each } x \in (b-\varepsilon_0, b].$$

This with (27) implies that

$$u(b) < u(x) \le u(b) \quad \text{for each } x \in (b-\varepsilon_0, b],$$

which is a contradiction. □

Now we prove the positivity result on (9) subject to the BCs $(B_1)$–$(B_3)$.

**Theorem 5.** $(i)$ *If $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (9) with $(B_1)$, then*

$$0 \le \min\{u(a), u(b)\} < u(x) \quad \textit{for each } x \in (a, b). \tag{28}$$

$(ii)$ *If $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (9) with $(B_2)$, then $u$ is increasing on $[a, b]$, and*

$$0 \le u(a) < u(x) \le u(b) \quad \textit{for each } x \in (a, b). \tag{29}$$

$(iii)$ *If $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (9) with $(B_3)$, then $u$ is decreasing on $[a, b]$, and*

$$0 \le u(b) < u(x) \le u(a) \quad \textit{for each } x \in (a, b). \tag{30}$$

**Proof.** Let $u : [a, b] \to \mathbb{R}$ be a nonconstant solution of (9). By Theorem 1, we have

$$\min\{u(a), u(b)\} < u(x) \quad \text{for each } x \in (a, b). \tag{31}$$

$(i)$ Since $(B_1)$ holds, if $u(a) \le u(b)$, then it follows from $u(a) - \beta u'(a) \ge 0$ and Lemma 2 $(i)$ that $u(a) \ge 0$. If $u(a) \ge u(b)$, then by $u(b) - \beta u'(b) \ge 0$ and Lemma 2 $(ii)$, we have $u(b) \ge 0$. This with (31) implies (28).

$(ii)$ By $(B_2)$, we have $u'(b) \ge 0$. By Lemma 1 $(ii)$, $u$ is increasing on $[a, b]$, and

$$u(a) \le u(x) \le u(b) \quad \text{for each } x \in [a, b].$$

This with (31) implies (29).

$(iii)$ By $(B_3)$, we have $u'(a) \le 0$. By Lemma 1 $(i)$, $u$ is decreasing on $[a, b]$, and

$$u(b) \le u(x) \le u(a) \quad \text{for each } x \in [a, b].$$

This with (31) implies (30). □

As an application of Theorem 5, we consider the S-L inequality (17), that is,

$$-(x^2 u'(x))' \ge 0 \quad \text{for each } x \in [0, 1] \tag{32}$$

subject to the IBC

$$u(0) - \beta u'(0) \geq 0 \quad \text{and } u(1) + \delta u'(1) \geq 0, \tag{33}$$

where $\beta, \delta \geq 0$ are given.

We first provide an example of a nonconstant solution $u$ of (32), which is not a solution of (32)–(33).

**Example 2.** *Let $u$ be the same as in (18). Then, $u$ is a solution of (32) but is not a solution of (32)–(33).*

**Proof.** The first result follows from Example 1. By (18) and (19), we have

$$u(1) - \delta u'(1) = 0 - \delta \frac{\pi}{2} < 0.$$

Hence, $u$ does not satisfy (33). □

Next, we provide an example of a nonconstant solution of (32)–(33).

**Example 3.** *Let $A \geq \frac{\pi}{2}\delta$, and*

$$u(x) = A + \cos \frac{\pi}{2} x \quad \text{for } x \in [0,1]. \tag{34}$$

*Then, the following assertions hold:*
*(1) $u$ is a solution of (32)–(33);*
*(2) $0 \leq \min\{u(0), u(1)\} < u(x)$ for each $x \in (0,1)$.*

**Proof.** (1) Differentiating both sides of (34), we have

$$u'(x) = -\frac{\pi}{2} \sin \frac{\pi}{2} x \quad \text{for } x \in [0,1]. \tag{35}$$

By (35), we have

$$x^2 u'(x) = -x^2 \frac{\pi}{2} \sin \frac{\pi}{2} x \quad \text{for } x \in [0,1].$$

Taking derivatives on both sides of the above equation implies that

$$-\left(x^2 u'(x)\right)' = \frac{\pi}{2}\left[2x \sin \frac{\pi}{2} x + x^2 \frac{\pi}{2} \cos \frac{\pi}{2} x\right] \geq 0 \quad \text{for } x \in [0,1]$$

and $u$ satisfies (32). By (34) and (35), we have

$$u(0) - \beta u'(0) = (1 + A) - \beta[-\frac{\pi}{2} \sin \frac{\pi}{2}(0)] = 1 + A \geq 0.$$

and, since $A \geq \frac{\pi}{2}\delta$, we have

$$u(1) + \delta u'(1) = A - \delta \frac{\pi}{2} \geq 0.$$

Hence, $u$ is a solution of (32)–(33).

(2) By the result (1), $u$ is a nonconstant solution of (32)–(33). The result follows from Theorem 5 $(i)$. □

Now, by applying Theorems 2 and 5, we give the Hopf's boundary minimum principle.

**Theorem 6.** *$(i)$ If $u : [a,b] \to \mathbb{R}$ is a nonconstant solution of (9) with $(B_1)$, then the following assertions hold:*
*(1) If $u(a) = 0$, then $u'(a) > 0$;*
*(2) If $u(b) = 0$, then $u'(b) < 0$.*

$(ii)$ *If* $u : [a,b] \to \mathbb{R}$ *is a nonconstant solution of* (9) *with* $(B_2)$, *then* $u'(a) > 0$;
$(iii)$ *If* $u : [a,b] \to \mathbb{R}$ *is a nonconstant solution of* (9) *with* $(B_3)$, *then* $u'(b) < 0$.

**Proof.** (1) Since $u(a) = 0$, it follows from Theorem 5 $(i)$ that $u(a) = 0 \leq u(b)$. By Theorem 2 (1), $u'(a) > 0$.

(2) Since $u(b) = 0$, it follows from Theorem 5 $(i)$ that $u(b) = 0 \leq u(a)$. By Theorem 2 (2), $u'(b) < 0$.
$(ii)$ By Theorem 5 $(ii)$, we have $u(a) < u(b)$. By Theorem 2 (1), $u'(a) > 0$.
$(iii)$ By Theorem 5 $(iii)$, we have $u(b) < u(a)$. By Theorem 2 (2), $u'(b) < 0$. □

**Remark 1.** *The Hopf's boundary minimum principle for some S-L inequalities with the BCs* $u(a) \geq 0$ *and* $u(b) \geq 0$ *was studied in [1] (p. 1072). Hence, Theorem 6* $(i)$ *with* $\beta > 0$ *or* $\delta > 0$ *and* $(ii)$ *and* $(iii)$ *are new. Our method is different from that used in [1] (p. 1072).*

Applying Theorem 5, we study the uniqueness of the solutions of the boundary value problem (BVP) of the S-L equation

$$-(p(x)u'(x))' = 0 \quad \text{for each } x \in [a,b] \tag{36}$$

subject to the separated BC

$$\alpha u(a) - \beta u'(a) = 0 \quad \text{and } \gamma u(b) + \delta u'(b) = 0, \tag{37}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ satisfy (24).

**Theorem 7.** *Equation* (36) *with Equation* (37) *has only a zero solution.*

**Proof.** It is obvious that $\hat{0}$ is a solution of (36) with (37). Let $u$ be a nonconstant solution of (36) with (37). Then it is easy to see that $-u$ is a solution of (36) with (37). By Theorem 5, we obtain $u = 0$. Hence, (36) with (37) has no nonconstant solutions. By Theorem 3 $(i)$, (36) with (37) has no nonzero constant solutions. The result follows. □

We study the uniqueness of the solutions of the BVP of the S-L equation

$$-(p(x)u'(x))' = v(x) \quad \text{for each } x \in [a,b] \tag{38}$$

subject to the separated BC

$$\alpha u(a) - \beta u'(a) = c_0 \quad \text{and } \gamma u(b) + \delta u'(b) = c_1, \tag{39}$$

where $v : [a,b] \to \mathbb{R}$ is a function, $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ satisfy (24) and $c_0, c_1 \in \mathbb{R}$.

**Definition 3.** *A function* $u : [a,b] \to \mathbb{R}$ *is said to be a solution of* (38) *if* $u \in C[a,b]$, $u'(x)$ *and* $(p(x)u'(x))'$ *exist for each* $x \in [a,b]$ *and* $u$ *satisfies* (38).

**Theorem 8.** *Equation* (38) *with Equation* (39) *has at most one solution.*

**Proof.** Assume that (38) and (39) have a solution $u_j$ for each $j \in \{1,2\}$. Let

$$u(x) = u_1(x) - u_2(x) \quad \text{for each } x \in [a,b].$$

It is easy to see that $u$ is a solution of (36) with (37). By Theorem 7, (36) with (37) has only a zero solution. Hence, $u = 0$ and $u_1 = u_2$. □

## 3. Second-Order Linear Differential Inequalities

Closely related to the S-L inequality is the second-order differential inequality

$$-u''(x) - r(x)u'(x) \geq 0 \quad \text{for each } x \in [a,b], \tag{40}$$

where $r : [a, b] \to \mathbb{R}$ is assumed to be a continuous function. Note that $r$ is not necessarily nonnegative. We can apply the results obtained in Section 2 to derive results on (40).

**Definition 4.** *A function $u : [a, b] \to \mathbb{R}$ is said to be a solution of (40) if $u \in C^1[a, b]$, $u''(x)$ exists for each $x \in [a, b]$ and $u$ satisfies (40).*

In Definition 4, $u$ is required to satisfy that $u''(x)$ exists for each $x \in [a, b]$, but $u''$ is not required to be continuous on $(a, b)$. This is different from the the classical solutions, that is, $u \in C^2(a, b) \cap C^1[a, b]$ studied in (p. 634, [19]), where the one-dimensional strongly uniformly elliptic equations were considered.

The inequality (40) can be studied via the following S-L inequality:

$$-(p(x)u'(x))' \geq 0 \quad \text{for each } x \in [a, b], \tag{41}$$

where $p : [a, b] \to (0, \infty)$ is a function defined by

$$p(x) = e^{\int_a^x r(s)\,ds}. \tag{42}$$

We note that the function $p$ satisfies $p(x) > 0$ for each $x \in [a, b]$ and is continuous on $[a, b]$. By (42) and continuity of $r$, we obtain

$$p'(x) = p(x)r(x) \quad \text{for each } x \in [a, b]. \tag{43}$$

**Lemma 3.** *If $u$ is a solution of (40), then $u$ is a solution of (41).*

**Proof.** Let $u$ be a solution of (40). By Definition 4, $u \in C^1[a, b]$ and $u''(x)$ exist for each $x \in [a, b]$. By (43), we have

$$\begin{aligned}(p(x)u'(x))' &= p(x)u''(x) + p'(x)u'(x) = p(x)u''(x) + p(x)r(x)u'(x)\\ &= p(x)[u''(x) + r(x)u'(x)] \leq 0 \quad \text{for each } x \in [a, b]\end{aligned} \tag{44}$$

and $(p(x)u'(x))'$ exist for each $x \in [a, b]$. It follows from Definition 1 that $u$ is a solution of (41). □

Similar to (9), we have the following minimum principle for nonconstant solutions of (40).

**Theorem 9.** *If $u$ is a nonconstant solution of (40), then*

$$\min\{u(a), u(b)\} < u(x) \quad \textit{for each } x \in (a, b).$$

**Proof.** Let $u$ be a nonconstant solution of (40). By Lemma 3, $u$ is a nonconstant solution of (41). The results follow from Theorem 1. □

**Definition 5.** *A function $u : [a, b] \to \mathbb{R}$ is said to be a solution of (40) with (22) if $u \in C^1[a, b]$ is a solution of (40) and satisfies (22).*

Similar to (9), we have the following positivity result on (40) subject to the BCs $(B_1)$–$(B_3)$.

**Theorem 10.** *$(i)$ If $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (40) with $(B_1)$, then*

$$0 \leq \min\{u(a), u(b)\} < u(x) \quad \textit{for each } x \in (a, b). \tag{45}$$

*$(ii)$ If $u : [a, b] \to \mathbb{R}$ is a nonconstant solution of (40) with $(B_2)$, then $u$ is increasing on $[a, b]$, and*

$$0 \leq u(a) < u(x) \leq u(b) \quad \textit{for each } x \in (a, b). \tag{46}$$

(*iii*) *If* $u : [a, b] \to \mathbb{R}$ *is a nonconstant solution of* (40) *with* $(B_3)$, *then* $u$ *is decreasing on* $[a, b]$, *and*

$$0 \le u(b) < u(x) \le u(a) \quad \text{for each } x \in (a, b). \tag{47}$$

**Proof.** Let $u$ be a nonconstant solution of (40). By Lemma 3, $u$ is a nonconstant solution of (41). The result follows from Theorem 5. □

Similar to S-L equations, we obtain the following uniqueness results.

**Theorem 11.** *The BVP of the second-order differential equation*

$$-u''(x) - r(x)u'(x) = 0 \quad \text{for each } x \in [a, b] \tag{48}$$

*subject to the separated BC* (37) *has only a zero solution.*

**Theorem 12.** *The BVP of the second-order differential equation*

$$-u''(x) - r(x)u'(x) = v(x) \quad \text{for each } x \in [a, b] \tag{49}$$

*subject to the BC* (39) *has at most one solution, where* $v : [a, b] \to \mathbb{R}$ *is a function.*

## 4. Applications

We consider the model arising from the source distribution of the human head governed by the following nonlinear S-L equation:

$$-(x^2u'(x))' = \lambda x^2 e^{-qu(x)} \quad \text{for each } x \in [0, 1] \tag{50}$$

subject to the BC

$$u(0) = 0 \quad \text{and } \gamma u(1) + u'(1) = 0, \tag{51}$$

where $\lambda, q > 0$ (see [11]). Some related models can be found in [13,14]. The exact solutions of the BVP (50)–(51) are not available, and there are no results on the existence of solutions of (50)–(51); therefore, there are extensive studies on computation of solutions of (50)–(51) (see [12,14–16] and the reference therein). There are generalizations on the computation of solutions to fractional differential equations [8]. Since $p : [0, 1] \to \mathbb{R}_+$ defined by $p(x) = x^2$ for $x \in [0, 1]$ does not satisfy $1/p \in L^1[0, 1]$, the Green's function does not exist. Hence, the previous results on the existence of solutions of the BVP for the S-L equations obtained, for example, in [3–5,7,20–22], via Green's functions cannot be used to deal with (50)–(51).

Here, we use Theorem 1 to obtain the following new result.

**Theorem 13.** (1) *If* $u$ *is a solution of* (50), *then*

$$\min\{u(0), u(1)\} < u(x) \quad \text{for each } x \in (0, 1). \tag{52}$$

(2) *If* $u$ *is a solution of the BVP* (50)–(51), *then* $u$ *is decreasing on* $[0, 1]$, *and*

$$0 \le u(1) < u(x) \le u(0) \quad \text{for each } x \in (0, 1). \tag{53}$$

**Proof.** Let $u \in C[0, 1]$ be a solution of (50). By (50), it is easy to see that

$$-(x^2u'(x))' = \lambda x^2 e^{-qu(x)} \ge 0 \quad \text{for each } x \in [0, 1] \tag{54}$$

and $u$ is a nonconstant solution of (54).

(1) The result (1) follows from (54) and Theorem 1.

(2) Let $u \in C[0, 1]$ be a solution of (50)–(51). Note that the BC (51) is a special case of $(B_3)$. It follows from (54) and Theorem 10 (*iii*) that $u$ is decreasing on $[0, 1]$, and (53) holds. □

The second model we consider is the following BVP of the second-order differential equation

$$-\beta u''(x) + u'(x) = f(u(x)) \quad \text{for each } x \in [0,1] \tag{55}$$

subject to the BC

$$\alpha u(0) - \beta u'(0) = 0 \quad \text{and } u'(1) = 0, \tag{56}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$f(u) = \lambda(q-u)e^{\frac{-k}{1+u}}. \tag{57}$$

The BVP (55)–(56) arises in chemical reactor theory. The function $u$ represents the dimensionless temperature in the reactor, and $\lambda, q, k > 0$ are known constants. The function $f(u)$ in (57) is the Arrhenius reaction rate, which essentially represents the rates of chemical production of the species (or the rate of heat generation) in the reactor (see [17,18] and the references therein).

It is proved in Theorem 3.5 of [17] that, if $\left(\frac{f(u)}{u}\right)' < 0$ for $u \in [0,q]$, then (55)–(56) have a unique solution. When $k > 4(1+1/q)$, it is proved in [18] that, under suitable conditions on $\lambda$, (55)–(56) have at least two or three nonnegative solutions. However, these results do not show that these solutions are strictly positive solutions.

By Theorems 9 and 10, we prove the following result which shows that all the solutions of (55)–(56) are strictly positive.

**Theorem 14.** (1) *If $u \in C[0,1]$ is a solution of* (55)*, then*

$$\min\{u(0), u(1)\} < u(x) \quad \textit{for each } x \in (0,1). \tag{58}$$

(2) *If $u \in C[0,1]$ is a solution of the BVP* (55)–(56)*, then $u$ is increasing on $[0,1]$, $u$ is a strictly positive solution and*

$$0 \le u(0) < u(x) \le u(1) \quad \textit{for each } x \in (0,1). \tag{59}$$

**Proof.** Let $u$ be a solution of (55). By (55) and (57), $u$ is a nonconstant solution. By Theorem 3.1 of [17], $u(x) \le q$ for $x \in [0,1]$. By (57), we obtain

$$f(u(x)) \ge 0 \quad \text{for each } x \in [0,1].$$

This with (55) implies that

$$-\beta u''(x) + u'(x) = f(u(x)) \ge 0 \quad \text{for } x \in [0,1]. \tag{60}$$

(1) The result (1) follows from Theorem 9.

(2) Let $u$ be a solution of (55)–(56). Note that the BC (51) is a special case of $(B_2)$. The result (2) follows from Theorem 10 $(ii)$. □

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