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Suzuki–Ćirić-Type Nonlinear Contractions Employing a Locally ζ-Transitive Binary Relation with Applications to Boundary Value Problems

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Abstract: This article is devoted to enhancing a class of generalized Suzuki-type nonlinear contractions following Pant to a class of Suzuki–Ćirić-type nonlinear contractions via comparison functions via a locally ζ-transitive relation and implemented the same to ascertain certain fixed-point results. The outcomes presented herewith unify and generalize a few existing findings. An illustrative examples is offered to explain our findings. Our outcomes assist us in figuring out the unique solution to a boundary value problem.

Keywords: nonlinear contractions; comparison functions; ϱ-comparative elements; fixed points; upper and lower solutions

MSC: 47H10; 34B15; 54H25

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1. Introduction

Throughout this manuscript, the following notations and abbreviations will be used:

- $\mathbb{N}$: the set of natural numbers;
- $\mathbb{R}$: the set of real numbers;
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- $\mathbb{R}^+ := [0, \infty)$;
- $\mathcal{C}[0, L]$: the space of continuous; functions from $[0, L]$ into $\mathbb{R}$;
- $\mathcal{C}^1[0, L]$: the space of continuously differentiable functions from $[0, L]$ into $\mathbb{R}$;
- BCP: Banach contraction principle;
- MS: metric space;
- CMS: complete metric space;
- BVP: boundary value problem.

Given a map $\zeta : \mathbb{M} \to \mathbb{M}$, we have the following:

- $F(\zeta)$: fixed-point set of a self-map ζ;
- $\mathcal{N}(\mathrm{w}, \mathrm{z}) := \max\{\varsigma(\mathrm{w}, \mathrm{z}), \varsigma(\mathrm{w}, \zeta\mathrm{w}), \varsigma(\mathrm{z}, \zeta\mathrm{z})\}$;
- $\mathcal{M}(\mathrm{w}, \mathrm{z}) := \max\{\varsigma(\mathrm{w}, \mathrm{z}), \varsigma(\mathrm{w}, \zeta\mathrm{w}), \varsigma(\mathrm{z}, \zeta\mathrm{z}), \frac{1}{2}\{\varsigma(\mathrm{w}, \zeta\mathrm{z}) + \varsigma(\mathrm{z}, \zeta\mathrm{w})\}\}$.

The classical BCP [1] serves as a key concept in nonlinear analysis. The BCP has been expanded and generalized in numerous ways. There is a number of extrapolations of the BCP by means of ϕ-contraction in the extant literature, e.g., [2–6]. A function $\phi : [0, \infty) \to [0, \infty)$ that verifies for every $t > 0$ that $\phi(t) < t$ is mentioned to as a control function. A self-map ζ on an MS $(\mathbb{M}, \varsigma)$ is termed as a nonlinear contraction relative to a control mapping ϕ (in short: ϕ-contraction) if $\varsigma(\zeta\mathrm{w}, \zeta\mathrm{z}) \leq \phi(\varsigma(\mathrm{w}, \mathrm{z}))$ for all $\mathrm{w}, \mathrm{z} \in \mathbb{M}$. Historically, Browder [2] presented a first generalization for a class of nonlinear contractions. The finding of Browder [2] was further strengthened by Boyd and Wong [3], Mukherjea

[4], Jotić [5] and Matkowski [6]. Inspired by Suzuki [7], Pant [8] presented a recent finding under nonlinear contraction, which is presented as follows.

Theorem 1 ([8]). *If $(\mathbb{M}, \varsigma)$ is a CMS, $\zeta : \mathbb{M} \to \mathbb{M}$ remains a map and ϕ remains a strictly increasing, right continuous control function verifying*

$$\frac{1}{2}\varsigma(w, \zeta w) \leq \varsigma(w, z) \implies \varsigma(\zeta w, \zeta z) \leq \phi(\mathcal{N}(w, z)), \quad \forall\, w, z \in \mathbb{M},$$

where $\mathcal{N}(w, z) := \max\{\varsigma(w, z), \varsigma(w, \zeta w), \varsigma(z, \zeta z)\}$. Then, ζ admits a unique fixed point.

During the past few years, several researchers have developed and broadened the BCP to ordered metric spaces, c.f. [9–12]. Subsequently, in 2015, the BCP was extended to a MS imposed with an arbitrary relation (rather than a partial order) by Alam and Imdad [13]. The outcomes of this type can be found in [14–21] and references throughout. Such findings are used to solve elastic beam equations, nonlinear elliptic problems, ordinary differential equations, integral equations, fractional differential equations and matrix equations verifying certain auxiliary conditions, e.g., [22–30].

This paper's objective is to provide a relation-theoretic version of Suzuki–Ćirić-type nonlinear contractions through making use of comparison functions and implemented the same to determine the outcomes on fixed points employing a locally ζ-transitive relation. Our established outcomes generalize various existing findings, notably those reported by Alam and Imdad [13], Pant [8], Agarwal et al. [12], Arif et al. [17] and others. We offer an illustrative example that demonstrates the reliability of established findings. Employing our outcomes, we investigate the presence of a unique solution to an order-one periodic BVP.

As was previously noted, in comparison to conditions in the current literature, a considerably sharp contractive condition is used. Due to this limitation, the results presented here and in subsequent studies can be used for the BVP and the areas of nonlinear elliptic issues, fractional differential equations, matrix equations, Fredholm integral equations and delayed hematopoiesis models.

2. Preliminaries

Given a nonempty set $\mathbb{M}$, a subset ϱ of $\mathbb{M}^2$ is termed as a binary relation (or simply, a relation) on $\mathbb{M}$. In the following, let $(\mathbb{M}, \varsigma)$ be an MS endued with a relation ϱ and a map $\zeta : \mathbb{M} \to \mathbb{M}$. We collect the following.

Definition 1 ([13]). *Elements w and $z \in \mathbb{M}$ are ϱ-comparative if either $(w, z) \in \varrho$ or $(z, w) \in \varrho$. We denote this by $[w, z] \in \varrho$.*

Definition 2 ([31]). *The relation $\varrho^{-1} = \{(w, z) \in \mathbb{M}^2 : (z, w) \in \varrho\}$ is an inverse of ϱ.*

Definition 3 ([31]). *$\varrho^s := \varrho \cup \varrho^{-1}$ being a symmetric relation is a symmetric closure of ϱ.*

Proposition 1 ([13]). *$(w, z) \in \varrho^s \iff [w, z] \in \varrho$.*

Definition 4 ([31]). *ϱ is complete if for every $w, z \in \mathbb{M}$, we have $[w, z] \in \varrho$.*

Definition 5 ([32]). *Given $\mathbb{P} \subset \mathbb{M}$, the set $\varrho|_{\mathbb{P}} := \varrho \cap \mathbb{P}^2$ (a relation on $\mathbb{P}$) remains a restriction of $\mathbb{M}$ on $\mathbb{P}$.*

Inspired by Roldán-López-de-Hierro et al. [33] and Turinici [34], Alam and Imdad [14] introduced the next concept.

Definition 6 ([14]). *ϱ is locally ζ-transitive if for every enumerable set $\mathbb{P} \subset \zeta(\mathbb{M})$, the restriction $\varrho|_{\mathbb{P}}$ is transitive.*

Definition 7 ([13]). $\{w_n\} \subset \mathbb{M}$ *is* ϱ*-preserving if* $(w_n, w_{n+1}) \in \varrho \ \ \forall\, n \in \mathbb{N}_0$.

Definition 8 ([15]). $(\mathbb{M}, \varsigma)$ *remains* ϱ*-complete if every* ϱ*-preserving Cauchy sequence in* $\mathbb{M}$ *converges.*

Definition 9 ([13]). ϱ *remains* ζ*-closed if for every* $w, z \in \mathbb{M}$, $(w, z) \in \varrho \Rightarrow (\zeta w, \zeta z) \in \varrho$.

Proposition 2 ([14]). *For all* $n \in \mathbb{N}_0$, ϱ *is* ζ^n*-closed when* ϱ *is* ζ*-closed.*

Definition 10 ([15]). ζ *is* ϱ*-continuous if for each* $w \in \mathbb{M}$ *and for any* ϱ*-preserving sequence* $\{w_n\}$ *such that* $w_n \xrightarrow{\varsigma} w$, *we have* $\zeta(w_n) \xrightarrow{\varsigma} \zeta(w)$.

Definition 11 ([13]). ϱ *is* ς*-self-closed if any* ϱ*-preserving sequence* $\{w_n\}$ *with* $w_n \xrightarrow{\varsigma} w$ *admits a subsequence* $\{w_{n_k}\}$ of $\{w_n\}$ with $[w_{n_k}, w] \in \varrho \ \ \forall\, k \in \mathbb{N}_0$.

Remark 1. *Evidently, we have* $\mathcal{N}(w, z) \leq \mathcal{M}(w, z), \quad \forall\, w, z \in \mathbb{M}$.

Proposition 3 ([35]). *Every comparison function is a control function.*

3. Main Results

We begin this section with the idea of Suzuki–Ćirić-type nonlinear contraction mappings via comparison functions as follows.

Definition 12. *Suppose that* $(\mathbb{M}, \varsigma)$ *is an MS imposed with a relation* ϱ *and* ζ *is a self-map on* $\mathbb{M}$. *If* ϕ *is a comparison function that verifies for every* $(w, z) \in \varrho$ *that*

$$\frac{1}{2}\varsigma(w, \zeta w) \leq \varsigma(w, z) \Longrightarrow \varsigma(\zeta w, \zeta z) \leq \phi(\mathcal{M}(w, z)),$$

then ζ *is called a Suzuki–Ćirić-type nonlinear* ϱ*-contraction map.*

Metric symmetricity proposes the following fact.

Proposition 4. ζ *is a Suzuki–Ćirić-type nonlinear* ϱ*-contraction* $\Longleftrightarrow$ ζ *is a Suzuki–Ćirić-type nonlinear* ϱ^s*-contraction.*

Proof. Since every Suzuki–Ćirić-type nonlinear ϱ-contraction $\Longrightarrow$ Suzuki–Ćirić-type nonlinear ϱ^s-contraction (due to Proposition 1). To assert that ζ is a Suzuki–Ćirić-type nonlinear ϱ^s-contraction $\Longrightarrow$ ζ is a Suzuki–Ćirić-type nonlinear ϱ-contraction, assume that (I) ζ is a Suzuki–Ćirić-type nonlinear ϱ^s-contraction. Choose $w, z \in \mathbb{M}$ such that $(w, z) \in \varrho$. Then, the conclusion follows from assumption (I). Otherwise, if $(z, w) \in \varrho$, then by assumption (I) and the symmetricity of d, we obtain the same conclusion again. □

The following notation will be utilised in our finding:

$$\mathbb{M}(\zeta, \varrho) := \{w \in \mathbb{M} : (w, \zeta w) \in \varrho\}.$$

We present the outcome on existence for fixed points as follows.

Theorem 2. *Suppose that* $(\mathbb{M}, \varsigma)$ *is an MS imposed with a relation* ϱ *and* ζ *is a self-map on* $\mathbb{M}$. *Then, we also have the following:*

(a) $(\mathbb{M}, \varsigma)$ *is* ϱ*-complete;*
(b) $\mathbb{M}(\zeta, \varrho) \neq \emptyset$;
(c) ϱ *is* ζ*-closed and locally* ζ*-transitive;*
(d) ζ *is a Suzuki–Ćirić-type nonlinear* ϱ*-contraction relative to some comparison function* ϕ;
(e) *either* ζ *remains* ϱ*-continuous or* ϱ *serves as* ς*-self-closed.*

Then, ζ has a fixed point.

Proof. Owing to (b), let $w_0 \in \mathbb{M}(\zeta, \varrho)$. Then, $(w_0, \zeta w_0) \in \varrho$. Set $w_n := \zeta^n w_0 = \zeta w_{n-1}$. As ϱ is ζ-closed and using Proposition 2, we have

$$(w_n, w_{n+1}) \in \varrho \ \ \forall n \in \mathbb{N}_0. \tag{1}$$

Therefore, the sequence $\{w_n\}$ is ϱ-preserving. If $\varsigma(w_{n_0+1}, w_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then we have $\zeta(w_{n_0}) = w_{n_0}$ so that $w_{n_0} \in F(\zeta)$ and so we are done.

If $\varsigma(w_{n+1}, w_n) > 0 \ \forall \ n \in \mathbb{N}_0$, then $\frac{1}{2}\varsigma(w_n, w_{n+1}) < \varsigma(w_n, w_{n+1})$. By (1), (d) and the monotone property of ϕ, one can find that for each $n \in \mathbb{N}_0$,

$$\begin{aligned}
\varsigma(w_{n+1}, w_n) = \varsigma(\zeta w_n, \zeta w_{n-1}) &\leq \phi(\mathcal{M}(w_n, w_{n-1})) \\
&= \phi\Big(max\{\varsigma(w_n, w_{n-1}), \varsigma(w_n, w_{n+1}), \varsigma(w_{n-1}, w_n), \\
&\quad \frac{1}{2}\{\varsigma(w_n, w_n) + \varsigma(w_{n-1}, w_{n+1})\}\Big) \\
&= \phi\Big(max\{\varsigma(w_n, w_{n-1}), \varsigma(w_n, w_{n+1}), \frac{1}{2}\{\varsigma(w_{n-1}, w_{n+1})\}\}\Big) \\
&\leq \phi\Big(max\{\varsigma(w_n, w_{n-1}), \varsigma(w_n, w_{n+1}), \\
&\quad \frac{1}{2}\{\varsigma(w_{n-1}, w_n) + \varsigma(w_n, w_{n+1}))\}\}\Big) \\
&\leq \phi\Big(max\{\varsigma(w_n, w_{n-1}), \varsigma(w_n, w_{n+1})\}\Big)
\end{aligned}$$

so that

$$\varsigma(w_{n+1}, w_n) \leq \phi(max\{\varsigma(w_n, w_{n-1}), \varsigma(w_n, w_{n+1})\}). \tag{2}$$

In the case that $\max\{\varsigma(w_n, w_{n-1}), \varsigma(w_n, w_{n+1})\} = \varsigma(w_{n+1}, w_n)$, by the control property of ϕ and (2), we obtain $\varsigma(w_{n+1}, w_n) < \varsigma(w_{n+1}, w_n)$, which raises a contradiction, and so (2) becomes

$$\varsigma(w_{n+1}, w_n) \leq \phi(\varsigma(w_n, w_{n-1})), \tag{3}$$

which, in employing (1), (d) and the monotone property of ϕ, becomes

$$\varsigma(w_{n+1}, w_n) \leq \phi^n(\varsigma(w_1, w_0)). \tag{4}$$

Taking $n \to \infty$ in (4) and by the property of ϕ, we find

$$\lim_{n\to\infty} \varsigma(w_{n+1}, w_n) = 0. \tag{5}$$

For a given $\epsilon > 0$ due to (5), we can determine $n \in \mathbb{N}_0$ verifying

$$\varsigma(w_{n+1}, w_n) < \epsilon - \phi(\epsilon). \tag{6}$$

To verify that $\{W_n\}$ is Cauchy, the monotone-property of ϕ, (1) and (5) are employed so that

$$\begin{aligned}\varsigma(W_{n+2}, W_n) &\leq \varsigma(W_{n+2}, W_{n+1}) + \varsigma(W_{n+1}, W_n)\\ &< \varsigma(\zeta W_{n+1}, \zeta W_n) + \epsilon - \phi(\epsilon)\\ &\leq \phi(\varsigma(W_{n+1}, W_n)) + \epsilon - \phi(\epsilon)\\ &\leq \phi(\epsilon - \phi(\epsilon)) + \epsilon - \phi(\epsilon)\\ &\leq \phi(\epsilon) + \epsilon - \phi(\epsilon)\\ &= \epsilon.\end{aligned}$$

Again, owing to monotone property ϕ, (1) and the local ζ-transitivity of ϱ, we obtain

$$\begin{aligned}\varsigma(W_{n+3}, W_n) &\leq \varsigma(W_{n+3}, W_{n+1}) + \varsigma(W_{n+1}, W_n)\\ &< \varsigma(\zeta W_{n+2}, \zeta W_n) + \epsilon - \phi(\epsilon)\\ &\leq \phi(\varsigma(W_{n+2}, W_n)) + \epsilon - \phi(\epsilon)\\ &\leq \phi(\epsilon - \phi(\epsilon)) + \epsilon - \phi(\epsilon)\\ &\leq \phi(\epsilon) + \epsilon - \phi(\epsilon)\\ &= \epsilon.\end{aligned}$$

An easy induction on the above yields

$$\varsigma(W_{n+k}, W_n) < \epsilon \text{ for all } k \in \mathbb{N},$$

which verifies that $\{W_n\}$ is Cauchy. By the ϱ-completeness of $(\mathbb{M}, \varsigma)$, $\exists\ \omega \in \mathbb{M}$ with $W_n \xrightarrow{\varsigma} \omega$.

If ζ is ϱ-continuous, then $W_{n+1} = \zeta(W_n) \xrightarrow{\varsigma} \zeta(\omega)$. So, we find $\zeta(\omega) = \omega$. Alternately, when ϱ remains ς-self-closed, $\exists$ a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ with $[W_{n_k}, \omega] \in \varrho \quad (\forall\, k \in \mathbb{N}_0)$.

We assert that (for all $k \in \mathbb{N}_0$)

$$\frac{1}{2}\varsigma(W_{n_k}, W_{n_k+1}) \leq \varsigma(W_{n_k}, \omega) \text{ or } \frac{1}{2}\varsigma(W_{n_K+1}, W_{n_k+2}) \leq \varsigma(W_{n_k+1}, \omega). \tag{7}$$

On the contrary, for some $k_o \in \mathbb{N}_0$, let us assume that

$$\frac{1}{2}\varsigma(W_{n_{k_o}}, W_{n_{k_o}+1}) > \varsigma(W_{n_{k_o}}, \omega) \text{ and } \frac{1}{2}\varsigma(W_{n_{k_o}+1}, W_{n_0+2}) > \varsigma(W_{n_{k_o}+1}, \omega)$$

Applying the triangle inequality, we obtain

$$\begin{aligned}\varsigma(W_{n_{k_o}}, W_{n_{k_o}+1}) &\leq \varsigma(W_{n_{k_o}}, \omega) + \varsigma(W_{n_{k_o}+1}, \omega)\\ &< \frac{1}{2}\varsigma(W_{n_{k_o}}, W_{n_{k_o}+1}) + \frac{1}{2}\varsigma(W_{n_{k_o}+1}, W_{n_{k_o}+2})\\ &< \frac{1}{2}\{\varsigma(W_{n_{k_o}}, W_{n_{k_o}+1}) + \varsigma(W_{n_{k_o}}, W_{n_{k_o+1}})\} = \varsigma(W_{n_{k_o}}, W_{n_{k_o}+1}),\end{aligned}$$

which is a contradiction. Consequently, (7) holds for all $k \in \mathbb{N}_0$.

By (d) (in view of (7)), Proposition 4 and $[\mathrm{W}_{n_k}, \omega] \in \varrho \ \ (\forall\, k \in \mathbb{N}_0)$, we find

$$\begin{aligned} \varsigma(\mathrm{W}_{n_k+1}, \zeta\omega) = \varsigma(\zeta \mathrm{W}_{n_k}, \zeta\omega) &\leq \phi\Big(\mathcal{M}(\mathrm{W}_{n_k}, \omega)\Big) \\ &= \phi\Big(max\Big\{\varsigma(\mathrm{W}_{n_k}, \omega), \varsigma(\omega, \zeta\omega), \varsigma(\mathrm{W}_{n_k}, \mathrm{W}_{n_k+1}), \\ &\quad \frac{1}{2}\big\{\varsigma(\mathrm{W}_{n_k}, \zeta\omega) + \varsigma(\mathrm{W}_{n_k+1}, \omega)\big\}\Big\}\Big). \end{aligned}$$

Set $\delta := \varsigma(\omega, \zeta\omega) \geq 0$. On the contrary, let us assume that $\delta > 0$. Employing (d), Proposition 4 and $[\zeta \mathrm{W}_{n_k}, \omega] \in \varrho$, for all $k \in \mathbb{N}_0$, we obtain

$$\varsigma(\mathrm{W}_{n_k+1}, \zeta\omega) = \varsigma(\zeta \mathrm{W}_{n_k}, \zeta\omega) \leq \phi(\mathcal{M}(\mathrm{W}_{n_k}, \omega)), \tag{8}$$

wherein

$$\mathcal{M}(\mathrm{W}_{n_k}, \omega) = max\Big\{\varsigma(\mathrm{W}_{n_k}, \omega), \varsigma(\omega, \zeta\omega), \varsigma(\mathrm{W}_{n_k}, \mathrm{W}_{n_k+1}), \frac{1}{2}\big\{\varsigma(\mathrm{W}_{n_k}, \zeta\omega) + \varsigma(\mathrm{W}_{n_k+1}, \omega)\big\}\Big\}.$$

If $\mathcal{M}(\mathrm{W}_{n_k}, \omega) = \varsigma(\zeta\omega, \omega) = \delta$, then (8) reduces to

$$\mathcal{M}(\mathrm{W}_{n_k}, \omega) \leq \phi(\delta),$$

which, in using $k \to \infty$, reduces to

$$\delta \leq \varphi(\delta),$$

which raises a contradiction.

Otherwise, if $\mathcal{M}(\mathrm{W}_{n_k}, \omega) = max\Big\{\varsigma(\mathrm{W}_{n_k}, \omega), \varsigma(\mathrm{W}_{n_k}, \mathrm{W}_{n_k+1}), \frac{1}{2}\big\{\varsigma(\mathrm{W}_{n_k}, \zeta\omega) + \varsigma(\mathrm{W}_{n_k+1}, \omega)\big\}\Big\}$,

then due to the fact that $\zeta \mathrm{W}_n \xrightarrow{\varsigma} \omega$, we can determine $N = N(\delta) \in \mathbb{N}$ with

$$\mathcal{M}(\mathrm{W}_{n_k}, \omega) \leq \frac{5}{6}\delta, \text{ for all } k \geq N.$$

By the monotone property of ϕ, we have

$$\phi(\mathcal{M}(\mathrm{W}_{n_k}, \omega)) \leq \varphi(\frac{5}{6}\delta), \text{ for all } k \geq N. \tag{9}$$

By (8) and (9), we obtain

$$\varsigma(\mathrm{W}_{n_k+1}, \zeta\omega) = \varsigma(\mathrm{W}_{n_k}, \zeta\omega) \leq \phi(\frac{5}{6}\delta), \text{ for all } k \geq N.$$

Making $k \to \infty$ and employing Proposition 3, we find

$$\delta \leq \varphi(\frac{5}{6}\delta) < \frac{5}{6}\delta < \delta,$$

which raises a contradiction implying that $\delta = 0$. Thus, we obtain $\varsigma(\zeta\omega, \omega) = \delta = 0 \Rightarrow \zeta(\omega) = \omega$.

Hence, in both cases, $\omega \in F(\zeta)$, which ends the proof. □

Corollary 1. *The conclusion of Theorem 2 remains true if ϕ is a strictly increasing, right continuous and control function instead of a comparison function.*

Corollary 2. *Theorem 2 remains true if ϱ remains a transitive relation instead of locally ζ-transitive.*

The following is an outline of our uniqueness result.

Theorem 3. *Under the assumptions of Theorem 2, if ϱ is a complete relation then ζ owns a unique fixed-point.*

Proof. Since $F(\zeta) \neq \emptyset$ (in lieu of Theorem 2), we can therefore choose $\mathrm{w}, \mathrm{z} \in F(\zeta)$, i.e., $\mathrm{w} = \zeta(\mathrm{w})$ and $\mathrm{z} = \zeta(\mathrm{z})$. We prove that

$$\mathrm{w} = \mathrm{z}.$$

If $\mathrm{w} \neq \mathrm{z}$, then the completeness of ϱ implies that $[\mathrm{w}, \mathrm{z}] \in \varrho$. As $0 = \frac{1}{2}\varsigma(\mathrm{w}, \zeta\mathrm{w}) < \varsigma(\mathrm{w}, \mathrm{z})$, employing (d), we find

$$\begin{aligned} \varsigma(\mathrm{w}, \mathrm{z}) = \varsigma(\zeta\mathrm{w}, \zeta\mathrm{z}) &\leq \phi(\mathcal{M}(\mathrm{w}, \mathrm{z})) \\ &\leq \phi(\max\{\varsigma(\mathrm{w}, \mathrm{z}), \varsigma(\mathrm{w}, \zeta\mathrm{w}), \varsigma(\mathrm{z}, \zeta\mathrm{z}), \frac{1}{2}\{\varsigma(\mathrm{w}, \zeta\mathrm{z}) + \varsigma(\mathrm{z}, \zeta\mathrm{w})\}\}) \\ &< \varsigma(\mathrm{w}, \mathrm{z}), \end{aligned}$$

which contradicts our supposition. Hence, we have $\mathrm{w} = \mathrm{z}$. □

Remark 2. *We conclude certain special demonstrations in the lines that follow. These are enhanced varieties of many famous fixed-point statements.*

(1) If we choose $\mathcal{M}(\mathrm{w}, \mathrm{z})$ to be $\mathcal{N}(\mathrm{w}, \mathrm{z})$, $\varrho = \mathbb{M}^2$ (universal relation) and ϕ to be an increasing continuous control function, then Theorem 3 reduces to the corresponding result of Pant [8].

(2) In taking $\mathcal{M}(\mathrm{w}, \mathrm{z})$ to be $\varsigma(\mathrm{w}, \mathrm{z})$ and ϱ to be a partial order relation $\preceq$ in Theorem 3, we determine a sharpened variety of the main result of Agarwal et al. [12] in the context of the Suzuki condition.

(3) In taking $\mathcal{M}(\mathrm{w}, \mathrm{z})$ to be $\varsigma(\mathrm{w}, \mathrm{z})$ in Theorem 2, we obtain sharpened version of the main result of Arif et al. [17] in the context of Suzuki condition.

(4) Under the Suzuki condition and the Ćirić contractive condition, Theorem 3 offers an enhanced version of Theorem 3.1 by Alam and Imdad [13] for $\phi(t) = ct$ (where $c \in [0, 1)$). The locally ζ-transitivity criterion on the involved relation is relaxed, still in this case.

(5) If we choose $\mathcal{M}(\mathrm{w}, \mathrm{z})$ to be $\zeta(\mathrm{w}, \mathrm{z})$, $\varrho = \mathbb{M}^2$ (universal relation) and $\phi(t) = ct$ (where $c \in [0, 1)$), then Theorem 3 reduces to the BCP [1] (without using the Suzuki condition).

4. An Illustrative Example

Now, we provide an example to illustrate the significance of our findings.

Example 1. *Let $\mathbb{M} = [0, 4)$ be provided with the relation $\varrho = \{(0,0), (0,1), (1,0), (1,1), (3,0)\}$ and the standard metric ς. Although ϱ is locally ζ-transitive, it is not transitive. Define a self-mapping ζ on $\mathbb{M}$ by*

$$\zeta(\mathrm{w}) = \begin{cases} 0, & \mathrm{w} \in [0, 1], \\ 1, & \mathrm{w} \in (1, 4), \end{cases}$$

Clearly, ϱ is ζ-closed, and $(\mathbb{M}, \varsigma)$ is a ϱ-complete MS. Define a comparison function $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{2t}{1+t}$. For any $(\mathrm{w}, \mathrm{z}) \in \varrho$, the contraction condition (d) of Theorem 2 is easily verified, with the exception of $(\mathrm{w}, \mathrm{z}) = (3, 0)$. So, we need to check the condition (d) at $(3, 0) \in \varrho$. For $\mathrm{w} = 3$, we have $\frac{1}{2}\varsigma(3, \zeta 3) = 1 < 3 = \varsigma(3, 0)$, and hence,

$$\begin{aligned} \varsigma(\zeta 3, \zeta 0) = 1 &\leq \phi(max\{\varsigma(3,0), \varsigma(3, \zeta 3), \varsigma(0, \zeta 0), \frac{1}{2}\{\varsigma(3, \zeta 0) + \varsigma(0, \zeta 3)\}\}) \\ &= \frac{6}{4} = \phi(3) \quad for \quad (3,0) \in \varrho. \end{aligned}$$

Secondly, if we choose $\mathrm{w} = 0$, *then*

$$\frac{1}{2}\varsigma(0,\zeta 0) = 0 \leq 0 = \varsigma(0,0) \implies \varsigma(\zeta 0,\zeta 0) = 0 \leq 0 = \phi(\mathcal{M}(0,0)), \text{ for } (0,0) \in \varrho$$

and

$$\frac{1}{2}\varsigma(0,\zeta 0) = 0 < 1 = \varsigma(0,1) \implies \varsigma(\zeta 0,\zeta 1) = 0 \leq \frac{1}{2} = \phi(\mathcal{M}(0,1)), \text{ for } (0,1) \in \varrho.$$

Taking any ϱ*-preserving sequence* $\{\mathrm{w}_n\}$ *such that* $\mathrm{w}_n \xrightarrow{\varsigma} \mathrm{w}$. *As* $(\mathrm{w}_n, \mathrm{w}_{n+1}) \in \varrho$, *for each* $n \in \mathbb{N}$, *one can determine* $N \in \mathbb{N}$ *verifying* $\mathrm{w}_n = \mathrm{w} \in \{0,1\}$, for each $n \geq N$; *This indicates that* ς*-self closed is* ϱ. *Theorem 2 holds in all of its assumptions, and as an outcome,* ζ *possesses a fixed point. Since* $(\mathbb{M}, \varsigma)$ *is not a complete MS in this case and* ϱ *is not a partial order, the findings of Pant [8] and Agarwal et al. [12] do not apply to this situation. This shows the utility of our findings.*

5. An Application to BVP

This section concludes with studying the existence and uniqueness of the solution for following one-order periodic BVP:

$$\begin{cases} \vartheta'(\tau) = \Phi(\tau, \vartheta(\tau)), & \tau \in [0, L] \\ \vartheta(0) = \vartheta(L) \end{cases} \tag{10}$$

where $L > 0$, and $\Phi : [0, L] \times \mathbb{R} \to \mathbb{R}$ remains a known continuous function.

Definition 13 ([10]). $\varpi \in \mathcal{C}^1[0, L]$ *is termed as a lower solution of* (10) *if*

$$\begin{cases} \varpi'(\tau) \leq \Phi(\tau, \varpi(\tau)), & \tau \in [0, L] \\ \varpi(0) \leq \varpi(L). \end{cases}$$

Definition 14 ([10]). $\varpi \in \mathcal{C}^1[0, L]$ *is termed an upper solution of* (10) *if*

$$\begin{cases} \varpi'(\tau) \geq \Phi(\tau, \varpi(\tau)), & \tau \in [0, L] \\ \varpi(0) \geq \varpi(L). \end{cases}$$

The primary outcome of this section is now presented.

Theorem 4. *Together with* (10), *if* $\exists\, k > 0$ *and a comparison function* ϕ *verifying* $\forall\, \alpha, \beta \in \mathbb{R}$ *and* $\alpha \leq \beta$, *then*

$$0 \leq \Phi(\tau, \beta) + k\beta - [\Phi(\tau, \alpha) + k\alpha] \leq k\phi(\beta - \alpha). \tag{11}$$

Moreover, if there is a lower solution for (10), *then there is a unique solution for* (10).

Proof. Problem (10) can be rearranged as

$$\begin{cases} \vartheta'(\tau) + k\vartheta(\tau) = \Phi(\tau, \vartheta(\tau)) + k\vartheta(\tau), & \forall\, \tau \in [0, L] \\ \vartheta(0) = \vartheta(L). \end{cases} \tag{12}$$

which is equivalent to the integral equation

$$\vartheta(\tau) = \int_0^L G(\tau, \xi)[\Phi(\xi, \vartheta(\xi)) + k\vartheta(\xi)]d\xi \tag{13}$$

where the Green function $G(\tau,\xi)$ is defined as

$$G(\tau,\xi)=\begin{cases}\frac{e^{k(L+\xi-\tau)}}{e^{kL}-1}, & 0\le\xi<\tau\le L\\ \frac{e^{k(\xi-\tau)}}{e^{kL}-1}, & 0\le\tau<\xi\le L.\end{cases}$$

Set $\mathbb{M}:=\mathcal{C}[0,L]$. Define the function $\zeta:\mathbb{M}\to\mathbb{M}$ as

$$(\zeta\vartheta)(\tau)=\int_0^L G(\tau,\xi)[\Phi(\xi,\vartheta(\xi))+k\vartheta(\xi)]d\xi,\quad\forall\,\tau\in[0,L]. \tag{14}$$

Thus, $\vartheta\in\mathbb{M}$ remains a fixed point of ζ iff $\vartheta\in\mathcal{C}^1[0,L]$ remains a solution of (13) and hence of (10).

Define a metric ς on $\mathbb{M}$ by

$$\varsigma(\vartheta,\mu)=\sup_{\tau\in[0,L]}|\vartheta(\tau)-\mu(\tau)|,\quad\forall\,\vartheta,\mu\in\mathbb{M}. \tag{15}$$

Consider the following relation ϱ on $\mathbb{M}$:

$$\varrho=\{(\vartheta,\mu):\vartheta(\tau)\le\mu(\tau),\ \forall\,\tau\in[0,L]\}. \tag{16}$$

Now, we verify all conditions of Theorem 2:
(a) Obviously, $(\mathbb{M},\varrho)$ remains a ϱ-complete MS.
(b) Let $\varpi\in\mathcal{C}^1[0,L]$ be a lower solution of (10). Then, one has

$$\varpi'(\tau)+k\varpi(\tau)\le\Phi(\tau,\varpi(\tau))+k\varpi(\tau)\ \ \forall\,\tau\in[0,L].$$

Multiplying with $e^{k\tau}$, we obtain

$$(\varpi(\tau)e^{k\tau})'\le[\Phi(\tau,\varpi(\tau))+k\varpi(\tau)]e^{k\tau}\ \ \forall\,\tau\in[0,L]$$

thereby yielding

$$\varpi(\tau)e^{k\tau}\le\varpi(0)+\int_0^\tau[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]e^{k\xi}d\xi,\ \ \forall\,\tau\in[0,L]. \tag{17}$$

Owing to $\varpi(0)\le\varpi(L)$, one obtains

$$\varpi(0)e^{kL}\le\varpi(L)e^{kL}\le\varpi(0)+\int_0^L[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]e^{k\xi}d\xi$$

so that

$$\varpi(0)\le\int_0^L\frac{e^{k\xi}}{e^{kL}-1}[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]d\xi. \tag{18}$$

By (17) and (18), one finds

$$\begin{aligned}\varpi(\tau)e^{k\tau}&\le\int_0^L\frac{e^{k\xi}}{e^{kL}-1}[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]d\xi+\int_0^\tau e^{k\xi}[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]d\xi\\&=\int_0^\tau\frac{e^{k(L+\xi)}}{e^{kL}-1}[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]d\xi+\int_\tau^L\frac{e^{k\xi}}{e^{kL}-1}[\Phi(\xi,\varpi(\xi))+k\varpi(\xi)]d\xi\end{aligned}$$

so that

$$\begin{aligned}\omega(\tau) &\leq \int_0^{\tau} \frac{e^{k(L+\xi-\tau)}}{e^{kL}-1}[\Phi(\xi,\omega(\xi))+k\omega(\xi)]d\xi + \int_{\tau}^{L} \frac{e^{k(\xi-\tau)}}{e^{kL}-1}[\Phi(\xi,\omega(\xi))+k\omega(\xi)]d\xi \\ &= \int_0^{L} G(\tau,\xi)[\Phi(\xi,\omega(\xi))+k\omega(\xi)]d\xi \\ &= (\zeta\omega)(\tau),\ \forall\ \tau\in[0,L]\end{aligned}$$

which yields that $(\omega,\zeta\omega)\in\varrho$ so that $\mathcal{M}(\zeta,\varrho)$ remains nonempty.

(c) Take $(\vartheta,\mu)\in\varrho$. Using (11), one obtains

$$\Phi(\tau,\vartheta(\tau))+k\vartheta(\tau)\leq\Phi(\tau,\mu(\tau))+k\mu(\tau),\quad\forall\ \tau\in[0,L]. \tag{19}$$

By (14), (19) and $G(\tau,\xi)>0,\ \forall\ (\tau,\xi)\in[0,L]\times[0,L]$, one finds

$$\begin{aligned}(\zeta\vartheta)(\tau) &= \int_0^{L} G(\tau,\xi)[\Phi(\xi,\vartheta(\xi))+k\vartheta(\xi)]d\xi \\ &\leq \int_0^{L} G(\tau,\xi)[\Phi(\xi,\mu(\xi))+k\mu(\xi)]d\xi \\ &= (\zeta\mu)(\tau)\quad\forall\ \tau\in[0,L],\end{aligned}$$

which, in utilizing (16), yields that $(\zeta\vartheta,\zeta\mu)\in\varrho$; so, ϱ is ζ-closed.

(d) Take $(\vartheta,\mu)\in\varrho$ with $\frac{1}{2}\varsigma(\vartheta,\zeta\vartheta)\leq\varsigma(\vartheta,\mu)$. Then, by (11), (14) and (15), one has

$$\begin{aligned}\varsigma(\zeta\vartheta,\zeta\mu) &= \sup_{\tau\in[0,L]}|(\zeta\vartheta)(\tau)-(\zeta\mu)(\tau)| = \sup_{\tau\in[0,L]}\big((\zeta\vartheta)(\tau)-(\zeta\mu)(\tau)\big) \\ &\leq \sup_{\tau\in[0,L]}\int_0^{L} G(\tau,\xi)[\Phi(\xi,\vartheta(\xi))+k\vartheta(\xi)-\Phi(\xi,\mu(\xi))-k\mu(\xi)]d\xi \\ &\leq \sup_{\tau\in I}\int_0^{L} G(\tau,\xi)k\phi(\vartheta(\xi)-\mu(\xi))d\xi.\end{aligned} \tag{20}$$

Now, $0\leq\vartheta(\xi)-\mu(\xi)\leq\varsigma(\vartheta,\mu))$. By the monotone property of ϕ, we find $\phi(\vartheta(\xi)-\mu(\xi))\leq\phi(\varsigma(\vartheta,\mu))$, and hence, (20) reduces to

$$\begin{aligned}\varsigma(\zeta\vartheta,\zeta\mu) &\leq k\phi(\varsigma(\vartheta,\mu))\sup_{\tau\in[0,L]}\int_0^{L} G(\tau,\xi)d\xi \\ &= k\phi(\varsigma(\vartheta,\mu))\sup_{\tau\in[0,L]}\frac{1}{e^{kL}-1}\Big(\frac{1}{k}e^{k(L+\xi-\tau)}\Big]_0^{t}+\frac{1}{k}e^{k(\xi-\tau)}\Big]_{\tau}^{L}\Big) \\ &= k\phi(\varsigma(\vartheta,\mu))\frac{1}{k(e^{kL}-1)}(e^{kL}-1) \\ &= \phi(\varsigma(\vartheta,\mu))\end{aligned}$$

so that

$$\varsigma(\zeta\vartheta,\zeta\mu)\leq\phi(\varsigma(\vartheta,\mu)),\quad\forall\ \vartheta,\mu\in\mathbb{M}\text{ such that }(\vartheta,\vartheta)\in\varrho.$$

(e) Let $\{\vartheta_n\}\subset\mathbb{M}$ be an ϱ-preserving sequence converging to $\vartheta\in\mathbb{M}$. Then, for every $\tau\in[0,L]$, $\{\vartheta_n(\tau)\}$ remains an increasing sequence in $\mathbb{R}$ that converges to $\vartheta(\tau)$. So, $\forall\ n\in\mathbb{N}_0$ and $\forall\ \tau\in[0,L]$, we have $\vartheta_n(\tau)\leq\vartheta(\tau)$. As before, $(\vartheta_n,\vartheta)\in\varrho,\ \forall\ n\in\mathbb{N}$, is the result of (16), and as such, ϱ is ς-self-closed.

Let $\vartheta,\mu\in\mathbb{M}$ be arbitrary. Then, for every $\tau\in[0,L]$, we have $\vartheta(\tau)\leq\mu(\tau)$ or $\vartheta(\tau)\leq\mu(\tau)$. Consequently, we have $[\vartheta,\mu]\in\varrho$, and hence, ϱ is complete. As of right now, ζ possesses a unique fixed point according to Theorem 3, which preserves it a unique solution of problem (10). □

6. Conclusions

In this work, we have proven the existence as well as uniqueness of the fixed point of a self-map under a Suzuki–Ćirić-type nonlinear ϱ-contraction map utilizing a locally ζ-transitive relation ϱ. Also, an illustrative example is offered to demonstrate the usefulness of our present work over corresponding known findings. We also used our results to determine a unique solution of certain BVP when a lower solution is provided. Analogously, one can find similar application in the presence of an upper solution. Moreover, the development of our findings is in generalizing our findings under various types of nonlinear contractions, namely, weak contractions, (ϕ, ψ)-contractions, weaker versions of Matkoski contractions and several auxiliary functions. In the future, these can be extended to utilize our findings through different types of frameworks such as, symmetric spaces, partial MS, b-MS, cone MS, etc.

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