mathematics

MDPI

Article

Alternative View of Inextensible Flows of Curves and Ruled Surfaces via Alternative Frame

Ana Savić [1], Kemal Eren [2], Soley Ersoy [3,*] and Vladimir Baltić [1]

[1] School of Electrical and Computer Engineering, Academy of Technical and Art Applied Studies, 11000 Belgrade, Serbia; ana.savic@viser.edu.rs (A.S.); baltic@gs.viser.edu.rs (V.B.)
[2] Sakarya University Technology Developing Zones Manager Company, 54050 Sakarya, Turkey; kemal.eren1@ogr.sakarya.edu.tr
[3] Department of Mathematics, Faculty of Sciences, Sakarya University, 54050 Sakarya, Turkey
* Correspondence: sersoy@sakarya.edu.tr

Abstract: In this paper, we present the evolutions of ruled surfaces generated by the principal normal, the principal normal's derivative, and the Darboux vector fields along a space curve that are the elements of an alternative frame. The comprehension of an object's rotational behavior is crucial knowledge relevant to various realms, and this can be accomplished by analyzing the Darboux vector along the path of a point on the object as it moves through space. In that regard, examining the evolutions of the ruled surfaces based on the changes in their directrices, including the Darboux vector in the alternative frame along a space curve, is significant. As the first step of this study, we express the evolution of the alternative frame elements of a space curve. Subsequently, the conditions for the ruled surfaces generated by them to be minimal, developable, and inextensible are investigated. These findings can allow some physical phenomena to be well understood through surface evolutions satisfying these conditions. In the final step, we provide graphical representations of some examples of inextensible ruled surfaces and curve evolutions.

Keywords: ruled surface; alternative frame; evolution of curve

MSC: 53A05; 53A25; 58C27

check for updates

Citation: Savić, A.; Eren, K.; Ersoy, S.; Baltić, V. Alternative View of Inextensible Flows of Curves and Ruled Surfaces via Alternative Frame. *Mathematics* **2024**, *12*, 2015. https://doi.org/10.3390/math12132015

Academic Editor: Mića S. Stanković

Received: 1 June 2024
Revised: 22 June 2024
Accepted: 25 June 2024
Published: 28 June 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Differential geometry places significant emphasis on the study of curves and surfaces, particularly their time evolution within Euclidean 3-space E^3. The transformation of curves and surfaces over time is described through their corresponding flows. When examining the time variation of a curve, maintaining constant arc length (zero arc-length variation) in its initial and final positions classifies it as an inelastic curve [1]. Similarly, for surfaces, preserving the main curvature at both initial and final positions designates them as inelastic surfaces [2]. Physically, inelastic curves and surfaces arise in situations where movements occur without tension energy, leading to the flow of surfaces. Examples include the oscillating motion of a fixed-length cable or the fluttering of a piece of paper in the wind, both of which are represented by inelastic curves and surface flows [1–4]. In a mathematical sense, the evolution of a surface $X(s,u,t)$ for all time points t is expressed as the isometric image of the original surface $X(s,u,t_0)$ defined at the initial time t_0. Research in the field of differential geometry has enlightened inelastic curves, and surface flows in various domains, such as computer imaging [5,6], computer animation [7], and mechanical motion science [8]. The concepts of surface evolution, particularly the minimization of mean curvature flow and surface energy, provide a clear understanding of physical phenomena, such as the behavior of soap films and bubbles [9,10]. While there is a considerable body of research on curve flows, surface flows have not received as much attention in the existing literature. In this study, we obtained ruled surfaces with the help of a space evaluation

using the alternative frame. This alternative frame consists of the principal normal vector of the Frenet frame, the derivative of the principal normal, and the Darboux vectors. In 2016, Uzunoğlu et al. defined an alternative moving frame of a space curve [11]. This research has caught the attention of many researchers, and the new approach affects studies on curves [12–14] and surfaces [15–21] based on the alternative frame.

In this work, we intend to present the evolutions of the ruled surfaces constructed by the principal normal, the derivative in the direction of the principal normal, and the Darboux vector fields of the space curves that are the elements of the alternative frame. It is a fact that understanding the rotational behavior of objects is crucial in various fields, and this can be achieved by analyzing the Darboux vector along a curve to find out how the curve rotates as it moves through space. Here, the evolutions of the ruled surfaces depend on the evolutions of their directrices, including the Darboux vector in the alternative frame along a space curve. In that regard, some geometric properties of the ruled surface evolutions are investigated based on the benefit of knowing in which direction the curve flows are rotating by their Darboux vectors. Examples of the curve evolutions and the surfaces constructed by their alternative frame elements are given, and their graphics are drawn.

2. Preliminaries

Let $\alpha : I \subset R \to E^3$ be a unit-speed curve parametrized by an arc-length s. The Frenet apparatus of α is $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ where $T(s) = \alpha'(s)$, $N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$, and $B(s) = T(s) \times N(s)$ are the unit tangent, principal normal, and binormal vector fields of the curve α. The functions $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = \langle N'(s), B(s)\rangle$ are the curvature and torsion of it, respectively.

On the other hand, the alternative moving frame along the curve α is given by $\{N(s), C(s), W(s)\}$ where $N(s)$, $C(s) = \frac{N'(s)}{\|N'(s)\|}$, and $W(s) = \frac{\tau(s)T(s)+\kappa(s)B(s)}{\sqrt{\kappa(s)^2+\tau(s)^2}}$ are the unit principal normal, the derivative of the principal normal, and the Darboux vector fields, respectively [11]. In addition, the alternative moving frame derivatives are given in matrix form

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix}_s = \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & \mu \\ 0 & -\mu & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}. \tag{1}$$

Here, for $\eta(s) = \frac{\kappa(s)^2}{\left(\kappa(s)^2+\tau(s)^2\right)^{3/2}} \left(\frac{\tau(s)}{\kappa(s)}\right)'$, the functions

$$\lambda(s) = \sqrt{\kappa(s)^2 + \tau(s)^2} \text{ and } \mu(s) = \eta(s)\lambda(s)$$

are the first and second curvature of α with respect to alternative moving frame, respectively [11].

Therefore, there is the following relationship between the new alternative frame $\{N, C, W\}$ and Frenet frame $\{T, N, B\}$:

$$T(s) = \frac{-\kappa(s)C(s) + \tau(s)W(s)}{\sqrt{\kappa(s)^2 + \tau(s)^2}} \text{ and } B(s) = \frac{\tau(s)C(s) + \kappa(s)W(s)}{\sqrt{\kappa(s)^2 + \tau(s)^2}},$$

and the principal normal vector $N(s)$ is also common in each frame for all $s \in I$.

Lemma 1. *Let α be a unit speed curve with the alternative apparatus $\{N, C, W, \lambda, \mu\}$ in Euclidean 3-space, then the following expressions are satisfied:*

i. If the second curvature function $\mu(s)$ vanishes for $\kappa(s) \neq 0$, then the curve α is a helix.
ii. If the function $\eta(s)$ is a constant, then the curve α is a slant helix [11].

Let $X(s,u,t)$ be a ruled surface evolution in E^3, then some quantities associated with the ruled surface evolution at any value of t are given below

i. The unit normal vector field is

$$U(s,u,t) = \frac{X_s \times X_u}{\|X_s \times X_u\|}, \tag{2}$$

where the tangent vectors of $X(s,u,t)$ are $X_s = \frac{\partial X}{\partial s}$ and $X_u = \frac{\partial X}{\partial u}$.

ii. The first fundamental form is $I(s,u,t) = Eds^2 + 2Fdsdu + Gdu^2$, where coefficients of I are

$$E(s,u,t) = \langle X_s, X_s \rangle,\ F(s,u,t) = \langle X_s, X_u \rangle,\ G(s,u,t) = \langle X_u, X_u \rangle. \tag{3}$$

iii. The second fundamental form is $II(s,u,t) = eds^2 + 2fdsdu + gdu^2$, where coefficients of II are

$$e(s,u,t) = \langle X_{ss}, U \rangle,\ f(s,u,t) = \langle X_{su}, U \rangle,\ g(s,u,t) = \langle X_{uu}, U \rangle. \tag{4}$$

iv. The Gaussian and mean curvatures are

$$K(s,u,t) = \frac{eg - f^2}{EG - F^2} \text{ and } H(s,u,t) = \frac{Eg - 2Ef + Ge}{2(EG - F^2)}, \tag{5}$$

respectively.

Also, it is well-known that any surface with vanishing Gaussian curvature is called developable, and with vanishing mean curvature is called minimal [15].

3. Inextensible Flows of Curves with Alternative Frame

In this section, first, we recall the basic concepts about the evolution of a moving curve in Euclidean 3-space E^3. Then, we give the intrinsic equations expressing its curvatures corresponding with a moving alternative frame $\{N, C, W\}$.

Let α be a moving space curve described in parametric form by a position vector $\alpha(s,t)$ for a time parameter t. Thus, the partial derivatives of the elements of the moving alternative frame $\{N, C, W\}$ of α with respect to t are given in the matrix form as follows

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix}_t = \begin{bmatrix} 0 & f_1 & f_2 \\ -f_1 & 0 & f_3 \\ -f_2 & -f_3 & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix},$$

where f_1, f_2, and f_3 are smooth function of s and t. If the curve flow $\alpha(s,t)$ is inextensible, then there exist $N_{st} = N_{ts}$, $C_{st} = C_{ts}$, and $W_{st} = W_{ts}$. Thus, considering the matrices above and (1) gives us the following relationships

$$\begin{cases} \lambda_t C + \lambda C_t = f_{1s} C + f_1 C_s + f_{2s} W + f_2 W_s, \\ \mu_t W + \mu W_t - \lambda_t N - \lambda N_t = f_{3s} W + f_3 W_s - f_{1s} N - f_1 N_s, \\ \mu_t C + \mu C_t = f_{2s} N + f_2 N_s + f_{3s} C + f_3 C_s, \end{cases}$$

such that

$$\begin{cases} f_1 = \dfrac{\lambda f_3 - f_{2s}}{\mu}, \\ f_2 = \dfrac{f_{1s} - \lambda_t}{\mu}, \\ f_3 = \dfrac{\mu f_1 + f_{2s}}{\lambda}. \end{cases}$$

Moreover, the curvatures of the evolving space curve satisfy the relations

$$\begin{cases} \lambda_t = f_{1_s} - \mu f_2, \\ \mu_t = \lambda f_2 - f_{3s}. \end{cases}$$

So, we can say that these equations represent the motion of the space curve with the alternative frame.

4. Inextensible Flows of Ruled Surfaces

In this section, we investigate the ruled surface constructed by the evolution of a space curve with the alternative frame. The parametric equation of a ruled surface is presented by

$$X(s,u) = \alpha(s) + ur(s), \tag{6}$$

where $\alpha(s)$ is called the base curve and $r(s)$ is the director curve of the ruled surface. By moving with time t curves $\alpha(s)$ and $r(s)$, the evolution equation of this ruled surface can be written as

$$X(s,u,t) = \alpha(s,t) + ur(s,t).$$

Now, let's give the definition of inextensibility.

Definition 1. *Let $X(s,u,t)$ be a surface evolution of the surface $X(s,u)$ given by Equation (6) in E^3. Then the flow of the surface evolution $X(s,u,t)$ is inextensible if the coefficients E, F, and G of the first fundamental form of $X(s,u,t)$ satisfy*

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0,$$

for all time t [1].

4.1. Inextensible Flows of N-Ruled Surface

Let X_N be an N-ruled surface which is generated by the motion of the vector N in the alternative frame of the curve α, then the evolution equation of this surface is represented by

$$X_N(s,u,t) = \alpha(s,t) + uN(s,t). \tag{7}$$

So, the subscripts s and u represent partial derivatives of the equation of the surface X_N are easily calculated in the following form

$$\begin{aligned} \frac{\partial X_N(s,u,t)}{\partial s} &= (p + u\lambda)C(s,t) + qW(s,t), \\ \frac{\partial X_N(s,u,t)}{\partial u} &= N(s,t), \end{aligned} \tag{8}$$

where the functions $p = \frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}$ and $q = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}$ depend on the parameters s and t. By substituting Equation (8) into Equation (2), one can get the normal vector field of the ruled surface X_N given by Equation (7) as

$$U_N(s,u,t) = \frac{qC(s,t) - (p+u\lambda)W(s,t)}{\sqrt{q^2 + (p+u\lambda)^2}}$$

for $q^2 + (p+u\lambda)^2 \neq 0$. On the other hand, by considering Equations (3) and (8), the coefficients E_N, F_N, and G_N of the first fundamental form of the ruled surface X_N are obtained as follows

$$E_N(s,u,t) = q^2 + (p+u\lambda)^2,\ F_N(s,u,t) = 0,\ G_N(s,u,t) = 1. \tag{9}$$

Theorem 1. *Let $X_N(s,u,t)$ be an $N-$ruled surface evolution of X_N with the alternative frame $\{N,C,W\}$. The ruled surface evolution $X_N(s,u,t)$ is inextensible provided that*

$$\frac{\partial \lambda}{\partial t}=0,\ \frac{\partial p}{\partial t}=0,\ \text{and}\ \frac{\partial q}{\partial t}=0,$$

for all time t.

Proof. Let $X_N(s,u,t)$ be an $N-$ruled surface evolution of X_N with the alternative frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Equation (9) with respect to t, we get

$$\frac{\partial E_N}{\partial t}=2q\frac{\partial q}{\partial t}+2(p+u\lambda)\left(\frac{\partial p}{\partial t}+u\frac{\partial \lambda}{\partial t}\right),\ \frac{\partial F_N}{\partial t}=0,\ \ \frac{\partial G_N}{\partial t}=0.$$

If there exists $\frac{\partial \lambda}{\partial t}=\frac{\partial p}{\partial t}=\frac{\partial q}{\partial t}=0$ (in other words, if the curve α is a circular helix), the conditions given in Definition 1 are satisfied. This completes the proof. □

Theorem 2. *Let $X_N(s,u,t)$ be an $N-$ruled surface evolution of X_N with the alternative frame, then the Gaussian curvature K_N and the mean curvature H_N of X_N are*

$$K_N(s,u,t)=-\frac{q^2\lambda^2}{\left(q^2+(p+u\lambda)^2\right)^2}$$

and

$$H_N(s,u,t)=\frac{q\left(-q\mu+\frac{\partial p}{\partial s}+u\frac{\partial \lambda}{\partial s}\right)-(p+u\lambda)\left((p+u\lambda)\mu+\frac{\partial q}{\partial s}\right)}{2\left(q^2+(p+u\lambda)^2\right)^{3/2}},$$

respectively.

Proof. Let $X_N(s,u,t)$ be a $N-$ruled surface evolution of X_N with the alternative frame. By differentiating Equation (8), one can get

$$\begin{aligned}
\frac{\partial^2 X_N(s,u,t)}{\partial s^2}&=-\left(p\lambda+u\lambda^2\right)N(s,t)+\left(\frac{\partial p}{\partial s}-q\mu+u\frac{\partial \lambda}{\partial s}\right)C(s,t)+\left(\frac{\partial q}{\partial s}+p\mu+u\lambda\mu\right)W(s,t),\\
\frac{\partial^2 X_N(s,u,t)}{\partial s\partial u}&=\lambda C(s,t),\\
\frac{\partial^2 X_N(s,u,t)}{\partial u^2}&=0.
\end{aligned}$$

Considering these last equations together with the principal normal vector field of X_N, the coefficients of the second fundamental form of X_N are given by

$$\begin{aligned}
e_N(s,u,t)&=\frac{q\left(-q\mu+\frac{\partial p}{\partial s}+u\frac{\partial \lambda}{\partial s}\right)-(p+u\lambda)\left((p+u\lambda)\mu+\frac{\partial q}{\partial s}\right)}{\sqrt{q^2+(p+u\lambda)^2}},\\
f_N(s,u,t)&=\frac{q\lambda}{\sqrt{q^2+(p+u\lambda)^2}},\ g_N(s,u,t)=0.
\end{aligned}\tag{10}$$

By substituting Equations (9) and (10) into Equation (5), one can obtain the Gaussian curvature and the mean curvature of the surface X_N as above. □

Corollary 1. *Let $X_N(s,u,t)$ be an $N-$ruled surface evolution of X_N with the alternative frame, then the surface X_N is*

- *developable if q vanishes,*
- *minimal if q vanishes, or $\mu = \lambda_s = 0$ and p, q are constants.*

4.2. Inextensible Flows of C-Ruled Surface

Let X_C be a C-ruled surface which is generated by the motion of the vector C of the curve α with the alternative frame, then the evolution equation of this surface is represented by

$$X_C(s,u,t) = \alpha(s,t) + uC(s,t). \tag{11}$$

So, the partial derivatives of the ruled surface X_C with respect to s and u are easily calculated in the following form

$$\begin{aligned} \frac{\partial X_C(s,u,t)}{\partial s} &= -\lambda u N(s,t) + pC(s,t) + (q+u\mu)W(s,t), \\ \frac{\partial X_C(s,u,t)}{\partial u} &= C(s,t), \end{aligned} \tag{12}$$

where the functions p and q depend on the parameters s and t. By considering the partial derivatives of the surface X_C given by Equations (2) and (11), one can get the normal vector field of the ruled surface X_C as

$$U_C(s,u,t) = -\frac{(q+u\mu)N(s,t) + u\lambda W(s,t)}{\sqrt{u^2\lambda^2 + (q+u\mu)^2}}$$

for $u^2\lambda^2 + (q+u\mu)^2 \neq 0$. On the other hand, by considering Equations (3) and (12), the coefficients E_C, F_C, and G_C of the first fundamental form of the ruled surface X_C are obtained as follows

$$E_C(s,u,t) = p^2 + u^2\lambda^2 + (q+u\mu)^2,\ F_C(s,u,t) = 0,\ G_C(s,u,t) = 1. \tag{13}$$

Theorem 3. *Let $X_C(s,u,t)$ be a C-ruled surface evolution of X_C with the alternative frame $\{N, C, W\}$. The ruled surface evolution $X_C(s,u,t)$ is inextensible if and only if*

$$\frac{\partial \lambda}{\partial t} = \frac{\partial \mu}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = 0 \text{ (or the curve } \alpha \text{ is a circular helix).}$$

Proof. Let $X_C(s,u,t)$ be a C-ruled surface evolution with the alternative frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Equation (13) with respect to t, we have

$$\begin{aligned} \frac{\partial E_C}{\partial t} &= 2p\frac{\partial p}{\partial t} + 2u^2\lambda\frac{\partial \lambda}{\partial t} + 2(q+u\mu)\left(\frac{\partial q}{\partial t} + u\frac{\partial \mu}{\partial t}\right), \\ \frac{\partial F_C}{\partial t} &= \frac{\partial p}{\partial t},\ \frac{\partial G_C}{\partial t} = 0. \end{aligned}$$

If there exists $\frac{\partial \lambda}{\partial t} = \frac{\partial \mu}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = 0$, or if the curve α is a circular helix, the conditions given in Definition 1 are satisfied. Thus, the proof is completed. □

Theorem 4. *Let $X_C(s,u,t)$ be a C-ruled surface evolution of X_C with the alternative frame, then the Gaussian curvature K_C and the mean curvature H_C of the surface X_C are*

$$K_C(s,u,t) = -\frac{q^2\lambda^2}{\left(u^2\lambda^2 + (q+u\mu)^2\right)^2}$$

and

$$H_C(s,u,t) = \frac{u\left((q+u\mu)\frac{\partial\lambda}{\partial s} - \lambda\left(\frac{\partial q}{\partial s} + u\frac{\partial\mu}{\partial s}\right)\right) - pq\lambda}{2\left(u^2\lambda^2 + (q+u\mu)^2\right)^{3/2}},$$

respectively.

Proof. Let $X_C(s,u,t)$ be a C-ruled surface evolution of X_C with the alternative frame. By differentiating Equation (12), one can get

$$\begin{aligned}
\frac{\partial^2 X_C(s,u,t)}{\partial s^2} &= \left(-p\lambda - u\frac{\partial\lambda}{\partial s}\right)N(s,t) + \left(\frac{\partial p}{\partial s} - u\lambda^2 - q\mu - u\mu^2\right)C(s,t) \\
&\quad + \left(p\mu + \frac{\partial q}{\partial s} + u\frac{\partial\mu}{\partial s}\right)W(s,t), \\
\frac{\partial^2 X_C(s,u,t)}{\partial s\partial u} &= -\lambda N(s,t) + \mu W(s,t), \\
\frac{\partial^2 X_C(s,u,t)}{\partial u^2} &= 0.
\end{aligned}$$

Considering these last equations together with the normal vector field of X_C, the coefficients of the second fundamental form of X_C are calculated as

$$\begin{aligned}
e_C(s,u,t) &= \frac{pq\lambda + u\left((q+u\mu)\frac{\partial\lambda}{\partial s} - \lambda\left(\frac{\partial q}{\partial s} + u\frac{\partial\mu}{\partial s}\right)\right)}{\sqrt{u^2\lambda^2 + (q+u\mu)^2}}, \\
f_C(s,u,t) &= \frac{q\lambda}{\sqrt{u^2\lambda^2 + (q+u\mu)^2}},\ g_C(s,u,t) = 0.
\end{aligned} \tag{14}$$

By substituting Equations (13) and (14) into Equation (4), one can obtain the Gaussian curvature and the mean curvature of the C-ruled surface as above. □

Corollary 2. *Let X_C be a C-ruled surface with the alternative frame. The surface X_C is developable and minimal, provided that q vanishes.*

4.3. Inextensible Flows of W-Ruled Surface

Let X_W be a W-ruled surface that is generated by the motion of the Darboux vector W of the curve α with the alternative frame, then the evolution equation of this surface is represented by

$$X_W(s,u,t) = \alpha(s,t) + uW(s,t). \tag{15}$$

So, the partial derivatives of the equation of W-ruled surface X_W with respect to s and u are easily calculated in the following form

$$\begin{aligned}
\frac{\partial X_W(s,u,t)}{\partial s} &= (p-u\mu)C(s,t) + qW(s,t), \\
\frac{\partial X_W(s,u,t)}{\partial u} &= W(s,t),
\end{aligned} \tag{16}$$

where the functions p and q depend on the parameters s and t. By considering the partial derivatives of the surface X_W and Equation (2), one can get the normal vector field of the W-ruled surface as

$$U_W(s,u,t) = N(s,t),$$

where $p > u\mu$. On the other hand, by considering Equation (16), the coefficients E_W, F_W, and G_W of the first fundamental form of the W-ruled surface given by Equation (15) are obtained as follows

$$E_W(s,u,t) = q^2 + (p - u\mu)^2,\ F_W(s,u,t) = q,\ G_W(s,u,t) = 1. \tag{17}$$

Theorem 5. *Let $X_W(s,u,t)$ be a W-ruled surface evolution of X_W with the alternative frame $\{N, C, W\}$. The W-ruled surface evolution $X_W(s,u,t)$ is inextensible if and only if*

$$\frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = \frac{\partial \mu}{\partial t} = 0,\ \textit{or the curve } \alpha \textit{ is a circular helix.}$$

Proof. Let $X_W(s,u,t)$ be a W-ruled surface evolution of X_W with the alternative frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Equation (17) with respect to t, we have

$$\frac{\partial E_W(s,u,t)}{\partial t} = 2q\frac{\partial q}{\partial t} + 2(p - u\mu)\left(\frac{\partial p}{\partial t} - u\frac{\partial \mu}{\partial t}\right),$$
$$\frac{\partial F_W(s,u,t)}{\partial t} = \frac{\partial q}{\partial t},\ \frac{\partial G_W(s,u,t)}{\partial t} = 0.$$

If there exists $\frac{\partial p}{\partial t} = \frac{\partial q}{\partial t} = \frac{\partial \mu}{\partial t} = 0$, or the curve α is a circular helix, the conditions given in Definition 1 are satisfied. Thus, the proof is completed. □

Theorem 6. *Let $X_W(s,u,t)$ be a $W-$ruled surface evolution of X_W with the alternative frame, then the Gaussian curvature K_W and the mean curvature H_W of the $W-$ruled surface are*

$$K_W(s,u,t) = 0$$

and

$$H_W(s,u,t) = \frac{\lambda}{2(u\mu - p)},$$

respectively.

Proof. Let $X_W(s,u,t)$ be a W-ruled surface evolution of X_W with the alternative frame. By differentiating Equation (16), we find that

$$\frac{\partial^2 X_W(s,u,t)}{\partial s^2} = (u\lambda\mu - p\lambda)N(s,t) + \left(p\mu - u\mu^2 + \frac{\partial q}{\partial t}\right)W(s,t) + \left(\frac{\partial p}{\partial t} - u\frac{\partial \mu}{\partial t} - q\mu\right)C(s,t),$$
$$\frac{\partial^2 X_W(s,u,t)}{\partial s \partial u} = -\mu C(s,t),$$
$$\frac{\partial^2 X_W(s,u,t)}{\partial u^2} = 0.$$

Considering these last equations together with the normal vector field of X_W, the coefficients of the second fundamental form of X_W are calculated by

$$e_W(s,u,t) = \lambda(u\mu - p), f_W(s,u,t) = 0,\ g_W(s,u,t) = 0. \tag{18}$$

By substituting Equations (17) and (18) into Equation (5), one can obtain the Gaussian curvature and the mean curvature of the W-ruled surface as above. □

Corollary 3. *Let X_W be a W-ruled surface with the alternative frame, then the surface X_W is*

- *developable,*
- *not minimal.*

Example 1. *Let's take a curve defined in parametric form*

$$\alpha(s) = (s - 2\tanh s, -2\,\mathrm{sech}\ s, -2\,\mathrm{sech}\ s), \tag{19}$$

see Figure 1. Let its evolution be given in the form

$$\alpha(s,t) = (s - 2\tanh s, -2\,\mathrm{sech}\, s\cos t, -2\,\mathrm{sech}\, s\sin t). \tag{20}$$

see Figure 2. The Frenet vectors and the curvatures of the curve evolution $\alpha(s,t)$ at any time t are

$$T(s,t) = \left(1 - 2\mathrm{sech}\, s^2, 2\cos t\,\mathrm{sech}\, s\tanh s, 2\mathrm{sech}\, s\sin t\tanh s\right),$$
$$N(s,t) = \left(2\mathrm{sech}\, s\tanh s, \cos t\left(2 - \cosh s^2\right)\mathrm{sech}\, s^2, \left(2 - \cosh s^2\right)\mathrm{sech}\, s^2\sin t\right),$$
$$B(s,t) = (0, \sin t, -\cos t),\ \kappa(s,t) = 2\mathrm{sech}\, s,\ \tau(s,t) = 0.$$

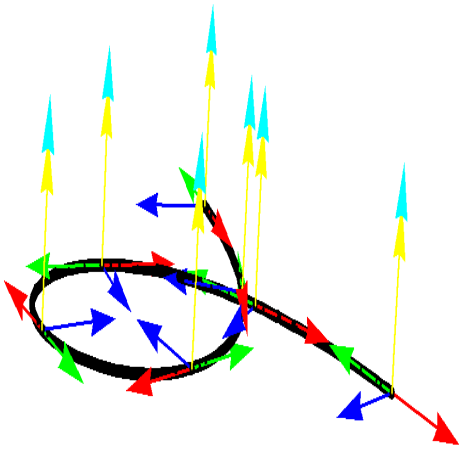

Figure 1. The tangent vectors (green), principal normal vectors (blue), binormal vectors (cyan), derivative of the principal normal vectors (red), and Darboux vectors (yellow) of $\alpha(s)$ given by Equation (19) where $s \in (-3,3)$.

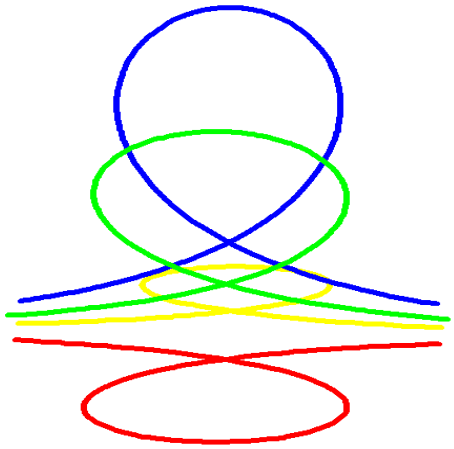

Figure 2. The curve evolution $\alpha(s,t)$ given by Equation (20) for $t = 0$ (red), $t = \pi/4$ (green), $t = \pi/2$ (blue), and $t = \pi$ (yellow) where $s \in (-3,3)$.

The alternative frame vectors and the alternative curvatures of $\alpha(s,t)$ *are*

$$N(s,t) = \left(2\text{sech}\, s \tanh s, \text{sech}\, s^2\left(2-\cosh s^2\right)\cos t, \text{sech}\, s^2\left(2-\cosh s^2\right)\sin t\right),$$
$$C(s,t) = \left(\frac{1}{2}(3-\cosh 2s)\text{sech}\, s^2, -2\,\text{sech}\, s \tanh s \cos t, -2\text{sech}\, s \tanh s \sin t\right),$$
$$W(s,t) = (0, \sin t, -\cos t),\ \lambda(s,t) = 2\text{sech}\, s,\ \mu(s,t) = 0.$$

Thus, the evolution equations of the ruled surfaces, which are generated by the motion of the vector N, C, *and* W *along the curve* α, *are represented by*

$$\begin{cases} X_N(s,u,t) = \begin{pmatrix} s-2(1-u\text{sech}\, s)\tanh\ s, \cos t(u-2\text{sech}\, s(1-u\text{sech}\, s)), \\ (u-2\text{sech}\, s(1-u\text{sech}\, s))\sin t \end{pmatrix}, \\ X_C(s,u,t) = \begin{pmatrix} s-u+2u\text{sech}\, s^2-2\tanh s, -2\cos t\,\text{sech}\, s(1+u\tanh s), \\ -2\text{sech}\, s \sin t(1+u\tanh s) \end{pmatrix}, \\ X_W(s,u,t) = (s-2\tanh s, -2\cos t\,\text{sech}\, s + u\sin t, -u\cos t - 2\text{sech}\, s \sin t), \end{cases} \tag{21}$$

see Figure 3, Figure 4, and Figure 5, respectively.

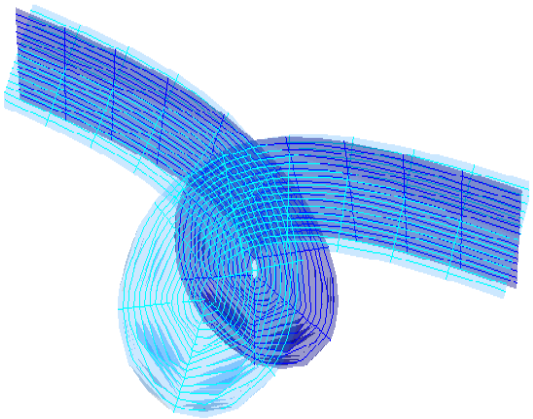

Figure 3. The surface evolution $X_N(s,u,t)$ given in Equations (21) for $t = \pi/4$ (cyan) and $t = \pi/2$ (blue) where $s \in (-5,5)$ and $u \in (-0.5, 0.5)$.



Figure 4. The surface evolution $X_C(s,u,t)$ given in Equations (21) for $t = \pi/4$ (red) and $t = \pi/2$ (yellow) where $s \in (-5,5)$ and $u \in (-2, 2)$.

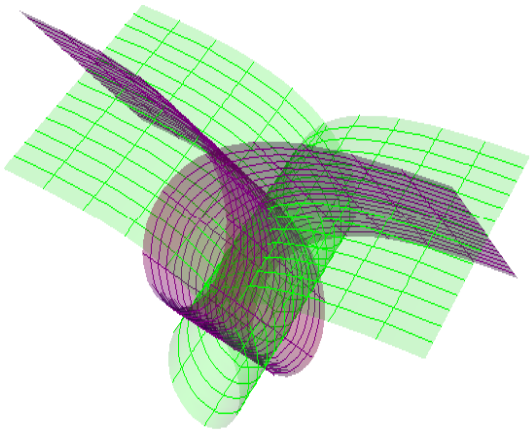

Figure 5. The surface evolution $X_W(s,u,t)$ given in Equations (21) for $t = \pi/4$ (purple) and $t = \pi/2$ (green) where $s \in (-5,5)$ and $u \in (-2,2)$.

Thus, one can get the normal vector fields of the ruled surfaces as:

$$\begin{cases} U_N(s,u,t) = \left(0, \dfrac{(\cosh s - 2u)\operatorname{sech} s \sin t}{\sqrt{(1-2u\operatorname{sech} s)^2}}, \dfrac{\cos t(2u\operatorname{sech} s - 1)}{\sqrt{(1-2u\operatorname{sech} s)^2}}\right), \ u\operatorname{sech} s \neq \dfrac{1}{2}, \\ U_C(s,u,t) = (0, -\sin t, \cos t), \\ U_W(s,u,t) = \left(-2\operatorname{sech} s \tanh s, \dfrac{\cos t(\cosh 2s - 3)\operatorname{sech} s^2}{2}, \dfrac{(\cosh 2s - 3)\operatorname{sech} s^2 \sin t}{2}\right). \end{cases}$$

The coefficients of the first fundamental form of the ruled surfaces are obtained as follows

$$\begin{cases} E_N(s,u,t) = (1-2u\operatorname{sech} s)^2, \ F_N(s,u,t) = 0, \ G_N(s,u,t) = 1, \\ E_C(s,u,t) = 1 + 4u^2\operatorname{sech} s^2, \ F_C(s,u,t) = -1, \ G_C(s,u,t) = 1, \\ E_W(s,u,t) = 1, \ F_W(s,u,t) = 0, \ G_W(s,u,t) = 1. \end{cases}$$

Taking the partial derivatives of the coefficients of the first fundamental form of X_N, X_C, and X_W with respect to t, we get

$$\begin{cases} \dfrac{\partial E_N}{\partial t} = \dfrac{\partial F_N}{\partial t} = \dfrac{\partial G_N}{\partial t} = 0, \\ \dfrac{\partial E_C}{\partial t} = \dfrac{\partial F_C}{\partial t} = \dfrac{\partial G_C}{\partial t} = 0, \\ \dfrac{\partial E_W}{\partial t} = \dfrac{\partial F_W}{\partial t} = \dfrac{\partial G_W}{\partial t} = 0. \end{cases}$$

Therefore, it is said that the ruled surface evolutions are inextensible. The Gaussian curvatures and the mean curvatures of X_N, X_C and X_W are

$$\begin{cases} K_N(s,u,t) = 0, \ H_N(s,u,t) = 0, \\ K_C(s,u,t) = 0, \ H_C(s,u,t) = 0, \\ K_W(s,u,t) = 0, \ H_W(s,u,t) = -\operatorname{sech} s. \end{cases}$$

Consequently, the surface evolutions

- *X_N, X_C, and X_W are developable,*
- *X_N and X_C are minimal but X_W is not minimal.*

Example 2. *Let us consider a curve defined in parametric form,*

$$\begin{aligned} \beta : (-\pi, \pi) \subset R &\to E^3 \\ s &\to \beta(s) = \frac{1}{\sqrt{2}}(\sin s, \cos s, s), \end{aligned} \tag{22}$$

see Figure 6, and its evolution is defined in the form

$$\begin{aligned} \beta : (-\pi, \pi) \times (0, w) &\to E^3 \\ (s,t) &\to \beta(s,t) = \frac{1}{\sqrt{2}}\left(\frac{\sin s(t+1)}{t+1}, \frac{\cos s(t+1)}{t+1}, s\right), \end{aligned} \tag{23}$$

see Figure 7. The Frenet vectors and the curvatures of the curve evolution $\beta(s,t)$ at any time t are found as

$$\begin{aligned} T(s,t) &= \left(\frac{\cos s(1+t)}{\sqrt{2}}, -\frac{\sin s(1+t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ N(s,t) &= (-\sin s(1+t), -\cos s(1+t), 0), \\ B(s,t) &= \left(\frac{\cos s(1+t)}{\sqrt{2}}, -\frac{\sin s(1+t)}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\ \kappa(s,t) &= \frac{1+t}{\sqrt{2}}, \ \tau(s,t) = -\frac{1+t}{\sqrt{2}}, \end{aligned}$$

and the alternative frame vectors and the alternative curvatures of $\beta(s,t)$ are

$$\begin{aligned} N(s,t) &= (-\sin s(1+t), -\cos s(1+t), 0), \\ C(s,t) &= (-\cos s(1+t), \sin s(1+t), 0), \\ W(s,t) &= (0,0,-1), \\ \lambda(s,t) &= 1+t, \ \mu(s,t) = 0. \end{aligned}$$

Thus, the evolution equations of the ruled surfaces, which are generated by the motion of the vectors N, C, and W in the alternative frame of the curve β, are represented by

$$\begin{cases} Y_N(s,u,t) = \left(\dfrac{\left(\sqrt{2}-2(1+t)u\right)\sin s(1+t)}{2(1+t)}, \dfrac{\left(\sqrt{2}-2(1+t)u\right)\cos s(1+t)}{2(1+t)}, \dfrac{s}{\sqrt{2}}\right), \\ Y_C(s,u,t) = \left(\dfrac{\sin s(1+t)}{\sqrt{2}(1+t)} - u\cos s(1+t), \dfrac{\cos s(1+t)}{\sqrt{2}(1+t)} + u\sin s(1+t), \dfrac{s}{\sqrt{2}}\right), \\ Y_W(s,u,t) = \left(\dfrac{\sin s(1+t)}{\sqrt{2}(1+t)}, \dfrac{\cos s(1+t)}{\sqrt{2}(1+t)}, \dfrac{s}{\sqrt{2}} - u\right), \end{cases} \tag{24}$$

see Figure 8, Figure 9, and Figure 10, respectively.

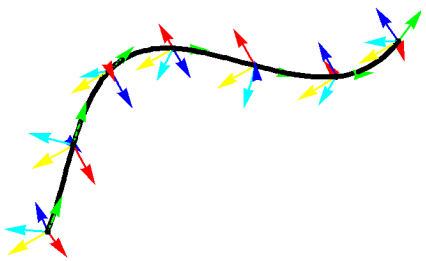

Figure 6. The tangent vectors (green), principal normal vectors (blue), binormal vectors (cyan), derivative of the principal normal vectors (red) and Darboux vectors (yellow) of $\beta(s)$ given by Equation (22) where $s \in (-\pi, \pi)$.

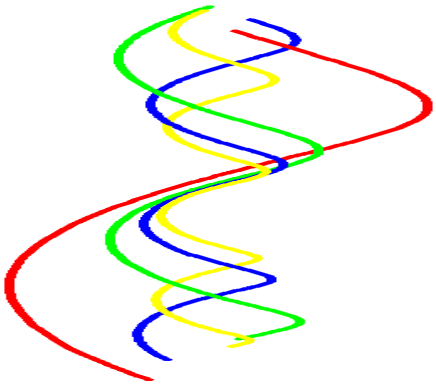

Figure 7. The curve evolution $\beta(s,t)$ given by Equation (23) for $t = 0$ (red), $t = 1$ (green), $t = 2$ (blue), $t = 3$ (yellow) where $s \in (-\pi, \pi)$.

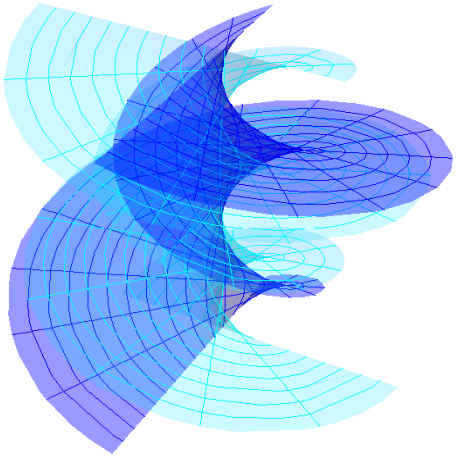

Figure 8. The surface evolution $Y_N(s,u,t)$ given by Equations (24) for $t = \pi/4$ (blue) and $t = \pi/2$ (cyan) where $s \in (-\pi/2, \pi/2)$ and $u \in (-1, 1)$.

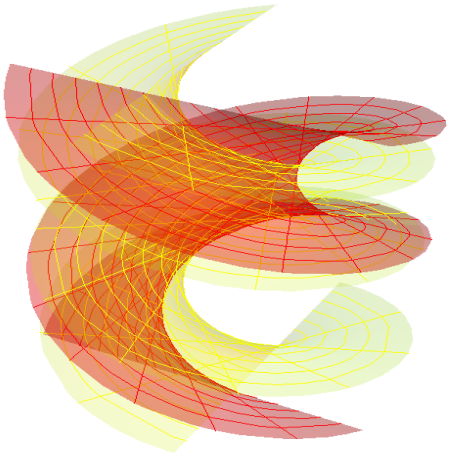

Figure 9. The surface evolution $Y_C(s,u,t)$ given by Equations (24) for $t = \pi/4$ (red) and $t = \pi/2$ (yellow) where $s \in (-\pi/2, \pi/2)$ and $u \in (-1, 1)$.

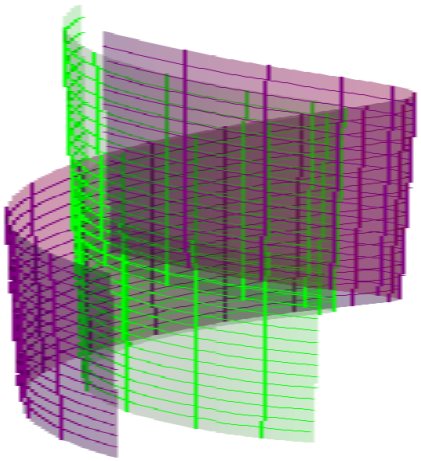

Figure 10. The surface evolution $T_W(s,u,t)$ given by Equations (24) for $t = \pi/4$ (purple) and $t = \pi/2$ (green) with $s \in (-\pi/2, \pi/2)$ and $u \in (-1, 1)$.

The equations of the normal vector fields of the ruled surfaces are found

$$\begin{cases} U_N(s,u,t) = \left(\dfrac{\sqrt{2}\cos s(1+t)}{2\sqrt{1+(1+t)\left((1+t)u-\sqrt{2}\right)u}}, \dfrac{-\sqrt{2}\sin s(1+t)}{2\sqrt{1+(1+t)\left((1+t)u-\sqrt{2}\right)u}}, \dfrac{\sqrt{2}u(1+t)-1}{2\sqrt{1+(1+t)\left((1+t)u-\sqrt{2}\right)u}} \right), \\ U_C(s,u,t) = \left(-\dfrac{\sin s(1+t)}{\sqrt{1+2(1+t)^2u^2}}, -\dfrac{\cos s(1+t)}{\sqrt{1+2(1+t)^2u^2}}, \dfrac{\sqrt{2}(1+t)u}{\sqrt{1+2(1+t)^2u^2}} \right), \\ U_W(s,u,t) = (\sin s(1+t), \cos s(1+t), 0), \\ 1+(1+t)\left((1+t)u-\sqrt{2}\right)u \neq 0. \end{cases}$$

The coefficients of the first fundamental form of the ruled surfaces are obtained as follows

$$\begin{cases} E_N(s,u,t) = 1+(1+t)\left((1+t)u-\sqrt{2}\right)u,\ F_N(s,u,t) = 0,\ G_N(s,u,t) = 1, \\ E_C(s,u,t) = 1+(1+t)^2u^2,\ F_C(s,u,t) = -\dfrac{1}{\sqrt{2}},\ G_C(s,u,t) = 1, \\ E_W(s,u,t) = 1,\ F_W(s,u,t) = -\dfrac{1}{\sqrt{2}},\ G_W(s,u,t) = 1. \end{cases}$$

Taking the partial derivatives of the coefficients of the first fundamental form of Y_N, Y_C, and Y_W with respect to t, we get

$$\begin{cases} \dfrac{\partial E_N}{\partial t} = \left(2(1+t)u-\sqrt{2}\right)u, \dfrac{\partial F_N}{\partial t} = \dfrac{\partial G_N}{\partial t} = 0, \\ \dfrac{\partial E_C}{\partial t} = 2(1+t)u^2, \dfrac{\partial F_C}{\partial t} = \dfrac{\partial G_C}{\partial t} = 0, \\ \dfrac{\partial E_W}{\partial t} = \dfrac{\partial F_W}{\partial t} = \dfrac{\partial G_W}{\partial t} = 0. \end{cases}$$

So, it is said that the ruled surface Y_W is inextensible, but the ruled surface Y_N and Y_C are not inextensible.

The Gaussian curvatures and the mean curvatures of Y_N, Y_C, and Y_W are

$$\begin{cases} K_N(s,u,t) = -\dfrac{(1+t)^2}{2\Big(1+(1+t)\Big(-\sqrt{2}+(1+t)u\Big)u\Big)^2},\ H_N(s,u,t)=0, \\ K_C(s,u,t) = -\dfrac{2(1+t)^2}{\Big(1+2(1+t)^2u^2\Big)^2},\ H_C(s,u,t) = -\dfrac{1+t}{\sqrt{2}\Big(1+2(1+t)^2u^2\Big)^{3/2}}, \\ K_W(s,u,t)=0,\ H_W(s,u,t) = -\dfrac{1+t}{\sqrt{2}}. \end{cases}$$

As a result, the surface

- *Y_N is not developable but minimal,*
- *Y_C is not both developable and minimal,*
- *Y_W is developable but not minimal.*

5. Conclusions

This paper presents the evolution of ruled surfaces formed by the vector fields in the alternative frame, such as the principal normal vector, its derivative, and the Darboux vector. Since analyzing the Darboux vector along a curve allows us to determine how a curve is rotating as it moves through space, the obtained results may have a crucial role in various fields like robotics, biomechanics, and mechanical engineering, where understanding the rotational behavior of objects is essential. Therefore, as the first step of this study, the evolution of a space curve is expressed with an alternative frame. Then, the conditions for these ruled surfaces to be minimal, developable, and inextensible are investigated. The evolution of these surfaces has not been investigated so far. It is thought that this subject may have important applications as well as in the aforementioned areas and in the field of physics.

Author Contributions: Conceptualization, K.E.; Formal analysis, A.S., K.E., S.E. and V.B.; Investigation, A.S., K.E., S.E. and V.B.; Methodology; A.S., K.E., S.E. and V.B.; Supervision, S.E.; Writing—original draft, K.E.; Writing—review & editing, A.S., K.E., S.E. and V.B. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the Academy of Technical and Art Applied Studies in Belgrade.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: Author Dr. Kemal Eren was employed by the company Sakarya University Technology Developing Zones Manager CO. The remaining authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

References

1. Kwon, D.Y.; Park, F.C. Evolution of inelastic plane curves. *Appl. Math. Lett.* **1999**, *12*, 115–119. [CrossRef]
2. Kwon, D.Y.; Park, F.C.; Chi, D.P. Inextensible flows of curves and developable surfaces. *Appl. Math. Lett.* **2005**, *18*, 1156–1162. [CrossRef]
3. Gürbüz, N.; Yoon, D.W. Geometry of curve flows in isotropic spaces. *AIMS Math.* **2020**, *5*, 3434–3445. [CrossRef]
4. Yıldız, Ö.G.; Ersoy, S.; Masal, M. A note on inextensible flows of curves on oriented surface. *Cubo (Temuco)* **2014**, *16*, 11–19. [CrossRef]
5. Kass, M.; Witkin, A.; Terzopoulos, D. Snakes: Active contour models. *Int. J. Comput. Vision* **1988**, *1*, 321–331. [CrossRef]
6. Lu, H.Q.; Todhunter, J.S.; Sze, T.W. Congruence conditions for nonplanar developable surfaces and their application surface recognition. *CVGIP Image Underst.* **1993**, *56*, 265–285. [CrossRef]
7. Desbrun, M.; Cani-Gascuel, M.P. Active Implicit Surface for Animation. In Proceedings of the Graphics Interface 1998 Conference, Vancouver, BC, Canada, 18–20 June 1998; pp. 143–150. [CrossRef]
8. Chirikjian, G.; Burdick, J. A modal approach to hyper-redundant manipulator kinematics. *IEEE Trans. Robot. Autom.* **1994**, *10*, 343–354. [CrossRef]

9. Courant, R. Soap film experiments with minimal surfaces. *Am. Math. Mon.* **1940**, *47*, 167–174. [CrossRef]
10. Raymond, G.; Adriana, I.P.; Christophe, R.; James, S. Geometry of catenoidal soap film collapse induced by boundary deformation. *Phys. Rev. E* **2021**, *104*, 035105. [CrossRef]
11. Uzunoğlu, B.; Gök, İ.; Yaylı, Y. A new approach on curves of constant precession. *Appl. Math. Comput.* **2016**, *275*, 317–323. [CrossRef]
12. Tuncer, Y. Vectorial moments of curves in Euclidean 3-space. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750020. [CrossRef]
13. Kaya, O.; Önder, M. New partner curves in the Euclidean 3-space. *Int. J. Geom.* **2017**, *6*, 41–50.
14. Kızıltuğ, S.; Erişir, T.; Mumcu, G.; Yaylı, Y. C^*-partner curves with modified adapted frame and their applications. *AIMS Math.* **2023**, *8*, 1345–1359. [CrossRef]
15. Ouarab, S. NC-Smarandache ruled surface and NW-Smarandache ruled surface according to alternative moving frame in E^3. *J. Math.* **2021**, *2021*, 9951434.
16. Ouarab, S. Simultaneous developability of partner-ruled surfaces according to Darboux frame in E^3. *Abstr. Appl. Anal.* **2021**, *2021*, 3151501. [CrossRef]
17. Yüksel, N.; Saltık, B. On inextensible ruled surfaces generated via a curve derived from a curve with constant torsion. *AIMS Math.* **2023**, *8*, 11312–11324. [CrossRef]
18. Solouma, E.; Al-Dayel, I.; Khan, M.A.; Lazer, Y.A.A. Characterization of imbricate-ruled surfaces via rotation-minimizing Darboux frame in Minkowski 3-space E_1^3. *AIMS Math.* **2024**, *9*, 13028–13042. [CrossRef]
19. Hano, J.; Nomizu, K. Surfaces of revolution with constant mean curvature in Lorentz-Minkowski space. *Tohoku Math. J.* **1984**, *36*, 427–437. [CrossRef]
20. Li, Y.; Pei, D. Evolutes of dual spherical curves for ruled surfaces. *Math. Methods Appl. Sci.* **2016**, *39*, 3005–3015. [CrossRef]
21. Li, Y.; Eren, K.; Ayvacı, K.H.; Ersoy, S. Simultaneous characterizations of partner-ruled surfaces using Flc frame. *AIMS Math.* **2022**, *7*, 20213–20229. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.