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The α_σ-Approximation Property and Its Related Operator Ideals

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Abstract: In this paper, we study the σ-tensor norm (α_σ), the absolutely τ-summing operator and the σ-nuclear operator. We characterize the α_σ-approximation property in terms of some density of the space of absolutely τ-summing operators. When X^* or Y^{***} has the approximation property, we prove that an operator T from X to Y is σ-nuclear if the adjoint of T is σ-nuclear.

Keywords: tensor norm; approximation property; operator ideal

MSC: 46B28; 46B45; 47L20

1. Introduction

One of the most important properties in Banach space theory is the *approximation property* (AP), which was systematically investigated by Grothendieck [1]. It not only deserves to be studied independently in itself, but also to be studied along with the theory of tensor norms and operator ideals for Banach spaces. The main notions of this paper originate from the AP, the *injective tensor norm* ε, the *projective tensor norm* π, the deal of *nuclear operators* and the ideal of *absolutely summing operators*.

Throughout this paper, Banach spaces will be denoted by X and Y over $\mathbb{R}$ or $\mathbb{C}$, with dual spaces X^* and Y^*, and the closed unit ball of X will be denoted by B_X. Let $X \otimes Y$ be the algebraic tensor product of X and Y. We say that a Banach space X has the AP if for every compact subset K of X and every $\delta > 0$, there exists a finite rank operator $S : X \to X$ such that $\sup_{x\in K} \|Sx - x\| \leq \delta$. For $u \in X \otimes Y$,

$$\varepsilon(u; X, Y) := \sup \left\{ \left| \sum_{n=1}^{m} x^*(x_n) y^*(y_n) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $\sum_{n=1}^{m} x_n \otimes y_n$ is any representation of u, and

$$\pi(u; X, Y) := \inf \left\{ \sum_{n=1}^{m} \|x_n\| \|y_n\| : u = \sum_{n=1}^{m} x_n \otimes y_n, m \in \mathbb{N} \right\}.$$

The normed space $X \otimes Y$ equipped with a norm α will be denoted by $X \otimes_\alpha Y$ and its completion is $X \hat{\otimes}_\alpha Y$. Grothendieck [1] proved that X has the AP if and only if for every Banach space Y, the canonical inclusion map

$$J_\pi : Y \hat{\otimes}_\pi X \hookrightarrow Y \hat{\otimes}_\varepsilon X$$

is injective (cf. [2], Theorem 5.6). Let α be a tensor norm. For a general background of the theory of tensor norms, we refer to [2,3]. In view of this criterion of the AP, it is natural to define that X has the *α-approximation property* (α-AP) if for every Banach space Y, the inclusion map

$$J_\alpha : Y \hat{\otimes}_\alpha X \hookrightarrow Y \hat{\otimes}_\varepsilon X$$

is injective (cf. [2], Section 21.7). Note that the Banach space Y can be replaced by dual spaces (see [2], Proposition 21.7(4)). For every tensor norm α, it is well known that X has the α-AP if X has the AP (cf. [2], Proposition 21.7(1)).

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Kim and Lee [4] defined the *σ-tensor norm* α_σ as follows. For $\sum_{n=1}^m x_n \otimes y_n \in X \otimes Y$, let

$$\Big| \sum_{n=1}^m x_n \otimes y_n \Big|_\sigma := \sup \Big\{ \sum_{n=1}^m |x^*(x_n) y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*} \Big\}.$$

For $u \in X \otimes Y$, let

$$\alpha_\sigma(u; X, Y) := \inf \Big\{ \Big| \sum_{n=1}^m x_n \otimes y_n \Big|_\sigma : u = \sum_{n=1}^m x_n \otimes y_n, m \in \mathbb{N} \Big\}.$$

The σ-tensor norm is closely related with the absolutely τ-summing operator and the σ-nuclear operator, which were introduced by Pietsch [5]. For a general background of the theory of operator ideals, we refer to [2,5,6].

For $1 \le p < \infty$, an operator $T : X \to Y$ is called *absolutely p-summing* if there exists a $C > 0$ such that for every finite sequence $(x_n)_{n=1}^m$ in X,

$$\Big(\sum_{n=1}^m \|Tx_n\|^p \Big)^{1/p} \le C \sup_{x^* \in B_{X^*}} \Big(\sum_{n=1}^m |x^*(x_n)|^p \Big)^{1/p}.$$

The tensor norm g_p is defined as follows. For $u \in X \otimes Y$,

$$g_p(u; X, Y) := \inf \Big\{ \Big(\sum_{n=1}^m \|x_n\|^p \Big)^{1/p} \sup_{x^* \in B_{X^*}} \Big(\sum_{n=1}^m |x^*(x_n)|^{p^*} \Big)^{1/p^*} : u = \sum_{n=1}^m x_n \otimes y_n, m \in \mathbb{N} \Big\},$$

where $1/p + 1/p^* = 1$. Saphar [7] systematically investigated the g_p-AP (*approximation property of order p*) and Bourgain and Reinov ([8], Lemma 7) proved that X has the g_p-AP if and only if for every reflexive Banach space Y, the space $\mathcal{F}(X, Y)$ of finite rank operators from X to Y is dense in some locally convex topology in the space of absolutely p^*-summing operators from X to Y. In the present paper, we consider the α_σ-AP and the absolutely τ-summing operator. An operator $T : X \to Y$ is called *absolutely τ-summing* if there exists a $C > 0$ such that

$$\sum_{n=1}^m |y_n^*(Tx_n)| \le C \sup \Big\{ \sum_{n=1}^m |x^*(x_n) y_n^*(y)| : x^* \in B_{X^*}, y \in B_Y \Big\}$$

for every finite sequences $(x_n)_{n=1}^m$ in X and $(y_n^*)_{n=1}^m$ in Y^*. We denote by $\mathcal{P}_\tau(X, Y)$ the space of all absolutely τ-summing operators from X to Y and for $T \in \mathcal{P}_\tau(X, Y)$, let $\|T\|_{\mathcal{P}_\tau} := \inf C$, where the infimum of $C > 0$ is taken over all such inequalities. Then, $[\mathcal{P}_\tau, \|\cdot\|_{\mathcal{P}_\tau}]$ is a Banach operator ideal ([5], Theorem 23.1.2). In Section 3, we introduce some locally convex topology τ_σ on $\mathcal{P}_\tau(X, Y)$ and prove that X has the α_σ-AP if and only if for every Banach space Y,

$$\mathcal{P}_\tau(X, Y) = \overline{\mathcal{F}(X, Y)}^{\tau_\sigma}.$$

For $1 \le p < \infty$, an operator $T : X \to Y$ is called *p-nuclear* if there exists an absolutely p-summable sequence $(x_n^*)_n$ in X^* and a weakly p^*-summable sequence $(y_n)_n$ in Y such that $T = \sum_{n=1}^\infty x_n^* \underline{\otimes} y_n$, where $x_n^* \underline{\otimes} y_n$ is an operator from X to Y defined by $(x_n^* \underline{\otimes} y_n)(x) = x_n^*(x) y_n$. An operator ideal $\mathcal{A}$ is called *regular* if for every Banach spaces X and Y, $T \in \mathcal{A}(X, Y)$ when $i_Y T \in \mathcal{A}(X, Y^{**})$, where $i_Y : Y \to Y^{**}$ is the canonical isometry. It is well known that the ideal of p-nuclear operators is not regular (cf. [9], Remark 3.7) and that for an operator $T : X \to Y$, if $i_Y T$ is p-nuclear, then T is p-nuclear whenever X^* or Y^{***} has the AP (cf. [9], Corollary 3.8). In the present paper, we consider the σ-nuclear operator. An operator $T : X \to Y$ is called *σ-nuclear* if there exists sequences $(x_n^*)_n$ in X^* and $(y_n)_n$ in Y, such that

$$T = \sum_{n=1}^\infty x_n^* \underline{\otimes} y_n$$

unconditionally converges in the Banach space $(\mathcal{L}(X,Y), \|\cdot\|)$ of all operators from X to Y. We denote by $\mathcal{N}_\sigma(X,Y)$ the space of all σ-nuclear operators from X to Y, and for $T \in \mathcal{N}_\sigma(X,Y)$, let

$$\|T\|_{\mathcal{N}_\sigma} := \inf\Big\{ \Big| \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big|_\sigma : T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big\},$$

where $|\sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n|_\sigma := \sup\{\sum_{n=1}^{\infty} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*}\}$ and the infimum is taken over all the σ-nuclear representations. Then, $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is a Banach operator ideal ([5], Theorem 23.2.2). Pietsch ([5], Remark 23.2.7) conjectured that $[\mathcal{N}_\sigma, \|\cdot\|_{\mathcal{N}_\sigma}]$ is not regular. In Section 4, we prove that for an operator $T : X \to Y$, if the adjoint of T is σ-nuclear, then T is σ-nuclear whenever X^* or Y^{***} has the AP. As a consequence, it follows that for an operator $T : X \to Y$, if $i_Y T$ is σ-nuclear, then T is σ-nuclear whenever X^* or Y^{***} has the AP. In the next section, we will summarize some basic tools to prove our main results.

2. Preliminaries

It is well known that a series $\sum_{n=1}^{\infty} z_n$ in a Banach space Z unconditionally converges if and only if

$$\lim_{l\to\infty} \sup_{z^* \in B_{Z^*}} \sum_{n\geq l} |z^*(z_n)| = 0$$

(see, e.g., ([2], Proposition 8.3) and ([6], the proof of 1.6, p. 5)). Then, we have the following.

Lemma 1. *Let $(x_n^*)_n$ in X^* and $(y_n)_n$ in Y be sequences. Then, the following statements are equivalent.*

(a) $T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \in \mathcal{N}_\sigma(X,Y)$.
(b) $\lim_{l\to\infty} \sup\{\sum_{n\geq l} |x^{**}(x_n^*) y^*(y_n)| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*}\} = 0$.
(c) $\lim_{l\to\infty} \sup\{\sum_{n\geq l} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*}\} = 0$.

Moreover,

$$\|T\|_{\mathcal{N}_\sigma} = \inf\Big\{ \sup\Big\{ \sum_{n=1}^{\infty} |x^{**}(x_n^*) y^*(y_n)| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*} \Big\} : T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big\},$$

where the infimum is taken over all the σ-nuclear representations.

Proof. (b)⇒(c) is trivial.

(a)⇒(b): Let us consider the linear functional $x^{**} \otimes y^*$ on $\mathcal{L}(X,Y)$ defined by $x^{**} \otimes y^*(T) = x^{**}(T^* y^*)$. Then, we have

$$\begin{aligned}
&\lim_{l\to\infty} \sup\Big\{ \sum_{n\geq l} |x^{**}(x_n^*) y^*(y_n)| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*} \Big\} \\
&= \lim_{l\to\infty} \sup\Big\{ \sum_{n\geq l} |x^{**} \otimes y^*(x_n^* \underline{\otimes} y_n))| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*} \Big\} \\
&\leq \lim_{l\to\infty} \sup\Big\{ \sum_{n\geq l} |\varphi(x_n^* \underline{\otimes} y_n)| : \varphi \in B_{(\mathcal{L}(X,Y), \|\cdot\|)^*} \Big\} = 0.
\end{aligned}$$

(c)⇒(a): Let $\delta > 0$ be given. Then, by (c), there exists an $l_\delta \in \mathbb{N}$ such that

$$\sup\Big\{ \sum_{n\geq l_\delta} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*} \Big\} \leq \delta.$$

Then, for every finite subset F of $\mathbb{N}$ with $\min F > l_\delta$,

$$\Big\| \sum_{n\in F} x_n^* \underline{\otimes} y_n \Big\| \leq \sup\Big\{ \sum_{n\geq l_\delta} |x_n^*(x) y^*(y_n)| : x \in B_X, y^* \in B_{Y^*} \Big\} \leq \delta.$$

Hence, $\sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \in \mathcal{N}_\sigma(X,Y)$.

To show the last part, let $T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$ be an arbitrary σ-nuclear representation. Let $\delta > 0$ be given. Then, there exists an $l_\delta \in \mathbb{N}$ such that

$$\sup\Big\{ \sum_{n>l_\delta} |x^{**}(x_n^*)y^*(y_n)| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*} \Big\} \le \delta.$$

Now, let $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$. By Helly's lemma (cf. [6], (8.15)), there exists an $x \in X$ with $\|x\| \le 1+\delta$ such that for all $n = 1, \dots, l_\delta$, $x^{**}(x_n^*) = x_n^*(x)$. Then, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |x^{**}(x_n^*)y^*(y_n)| &\le \sum_{n=1}^{l_\delta} |x^{**}(x_n^*)y^*(y_n)| + \delta \\ &= \sum_{n=1}^{l_\delta} |x_n^*(x)y^*(y_n)| + \delta \\ &\le (1+\delta)\Big| \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big|_\sigma + \delta. \end{aligned}$$

Thus,

$$\sup\Big\{ \sum_{n=1}^{\infty} |x^{**}(x_n^*)y^*(y_n)| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*} \Big\} \le (1+\delta)\Big| \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n \Big|_\sigma + \delta.$$

Since $\delta > 0$ was arbitrary, we have the conclusion. □

As in the proof of Lemma 1, we have

Lemma 2 ([4], Lemma 5)**.** *Let $(x_n)_n$ in X and $(y_n)_n$ in Y be sequences. Then*

$$\lim_{l\to\infty} \sup\Big\{ \sum_{n\ge l} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*} \Big\} = 0$$

if and only if the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ unconditionally converges in $X \hat{\otimes}_{\alpha_\sigma} Y$.

It was shown in ([4], Proposition 2) that α_σ is a finitely generated tensor norm and by definition, we see that the transpose $\alpha_\sigma^t = \alpha_\sigma$.

Lemma 3 ([4], Proposition 3)**.** *If $u \in X \hat{\otimes}_{\alpha_\sigma} Y$, then there exist, sequences $(x_n)_n$ in X and $(y_n)_n$ in Y such that*

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converges in $X \hat{\otimes}_{\alpha_\sigma} Y$ and

$$\alpha_\sigma(u; X, Y) = \inf\Big\{ \sup\Big\{ \sum_{n=1}^{\infty} |x^*(x_n)y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*} \Big\} : u = \sum_{n=1}^{\infty} x_n \otimes y_n \Big\}.$$

It was shown in ([4], Corollary 2) that for every Banach space X and Y,

$$\mathcal{P}_\tau(X, Y^*) = (X \otimes_{\alpha_\sigma} Y)^*$$

holds isometrically with the dual action $[T, \sum_{n=1}^m x_n \otimes y_n] = \sum_{n=1}^m (Tx_n)(y_n)$ for $T \in \mathcal{P}_\tau(X, Y^*)$ and $\sum_{n=1}^m x_n \otimes y_n \in X \otimes_{\alpha_\sigma} Y$. Using Lemma 3, we can obtain the following dual action in the completion of $X \otimes_{\alpha_\sigma} Y$.

Lemma 4. *For every Banach space X and Y,*

$$\mathcal{P}_\tau(X, Y^*) = (X \hat{\otimes}_{\alpha_\sigma} Y)^*$$

holds isometrically with the dual action $[T, u] = \sum_{n=1}^{\infty} (Tx_n)(y_n)$ for $T \in \mathcal{P}_\tau(X, Y^*)$ and $u = \sum_{n=1}^{\infty} x_n \otimes y_n \in X \hat{\otimes}_{\alpha_\sigma} Y$.

3. The Absolutely τ-Summing Operator

In this section, we establish a representation of the dual space of $\mathcal{P}_\tau(X, Y)$ equipped with our topology to characterize the α_σ-AP. Let $T \in \mathcal{P}_\tau(X, Y)$. If $\sum_{n=1}^{\infty} y_n^* \underline{\otimes} x_n \in \mathcal{N}_\sigma(Y, X)$, then, by Lemma 1,

$$\begin{aligned} &\lim_{l,m\to\infty} \sum_{n=l}^{m} |y_n^*(Tx_n)| \\ &\leq \|T\|_{\mathcal{P}_\tau} \lim_{l,m\to\infty} \sup \left\{ \sum_{n=l}^{m} |y_n^*(y)x^*(x_n)| : y \in B_Y, x^* \in B_{X^*} \right\} = 0. \end{aligned}$$

Consequently, $(y_n^*(Tx_n))_n \in \ell_1$. Then, we can define a locally convex topology, which will be denoted by τ_σ, on $\mathcal{P}_\tau(X, Y)$ generated by the seminorms

$$\|(y_n^*(Tx_n))_n\|_{\ell_1}$$

for every $\sum_{n=1}^{\infty} y_n^* \underline{\otimes} x_n \in \mathcal{N}_\sigma(Y, X)$. The topology τ_σ is motivated from a locally convex topology in ([10], p. 220).

Proposition 1. *For every Banach space X and Y,*

$$(\mathcal{P}_\tau(X, Y), \tau_\sigma)^* = \left\{ f(T) = \sum_{n=1}^{\infty} \lambda_n y_n^*(Tx_n) : \sum_{n=1}^{\infty} y_n^* \underline{\otimes} x_n \in \mathcal{N}_\sigma(Y, X), (\lambda_n)_n \in \ell_\infty \right\}.$$

Proof. Suppose that for every $T \in \mathcal{P}_\tau(X, Y)$,

$$f(T) = \sum_{n=1}^{\infty} \lambda_n y_n^*(Tx_n),$$

where $\sum_{n=1}^{\infty} y_n^* \underline{\otimes} x_n \in \mathcal{N}_\sigma(Y, X)$ and $(\lambda_n)_n \in \ell_\infty$. Then, for every $T \in \mathcal{P}_\tau(X, Y)$, we have

$$|f(T)| \leq \sum_{n=1}^{\infty} |\lambda_n y_n^*(Tx_n)| \leq (\sup_n |\lambda_n|) \|(y_n^*(Tx_n))_n\|_{\ell_1}.$$

Hence, $f \in (\mathcal{P}_\tau(X, Y), \tau_\sigma)^*$.

Conversely, suppose that $f \in (\mathcal{P}_\tau(X, Y), \tau_\sigma)^*$. Then, there exists $\sum_{n=1}^{\infty} y_n^* \underline{\otimes} x_n \in \mathcal{N}_\sigma(Y, X)$ such that

$$|f(T)| \leq \|(y_n^*(Tx_n))_n\|_{\ell_1}$$

for every $T \in \mathcal{P}_\tau(X, Y)$. Let us consider the linear subspace

$$\{(y_n^*(Tx_n))_n : T \in \mathcal{P}_\tau(X, Y)\}$$

of ℓ_1 and the linear functional φ on $\{(y_n^*(Tx_n))_n : T \in \mathcal{P}_\tau(X, Y)\}$ given by

$$\varphi((y_n^*(Tx_n))_n) = f(T).$$

We see that φ is well defined and linear, and $\|\varphi\| \leq 1$. Let $\hat{\varphi} \in \ell_1^* = \ell_\infty$ be a Hahn–Banach extension of φ. Let

$$\hat{\varphi} := (\lambda_n)_n.$$

Then, for every $T \in \mathcal{P}_\tau(X,Y)$, we have

$$f(T) = \hat{\varphi}((y_n^*(Tx_n))_n) = \sum_{n=1}^{\infty} \lambda_n y_n^*(Tx_n).$$

□

Theorem 1. *A Banach space X has the α_σ-AP if and only if for every Banach space Y,*

$$\mathcal{P}_\tau(X,Y) = \overline{\mathcal{F}(X,Y)}^{\tau_\sigma}.$$

Proof. Suppose that X has the α_σ-AP. Let Y be a Banach space and let $T \in \mathcal{P}_\sigma(X,Y)$. We use the Hahn–Banach separation theorem to show that $T \in \overline{\mathcal{F}(X,Y)}^{\tau_\sigma}$. Let $f \in (\mathcal{P}_\tau(X,Y),\tau_\sigma)^*$ be such that $f(S) = 0$ for every $S \in \mathcal{F}(X,Y)$. By Proposition 1, there exists $\sum_{n=1}^{\infty} y_n^* \underline{\otimes} x_n \in \mathcal{N}_\sigma(Y,X)$ and $(\lambda_n)_n \in \ell_\infty$ such that

$$f(R) = \sum_{n=1}^{\infty} \lambda_n y_n^*(Rx_n)$$

for every $R \in \mathcal{P}_\tau(X,Y)$. By Lemma 1, we see that

$$\lim_{l\to\infty} \sup \Big\{ \sum_{n\geq l} |y^{**}(\lambda_n y_n^*)x^*(x_n)| : y^{**} \in B_{Y^{**}}, x^* \in B_{X^*} \Big\} = 0.$$

By Lemma 2, $\sum_{n=1}^{\infty} \lambda_n y_n^* \otimes x_n$ unconditionally converges in $Y^* \hat{\otimes}_{\alpha_\sigma} X$.

Now, since for every $x^* \in X^*$ and $y \in Y$,

$$0 = f(x^* \underline{\otimes} y) = \sum_{n=1}^{\infty} \lambda_n y_n^*(y)x^*(x_n) = x^*\Big(\sum_{n=1}^{\infty} \lambda_n y_n^*(y)x_n \Big),$$

$\sum_{n=1}^{\infty} \lambda_n y_n^* \otimes x_n = 0$ in $Y^* \hat{\otimes}_\varepsilon X$. Since X has the α_σ-AP,

$$\sum_{n=1}^{\infty} \lambda_n y_n^* \otimes x_n = 0 \text{ in } Y^* \hat{\otimes}_{\alpha_\sigma} X.$$

By Lemma 4,

$$\mathcal{P}_\tau(X,Y^{**}) = (X \hat{\otimes}_{\alpha_\sigma} Y^*)^* = (Y^* \hat{\otimes}_{\alpha_\sigma} X)^*$$

holds isometrically. Since $i_Y T \in \mathcal{P}_\tau(X,Y^{**})$,

$$0 = \sum_{n=1}^{\infty} (i_Y Tx_n)(\lambda_n y_n^*) = \sum_{n=1}^{\infty} \lambda_n y_n^*(Tx_n) = f(T).$$

Hence, $T \in \overline{\mathcal{F}(X,Y)}^{\tau_\sigma}$.

In order to show the other part, let Y be a Banach space. Then, by assumption,

$$\mathcal{P}_\tau(X,Y^{**}) = \overline{\mathcal{F}(X,Y^{**})}^{\tau_\sigma}.$$

Let $u \in Y^* \hat{\otimes}_{\alpha_\sigma} X$ with $u = 0$ in $Y^* \hat{\otimes}_\varepsilon X$. By Lemma 3, there exists sequences $(y_n^*)_n$ in Y^* and $(x_n)_n$ in X such that

$$u = \sum_{n=1}^{\infty} y_n^* \otimes x_n$$

unconditionally converges in $Y^*\hat{\otimes}_{\alpha_\sigma}X$. To show that $u=0$ in $Y^*\hat{\otimes}_{\alpha_\sigma}X$, let $T\in\mathcal{P}_\tau(X,Y^{**})=(Y^*\hat{\otimes}_{\alpha_\sigma}X)^*$. Let us consider the linear functional $f:=\sum_{n=1}^\infty i_{Y^*}(y_n^*)(\cdot\ x_n)$. Since by Lemma 2,

$$\lim_{l\to\infty}\sup\Big\{\sum_{n\geq l}|y^{**}(y_n^*)x^*(x_n)|:y^{**}\in B_{Y^{**}},x^*\in B_{X^*}\Big\}=0,$$

we see that $\sum_{n=1}^\infty i_{Y^*}(y_n^*)\underline{\otimes}x_n\in\mathcal{N}_\sigma(Y^{**},X)$. Then, by Proposition 1,

$$f\in(\mathcal{P}_\tau(X,Y^{**}),\tau_\sigma)^*.$$

Since for every $S:=\sum_{k=1}^m x_k^*\otimes y_k^{**}\in\mathcal{F}(X,Y^{**})$,

$$f(S)=\sum_{n=1}^\infty i_{Y^*}(y_n^*)(Sx_n)=\sum_{n=1}^\infty\sum_{k=1}^m x_k^*(x_n)y_k^{**}(y_n^*)=\sum_{k=1}^m\sum_{n=1}^\infty x_k^*(x_n)y_k^{**}(y_n^*)=0,$$

$$0=f(T)=\sum_{n=1}^\infty i_{Y^*}(y_n^*)(Tx_n)=\sum_{n=1}^\infty(Tx_n)(y_n^*).$$

Hence, $u=\sum_{n=1}^\infty y_n^*\otimes x_n=0$ in $Y^*\hat{\otimes}_{\alpha_\sigma}X$. □

4. The σ-Nuclear Operator

Let α be a tensor norm. Then, the inclusion map $J_\alpha: X^*\hat{\otimes}_\alpha Y\hookrightarrow X^*\hat{\otimes}_\varepsilon Y$ can be viewed by

$$J_\alpha: X^*\hat{\otimes}_\alpha Y\to\mathcal{L}(X,Y)$$

because $X^*\hat{\otimes}_\varepsilon Y$ isometrically embedded in $\mathcal{L}(X,Y)$ by the natural map. We give the quotient norm of $X^*\hat{\otimes}_\alpha Y/\ker J_\alpha$ on the space $J_\alpha(X^*\hat{\otimes}_\alpha Y)$, which will be denoted by $\|\cdot\|_{J_\alpha}$. Since $\varepsilon^t=\varepsilon$ and $\alpha_\sigma^t=\alpha_\sigma$, if X^* or Y has the α_σ-AP, then J_{α_σ} is injective; hence, $X^*\hat{\otimes}_{\alpha_\sigma}Y$ is isometric to $(J_{\alpha_\sigma}(X^*\hat{\otimes}_{\alpha_\sigma}Y),\|\cdot\|_{J_{\alpha_\sigma}})$.

Lemma 5. *If X^* or Y has the α_σ-AP, then*

$$(J_{\alpha_\sigma}(X^*\hat{\otimes}_{\alpha_\sigma}Y),\|\cdot\|_{J_{\alpha_\sigma}})=(\mathcal{N}_\sigma(X,Y),\|\cdot\|_{\mathcal{N}_\sigma}).$$

Proof. Let $J_{\alpha_\sigma}(u)\in J_{\alpha_\sigma}(X^*\hat{\otimes}_{\alpha_\sigma}Y)$. Let $u=\sum_{n=1}^\infty x_n^*\otimes y_n$ be an arbitrary representation in Lemma 3. Then, by Lemmas 1 and 2,

$$J_{\alpha_\sigma}(u)=\sum_{n=1}^\infty x_n^*\underline{\otimes}y_n\in\mathcal{N}_\sigma(X,Y)$$

and $\|J_{\alpha_\sigma}(u)\|_{\mathcal{N}_\sigma}\leq\sup\{\sum_{n=1}^\infty|x_n^*(x)y^*(y_n)|:x\in B_X,y^*\in B_{Y^*}\}$. Since the representation of u was arbitrary, $\|J_{\alpha_\sigma}(u)\|_{\mathcal{N}_\sigma}\leq\alpha_\sigma(u;X^*,Y)=\|J_{\alpha_\sigma}(u)\|_{J_{\alpha_\sigma}}$.

Let $T\in\mathcal{N}_\sigma(X,Y)$. Let $T=\sum_{n=1}^\infty x_n^*\underline{\otimes}y_n$ be an arbitrary σ-nuclear representation. Then, by Lemmas 1 and 2, $\sum_{n=1}^\infty x_n^*\otimes y_n$ unconditionally converges in $X^*\hat{\otimes}_{\alpha_\sigma}Y$. Thus,

$$T=J_{\alpha_\sigma}\Big(\sum_{n=1}^\infty x_n^*\otimes y_n\Big)\in J_{\alpha_\sigma}(X^*\hat{\otimes}_{\alpha_\sigma}Y)$$

and

$$\begin{aligned}\|T\|_{J_{\alpha_\sigma}}&=\alpha_\sigma\Big(\sum_{n=1}^\infty x_n^*\otimes y_n;X^*,Y\Big)\\&\leq\sup\Big\{\sum_{n=1}^\infty|x^{**}(x_n^*)y^*(y_n)|:x^{**}\in B_{X^{**}},y^*\in B_{Y^*}\Big\}.\end{aligned}$$

Since the representation of T was arbitrary, $\|T\|_{J_{\alpha_\sigma}}\leq\|T\|_{\mathcal{N}_\sigma}$. □

For $T = \sum_{n=1}^{m} x_n^* \underline{\otimes} y_n \in \mathcal{F}(X,Y)$, let $u_T := \sum_{n=1}^{m} x_n^* \otimes y_n \in X^* \otimes Y$.

Proposition 2. *The following statements are equivalent:*

(a) *X has the α_σ-AP.*
(b) *For every Banach space Z, the map $J_{\alpha_\sigma} : Z^* \hat{\otimes}_{\alpha_\sigma} X \to \mathcal{N}_\sigma(Z,X)$ is an isometric isomorphism.*
(c) *For every Banach space Z, $\alpha_\sigma(u_T; Z^*, X) = \|T\|_{\mathcal{N}_\sigma}$ for every $T \in \mathcal{F}(Z,X)$.*
(d) *For every Banach space Z, there exists a $\lambda \geq 1$ such that $\alpha_\sigma(u_T; Z^*, X) \leq \lambda \|T\|_{\mathcal{N}_\sigma}$ for every $T \in \mathcal{F}(Z,X)$.*

Proof. (a)⇒(b) follows from Lemma 5. (b)⇒(c) and (c)⇒(d) are clear.

(d)⇒(a): Let Z be a Banach space and let $u = \sum_{n=1}^{\infty} z_n^* \otimes x_n \in Z^* \hat{\otimes}_{\alpha_\sigma} X$ with $u = 0$ in $Z^* \hat{\otimes}_\varepsilon X$. Then, $\sum_{n=1}^{\infty} z_n^* \underline{\otimes} x_n = 0$ in $\mathcal{N}_\sigma(Z,X)$. By (d),

$$\alpha_\sigma(u; Z^*, X) = \lim_{l\to\infty} \alpha_\sigma\Big(\sum_{n=1}^{l} z_n^* \otimes x_n; Z^*, X\Big) \leq \lambda \lim_{l\to\infty} \Big\| \sum_{n=1}^{l} z_n^* \underline{\otimes} x_n \Big\|_{\mathcal{N}_\sigma} = 0.$$

Hence, X has the α_σ-AP. □

Lemma 6 ([9], Theorem 2.4)**.** *Let α be a finitely generated tensor norm. Assume that X^{***} or Y has the AP.*

If $T \in J_\alpha(X^{**} \hat{\otimes}_\alpha Y) \subset \mathcal{L}(X^*, Y)$ and $T^*(Y^*) \subset X$, then $T \in \overline{J_\alpha(X \otimes Y)}^{\|\cdot\|_{J_\alpha}}$.

Theorem 2. *Assume that X^{***} or Y has the AP. If $T \in \mathcal{N}_\sigma(X^*, Y)$ and $T^*(Y^*) \subset X$, then for every $\delta > 0$, there exists sequences $(x_n)_n$ in X and $(y_n)_n$ in Y such that*

$$T = \sum_{n=1}^{\infty} x_n \underline{\otimes} y_n$$

unconditionally converges in $\mathcal{N}_\sigma(X^*, Y)$ and $|\sum_{n=1}^{\infty} x_n \underline{\otimes} y_n|_\sigma \leq (1+\delta)\|T\|_{\mathcal{N}_\sigma}$.

Proof. Let $T \in \mathcal{N}_\sigma(X^*, Y)$ with $T^*(Y^*) \subset X$. Let $\delta > 0$ be given. By Lemma 5, there exists an $u \in X^{**} \hat{\otimes}_{\alpha_\sigma} Y$ so that $T = J_{\alpha_\sigma}(u)$. By Lemma 6, $J_{\alpha_\sigma}(u) \in \overline{J_{\alpha_\sigma}(X \otimes Y)}^{\|\cdot\|_{J_{\alpha_\sigma}}}$. Since J_{α_σ} is an isometry and $X \hat{\otimes}_{\alpha_\sigma} Y$ is isometrically embedded in $X^{**} \hat{\otimes}_{\alpha_\sigma} Y$ (cf. [3], Proposition 6.4), we see that $u \in X \hat{\otimes}_{\alpha_\sigma} Y$. By Lemma 3, there exists sequences $(x_n)_n$ in X and $(y_n)_n$ in Y such that $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ unconditionally converges in $X \hat{\otimes}_{\alpha_\sigma} Y$ and

$$\sup\Big\{ \sum_{n=1}^{\infty} |x^*(x_n) y^*(y_n)| : x^* \in B_{X^*}, y^* \in B_{Y^*} \Big\} \leq (1+\delta)\alpha_\sigma(u; X, Y).$$

Hence,

$$T = J_{\alpha_\sigma}(u) = \sum_{n=1}^{\infty} x_n \underline{\otimes} y_n$$

unconditionally converges in $\mathcal{N}_\sigma(X^*, Y)$ and

$$\Big| \sum_{n=1}^{\infty} x_n \underline{\otimes} y_n \Big|_\sigma \leq (1+\delta)\alpha_\sigma(u; X, Y) = (1+\delta)\|T\|_{\mathcal{N}_\sigma}.$$

□

If $T \in \mathcal{N}_\sigma(X,Y)$, then clearly $T^* \in \mathcal{N}_\sigma(Y^*, X^*)$ and $\|T^*\|_{\mathcal{N}_\sigma} \leq \|T\|_{\mathcal{N}_\sigma}$ (cf. [5], Theorem 23.2.7). It is not known whether the converse statement is true.

Corollary 1. *Assume that X^* or Y^{***} has the AP. If $T^* \in \mathcal{N}_\sigma(Y^*, X^*)$, then $T \in \mathcal{N}_\sigma(X,Y)$ and $\|T\|_{\mathcal{N}_\sigma} \leq \|T^*\|_{\mathcal{N}_\sigma}$.*

Proof. Let $\delta > 0$ be given. Since T is a (weakly) compact operator, $T^{**}(X^{**}) \subset Y$. Thus, by Theorem 2, there exists sequences $(y_n)_n$ in Y and $(x_n^*)_n$ in X^* such that

$$T^* = \sum_{n=1}^{\infty} y_n \underline{\otimes} x_n^*$$

unconditionally converges in $\mathcal{N}_\sigma(Y^*, X^*)$ and $|\sum_{n=1}^{\infty} y_n \underline{\otimes} x_n^*|_\sigma \leq (1+\delta)\|T^*\|_{\mathcal{N}_\sigma}$. Since for every $x \in X$ and $y^* \in Y^*$,

$$y^*(Tx) = (T^*y^*)(x) = \sum_{n=1}^{\infty} y^*(y_n)x_n^*(x) = y^*\Big(\sum_{n=1}^{\infty} x_n^*(x)y_n\Big),$$

$$T = \sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n.$$

Since $\sum_{n=1}^{\infty} x_n^* \underline{\otimes} y_n$ also unconditionally converges in $\mathcal{L}(X,Y)$, $T \in \mathcal{N}_\sigma(X,Y)$ and $\|T\|_{\mathcal{N}_\sigma} \leq (1+\delta)\|T^*\|_{\mathcal{N}_\sigma}$. □

Corollary 2. *Assume that X^* or Y^{***} has the AP. If $i_Y T \in \mathcal{N}_\sigma(X, Y^{**})$, then $T \in \mathcal{N}_\sigma(X,Y)$ and $\|T\|_{\mathcal{N}_\sigma} \leq \|i_Y T\|_{\mathcal{N}_\sigma}$.*

Proof. Since $(i_Y T)^* \in \mathcal{N}_\sigma(Y^{***}, X^*)$,

$$T^* = T^* i_Y^* i_{Y^*} = (i_Y T)^* i_{Y^*} \in \mathcal{N}_\sigma(Y^*, X^*).$$

Hence, by Corollary 1, $T \in \mathcal{N}_\sigma(X,Y)$ and $\|T\|_{\mathcal{N}_\sigma} \leq \|(i_Y T)^* i_{Y^*}\|_{\mathcal{N}_\sigma} \leq \|i_Y T\|_{\mathcal{N}_\sigma}$. □

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