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Multi-Soliton, Soliton–Cnoidal, and Lump Wave Solutions for the Supersymmetric Boussinesq Equation

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Abstract: Based on the bosonization approach, the supersymmetric Boussinesq equation is converted into a coupled bosonic system. The symmetry group and the commutation relations of the corresponding bosonic system are determined through the Lie point symmetry theory. The group invariant solutions of the coupled bosonic system are analyzed by the symmetry reduction technique. Special traveling wave solutions are generated by using the mapping and deformation method. Some novel solutions, such as multi-soliton, soliton–cnoidal interaction solutions, and lump waves, are given by utilizing the Hirota bilinear and the consistent tanh expansion methods. The methods in this paper can be effectively expanded to study rich localized waves for other supersymmetric systems.

Keywords: supersymmetric Boussinesq equation; bosonization approach; Lie point symmetry theory; mapping and deformation method; consistent tanh expansion method

MSC: 35Q51; 35Q53; 37K40

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Citation: Wei, P.-F.; Zhang, H.-B..; Liu, Y.; Lin, S.-Y.; Chen, R.-Y.; Xu, Z.-Y.; Wang, W.-L.; Ren, B. Multi-Soliton, Soliton–Cnoidal, and Lump Wave Solutions for the Supersymmetric Boussinesq Equation. *Mathematics* **2024**, *12*, 2002. https://doi.org/10.3390/math12132002

Academic Editor: Baccouch Mahboub

Received: 18 May 2024
Revised: 20 June 2024
Accepted: 26 June 2024
Published: 28 June 2024

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1. Introduction

The $(1+1)$-dimensional Boussinesq equation, which describes long water waves, has been widely studied in the fields of atmospheric science and oceanography. The integrability and analytic solutions have been systematically explored by many methodologies, such as the inverse scattering method, the Hirota bilinear method [1], symmetry reduction [2], and so on [3]. Based on the $(1+1)$-dimensional Boussinesq equation, a novel $(2+1)$-dimensional nonlinear Boussinesq equation is derived using a deformation algorithm [4]. Multi-soliton solutions of the discrete Boussinesq system are constructed by using the Bäcklund transformation [5]. Supersymmetric integrable systems are currently a promising area of research in nonlinear science [6–9]. The supersymmetric Boussinesq equation can be formally obtained by introducing the noncommutative field into the Boussinesq equation, which is the topic of the current research.

Solitary waves, cnoidal periodic waves, and lump waves have attracted much attention as the typical waves in many physical fields [10,11]. The lump wave, which rationally localizes in all directions in space, plays an important role in nonlinear systems. These nonlinear localized waves are often found in classical integrable systems. The interaction among solitons and other complicated waves, especially the soliton–cnoidal periodic wave and the soliton–lump wave, has been found by using symmetry reduction along with nonlocal symmetry [12,13], the long-wave limit method [14], and the Kadomtsev–Petviashvili reduction method [15]. The dynamics of the lump solution and other complicated waves of the ill-posed Boussinesq equation have been studied by the Hirota bilinear method [16]. The study of these types of localized waves involves a lack of literature on supersymmetric integrable systems due to the presence of both commutative and noncommutative fields in

the systems [17,18]. In this paper, multi-soliton, cnoidal periodic waves, lump waves, and their interactions in the supersymmetric Boussinesq system are systematically studied.

By introducing a Grassmannian variable θ, the supersymmetric Boussinesq equation in superspace (x, θ) reads as follows [19]:

$$v_{tt} + \alpha v_{xxxx} + 6\alpha v_x^2 + 6\alpha v v_{xx} = 0, \tag{1a}$$

$$\xi_{tt} + \alpha \xi_{xxxx} + 6\alpha v \xi_{xx} + 6\alpha v \xi_x = 0, \tag{1b}$$

where ξ is the fermionic super field. The Lax pair and Painlevé property are analyzed by using a simplified ansatz [19].

To tackle the challenges associated with handling the anticommutative fermionic field ξ in (1), one introduces two Grassmann parameters ζ_1 and ζ_2 [20–22]. The component fields ξ and v are expanded as

$$\xi(x,t) = p\zeta_1 + q\zeta_2, \tag{2a}$$

$$v(x,t) = u + w\zeta_1\zeta_2, \tag{2b}$$

where the coefficients u, p, q, and w are four real or complex functions with respect to the x and t variables. Substituting (2) into (1) yields the pure bosonic Boussinesq (BB) system

$$u_{tt} + \alpha u_{xxxx} + 6\alpha u u_{xx} + 6\alpha u_x^2 = 0, \tag{3a}$$

$$r_{k,tt} + \alpha r_{k,xxxx} + 6\alpha u r_{k,xx} + 6\alpha u_x r_{k,x} = 0, \tag{3b}$$

$$w_{tt} + \alpha w_{xxxx} + 6\alpha u w_{xx} + 6\alpha w u_{xx} + 12\alpha u_x w_x = 0, \tag{3c}$$

where r_k represents p and q for simplicity. The BB Equation (3) includes the usual Boussinesq equation and linear partial differential equations. In principle, these equations can be readily solved due to the purely bosonic system (3). The above procedure is called the bosonization approach, which can be effectively used on the supersymmetric integrable systems [23,24].

The paper is organized as follows. In Section 2, the Lie point symmetry for the BB equation is investigated. The group invariant solutions and commutation relations are given by the Lie point theory. In Section 3, the traveling wave solutions of the BB equation are constructed using the mapping and deformation method. In Section 4, some novel solutions based on prior solutions and symmetry of the Boussinesq equation are obtained by combining the Hirota bilinear and consistent tanh expansion (CTE) approaches. Results and a discussion are presented in Section 5. The last section is a simple summary.

2. Similarity Reductions of the BB Equation

The fields of u and r_k satisfy the same forms according to (3). One takes $r_k = u$ for simplifying the following calculations. According to the symmetry reduction theory [25], a Lie point symmetry in the space $\{x, t, u, w\}$ with the vector form reads

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + W\frac{\partial}{\partial w}, \tag{4}$$

where X, T, U, and W are functions of x, t, u, and w.

The Lie point symmetry σ^m $(m = u, w)$ is given by using the linearized system (3) [18]

$$\sigma^u_{tt} + \alpha\sigma^u_{xxxx} + 12\alpha u_x\sigma^u_x + 6\alpha u\sigma^u_{xx} + 6\alpha u_{xx}\sigma^u = 0, \tag{5a}$$

$$\sigma^w_t + \alpha\sigma^w_{xxxx} + 6\alpha u\sigma^w_{xx} + 6\alpha w_{xx}\sigma^u + 6\alpha w\sigma^u_{xx} + 6\alpha u_{xx}\sigma^w + 12\alpha w_x\sigma^u_x + 12\alpha u_x\sigma^w_x = 0. \tag{5b}$$

The symmetry component σ^m has the forms

$$\sigma^u = Xu_x + Tu_t - U, \qquad \sigma^w = Xw_x + Tw_t - W. \tag{6}$$

By substituting (6) into the symmetry Equation (5) and setting u and w to satisfy (3), the over-determined equations consisting of the coefficients of the independent terms of polynomials u, w, and their partial derivatives are obtained. The over-determined equations can produce the infinitesimals X, T, U, and W. By solving the over-determined equations, the infinitesimals are given as

$$X = \frac{C_1}{2}x + C_3, \qquad T = C_1 t + C_2, \qquad U = -C_1 u, \qquad W = C_4 w, \tag{7}$$

where C_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

The Lie algebra of symmetries of (3) is generated by the following four vectors

$$V = C_1 X_1 + C_2 X_2 + C_3 X_3 + C_4 X_4, \tag{8}$$

with

$$X_1 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = w\frac{\partial}{\partial w}.$$

The commutation relations for X_1, X_2, X_3, and X_4 are given in the following table.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	$-\frac{1}{2}X_3$	0
X_2	X_2	0	0	0
X_3	$\frac{1}{2}X_3$	0	0	0
X_4	0	0	0	0

The group invariant solutions can be derived though the symmetry constraint condition $\sigma^m = 0$ defined by (6). The corresponding solutions can also be given by solving the characteristic equations [26,27]

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{dw}{W}. \tag{9}$$

Solving the characteristic equations results in two nontrivial cases as follows.

Case I . With all arbitrary constants being nonzero, the similarity solution takes the following form by solving the characteristic Equation (9):

$$u = (C_1 t + C_2)^{-1} U(\xi), \tag{10a}$$

$$w = (C_1 t + C_2)^{\left(\frac{C_4}{C_1}\right)} W(\xi), \tag{10b}$$

with the similarity variable $\xi = \frac{C_1 x + 2C_3}{C_1\sqrt{C_1 t + C_2}}$ and the group invariant functions $U = U(\xi)$ and $W = W(\xi)$. Substituting (10) into (3), the invariant functions U and W satisfy the reduction systems

$$U_{\xi\xi\xi\xi\xi} + 6UU_{\xi\xi} + 6U_\xi^2 + \frac{2C_1^2}{\alpha}U + \frac{7C_1^2}{4\alpha}\xi U_\xi + \frac{C_1^2}{4\alpha}\xi^2 U_{\xi\xi} = 0, \tag{11a}$$

$$W_{\xi\xi\xi\xi\xi} + 6WU_{\xi\xi} + 12U_\xi W_\xi + \frac{C_4(C_4 - C_1)}{\alpha}W + \frac{C_1(3C_1 - 4C_4)}{4\alpha}\xi W_\xi + (\frac{C_1^2\xi^2}{4\alpha} + 6U)W_{\xi\xi} = 0. \tag{11b}$$

Once the solutions U and W are solved by (11), u and w can be given by (10).

Case II. The arbitrary constant C_1 cannot equal zero based on the similarity variable in Case I. We should discuss this scenario of $C_1 = 0$ separately. With $C_1 = 0$, the similarity solution leads to

$$u = U(X), \tag{12a}$$

$$w = \exp\left(\frac{C_4 x}{C_3}\right) W(X), \tag{12b}$$

with the invariant variables $X = t - \frac{C_2}{C_3}x$ and the group invariant functions $U = U(X)$ and $W = W(X)$. Substituting (12) into (3), the invariant functions U and W satisfy the reduction systems

$$U_{XXX} + \frac{6C_2^2}{C_3^2} U U_X + \frac{C_2^4}{C_3^4 \alpha} U_X = 0, \tag{13a}$$

$$W_{XXXX} - \frac{4C_4}{C_2} W_{XXX} + \frac{6C_3^2}{C_2^2} W U_{XX} + \left(\frac{6C_3^2}{C_2^2} U + \frac{6C_4^2}{C_2^2} + \frac{C_3^4}{C_2^4 \alpha}\right) W_{XX} -$$
$$\frac{4C_4}{C_2^3}(C_4^2 + 3C_3^2 U) W_X + \frac{12C_3^2}{C_2^3}(C_2 W_\xi - C_4 W) U_X + \frac{6C_4^2 C_3^2}{C_2^4} U W + \frac{C_4^4}{C_2^4} W = 0. \tag{13b}$$

While the invariant functions U and W satisfy the reduction system (13), u and w can be solved by (12). The function U satisfies the following elliptic equation by solving (13a):

$$U_X^2 = -\frac{2C_3^2}{C_2^2} U^3 - \frac{C_3^4}{\alpha C_2^4} U^2 + A_0, \tag{14}$$

which is just the Jacobi elliptic functional differential equation with the arbitrary constant A_0.

3. Traveling Wave Solutions with the Mapping and Deformation Method

Introducing the traveling wave variable $X = kx + \omega t + c_0$ with constants k, ω, and c_0, (3) is transformed into the ordinary differential equations (ODEs) as follows:

$$u_{XXXX} + \tfrac{\omega^2}{\alpha k^4} u_{XX} + \tfrac{6}{k^2} u u_{XX} + \tfrac{6}{k^2} u_X^2 = 0, \tag{15a}$$

$$r_{XXXX} + \tfrac{\omega^2}{\alpha k^4} r_{XX} + \tfrac{6}{k^2} u r_{XX} + \tfrac{6}{k^2} u_X r_X = 0, \tag{15b}$$

$$w_{XXXX} + \tfrac{\omega^2}{\alpha k^4} w_{XX} + \tfrac{6}{k^2} u w_{XX} + \tfrac{12}{k^2} u_X w_X + \tfrac{6}{k^2} w u_{XX} = 0. \tag{15c}$$

The expression of u_X is given by solving (15a):

$$u_X = a_0 \sqrt{-\frac{2}{k^2} u^3 - \frac{\omega^2}{\alpha k^4} u^2 + A_1 u + A_2}, \tag{16}$$

with $a_0^2 = 1$ and the arbitrary constants A_1 and A_2. The exact solutions of (15a) are well-known, and one attempts to construct the mapping and deformation relationship between (15a) and (15b), (15c).

In order to construct the mapping and deformation relationship, we introduce the variable transformation

$$r(X) = R(u(X)), \qquad w(X) = W(u(X)). \tag{17}$$

By using the transformation (17) and vanishing u_X with (16), the linear ODE (15c) becomes

$$\left(A_1 \alpha k^4 u + A_2 \alpha k^4 - 2\alpha k^2 u^3 - \omega^2 u^2\right) \tfrac{d^2 W}{du^2} + \left(\tfrac{A_1 \alpha k^4}{2} - 3\alpha k^2 u^2 - \omega^2 u\right) \tfrac{dW}{du} \tag{18}$$
$$+ 6\alpha k^2 u W + \omega^2 W = 0.$$

The specific form for $R(u(X))$ is ignored due to the complicated nature of the equation. The mapping and deformation relation between u and r, w is given by solving the above Equation (18):

$$r = R = \int^u \int^y \frac{A_3}{(A_1\alpha k^4 y_1 + A_2\alpha k^4 - 2\alpha k^2 y_1^3 - \omega^2 y_1^2)^{\frac{3}{2}}} dy_1 dy + A_4 u + A_5, \tag{19}$$

$$w = W = \sqrt{A_1\alpha k^4 u + A_2\alpha k^4 - 2\alpha k^2 u^3 - \omega^2 u^2}\left(A_7 + \int^u \frac{A_6}{(A_1\alpha k^4 y + A_2\alpha k^4 - 2\alpha k^2 y^3 - \omega^2 y^2)^{\frac{3}{2}}} dy\right), \tag{20}$$

with arbitrary constants $A_3, A_4, \cdots, A_7$. Once the solution for u is obtained, the solution for r and w can be derived by using (19). The solutions to the supersymmetric Boussinesq equation are not restricted to the traveling wave solutions. We shall generate some novel solutions to the supersymmetric Boussinesq equation through the solution and symmetry of the usual Boussinesq equation.

4. Novel Solutions Based on Prior Solutions and Symmetry of the Boussinesq Equation

Some novel solutions of the BB Equation (3) can be given by integrating the prior solution and symmetry of the usual Boussinesq equation

$$p = C_5 u, \qquad q = C_6 u, \qquad w = W = C_7 \sigma(u), \tag{21}$$

where $\sigma(u)$ is the symmetry of the Boussinesq equation, and C_5, C_6, and C_7 are arbitrary constants. Based on the Lie point theory, the Lie point symmetry of the Boussinesq equation reads as follows:

$$\sigma(u) = C_1\left(\frac{x}{2}\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} + u\right) + C_2\frac{\partial u}{\partial t} + C_3\frac{\partial u}{\partial x}. \tag{22}$$

One can construct not only traveling wave solutions but also some novel types of solutions of the bosonic Boussinesq system by using (21) and (22). We enumerate three cases to illustrate this point.

Case 1. Multi-soliton solutions of the Boussinesq equation can be obtained by using the Hirota bilinear method. By applying the following relation between u and f

$$u = 2(\ln f)_{xx} + C_0, \tag{23}$$

the field f satisfies the bilinear form

$$(D_t^2 + \alpha D_x^4 + 6C_0\alpha D_x^2) f \cdot f = 0, \tag{24}$$

where D_x and D_t are the bilinear derivative operators defined by [28]

$$D_x^l D_t^m [a(x,t) \cdot b(x,t)] = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x,t) \cdot b(x',t')|_{x'=x,t'=t}. \tag{25}$$

The functions $a(x,t)$ and $b(x,t)$ are differentiable functions, x' and t' are the independent variables, and l and m are nonnegative integers. Based on the Hirota bilinear method, the field $f(x,y,t)$ is expanded as

$$f(x,y,t) = 1 + f^{(1)}\epsilon^1 + f^{(2)}\epsilon^2 + \cdots + f^{(j)}\epsilon^j, \tag{26}$$

with the parameter ϵ. Selecting the form $f^{(1)}$ as $f^{(1)} = \exp(\eta_1)$, $\eta_1 = k_1 x + \omega_1 t + \phi_1$, and comparing the coefficients of ϵ with the same power, we get the dispersion relation

$$\omega_1^2 = -\alpha k_1^2 (k_1^2 + 6C_0). \tag{27}$$

The term of $\exp(A_{12})$ reads as

$$\exp(A_{12}) = \frac{2k_1^2 - 3k_1k_2 + 2k_2^2 + 6C_0 + \sqrt{-(k_1^2+6C_0)}\sqrt{-(k_2^2+6C_0)}}{2k_1^2 + 3k_1k_2 + 2k_2^2 + 6C_0 + \sqrt{-(k_1^2+6C_0)}\sqrt{-(k_2^2+6C_0)}}, \tag{28}$$

when selecting $f^{(1)} = \exp(\eta_1) + \exp(\eta_2), f^{(2)} = \exp(\eta_1 + \eta_2 + A_{12})$ and comparing the coefficients of ϵ.

The multi-soliton of f can be concluded as [29]

$$f = f_N = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i\eta_i + \sum_{1\le i\le j} \mu_i\mu_j A_{ij}\Big), \tag{29}$$

with $\mu_i, \mu_j = 0, 1$ and

$$\eta_i = k_i x + \omega_i t + \phi_i, \qquad \omega_i^2 = -\alpha k_i^2(k_i^2 + 6C_0), \tag{30}$$
$$\exp(A_{ij}) = \frac{2k_i^2 - 3k_ik_j + 2k_j^2 + 6C_0 + \sqrt{-(k_i^2+6C_0)}\sqrt{-(k_j^2+6C_0)}}{2k_i^2 + 3k_ik_j + 2k_j^2 + 6C_0 + \sqrt{-(k_i^2+6C_0)}\sqrt{-(k_j^2+6C_0)}},$$

with the arbitrary constants k_i, ω_i, and ϕ_i. The multi-soliton solution for w can be constructed in the following form by the above analysis:

$$w = (C_1x + 2C_3)(\ln f)_{xxx} + 2(C_1t + C_2)\ln f_{xxt} + 2C_1(\ln f)_{xx} + C_0C_1, \tag{31}$$

in which f satisfies (29).

Case 2. According to the CTE method [30], the expansion solution has the form

$$u = u_0 + u_1\tanh(f) + u_2\tanh(f)^2, \tag{32}$$

where u_0, u_1, u_2, and f are functions of (x, t) and can be determined by eliminating the powers of the tanh functions. The expansion solution (32) becomes

$$u = \frac{4}{3}f_x^2 + \frac{f_{xx}^2}{2f_x^2} - \frac{2f_{xxx}}{3f_x} - \frac{f_t^2}{6\alpha f_x^2} + 2f_{xx}\tanh(f) - 2f_x^2\tanh(f)^2, \tag{33}$$

and f satisfies

$$C_t + \alpha S_x + CC_x - 4\alpha f_x f_{xx} = 0, \tag{34}$$

with $C = \frac{f_t}{f_x}$ and Schwarzian derivative $S = \{f; x\} = \frac{f_{xxx}}{f_x} - \frac{3f_{xx}^2}{2f_x^2}$.

If we assume the trivial solution $f = k_1x + \omega_1t$, the single soliton u reads as follows:

$$u = -2k_1^2\tanh(k_1x + \omega_1t)^2 + \frac{4k_1^2}{3} - \frac{\omega_1^2}{6\alpha k_1^2}. \tag{35}$$

To obtain the interaction between solitons and cnoidal periodic waves, we assume that the solution is a linear solution $k_1x + \omega_1t$ and an undetermined traveling wave $G(X)$

$$f = k_1x + \omega_1t + G(X), \qquad X = kx + \omega t. \tag{36}$$

Substituting (36) into (34) transforms

$$G_{1,X}^2 = 4G_1^4 + 2\Big(\frac{4k_1}{k} - C_8k^2\Big)G_1^3 + 2\Big(\frac{2k_1^2}{k^2} + C_9k - 3C_8kk_1\Big)G_1^2 + \tag{37}$$
$$2\Big(\frac{\omega\omega_1}{\alpha k^4} - \frac{k_1\omega^2}{\alpha k^5} + 2C_9k_1 - 3C_8k_1^2\Big)G_1 - \frac{5\omega^2k_1^2}{3\alpha k^6} + \frac{4k_1\omega\omega_1}{3\alpha k^5} + \frac{\omega_1^2}{3\alpha k^4} - \frac{2C_8k_1^3}{k} + \frac{2C_9k_1^2}{k}, \quad G_1 = G_X,$$

where C_8 and C_9 are arbitrary constants. The solution to (37) can be solved in terms of Jacobi elliptic functions. The solution expressed by (36) is thus the explicit interaction between one soliton and cnoidal periodic waves. To describe this type of solution, we assume the solution to (37) is

$$G_1 = a_1 + a_2 \text{JacobiSN}(a_3 X, m), \tag{38}$$

where a_j $(j = 1, 2, 3)$ are arbitrary constants. Substituting (38) into (37) and vanishing the different powers of Jacobi elliptic functions leads to the nontrivial constants

$$\begin{gathered} C_8 = \frac{5C_9^2k^3}{8} - 4C_9a_1k - \frac{a_3^2m^2}{2k} + \frac{4a_1^2}{k} - \frac{a_3^2}{2k}, \quad a_2 = \frac{a_3m}{2}, \quad k_1 = \frac{C_9k^3}{4} - 2a_1k, \\ \alpha = \frac{3[(C_9k^2-4a_1)^2-4a_1^2][(C_9k^2-4a_1)^2-4a_3^2]}{k^4(C_2k^2-4a_1)^2(C_9^2k^4-8C_9a_1k^2-2a_3^2m^2+16a_1^2-2a_3^2)^2}, \\ \omega_1 = \frac{\omega[48a_3^4m^2-8(m^2+1)(C_9k^2-2a_1)(C_9k^2-4a_1)a_3^2+(C_9k^2+4a_1)(C_9k^2-4a_1)^3]}{8(4a_1-k^2C_9)(C_9^2k^4-8C_2a_1k^2-2a_3^2m^2+16a_1^2-2a_3^2)}. \end{gathered} \tag{39}$$

The solution f of (34) reads as

$$f = k_1x + \omega_1t + a_1X + \frac{a_2}{a_3m}\ln(\text{JacobiDN}(a_3X, m) - m\text{JacobiCN}(a_3X, m)). \tag{40}$$

The interaction between one soliton and cnoidal periodic waves can be derived by substituting (40) into (31) and (33). This type of solution is shown by detailed parameters as

$$m = \frac{7}{10}, \ C_9 = 1, \ a_1 = 1, \ a_3 = 1, \ k = 1, \ \omega = \frac{1}{2}, \ C_1 = 3, \ C_2 = \frac{1}{4}, \ C_3 = \frac{1}{2}. \tag{41}$$

The three-dimensional and density forms for u and w are plotted in Figures 1 and 2, respectively.

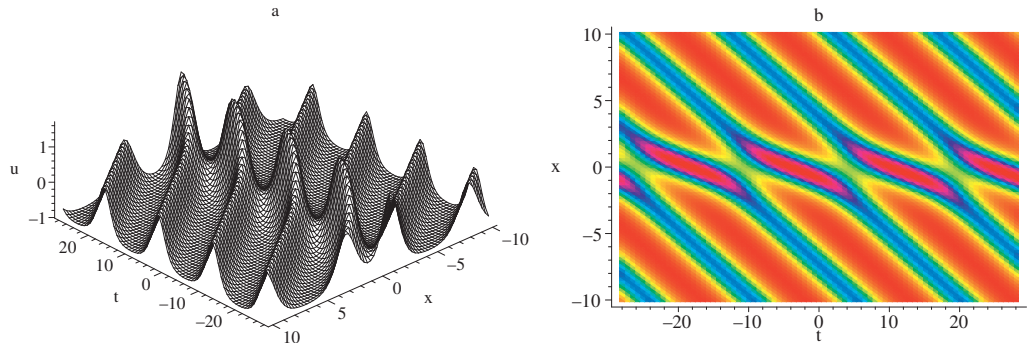

Figure 1. The interaction between one soliton and cnoidal periodic waves of u is plotted with parameters as (41). (**a**) The three-dimensional form. (**b**) The corresponding density form.

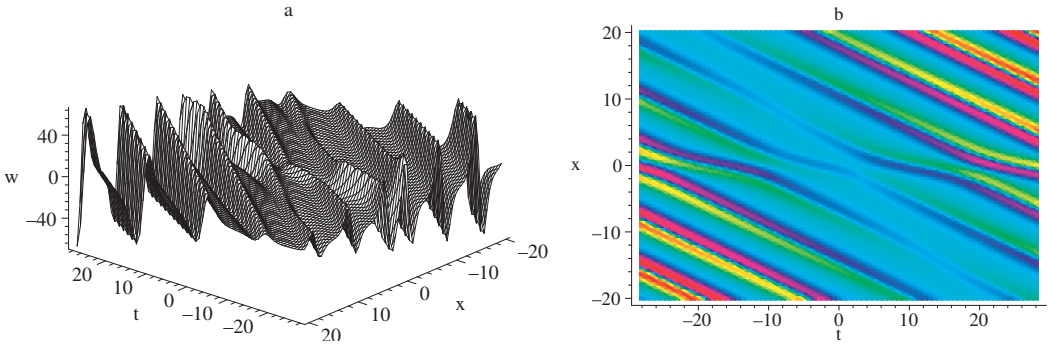

Figure 2. The interaction between one soliton and cnoidal quasi-periodic waves of w is plotted with parameters as (41). (**a**) The three-dimensional form. (**b**) The corresponding density form.

Case 3. Compared with the soliton solution, the lump wave is a type of rational function solution localized in all directions of space that has been widely studied by the Hirota bilinear and the long wave limit methods [31–34]. A lump wave can be constructed by using f in the following form:

$$f = (a_1x + a_2t + a_3)^2 + (a_4x + a_5t + a_6)^2 + a_7, \tag{42}$$

where $a_i, i = 1, 2 \cdots 7$ are arbitrary constants. Substituting (42) into (24) and vanishing the coefficients of the powers of t and x, the relations for the constants are

$$a_1 = -\frac{a_5}{\sqrt{6\alpha C_0}}, \quad a_2 = \sqrt{6\alpha C_0}a_4, \quad a_7 = -\frac{a_4^2}{2C_0} - \frac{a_5^2}{12\alpha C_0^2}. \tag{43}$$

To guarantee the analytic nature of the lump wave, the parameters should satisfy the following constraint conditions:

$$\alpha C_0 > 0, \qquad C_0(a_5^2 + 6\alpha C_0 a_4^2) < 0. \tag{44}$$

The lump wave is given by substituting (42) into (31) and (33). To realize the properties of the lump wave, we select the parameters as

$$C_0 = -\frac{1}{2},\ \alpha = -\frac{1}{4},\ a_3 = \frac{1}{2},\ a_4 = \frac{1}{4},\ a_5 = \frac{1}{3},\ a_6 = 1,\ C_1 = 3,\ C_2 = \frac{1}{4},\ C_3 = \frac{1}{2}. \tag{45}$$

The lump wave in three-dimensional and density forms for u and w is shown in Figures 3 and 4, respectively.

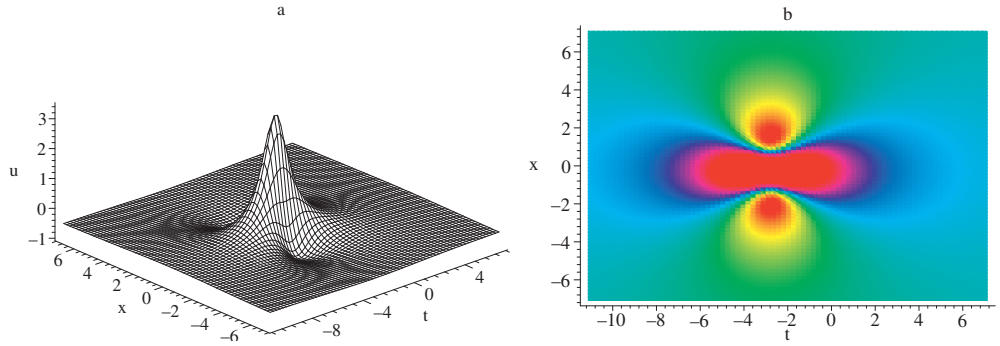

Figure 3. The lump wave of u is displayed with parameters as in (45). (**a**) The three-dimensional form. (**b**) The corresponding density form.

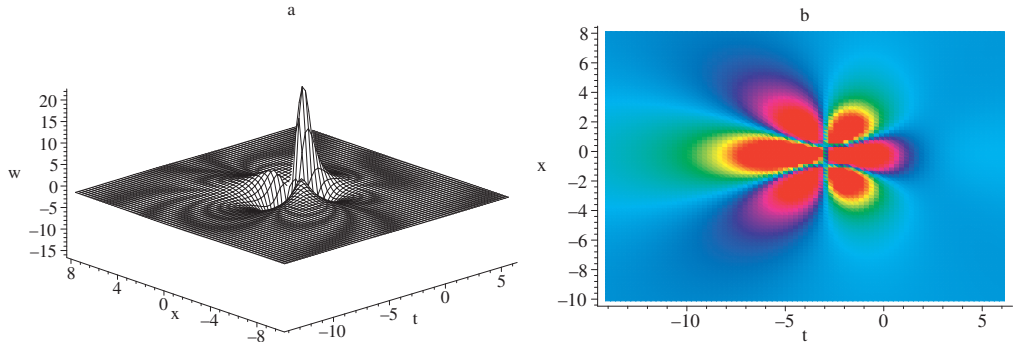

Figure 4. Evolution graph of the special lump wave of w is given by parameters as in (41). (**a**) The three-dimensional form. (**b**) The corresponding density form.

5. Results and Discussion

In this section, we establish a connection between our refinement work and previous results. Based on the mapping and deformation relationship method, the relation between

the usual Boussinesq equation and an unknown equation is constructed. Some novel traveling wave solutions are derived by using the mapping and deformation relationship. Additionally, the novel solutions to the BB equation are given through the prior solution and symmetry of the usual Boussinesq equation. By applying the Lie point symmetry, the explicit solutions are given as (21) and (22). By using the multi-soliton (29), the field w of the BB equation is derived as in (31). In the meantime, the interaction between soliton and cnoidal periodic waves is derived by combining a linear solution and a traveling wave. By selecting certain parameters, the interaction between soliton and cnoidal periodic waves of the BB equation are plotted in Figures 1 and 2. The lump wave is given by assuming a quadratic function. The lump wave for the $(1+1)$-dimensional nonlinear system has seldom been studied. By selecting the detail parameters, the lump wave and a special lump wave are plotted in Figures 3 and 4, respectively. Figures 3 and 4 demonstrate that the lump wave rationally localizes in space. The obtained results are novel and have potential applications in supersymmetric systems.

6. Conclusions

Based on the bosonization approach, the Lie point symmetry theory, the mapping and deformation method, the Hirota bilinear method, and the CTE method, some integrability properties and exact solutions for the supersymmetric Boussinesq system are systematically studied. The supersymmetric Boussinesq system is converted into a coupled bosonic system by introducing two additional fermionic parameters. The Lie point symmetry group of the BB equation is determined by using the classical Lie group method. The commutation relations of the infinitesimal generators are consolidated in a table. The group invariant solutions are derived by using symmetry reductions. A specific class of the traveling wave solution of the BB system is generated through the application of the the mapping and deformation method. Some novel solutions can be directly derived from the exact solutions and symmetries of the Boussinesq equation. In the meantime, by combining the Hirota bilinear and CTE approaches, the dynamics of the multi-soliton, soliton–cnoidal interaction solutions, and the lump wave are analyzed in detail. The lump wave has been extensively used in the study of the classical integrable systems. These results are highly meaningful due to the scarcity of such novel solutions in supersymmetric systems. Recently, the Grassmann number and the supersymmetric derivative have been extended to the R-number and the ren-symmetric derivatives. Based on the R-numbers and ren-symmetric derivatives, ren-integrable and ren-symmetric integrable systems have been formulated [35]. Studying the integrability and other complex wave phenomena in supersymmetric and ren-symmetric integrable systems remains a significant problem.

Author Contributions: P.-F.W.: Funding acquisition, software, supervision. H.-B.Z.: Formal analysis, software. Y.L.: Formal analysis, software. S.-Y.L.: Formal analysis, software. R.-Y.C.: Formal analysis, software. Z.-Y.X.: Formal analysis, software. W.-L.W.: Formal analysis, software. B.R.: Conceptualization, formal analysis, supervision, writing—original draft. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China under Grant Nos. 12375006, 12105243, and KYY-ZX-20230624, and by the Natural Science Foundation of Zhejiang Province of China under Grant No. LQ22A050002.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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