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The Intrinsic Characterization of a Fuzzy Consistently Connected Domain

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Abstract: The concepts of a fuzzy connected set (fc set) and a fuzzy consistently connected set (fcc set) are introduced on fuzzy posets, along with a discussion of their basic properties. Inspired by some equivalent conditions of crisp connected sets, the characterizations of the fc sets are given, and we also explore fuzzy completeness and fuzzy compactness in addition to defining a new fuzzy way-below relation based on fcc complete sets. Using this relationship as a basis, the fcc domain is also provided and studied, and its equivalent characterizations are obtained. In summary, we develop a method to establish fcc completeness from a continuous poset.

Keywords: fuzzy connected sets; fuzzy consistently connected sets; fuzzy connected complete sets; fuzzy connected continuous domain

MSC: 06A11; 06B35; 03E72; 54H10

1. Introduction and Related Work

In the 1960s, the birth of fuzzy mathematics [1] and the establishment of continuous lattices [2] aroused the research interest of a large number of scholars. Following its development, the theory of continuous lattices was successfully promoted to the theory of continuous domain by G Gierz [3]. References [4–6] innovatively combined domain theory with fuzzy mathematics, with Zhang Qiye and Fan Lei introducing the concept of fuzzy partial order, which, in turn, led to the emergence of fuzzy domain theory. A Chaudhuri and P Das [7] introduced a new concept of fuzzy set connectivity called cs-connectivity. This concept is different from other connectivity concepts, and they found that cs-connectivity is not equivalent to these existing definitions of connectivity; they examined the validity of the standard results under this new concept of connectivity. In references [8,9], the concept of connected sets was introduced to broaden the scope of continuous partial order theory, and the concept of the connected continuous domain was introduced using the concept of connected sets, with fruitful results obtained by Shang Yun and Zhao Bin, they introduce and explore the concept of a consistently connected continuous domain, and extend the application scope of continuous poset theory by exploiting the characteristics of connected sets, solved the limitations of continuous poset theory in the treatment of the real number set and the natural number set, and to characterize it through the properties of the principal ideal and connected closed sets, they also study the directional completeness of consistently connected complete posets and obtains good theoretical results. In [10–12], Tang Zhaoyong introduced and examined the connectivity on partially ordered sets from various perspectives using step sets, resulting in a series of significant findings. In [13], Tang Zhaoyong and Xu Luoshan deeply explore the connectivity and local connectivity of posets from the perspective of order and topology, especially the properties of multiple intrinsic

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topologies (such as Alexandrov topology and Scott topology); they try to prove the equivalence of the order connectivity of posets and its intrinsic topology and to show the properties of local connectivity. Moreover, by constructing counterexamples, they also reveal that the connectivity of the lower topology does not necessarily guarantee the sequential connectivity of the poset itself. In [14,15], they introduce and study the concept of connectivity, especially to explore the step set. They also explore the construction of connected branches and show that posets can be uniquely decomposed into the non-intersection union of these connected branches. Furthermore, they show that the connected relations of the posets constitute an equivalence relation. These results provide a new tool and theoretical framework for understanding and operating posets. Reference [16] examines and characterizes the notion of the prime neutrosophic ideal and prime neutrosophic filter. The structure of the neutrosophic open-set lattice on a topology generated by a neutrosophic relation is described in it. They have defined the concepts of neutrosophic ideals and neutrosophic filters on that lattice in terms of their level sets and meet and join operations. In addition, we have examined and defined the concepts of prime neutrosophic filters and ideals as fascinating subsets of neutrosophic ideals and filters. This work mostly discussed neutrosophic ideals and neutrosophic filters on the lattice structure of neutrosophic open sets. Reference [17] explores the fuzzy topology induced by fuzzy relations, extending classical concepts, and establishes necessary and sufficient conditions for its generation, along with characterizations involving fuzzy interval orders, preorders, and sequential fuzzy topologies. Furthermore, the fuzzy bi-topological space generated by the fuzzy relations is explored.

References [18–22] discussed related problems, such as fuzzy classification, fuzzy measurement, and fuzzy decision under fuzzy theory, obtaining good results.

The research motivation for exploring the fuzzy connected set lies in the desire to expand the application of the continuous poset theory. By defining a novel fuzzy way-below relation on the fcc complete set, we aim to deepen our understanding of this structure and enhance its utility. Furthermore, the introduction and investigation of the concept of fcc continuous domain serve to enrich the theoretical framework and broaden its potential applications. This line of inquiry offers significant insights into the properties and characteristics of fc sets, thereby contributing to the advancement of the field of fuzzy set theory and its applications.

However, in fuzzy domain theory, there is no concept of connected sets. Thus, we introduce the concepts of fuzzy connected sets and fuzzy consistently connected sets on fuzzy posets, and their basic properties are discussed. In the third section of this paper, a new fuzzy way-below relationship is defined on fuzzy consistently connected complete posets, which allows for the exploration of the fuzzy consistently connected continuous domain. In addition, its equivalent characterizations are determined.

For some basic notations and definitions, we refer the reader to [3,23–25].

2. Preliminaries

Below, we present some important terms and definitions used in this paper. Here, we describe the definitions of fuzzy sets, domain theory, and consistently connected theory.

Definition 1 ([1]). *Let $X = \{x_1, x_2, \cdots x_n\}$ be the universe of discourse. A fuzzy set A on X is characterized by $A = \{(x_i, \mu_A(x_i)); x_i \in X, i = 1, 2, \cdots n\}$, where $\mu_A : X \to [0, 1]$, the membership function, gives the grade of membership of each element $x_i \in X$ in A.*

Definition 2 ([1]). *Let X be a set, L be a complete lattice, and L^X be all mappings from X to L. Each $A \in L^X$ is called a fuzzy subset of X. For $A \subseteq X$, $\chi_A \in L^X$ is the characteristic function of A, defined as*

$$\chi_A(x) = 1, (x \in A)$$
$$\chi_A(x) = 0, (x \notin A)$$

where 0 and 1 represent the least and great elements of L.

Definition 3 ([3]). *A poset is said to be complete with respect to directed sets if every directed subset has a sup. A directed complete poset is abbreviated as a dcpo.*

Definition 4 ([3]). *Let L be a poset. We say that x is way $-$ below y, in symbols $x \ll y$ if for all directed subsets $D \subseteq L$ for which $supD$ exists, the relation $y \leq supD$ always implies the existence of $d \in D$ with $x \leq d$. An element satisfying $x \ll x$ is said to be compact or isolated from below.*

Definition 5 ([3]). *A poset L is deemed continuous if for all $x \in L$, the set $\Downarrow x = \{u \in L : u \ll x\}$ is directed and $x = sup\{u \in L : u \ll x\}$. A dcpo that is continuous as a poset is referred to as a domain.*

Definition 6 ([4]). *Let X be a set and $e : X \times X \to L$ be a mapping. Then, (X, e) is deemed a fuzzy poset if e satisfies the following:*
(1) $\forall x \in X, e(x, x) = 1$;
(2) $\forall x, y, z \in X, e(x, y) \wedge e(y, z) \leq e(x, z)$;
(3) $\forall x, y \in X, e(x, y) = e(y, x) = 1 \Rightarrow x = y$.
Then, e is referred to as a fuzzy partial order in X.

Definition 7 ([6]). *Let $f : X \to Y$ be a mapping from a set X to a fuzzy poset (Y, e_Y). Define $f^{\rightarrow} : L^X \to L^Y, \forall A \in L^X, y \in Y, f^{\rightarrow}(A)(y) = \vee_{x \in X} A(x) \wedge e_Y(y, f(x))$.*

Definition 8 ([8]). *Let X be a poset. $\emptyset \neq B \subseteq X$. B is considered connected if for all $x, y \in B$, there exists $x = x_1, x_2, \cdots, x_n = y$ such that $x_i \in B$, and x_i, x_{i+1} are comparable.*
If B is connected and $x, y \in B$, then x and y are considered connected in B.

Definition 9 ([4]). *Let (X, e) be a fuzzy poset, $x_0 \in X$ and $A \in L^X$. x_0 is said to be the supremum (resp. infimum) of A, written as $x_0 = \sqcup A$ (resp. $x_0 = \sqcap A$), if*
(1) $\forall x \in X, A(x) \leq e(x, x_0)$(resp.$A(x) \leq e(x_0, x)$);
(2) $\forall y \in X, \wedge_{x \in X}(A(x) \to e(x, y)) \leq e(x_0, y)$(resp.$\wedge_{x \in X}(A(x) \to e(y, x)) \leq e(y, x_0)$).

Definition 10 ([5]). *Let (X, e) be a fuzzy poset. $A \in L^X$. A is considered a fuzzy upper set (resp. fuzzy lower set) if $\forall x, y \in X, e(x, y) \wedge A(x) \leq A(y)$ (resp. $\forall x, y \in X, e(x, y) \wedge A(y) \leq A(x)$).*

Definition 11 ([5]). *Let (X, e) be a fuzzy poset. For all $A \in L^X, \downarrow A, \uparrow A \in L^X$:*

$$\forall x \in X, \downarrow A(x) = \vee_{y \in X} A(y) \wedge e(x, y);$$

$$\forall x \in X, \uparrow A(x) = \vee_{y \in X} A(y) \wedge e(y, x).$$

Definition 12 ([5]). *Let (X, e) be a fuzzy poset. For all $D \in L^X$, D is a fuzzy directed subset if*
(1) $\vee_{x \in X} D(x) = 1$;
(2) $\forall x, y \in X, D(x) \wedge D(y) \leq \vee_{z \in X} D(z) \wedge e(x, z) \wedge e(y, z)$.
A fuzzy direct subset $I \in L^X$ is considered a fuzzy ideal if it is a fuzzy lower set.

Definition 13 ([5]). *Let (X, e) be a fuzzy poset. For all $D \in L^X$, D is considered a fuzzy co-directed subset if*

(1) $\vee_{x\in X} D(x) = 1$;

(2) $\forall x, y \in X, D(x) \wedge D(y) \leq \vee_{z\in X} D(z) \wedge e(z, x) \wedge e(z, y)$.

A fuzzy co-direct subset $I \in L^X$ is considered a fuzzy filter if it is a fuzzy upper set.

Definition 14 ([5]). *Let (X, e) be a fuzzy poset. $Sub_X : L^X \times L^X \to L$ is considered a Subsethood operator if*

$$\forall A_1, A_2 \in L^X, Sub_X(A_1, A_2) = \wedge_{x\in X} A_1(x) \to A_2(x).$$

Remark 1. *The following statements are easy to verify.*

(1) Both the directed set and the co-directed set are connected sets.

(2) The image of a connected set under a homomorphic mapping is a connected set.

(3) Both the totally ordered set and the one-point set are connected sets.

Definition 15 ([9]). *Let X be a poset, $\emptyset \neq D \subseteq X$. D is a consistently connected set if the following are satisfied:*

(1) D is connected;

(2) $\exists p \in X$, such that $D \subseteq{\downarrow} p = \{x \in X : x \leq p\}$.

Definition 16 ([9]). *Let X be a poset. X is a consistently connected complete poset if, for all consistently connected sets $D \subseteq X$, $\sqcup D$ exists.*

Definition 17 ([9]). *Let X be a consistently connected complete poset. The consistently connected way-below relation $\ll_c$ of X is defined as follows:*
for $x, y \in X$, x is said to be compatible when less than or equal to y, in symbols $x \ll_c y$ if for all consistently connected set D, $y \leq supD$ implies $x \leq d$ for some $d \in D$. We write $\Downarrow_c x = \{u \in X : u \ll_c x\}$.

Definition 18 ([9]). *Let X be a consistently connected complete poset. X is considered a consistently connected domain if*

(1) $\forall x \in X, \Downarrow_c x$ is the consistently connected set in X;

(2) $\forall x \in X, x = sup \Downarrow_c x$.

3. Fuzzy Connected Sets and Fuzzy Consistently Connected Sets

In order to extend the connected sets on posets to fuzzy domain theory in this section, an equivalent definition in alternative form of connected sets is first provided.

Definition 19. *Let X be a poset, $\emptyset \neq B \subseteq X$. B is considered connected if for every $x, y \in B$, there exists $x = x_1, x_2, \cdots, x_n = y$ such that $x_i \in B$, and for all i, $\{x_{i-1}, x_i, x_{i+1}\}$ is a directed set or co-directed set.*

Definition 20. *Let (X, e) be a fuzzy poset. For every $D \in L^X$, the fuzzy subset D is considered a fuzzy connected set if*

(1) $\vee_{x\in X} D(x) = 1$;

(2) For all $a, b \in X$, there exists $D(a) = D(x_1), D(x_2), \cdots, D(x_{n-1}), D(x_n) = D(b)$, such that for every i,

$$\begin{cases} D(x_{i-1}) \wedge D(x_{i+1}) \leq D(x_i) \wedge e(x_{i-1}, x_i) \wedge e(x_{i+1}, x_i); \\ \qquad\qquad or \\ D(x_{i-1}) \wedge D(x_{i+1}) \leq D(x_i) \wedge e(x_i, x_{i-1}) \wedge e(x_i, x_{i+1}). \end{cases}$$

A fuzzy connected set is abbreviated as an fc set.

Lemma 1. *Let $(X, \leq)$ be a poset, and it is considered as a fuzzy poset (X, e), $L = 2$. Then, for all cases, $D \in X$ is a connected set in $(X, \leq)$ if and only if the characteristic function χ_D is an fc set in (X, e).*

Proof. $\Rightarrow$ Let D be a connected set.

(1) For every $D \in X, D \neq \emptyset$; hence, $\vee_{x \in D} \chi_D(x) = 1$.

(2) Because D is a connected set, then, for every $a, b \in D$, there exists $x_1, x_2, \cdots, x_n \in D$ such that

$$a = x_1, x_2, \cdots, x_n = b,$$

and for every i, x_i and x_{i+1} are comparable; that is, for every i,

$$x_i = sup_X\{x_{i-1}, x_{i+1}\} \text{ or } x_i = inf_X\{x_{i-1}, x_{i+1}\}.$$

Then,

$$\chi_D(a) = \chi_D(x_1), \chi_D(x_2), \cdots, \chi_D(x_n) = \chi_D(b)$$

Using Definition 2, we obtain

$$\chi_D(a) = \chi_D(x_1), \chi_D(x_2), \cdots, \chi_D(x_n) = \chi_D(b) = 1.$$

In addition, for every i,

$$e(x_{i-1}, x_i) = e(x_{i+1}, x_i) = 1 \text{ or } e(x_i, x_{i-1}) = e(x_i, x_{i+1}) = 1.$$

From this, for every i, we obtain

$$1 = \chi_D(x_{i-1}) \wedge \chi_D(x_{i+1}) \leq 1 = \chi_D(x_i) \wedge e(x_{i-1}, x_i) \wedge e(x_{i+1}, x_i)$$

or

$$1 = \chi_D(x_{i-1}) \wedge \chi_D(x_{i+1}) \leq 1 = \chi_D(x_i) \wedge e(x_i, x_{i-1}) \wedge e(x_i, x_{i+1}).$$

Hence, the character function χ_D is an fc set in (X, e).

$\Leftarrow$ Let χ_D be an fc set in (X, e).

(1) From $\vee_{x \in D} \chi_D(x) = 1$, we have $D \neq \emptyset$.

(2) Because χ_D is an fc set in (X, e), then for all $a, b \in D$, there exists

$$\chi_D(a) = \chi_D(x_1), \chi_D(x_2), \cdots, \chi_D(x_n) = \chi_D(b)$$

Therefore,

$$a = x_1, x_2, \cdots, x_n = b$$

exists. For every i,

$$\chi_D(x_{i-1}) \wedge \chi_D(x_{i+1}) \leq \chi_D(x_i) \wedge e(x_{i-1}, x_i) \wedge e(x_{i+1}, x_i)$$

or

$$\chi_D(x_{i-1}) \wedge \chi_D(x_{i+1}) \leq \chi_D(x_i) \wedge e(x_i, x_{i-1}) \wedge e(x_i, x_{i+1}).$$

Then, we have

$$x_{i-1} \leq x_i, x_{i+1} \leq x_i \text{ or } x_{i-1} \geq x_i, x_{i+1} \geq x_i,$$

which means that for every i, x_i and x_{i+1} are comparable, and hence, $D \in X$ is a connected set in $(X, \leq)$. □

Remark 2. *A fuzzy connected set may not necessarily be a fuzzy directed (co-directed) set.*

Example 1. *Let $(X, \leq)$ be a poset of Figure 1. Consider it as a fuzzy poset (X, e), $L = 2$, $D = \{a, b, c, d\}$. Then, D is a connected subset of X, and χ_D is an fc subset, but χ_D is not a fuzzy directed (co-directed) subset of (X, e).*

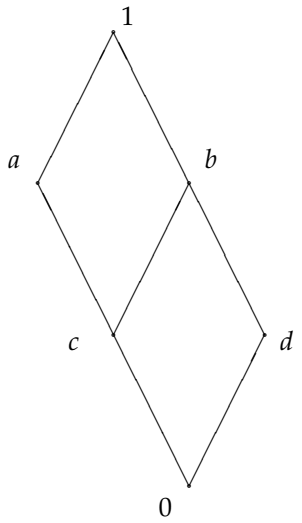

Figure 1. Graph of example 1.

Proposition 1. *Let (X, e) be a fuzzy poset, and a fuzzy directed (co-directed) subset $D \subseteq L^X$. Then, D is a fuzzy connected if there exists a point $x_0 \in X$ such that $D(x_0) = 1$.*

Proof. It follows from D being fuzzy directed that $\vee D(x) = 1$. Since D is a fuzzy directed set, by the condition that for all $a, b \in X, D(a) \wedge D(b) \leq \vee_{z \in X} D(z) \wedge e(a, z) \wedge e(b, z)$, then there exists $x_0 \in X$. Let $x_a = a, x_b = b$, such that $D(a) = D(x_a), D(x_0) = 1, D(x_b) = D(b)$. Then, we obtain

$$D(x_a) \wedge D(x_b) \leq \vee_{z \in X} D(z) \wedge e(x_a, z) \wedge e(x_b, z) \leq D(x_0) \wedge e(x_a, x_0) \wedge e(x_b, x_0).$$

Thus, D is an fc set. □

Proposition 2. *Let $(X, \leq)$ be a poset, and it is considered as a fuzzy poset (X, e), $L = 2$. For all $D \in X$, χ_D is an fc subset if χ_D is a fuzzy directed (co-directed) subset.*

Proof. It can be seen from the assumed conditions that there exists $x_0 \in X$ such that $\chi_D(x_0) = 1$, and based on Proposition 1, the conclusion is true. □

Definition 21. *Let (X, e) be a fuzzy poset, and a fuzzy subset $D \subseteq L^X$. D is considered a fuzzy consistently connected set if*

(1) D is a fuzzy connected set;

(2) there exists $p \in X$ such that $D \subseteq \downarrow p$ ($\downarrow p \in L^X, \forall x \in X, \downarrow p(x) = e(x, p)$).

A fuzzy consistently connected set is abbreviated as an fcc set.

All fcc sets in the fuzzy poset (X, e) are denoted by $C_F(X)$, and D is an fcc ideal if D is a fuzzy lower set. All fcc ideals in the fuzzy poset (X, e) are denoted by $CI_F(X)$.

Definition 22. *Let (X, e) be a fuzzy poset. (X, e) is an fcc complete poset if for all fcc subset D, $\sqcup D$ exists.*

Proposition 3. *Let $(X, \leq)$ be a poset, and consider it as a fuzzy poset (X, e), $L = 2$. $(X, \leq)$ is a consistently connected complete set if and only if (X, e) is an fcc complete poset.*

Proof. Suppose that $(X, \leq)$ is a consistently connected complete set. For all D is an fcc set, let $A = \{x \in X : D(x) = 1\}$, then $A \subseteq X$. According to Lemma 1, A is consistently connected, then there exists $x_0 \in X$ such that $x_0 = supA$, and we need to prove that $x_0 = \sqcup D$.

(1) $\forall x \in X, D(x) \leq e(x, x_0)$;

(2)

$$\begin{aligned} &\forall y \in X, \wedge_{x \in X}(D(x) \to e(x, y)) = 1 \\ \Leftrightarrow\ &\forall x \in X, D(x) \to e(x, y) = 1 \\ \Leftrightarrow\ &\forall x \in X, D(x) \leq e(x, y) \\ \Leftrightarrow\ &D(x) = 1 \\ \Rightarrow\ &e(x, y) = 1 \\ \Leftrightarrow\ &\forall x \in X, x \leq y \\ \Rightarrow\ &x_0 \leq y \\ \Leftrightarrow\ &e(x_0, y) = 1, \end{aligned}$$

Then, we obtain $\wedge_{x \in X}(D(x) \to e(x, y)) \leq e(x_0, y)$, and hence, $x_0 = \sqcup D$.

Conversely, suppose (X, e) is an fcc complete set and A is a consistently connected set. Then, according to Lemma 1, χ_A is fcc, and (X, e) is fcc complete; as such, there exists an $x_1 \in X$ such that $x_1 = \sqcup \chi_A$, and we need to demonstrate that $x_1 = \sqcup A$.

(1)

$$\begin{aligned} &\forall x \in A \Rightarrow \chi_A = 1 \\ \Rightarrow\ &e(x, x_1) \geq \chi_A(x) = 1 \\ \Rightarrow\ &x \leq x_1. \end{aligned}$$

(2) Suppose $y \in X$ such that for all $x \in A, x \leq y$, that is,

$$\begin{aligned} &\forall x \in X, \chi_A(x) = 1 \\ \Rightarrow\ &e(x, y) = 1 \\ \Rightarrow\ &\chi_A(x) \leq e(x, y) \\ \Rightarrow\ &\chi_A(x) \to e(x, y) = 1, \end{aligned}$$

Therefore, $e(x_1, y) \geq \wedge_{x \in X}(\chi_A(x) \to e(x, y)) = 1$, and hence $x_1 \leq y$. So, we have $x_1 = supA$. □

4. Fcc Way-Below and Fcc Domain

In this section, we give definitions of fcc way-below and a fcc continuous set, as well as equivalent characterizations of fcc domain and a discussion of related properties.

Definition 23. *Let (X, e) be an fcc directed complete poset. For all $y \in X, \Downarrow_{FC} y \in L^X$. This is deemed fcc way-below if*

$$\forall x \in X, \Downarrow_{FC} y(x) = \wedge_{I \in CI_F(X)}(e(y, \sqcup I) \to I(x)).$$

From Definition 21, we plot Figure 2 as an elaborated illustration of the Definition 23.

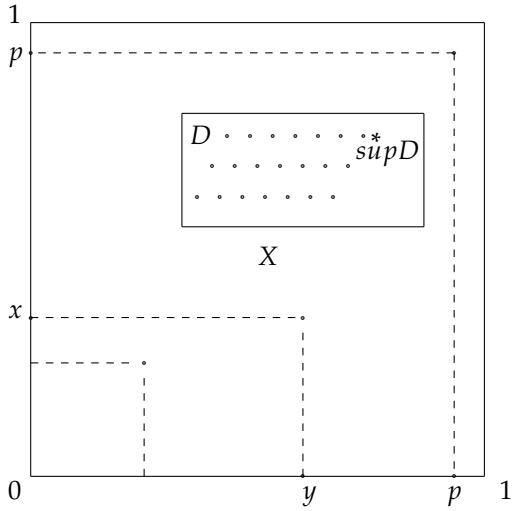

Figure 2. Graph of FC way-below relation.

X is a fuzzy poset, and the side length is unit length 1, where the projection coordinate value of any point is the matching degree of membership of x and y with respect to X. D is an fcc set, all the internal points have connectivity, and use $*$ to represent $supD$. As p is a point in the unit cube, D is below the projection coordinates of p, which satisfies the consistently connectivity of D.

Definition 24. *Let (X, e) be an fcc complete set. (X, e) is considered an fcc continuous set if $\forall x \in X, \Downarrow_{FC} x \in CI_F(X)$ and $x = \sqcup \Downarrow_{FC} x$.*

Definition 25. *An fcc directed complete poset that is an fcc continuous set is an fcc domain.*

Definition 26. *Let (X, e) be an fcc complete set. x is the fcc compact element in X if $\Downarrow_{FC} x(x) = 1$. All fcc compact elements in X are denoted $K_{FC}(X)$.*

Definition 27. *Let (X, e) be an fcc complete set, $x \in X$. Define a mapping $k_x : X \to L$:*

$$k_x(y) = \begin{cases} e(y, x), y \in K_{FC}(X); \\ 0, y \notin K_{FC}(X). \end{cases}$$

X is considered an fcc algebraic poset if k_x is an fcc subset of X and $\sqcup k_x = x$.

From Example 1.9 of [9] and Proposition 3, we have the following examples:

Example 2. *For the poset $\mathbb{R}$ of real numbers, $(\mathbb{R}, e_{\leq})$ is an fcc domain.*

Example 3. *For the poset $\mathbb{N}$ of natural numbers, $(\mathbb{N}, e_{\leq})$ is a fuzzy algebraic domain.*

Example 4. *Let A be an fcc domain. Then, the principal ideal of A is an fcc domain.*

Proposition 4. *Let (X, e) be an fcc complete set. Then, the conditions are as follows:*
(1) $\forall x \in X, I \in CI_F(X), \wedge_{y \in X} e(x, y) \leq I(x)$;
(2) $\forall x, y \in X, \wedge_{z \in X} e(x, z) \leq \Downarrow_{FC} y(x)$;
(3) $\forall x \in X, \Downarrow_{FC} \leq \downarrow x$;
(4) $\forall x, u, v, y \in X, e(u, x) \wedge \Downarrow_{FC} y(x) \wedge e(y, v) \leq \Downarrow_{FC} v(u)$.

Proof. (1) For all $x \in X, I \in CI_F(X)$, we have $\wedge_{y\in X} e(x,y) = (\wedge_{y\in X} e(x,y)) \wedge (\vee_{z\in X} I(z)) = \vee_{z\in X}(\wedge_{y\in X} e(x,y) \wedge I(z)) \leq \wedge_{z\in X} e(x,z) \wedge I(z) \leq I(x)$.

(2) For all $x, y \in X$, we have

$\Downarrow_{FC} y(x) = \wedge_{I\in CI_F(X)} e(y, \sqcup I) \to I(x) \geq \wedge_{I\in CI_F(X)} I(x) \geq \vee_{z\in X} e(x,z)$.

(3) For all $y \in X$,we obtain

$$\begin{aligned} \Downarrow_{FC} x(y) &= \wedge_{I\in CI_F(X.e)} e(x, \sqcup I) \to I(y) \leq e(x, \sqcup \downarrow x) \to \downarrow x(y) \\ &= e(x,x) \to \downarrow x(y) \\ &= \downarrow x(y). \end{aligned}$$

Hence $\Downarrow_{FC} x \leq \downarrow x$.

(4) For all $I \in CI_F(X,e)$, we have

$$\begin{aligned} & e(u,x) \wedge e(y,v) \wedge (e(y, \sqcup I) \to I(x)) \wedge e(v, \sqcup I) \\ & \leq e(u,x) \wedge e(y, \sqcup I) \wedge (e(y, \sqcup I) \to I(x)) \\ & \leq e(u,x) \wedge I(x) \\ & \leq I(u). \end{aligned}$$

Then, we obtain $e(u,x) \wedge e(y,v) \wedge (e(y, \sqcup I) \to I(x)) \leq e(v, \sqcup I) \to I(u)$. According to Definition 23,

$$\begin{aligned} & e(u,x) \wedge \Downarrow_{FC} y(x) \wedge e(y,v) \\ & = e(u,x) \wedge e(y,v) \wedge (e(y, \sqcup I) \to I(x)) \\ & \leq \wedge_{I\in CI_F(X,e)} e(u,x) \wedge e(y,v) \wedge (e(y, \sqcup I) \to I(x)) \\ & \leq \wedge_{I\in CI_F(X)} (e(v, \sqcup I) \to I(u)) \\ & = \Downarrow_{FC} v(u). \end{aligned}$$

Hence, $e(u,x) \wedge \Downarrow_{FC} y(x) \wedge e(y,v) \leq \Downarrow_{FC} v(u)$. □

Theorem 1. *Let (X,e) be an fcc domain. Then, for all $x, y \in X, \Downarrow_{FC} y(x) = \vee_{z\in X} \Downarrow_{FC} z(x) \wedge \Downarrow_{FC} y(z)$.*

Proof. On the one hand, according to Proposition 4, we have

$$\vee_{z\in X} \Downarrow_{FC} z(x) \wedge \Downarrow_{FC} y(z) \leq \Downarrow_{FC} y(x).$$

On the other hand, we simply need to prove that

$$\Downarrow_{FC} y(x) \leq \vee_{z\in X} \Downarrow_{FC} z(x) \wedge \Downarrow_{FC} y(z).$$

Suppose $D \in L^X$, for all $a \in X, D(a) = \vee_{z\in X} \Downarrow_{FC} z(a) \wedge \Downarrow_{FC} y(z)$, we shall prove that $\Downarrow_{FC} y(x) \leq D(x)$.

Firstly, $D(x)$ is an fcc ideal.

(1) For all $x \in X, \Downarrow_{FC} x$ is an fcc ideal.

$$\begin{aligned} \vee_{a\in X} D(a) &= \vee_{a\in X} \vee_{z\in X} \Downarrow_{FC} z(a) \wedge \Downarrow_{FC} y(z) \\ &= \vee_{z\in X} (\vee_{a\in X} \Downarrow_{FC} z(a)) \wedge \Downarrow_{FC} y(z) \\ &= \vee_{z\in X} \Downarrow_{FC} y(z) \\ &= 1. \end{aligned}$$

(2) For all $a, b \in X$, every fcc way-below lower set is an fcc ideal in fcc domain, so that it is fuzzy consistently connected, according to Definition 20,

$$D(a) \wedge D(b) = \vee_{m,n\in X} \Downarrow_{FC} m(a) \wedge \Downarrow_{FC} n(b) \wedge \Downarrow_{FC} y(m \wedge \Downarrow_{FC} y(n)).$$

There exists $c \in X$ such that

$$\begin{aligned}
&\vee_{m,n\in X} \Downarrow_{FC} m(a) \wedge \Downarrow_{FC} n(b) \wedge \Downarrow_{FC} y(m) \wedge \Downarrow_{FC} y(n)) \\
&\leq \vee_{m,n\in X} \Downarrow_{FC} m(a) \wedge \Downarrow_{FC} n(b) \wedge \Downarrow_{FC} y(c) \wedge e(m,c) \wedge e(n,c) \\
&= \vee_{m,n\in X} (\Downarrow_{FC} m(a) \wedge e(m,c)) \wedge (\Downarrow_{FC} n(b) \wedge e(n,c)) \wedge \Downarrow_{FC} y(c) \\
&\leq \vee_{m,n\in X} \Downarrow_{FC} c(a) \wedge \Downarrow_{FC} c(b) \wedge \Downarrow_{FC} y(c) \\
&= \Downarrow_{FC} c(a) \wedge \Downarrow_{FC} c(b) \wedge \Downarrow_{FC} y(c).
\end{aligned}$$

Furthermore, there exists $d \in X$ such that

$$\begin{aligned}
&\Downarrow_{FC} c(a) \wedge \Downarrow_{FC} c(b) \wedge \Downarrow_{FC} y(c) \\
&\leq \Downarrow_{FC} c(d) \wedge e(a,d) \wedge e(b,d) \wedge \Downarrow_{FC} y(c) \\
&= (\Downarrow_{FC} c(d) \wedge \Downarrow_{FC} y(c)) \wedge e(a,d)) \wedge e(b,d) \\
&= D(d) \wedge e(a,d) \wedge e(b,d).
\end{aligned}$$

Let $a = x_1, x_2, x_3 = b$. Then, there exists $D(a) = D(x_1), D(x_2), D(x_3) = D(b)$, such that $D(x_1) \wedge D(x_3) \leq D(x_2) \wedge e(x_1, x_2) \wedge e(x_3, x_2)$. Hence, D is fuzzy consistently connected.

(3) For all $a, b \in X$,

$$\begin{aligned}
D(a) \wedge e(b,a) &= \vee_{z\in X} \Downarrow_{FC} z(a) \wedge \Downarrow_{FC} y(z) \wedge e(b,a) \\
&\leq \vee \Downarrow_{FC} z(b) \wedge \Downarrow_{FC} y(z) \\
&= D(b).
\end{aligned}$$

Therefore, D is a fuzzy lower set.

(4) For all $x \in X, D(x) \leq \Downarrow_{FC} y(x) \leq \downarrow y(x)$, then we obtain $D \leq \downarrow y$, and thus D is fuzzy consistent.

Secondly, $y = \sqcup D$.

In fact, for all $a \in X$,

$$\begin{aligned}
\wedge_{z\in X} D(z) \to e(z,a) &= \wedge_{z\in X} (\wedge_{c\in X} (\Downarrow_{FC} c(z) \wedge \Downarrow_{FC} y(c)) \to e(z,a) \\
&= \wedge_{c\in X} (\Downarrow_{FC} y(c) \to \wedge_{z\in X} (\Downarrow_{FC} c(z) \to e(z,a)) \\
&= \wedge_{c\in X} \Downarrow_{FC} y(c) \to e(\sqcup \Downarrow_{FC} c, a) \\
&= \wedge_{c\in X} \Downarrow_{FC} y(c) \to e(c,a) \\
&= e(\sqcup \Downarrow_{FC} y, a) \\
&= e(y,a).
\end{aligned}$$

Finally,

$$\begin{aligned}
\Downarrow_{FC} y(x) &= \wedge_{I\in CI_F(X)} e(y, \sqcup I) \to I(x) \\
&\leq e(y, \sqcup D) \to D(x) \\
&= 1 \to D(x) \\
&= D(x).
\end{aligned}$$

□

Theorem 2. *Let (X, e) be an fcc complete poset. Then, (X, e) is an fcc domain if and only if $(\Downarrow_{FC}, \sqcup)$ is a fuzzy Galois adjunction between (X, e) and $(CI_F(X), Sub_X)$.*

Proof. Suppose that (X, e) is an fcc continuous poset. $\forall x, y \in X, Sub_X(\Downarrow_{FC} x, \Downarrow_{FC} y)$

$$\begin{aligned}
&= \wedge_{z\in X} \Downarrow_{FC} x(z) \to \Downarrow_{FC} y(z)\\
&= \wedge_{z\in X}(\wedge_{I\in CI_F(X)} e(x, \sqcup I) \to I(z)) \to (\wedge_{J\in CI_F(X)} e(y, \sqcup J) \to J(z))\\
&= \wedge_{z\in X}(\wedge_{J\in CI_F(X)}(\wedge_{I\in CI_F(X)} e(x, \sqcup I) \to I(z)) \to (e(y, \sqcup J) \to J(z)))\\
&\geq \wedge_{z\in X}(\wedge_{J\in CI_F(X)}(e(x, \sqcup J) \to J(z)) \to (e(y, \sqcup J) \to J(z)))\\
&\geq \wedge_{J\in CI_F(X)} e(y, \sqcup J) \to e(x, \sqcup J)\\
&\geq e(x, y).
\end{aligned}$$

Therefore, $Sub_X(\Downarrow_{FC} x, \Downarrow_{FC} y) \geq e(x, y)$; then, $\Downarrow_{FC}$ is fuzzy order-preserving.
For all $I, J \in CI_F(X)$,

$$e(\sqcup I, \sqcup J) = \wedge_{x\in X} I(x) \to e(x, \sqcup J) \geq \wedge_{x\in X} I(x) \to J(x) = Sub_X(I, J).$$

Then, $Sub_X(I, J) \leq e(\sqcup I, \sqcup J)$, which means that $\sqcup$ is fuzzy order-preserving. $\forall x \in X, I \in CI_F(X)$,

$$\begin{aligned}
Sub_X(I, J) &= \wedge_{y\in X} \Downarrow_{FC} x(y) \to I(y)\\
&= \wedge_{y\in X}(\wedge_{J\in CI_F(X)} e(x, \sqcup J) \to J(y)) \to I(y)\\
&\geq \wedge_{y\in X}((e(x, \sqcup I) \to I(y)) \to I(y)\\
&\geq e(x, \sqcup I).
\end{aligned}$$

From (X, e) is fcc poset, we have $x = \sqcup \Downarrow_{FC} x$, and thus

$$\begin{aligned}
e(x, \sqcup I) &= e(\sqcup \Downarrow_{FC} x, \sqcup I)\\
&= \wedge_{y\in X} \Downarrow_{FC} x(y) \to e(y, \sqcup I)\\
&\geq \wedge_{y\in X} \Downarrow_{FC} x(y) \to I(y)\\
&= Sub_X(\Downarrow_{FC} x, I).
\end{aligned}$$

Thus, $(\Downarrow_{FC}, \sqcup)$ is a fuzzy Galois adjunction between (X, e) and $(CI_F(X), Sub_X)$.
Conversely, suppose that $(\Downarrow_{FC}, \sqcup)$ is a fuzzy Galois adjunction between (X, e) and $(CI_F(X), Sub_X)$, then $\sqcup \Downarrow_{FC} \geq 1_X$. Thus, for all $x \in X, \sqcup \Downarrow_{FC} x \geq x$. Since $\sqcup \Downarrow_{FC} x \leq \sqcup \downarrow x = x$, then $\sqcup \Downarrow_{FC} x = x$. Hence (X, e) is an fcc continuous poset, then (X, e) is an fcc domain. □

Proposition 5. *Let (X, e) be an fcc continuous poset. $\forall x \in X, \downarrow_e x = \{y \in X :\downarrow x(y) = 1\}$ is an fcc complete set.*

Proof. Suppose that D is an fcc subset and $i :\downarrow_e x \to X$ is an embedded mapping. For all $a \in \downarrow_e x$,

$$\begin{aligned}
e(\sqcup \overrightarrow{i_C}(A), a) &= \wedge_{b\in X} \overrightarrow{i_C}(A)(b) \to e(b, a)\\
&= \wedge_{b\in X}(\vee_{i(c)=b} A(c)) \to e(b, a)\\
&= \wedge_{c\in Y} A(c) \to e(i(c), a)\\
&= \wedge_{c\in Y} A(c) \to e(c, a)\\
&= e(\sqcup A, a).
\end{aligned}$$

Then, $\sqcup A = \sqcup \overrightarrow{i_C}(A)$. Since $\forall b \in \downarrow_e x, e(b,x) = 1$, which means $A(b) \to e(b,x) = 1$, furthermore $\wedge_{b \in \downarrow_e x} A(b) \to e(b,x) = e(\sqcup D, x) = 1$. Hence, $\sqcup A \in \downarrow_e x$, i.e., $\downarrow_e x$ is an fcc complete set. □

5. Conclusions

We discuss fuzzy connectivity under a fuzzy partial order and provide the equivalence characterization of the connected set along with the definition of the fc set. A set is considered a connected poset if and only if its characterization functions are fuzzy-connected. The definition of the fcc set and its equivalent characterizations are obtained. In the last section, the definitions of fcc way-below and fcc domain are given, and the equivalence characterizations of fcc continuous poset and fcc complete set are then discussed. Finally, a method for deriving fcc completeness from an fcc continuous poset is established.

In this paper, we deeply explore the fuzzy connectivity under the fuzzy partial order and provide the equivalent characterization of the connected set by defining the fuzzy connected set (fc set). In the framework of fuzzy mathematics, connectivity is an important concept that helps us to understand and analyze the properties of complex mathematics structures. First, we define the connectivity of a set under a fuzzy partial order. Specifically, a set is considered to be a connected poset if and only if its characteristic function is fuzzy connected. This means that, under the fuzzy partial order, there is a continuous and uninterrupted relationship between the elements in the set, which makes the whole set present a holistic structure. In order to further understand the concept of a fuzzy connected set, we further explore the definition of the fuzzy consistently connected set (fcc set) and its equivalent characterization. The fcc set is a special set of fuzzy connectivity that satisfies more stringent conditions to enable a better description of certain specific types of fuzzy structure. Through the equivalent characterization of the fcc set, we can more clearly recognize its properties and characteristics, providing a basis for subsequent research and application. In the final section of this article, we introduce the concept of fcc way-below relation and fcc domain and discuss the equivalent characterization of fcc continuous posets and fcc complete sets. These concepts provide us with a new perspective to examine the application of fuzzy connectivity in complex systems. In particular, we propose a method to derive fcc completeness from the fcc continuous poset, which helps us to better understand and apply the theory of fuzzy consistent connectivity. Overall, this paper explores, in depth, the related concepts and properties of fuzzy consistent connectivity in the framework of fuzzy partial order. By introducing the concepts of the fc set, fcc set, fcc completed set, and fcc domain, we provide new ideas and methods for the study of fuzzy consistent connectivity. These results not only help us to have a deeper understanding of the intrinsic structure of fuzzy mathematics and complex mathematic structures but also provide strong support for subsequent research and applications.

It is worth mentioning that the fuzzy connectivity theory explored in this paper has broad prospects in practical applications. For example, in the fields of image processing, social network analysis, data mining, etc., fuzzy connectivity can be used to describe connected regions in an image, connected subgraphs in a social network, and connected clusters in a data set. Through the analysis and utilization of these connected structures, we can better understand and exploit the inherent laws and properties of these complex structures. Moreover, with the continuous development of fuzzy mathematics and complex structures, we believe that fuzzy connectivity theory will be more widely applied and developed. In the future, we can further explore the combination of fuzzy connectivity and other mathematical tools, such as fuzzy logic, fuzzy clustering, etc., to form a more perfect theoretical system and methodology. At the same time, we can also focus on the application cases of fuzzy connectivity in practical problems to promote its in-depth development and application in various fields. For example, we can take the fuzzy poset as the starting point and consider the connected proposition of its intrinsic topology. In conclusion, this

paper explores fuzzy connectivity in the framework of a fuzzy partial order and proposes a series of new concepts and methods. These achievements not only enrich the theoretical system of fuzzy mathematics but also provide strong support for subsequent research and application. We believe that in future studies, fuzzy connectivity theory and fuzzy consistently connected theory will play an increasingly important role in providing us with new ideas and methods to solve complex problems.

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