

Article

Ruin Probabilities with Investments in Random Environment: Smoothness

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Abstract: This paper deals with the ruin problem of an insurance company investing its capital reserve in a risky asset with the price dynamics given by a conditional geometric Brownian motion whose parameters depend on a Markov process describing random variations in the economic and financial environments. We prove a sufficient condition on the distribution of jumps of the business process ensuring the smoothness of the ruin probability as a function of the initial capital and obtain for this function an integro-differential equation.

Keywords: ruin probabilities; actuarial models with investments; smoothness of ruin probabilities; stochastic volatility; regime switching; integro-differential equations for ruin probabilities

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1. Introduction

Ruin models with risky investments (in other words, with stochastic interest rates) constitute one of the most active fields of the present day ruin theory. These models combine classical frameworks with models of asset price dynamics developed in mathematical finance. The goal of numerous studies is to provide information about asymptotes of ruin probabilities. For this there are several approaches. One of them is based on the integro-differential equations for ruin probabilities. An inspection of the literature reveals that in many cases these equations are derived assuming the smoothness of ruin probabilities as a function of the initial reserve; see, e.g., [1,2]. To our mind, the smoothness of ruin probabilities is a rather delicate property requiring a special (and rather involved) study but there are very few papers on it; see [3,4].

In the present note, we consider the price dynamics suggested by Di Masi, Kabanov, and Runggaldier [5], where the coefficients of a (conditional) geometric Brownian motion depend on a Markov process with a finite number of states. Such a setting (sometimes referred to as the stochastic volatility model, regime switching, or hidden Markov chain model) reflects the random dynamics of the economic environment and seems to be adequate for a context with long-term contracts typical in insurance. It was already considered in the actuarial literature; see, e.g., Refs. [6,7], where the analysis was based on using implicit renewal theory and Ref. [2], where a second-order integro-differential equation for the ruin probability was derived using intuitive arguments.

Our main result is the theorem asserting that if the distribution of jumps has a density with two continuous and integrable derivatives, then the ruin probability is twice continuously differentiable.

2. The Model

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbf{P})$ be a stochastic basis where there are given three independent stochastic processes, W , P , and θ^i , generating the filtration $\mathbf{F}$:

1. A standard Wiener process $W = (W_t)$, i.e., with $W_0 = 0$.
2. A compound Poisson process $P = (P_t)$ with drift:

$$P_t = ct + \sum_{k=1}^{N_t} \xi_k$$

where c is a constant, $N = (N_t)$ is a right-continuous Poisson process with intensity $\alpha > 0$, and (ξ_k) is an i.i.d. sequence independent of N with the distribution $F_\xi = F_\xi(dx)$. Sometimes it is more convenient to use an alternative description employing the notation of stochastic calculus, namely,

$$P_t = ct + x * \pi_t = ct + \int_0^t \int x \pi(ds, dx) \quad (1)$$

where $\pi = \pi(dt, dx)$ is the jump measure of P , that is, a Poisson random measure $\mathbb{R}_+ \times \mathbb{R}$ with the deterministic compensator $\tilde{\pi} = \tilde{\pi}(dt, dx) = \Pi(dx)dt$ (coinciding with the mean of π) and the Lévy measure $\Pi(dx) = \alpha F_\xi(dx)$. We denote by T_n , $n \geq 1$, the times of the consecutive jumps of the process $N_t = \pi([0, t] \times \mathbb{R})$ with the usual convention $T_0 = 0$.

3. A piecewise constant right-continuous Markov process $\theta^i = (\theta_t^i)$ with values in the finite set $E := \{0, 1, \dots, K-1\}$, the initial value $\theta_0^i = i$, and the transition intensity matrix (λ_{jk}) . We assume that all states are communicating.

Let D_0 be the space of trajectories of such processes θ^i , the character Z will stand for a generic point of D_0 . Note that D_0 can be considered as a subspace of the Skorohod space D .

Recall that $\lambda_{jj} = -\sum_{k \neq j} \lambda_{jk}$ and $\lambda_j := -\lambda_{jj} > 0$ for each j . We denote by τ_n^i , $n \geq 1$ the times of the consecutive jumps of θ^i with the convention $\tau_0^i = 0$. We consider also the embedded Markov chain $\vartheta_n^i := \theta_{\tau_n^i}^i$ with transition probabilities $p_{kl} = \lambda_{kl} / \lambda_k$, $k \neq l$, and $p_{kk} = 0$.

The conditional distribution of the length of interval $\tau_{k+1}^i - \tau_k^i$ given $\vartheta_k^i = j$ is exponential with parameter λ_j .

We consider the random integer-valued measure $p^i := \sum_n \varepsilon_{(\tau_n^i, \vartheta_n^i)}$, where ε_x is the unit mass at x . It can be also defined by the counting processes

$$p^i([0, t], k) := \sum_{\tau_n^i \leq t} \mathbf{1}_{\{\vartheta_n^i = k\}}, \quad k \in E.$$

The compensator $\tilde{p}^i$ (dual predictable projection) of the random measure p^i is given by the formula

$$\tilde{p}^i(dt, k) = \mathbf{1}_{\{\vartheta_t^i \neq k\}} \lambda_{\vartheta_t^i k} dt. \quad (2)$$

To alleviate formulae we shall omit the superscript i when it will not lead to an ambiguity and use for brevity the standard “dot” notation of the semimartingale stochastic calculus for integrals, with $L = (L_t)$ standing for the (deterministic) process with $L_t := t$.

With these conventions the Markov-modulated price process S can be given as the solution of the linear integral equation

$$S = 1 + S \cdot (a_\theta \cdot L + \sigma_\theta \cdot W),$$

where $a_k \in \mathbb{R}$, $\sigma_k > 0$, $k = 0, \dots, K-1$. In the traditional symbolical form of the Itô calculus it is usually written as the stochastic differential equation

$$dS_t = S_t(a_{\theta_t} dt + \sigma_{\theta_t} dW_t), \quad S_0 = 1.$$

Let $\kappa_k := a_k - (1/2)\sigma_k^2$. The solution of the above equation can be represented by the formula $S = \mathcal{E}(R)$, where $\mathcal{E}$ is the stochastic exponential and $R := a_\theta \cdot L + \sigma_\theta \cdot W$ is the relative price process, or, in a more explicit form, by the formula $S = e^V$, where

$$V := \kappa_\theta \cdot L + \sigma_\theta \cdot W$$

is the logprice process.

The process $X = X^{u,i}$ is defined by the formula

$$X = u + X \cdot R + P, \quad (3)$$

where $u > 0$. In the actuarial context this is interpreted as the capital reserve of an insurance company fully invested in a risky asset whose price S^i is a conditional geometric Brownian motion given the Markov process θ^i describing the financial environment.

The process P describes the business activity. It is a compound Poisson process with drift as in the classical Cramér–Lundberg model. There are two basic variants of the latter: of non-life insurance ($c > 0$, $F_\xi((-\infty, 0)) = 1$, and of annuity payments ($c < 0$, $F_\xi((0, \infty)) = 1$). In the literature, one can find also a mixed model where F_ξ charges both half-axes.

We consider here the setting where only the coefficients of the price process (i.e., the stochastic interest rate) depend on θ . The extension to the case where parameters of the business process also depend on θ is rather straightforward and has no effect on our main results.

Let $\tau^{u,i} := \inf\{t > 0: X_t^{u,i} \leq 0\}$ be the instant of ruin corresponding to the initial capital u and the initial regime i . Then, $\Psi_i(u) := \mathbf{P}[\tau^{u,i} < \infty]$ is the ruin probability and $\Phi_i(u) := 1 - \Psi_i(u)$ is the survival probability.

Our main result is a sufficient condition on the smoothness of survival (and ruin) probabilities as functions of the initial capital.

Theorem 1. *Let $\xi < 0$ be a random variable with a density f which is twice continuously differentiable on $(0, \infty)$ and such that $f', f'' \in L^1(\mathbb{R}_+)$. Then, the functions $\Phi_i(u)$ are twice continuously differentiable for $u > 0$.*

The proof is given in Section 3. Its idea is based on the following simple arguments. First, let $G(u) := \mathbf{E}g(u + W_1)$, where g is a bounded Borel function. Of course, we cannot differentiate inside the expectation. However, the function G is smooth. Indeed, rewriting G in terms of the distribution of W_1 and using the convolution formula (i.e., a change of variable) we obtain that

$$G(u) = \int_{\mathbb{R}} g(y) \varphi_{0,1}(y - u) dy, \quad \varphi_{0,t}(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t > 0.$$

Since the derivatives of $\varphi_{0,1}$ are polynomials multiplied by $\varphi_{0,1}$ we can differentiate under the sign of the integral an arbitrary number of times and obtain that $G \in C^\infty$.

Now, let us consider the function $G(u) := \mathbf{E}g(u + W_\zeta + \zeta)$, where the Wiener process W and the random variables $\zeta > 0$ and ζ with densities f_ζ and f_ζ are independent. Then

$$G(u) = \int_{\mathbb{R}} \mathbf{E}[g(z + W_\zeta) \mathbf{1}_{\{\zeta \leq t\}}] f_\zeta(z - u) dz + \int_2^\infty \int_{\mathbb{R}} \mathbf{E}[g(y + \zeta)] \varphi_{0,t}(y - u) f_\zeta(t) dy dt.$$

The second integral on the right-hand side of the above identity is a C^∞ -function because for each $n \geq 1$ there are constants C_n such that $\sup_x |(\partial^n / \partial t^n) \varphi_{0,t}(x)| \leq C_n t^{3/2}$ for $t \in [2, \infty)$. The smoothness of the first integral can be ensured by the smoothness of f_ζ and integrability of its derivatives.

The first step of the proof is to obtain an integral representation for the ruin probability. For this we use the strong Markov property of the process $(X^{u,i}, \theta^i)$. Although the

functionals we are dealing with in this paper are more complicated than the ones in the above examples and require a study of the growth rate of the derivatives of their densities (see Lemma 1), the argument we will use goes along the same lines as the one above.

3. Smoothness of the Survival Probability

To use the theory of Markov processes we should consider not a single process θ but a family of processes θ^i with initial values $\theta_0^i = i$, $i \in E$. Automatically, other related processes will depend on i , the latter symbol to be often omitted in the formulae below.

The basic idea to prove the smoothness of the survival probability is to use an integral representation. Define the continuous process $Y = Y^{u,i}$ with

$$Y_t^{u,i} := e^{V_t^i} \left(u + c \int_0^t e^{-V_s^i} ds \right), \quad (4)$$

coinciding with $X^{u,i}$ on $[0, T_1)$.

We consider the case of non-life insurance, i.e., with $c > 0$.

By virtue of the strong Markov property of $(X^{u,i}, \theta^i)$

$$\Phi_i(u) = \sum_{j \in E} \mathbf{E} \left[\Phi_j(X_{T_1}^{u,i} \mathbf{1}_{\{\theta_{T_1}^i = j\}} \right] = \sum_{j \in E} \mathbf{E} \left[\Phi_j(X_{T_1-}^{u,i} + \Delta X_{T_1}^{u,i} \mathbf{1}_{\{\theta_{T_1}^i = j\}} \right].$$

On the interval $[0, T_1)$ the process $X^{u,i}$ coincides with the continuous process $Y^{u,i}$ but $\Delta X_{T_1}^{u,i} = \xi_1 < 0$. Putting $v(dt) := \alpha e^{-\alpha t} \mathbf{1}_{\{t > 0\}} dt$ we obtain that

$$\Phi_i(u) = \sum_{j \in E} \mathbf{E} \left[\mathbf{1}_{\{\theta_{T_1}^i = j\}} \Phi_j(Y_{T_1}^{u,i} + \xi_1) \right] = \sum_{j \in E} (Y_j^{1i}(u) + Y_j^{2i}(u)),$$

where

$$\begin{aligned} Y_j^{1i}(u) &:= \int_0^2 \mathbf{E} \left[\mathbf{1}_{\{\theta_t^i = j\}} \Phi_j(Y_t^{u,i} + \xi_1) \right] v(dt), \\ Y_j^{2i}(u) &:= \int_2^\infty \mathbf{E} \left[\mathbf{1}_{\{\theta_t^i = j\}} \Phi_j(Y_t^{u,i} + \xi_1) \right] v(dt). \end{aligned}$$

Clearly,

$$\begin{aligned} Y_j^{1i}(u) &:= \int_0^2 \int_0^\infty \int_{D_0} \mathbf{E} [\Phi_j(Y_t^u(Z) - w)] \mathbf{1}_{\{Z_t = j\}} m^i(dZ) F_{|\xi_1|}(dw) v(dt), \\ Y_j^{2i}(u) &:= \int_2^\infty \int_{D_0} \mathbf{E} [\tilde{\Phi}_j(Y_t^u(Z))] \mathbf{1}_{\{Z_t = j\}} m^i(dZ) v(dt), \end{aligned}$$

where $m^i(dZ)$ is the distribution of the process θ^i in the space D_0 of the E -valued càdlàg functions with a generic point $Z = (Z_s)$ and $\tilde{\Phi}_j(y) := \mathbf{E} \Phi_j(y - |\xi_1|)$. Recall that we consider the case where $\xi_1 = -|\xi_1|$.

The notation $Y_t^u(Z) = Y_t^{u,i}(Z)$ is used for the process corresponding to a fixed trajectory Z of θ , i.e., constructed from the process $R(Z)$ with $dR_t(Z) = a_{Z_t} dt + \sigma_{Z_t} dW_t$ with deterministic characteristics. Then, $Y_t^u = Y_t^u(\theta)$. A similar meaning will have symbols $S(Z)$, $V(Z)$, etc.

3.1. Smoothness of Υ^2

In this case, the distribution F_ξ is not involved (and this is the reason why we introduce the function $\tilde{\Phi}_j$).

Lemma 1. Let $G : \mathbb{R} \rightarrow [0, 1]$ be a Borel function such that $G(y) = 0$ for $y \leq 0$. Let $J(u, t, Z) := \mathbf{E}[G(Y_t^u(Z))]$, $t \geq 2$. Then, $J(\cdot, t, Z) \in C^\infty((0, \infty))$ and for each $n \geq 0$ there is a constant $C(n)$ such that

$$(\partial^n / \partial u^n) J(u, t, Z) \leq C(n) \quad \forall t \geq 2, Z \in D_0, u > 0.$$

Proof. 1. To simplify formulae we use in this proof, in a slightly abusive way, the “local” notation: Y_t^u instead $Y_t^u(Z)$, V_t instead of $V_t(Z)$, etc., and a_s , σ_s , and κ_s instead of a_{Z_s} , σ_{Z_s} , and κ_{Z_s} . Using the representation

$$Y_t^u = e^{V_t - V_1} Y_1^u + c \int_1^t e^{V_t - V_s} ds, \quad t \geq 1,$$

and the independence of the process $(V_s - V_1)_{s \geq 1}$ and the random variable Y_1^u , we obtain that

$$\mathbf{E}[G(Y_t^u) | Y_1^u] = G(t, Y_1^u),$$

where

$$G(t, y) := \mathbf{E} \left[G \left(e^{V_t - V_1} y + c \int_1^t e^{V_t - V_s} ds \right) \right].$$

Substituting the expression for Y_1^u given by (4) and using abbreviated notation for the integrals

$$\kappa \cdot L_t := \int_0^t \kappa_s ds, \quad \sigma \cdot W_t = \int_0^t \sigma_s dW_s, \quad M_t := ce^{-\kappa \cdot L} \cdot L_t = c \int_0^t e^{-\int_0^s \kappa_r dr} ds,$$

we have

$$\begin{aligned} J(u, t) &:= \mathbf{E}[G(Y_t^u)] = \mathbf{E}[G(t, Y_1^u)] = \mathbf{E} \left[G \left(t, e^{\kappa \cdot L_1 + \sigma \cdot W_1} (u + e^{-\sigma \cdot W} \cdot M_1) \right) \right] \\ &= \mathbf{E} \left[\tilde{G} \left(t, e^{\sigma \cdot W_1} (u + e^{-\sigma \cdot W} \cdot M_1) \right) \right], \end{aligned}$$

where $\tilde{G}(t, y) := G(t, e^{\kappa \cdot L_1} y)$.

2. The next step is a reduction to the case of constant coefficients. The arguments are simple. The function $A = \sigma^2 \cdot L$ is strictly increasing. Let us denote A^{-1} its inverse. The law of the continuous process with independent increments $(\sigma \cdot W_s)_{s \geq 0}$ coincides with the law of the process $(W_{A_s})_{s \geq 0}$.

Put $m(s) := ce^{-\kappa \cdot L_s}$, $\tilde{m}(r) := m(A_r^{-1}) / \sigma_{A_r^{-1}}^2$. Changing the variable, we obtain that

$$\int_0^1 e^{-W_{A_s}} dM_s = \int_0^1 e^{-W_{A_s}} m(s) ds = \int_0^{A_1} e^{-W_r} \tilde{m}(r) dr.$$

It follows that

$$J(u, t) = \mathbf{E} \left[\tilde{G} \left(t, e^{W_{A_1}} (u + e^{-W} \tilde{m} \cdot L_{A_1}) \right) \right].$$

Let $U_1(s) := A_{A_1 s}^{-1}$ and $\bar{\sigma} := \sqrt{A_1}$. Then, $(\bar{\sigma} W_{t/\bar{\sigma}^2})_{t \geq 0}$ is a Wiener process and we obtain that

$$J(u, t) = \mathbf{E} \left[\tilde{G} \left(t, e^{\bar{\sigma} W_1} (u + e^{-\bar{\sigma} W} \tilde{m} \cdot L_1) \right) \right],$$

where $\bar{m}(s) := \tilde{m}(A_1 s) A_1 = m(U_1(s)) A_1 / \sigma_{U_1(s)}^2$.

Using the conditioning with respect to $W_1 = x$ and taking into account that the conditional law of $(W_t)_{t \in [0, 1]}$ in this case is that of the Brownian bridge on $[0, 1]$ ending at the level x , that is, the same as the law of the process $W_s + s(x - W_1)$, $s \in [0, 1]$, we obtain that

$$J(u, t) = \int \mathbf{E}[H(t, x, u + \eta_x)] \varphi_{0,1}(x) dx,$$

where $H(t, x, y) = \tilde{G}(t, e^{\bar{\sigma}x}y)$ is a function taking values in $[0, 1]$,

$$\eta_x := \int_0^1 e^{-\bar{\sigma}(W_s - sW_1)} m(s, x) ds, \quad (5)$$

$$m(s, x) := e^{-\bar{\sigma}sx} \bar{m}(s) = e^{-\bar{\sigma}sx} m(U_1(s)) \bar{\sigma}^2 / \sigma_{U_1(s)}^2. \quad (6)$$

3. Let us check that η_x has a density. To this end we consider on $[0, 1]$ the Gaussian process $D_s := W_s - sW_1 - \gamma_s W_{1/2}$, where $\gamma_s := (2s) \wedge 1 - s$. Note that

$$\mathbf{E}[D_s W_{1/2}] = \mathbf{E}[(W_s - sW_1 - \gamma_s W_{1/2}) W_{1/2}] = s \wedge 1/2 - s/2 - \gamma_s/2 = 0$$

and, hence, D and $W_{1/2}$ are independent. It follows that for any bounded Borel function g

$$\mathbf{E}[g(\eta_x)] = \mathbf{E}\left[\int g(B_x(v)) \varphi_{0,1/2}(v) dv\right],$$

where the strictly positive C^∞ -function

$$v \mapsto B_x(v) := \int_0^1 e^{-\bar{\sigma}(D_s + \gamma_s v)} m(s, x) ds$$

(depending on x and also on ω , omitted as usual) is strictly decreasing, $B_x(-\infty) = \infty$ and $B_x(\infty) = 0$. The inverse function $y \mapsto B_x^{-1}(y)$ is also strictly decreasing $B_x^{-1}(0+) = \infty$ and $B_x^{-1}(\infty) = -\infty$.

After the change of variable we obtain that

$$\mathbf{E}[g(\eta_x)] = \int_0^\infty g(y) \mathbf{E}\left[\frac{\varphi_{0,1/2}(B_x^{-1}(y))}{K_x(B_x^{-1}(y))}\right] dy$$

where

$$K_x(v) = B'_x(v) = -\bar{\sigma} \int_0^1 \gamma_s e^{-\bar{\sigma}(D_s + \gamma_s v)} m(s, x) ds. \quad (7)$$

Thus, η_x has the density

$$\rho_x(y) = \mathbf{E}\left[\frac{\varphi_{0,1/2}(B_x^{-1}(y))}{K_x(B_x^{-1}(y))}\right], \quad y > 0.$$

Changing the variable we obtain that

$$J(u, t) = \int \mathbf{E}[H(t, x, u + \eta_x)] \varphi_{0,1}(x) dx = \int_{\mathbb{R}^2} H(t, x, y) \rho_x(y - u) \varphi_{0,1}(x) dy dx.$$

4. It remains to show that $\rho_x(\cdot)$ belongs to C^∞ and its derivatives have at most exponential growth. More precisely, that

$$\sup_y \left| \frac{\partial^n}{\partial y^n} \rho_x(y) \right| \leq C_n e^{C_n |x|}. \quad (8)$$

For a function $f(x, v)$ we denote by $f_v(x, v)$ its partial derivative in v . Put

$$Q^{(0)}(x, v) := \frac{\varphi_{0,1/2}(v)}{K_x(v)}, \quad Q^{(n)}(x, v) := -Q_v^{(n-1)}(x, v) \frac{1}{K_x(v)}, \quad n \geq 1.$$

Then,

$$\begin{aligned}\frac{\partial}{\partial y} Q^{(0)}(x, B_x^{-1}(y)) &= Q_v^{(0)}(x, B_x^{-1}(y)) \frac{\partial B_x^{-1}(y)}{\partial y} = -Q_v^{(0)}(x, B_x^{-1}(y)) \frac{1}{K_x(B_x^{-1}(y))} \\ &= Q^{(1)}(x, B_x^{-1}(y))\end{aligned}$$

and, similarly,

$$\frac{\partial}{\partial y} Q^{(n-1)}(x, B_x^{-1}(y)) = -Q_v^{(n-1)}(x, B_x^{-1}(y)) \frac{1}{K_x(B_x^{-1}(y))} = Q^{(n)}(x, B_x^{-1}(y)).$$

It follows that

$$\frac{\partial^n}{\partial y^n} Q^{(0)}(x, B_x^{-1}(y)) = Q^{(n)}(x, B_x^{-1}(y)).$$

It is easily seen that

$$Q^{(n)}(x, v) = \frac{\varphi_{0,1/2}(v)}{K_x^{n+1}(v)} \sum_{k=0}^n P_k(v) R_{n-k}(x, v),$$

where $P_k(v)$ is a polynomial of order k and $R_{n-k}(x, v)$ is a linear combination of products of derivatives of $K_x(v)$ in variable v .

Recall that

$$m(s, x) = e^{-\bar{\sigma}sx} m(U_1(s)) \bar{\sigma}^2 / \sigma_{U_1(s)}^2 = c e^{-\bar{\sigma}sx - \kappa \cdot L_{U_1(s)}} \bar{\sigma}^2 / \sigma_{U_1(s)}^2.$$

It follows that for all $s \in [0, 1]$ and $x \in \mathbb{R}$ we have the bounds

$$m_* e^{-\bar{\sigma}|x|} \leq m(s, x) \leq m^* e^{\bar{\sigma}|x|},$$

where $m_*, m^* > 0$ are constants depending only on bounds on $|a|$ and σ . Let $\sigma^* := \sup_s \sigma_s$, $\kappa^* := \sup_s |\kappa_s|$, and $W_1^* := \sup_{s \in [0,1]} |W_s|$. Note that $\gamma \leq 1/2$ and

$$\gamma^n \cdot L_1 = 2^{-(n+1)} / (n+1), \quad n \geq 1.$$

Clearly,

$$e^{-\sigma^*(3W_1^*+v)} \leq e^{-\bar{\sigma}(D_s+\gamma_s v)} \leq e^{\sigma^*(3W_1^*+v)},$$

Combining this with the above bounds we easily obtain from (7) that

$$\begin{aligned}\frac{1}{4} \sigma_* m_* e^{-\sigma^*(|x|+|v|+3W_1^*)} &\leq |K_x(v)| \leq \frac{1}{4} \sigma^* m^* e^{\sigma^*(|x|+|v|+3W_1^*)}, \\ \frac{(\sigma_*/2)^{n+1}}{n+2} m_* e^{-\sigma^*(|x|+|v|+3W_1^*)} &\leq \left| \frac{\partial^n}{\partial v^n} K_x(v) \right| \leq \frac{(\sigma^*/2)^{n+1}}{n+1} m^* e^{\sigma^*(|x|+|v|+3W_1^*)}.\end{aligned}$$

It follows that there exists a constant $C_n > 0$ such that

$$|Q^{(n)}(x, v)| \leq C_n e^{C_n|x|} (1+v^n) e^{-v^2} e^{C_n W_1^*}. \quad (9)$$

Therefore,

$$\mathbf{E} \left[\sup_y |Q^{(n)}(x, B_x^{-1}(y))| \right] \leq \mathbf{E} \left[\sup_{v \in \mathbb{R}} |Q^{(n)}(x, v)| \right] \leq \tilde{C}_n e^{C_n|x|} \mathbf{E} \left[e^{C_n W_1^*} \right] < \infty$$

for some constant $\tilde{C}_n > 0$. Thus, for every x the derivative $(\partial^n / \partial y^n) Q^{(0)}(x, B_x^{-1}(y))$ admits an integrable bound not depending on y . This implies that the function $y \mapsto \rho_x(y)$ is infinitely differentiable and

$$\frac{\partial^n}{\partial y^n} \rho_x(y) = \mathbf{E} \left[\frac{\partial^n}{\partial y^n} Q^{(0)}(x, B_x^{-1}(y)) \right].$$

Moreover, increasing in the need the constant C_n we obtain the inequality (8).

Since $B_x^{-1}(0+) = \infty$, the inequality (9) implies that $Q^{(n)}(x, B_x^{-1}(0+)) = 0$. Hence, $(\partial^n / \partial y^n) \rho_x(0) = 0$.

Differentiating the function $u \mapsto J(u, t)$ we obtain

$$\frac{\partial^n}{\partial u^n} J(u, t) = \int \left[\int H(t, x, y) \frac{\partial^n}{\partial u^n} \rho_x(y - u) dy \right] \varphi_{0,1}(x) dx.$$

The lemma is proven. $\square$

3.2. Smoothness of Y^1

Now, we obtain a result on the smoothness of the function

$$Y^1(u) = \int_0^2 \int_{D_0} \mathbf{E}[\Phi(Y_t^u(Z) + \xi)] M(dZ) \nu(dt).$$

Recall that $c > 0$ and $\xi < 0$. For us it is more convenient to work with the random variable $\zeta := -\xi > 0$. The needed result follows from

Proposition 1. Let $Y^u = e^V(u + ce^{-V} \cdot L)$, where $V = \kappa \cdot L + \sigma \cdot W$, κ , and $\sigma > 0$ are measurable functions. Let $\zeta > 0$ be a random variable with a density f twice continuously differentiable on $(0, \infty)$ and such that $f', f'' \in L^1(\mathbb{R}_+)$. Then, the function $\phi_t(u) := \mathbf{E}[\Phi(Y_t^u - \zeta)]$ is twice continuously differentiable in u and there is a constant C depending only on κ^* and σ^* and such that

$$|\phi_t'(u)| \leq C \|f'\|_{L^1}, \quad |\phi_t''(u)| \leq C \|f''\|_{L^1} \quad (10)$$

for all $t \in [0, 2]$, $u > 0$.

Proof. Let $H : \mathbb{R} \rightarrow [0, 1]$ be a measurable function such that $H = 0$ on $\mathbb{R}_-$ and let $\gamma > 0$ be a constant. Then, the function

$$h(y) := \int_{\mathbb{R}_+} H(y - \gamma w) f(w) dw = (1/\gamma) \int_{\mathbb{R}_+} H(z) f((y - z)/\gamma) dz.$$

It follows that h is twice differentiable,

$$h'(y) = (1/\gamma^2) \int_{\mathbb{R}} H(z) f'((y - z)/\gamma) dz, \quad h''(y) = (1/\gamma^3) \int_{\mathbb{R}} H(z) f''((y - z)/\gamma) dz.$$

Clearly, $|h'(y)| \leq (1/\gamma) \|f'\|_{L^1}$ and $|h''(y)| \leq (1/\gamma^2) \|f''\|_{L^1}$.

Let us transform our problem to the above elementary framework. Using the definitions of Y^u and V we obtain that

$$\phi_t(u) = \mathbf{E}[\Phi(Y_t^u - \zeta)] = \mathbf{E}[\Phi_t(e^{\sigma \cdot W_t}(u - e^{-\sigma \cdot W_t} m \cdot L_t) - e^{-\kappa \cdot L_t} \zeta)],$$

where the functions $\Phi_t(y) := \Phi(e^{\kappa \cdot L_t} y)$, $m := ce^{-\kappa \cdot L_t}$. Thus,

$$\phi_t(u) = \int_{\mathbb{R}} \mathbf{E}[\Phi_t(e^{\sigma \cdot W_t}(u - e^{-\sigma \cdot W_t} m \cdot L_t) - e^{-\kappa \cdot L_t} w)] f(w) dw.$$

To get rid of stochastic integrals in the above formula we consider on $\mathbb{R}_+$ the strictly increasing function $A := \sigma^2 \cdot L$ with the inverse A^{-1} and observe that the law of the process

$\sigma \cdot W$ is the same as of the process W_A . Changing the variable in the integral with respect to L we obtain that $e^{W_A m \cdot L_t} = e^W \tilde{m} \cdot L_{A_t}$ where the function $\tilde{m}(r) := m(A_r^{-1})/\sigma_{A_r^{-1}}^2$.

It follows that

$$\mathbf{E}[\Phi_t(e^{\sigma \cdot W_t}(u - e^{-\sigma \cdot W} m \cdot L_t) - e^{-\kappa \cdot L_t} w)] = \mathbf{E}[\Phi_t(e^{W_{A_t}}(u - e^{-W} \tilde{m} \cdot L_{A_t}) - e^{-\kappa \cdot L_t} w)].$$

The change in variable $s \rightarrow st/A_t$ replaces the integration over the interval $[0, A_t]$ by the integration over the interval $[0, t]$ and the right-hand side of the above equality is equal to

$$\mathbf{E}[\Phi_t(e^{\sqrt{A_t/t}W_t}(u - e^{-\sqrt{A_t/t}W} \bar{m}_t \cdot L_t) - e^{-\kappa \cdot L_t} w)], \quad (11)$$

where $\bar{m}_t(s) := \tilde{m}(A_t s)A_t = m(U_t(s))A_t/\sigma_{U_t(s)}^2$ with the abbreviation $U_t(s) := A_{A_t s}^{-1}$.

Note that in the conventional notation

$$e^{-\sqrt{A_t/t}W} \bar{m}_t \cdot L_t := \int_0^t e^{-\sqrt{A_t/t}W_s} \bar{m}_t(s) ds.$$

Using the conditioning with respect to $W_t = x$ and taking into account that the conditional law of $(W_s)_{s \in [0, t]}$ is that of the Brownian bridge on $[0, t]$ ending at the level x , that is, the same as of the process $W_s + (s/t)(x - W_t)$, $s \in [0, t]$, we infer that (11) can be written as

$$\frac{1}{\sqrt{t}} \int_{\mathbb{R}} H(u - \gamma_{t,x} w) \varphi_{0,1}(x/\sqrt{t}) dx,$$

where $H(y) = H_{t,x}(y) := \mathbf{E}[\Phi_t(e^{\sqrt{A_t/t}x}(y - \eta_{t,x}))]$, $\gamma_{t,x} := e^{-\kappa \cdot L_t - \sqrt{A_t/t}x}$,

$$\eta_{t,x} := \int_0^t e^{-\sqrt{A_t/t}(W_s - (s/t)W_t)} m_{t,x}(s) ds, \quad m_{t,x}(s) := e^{(s/t)x - \sqrt{A_t/t}x} \bar{m}_t(s).$$

Thus,

$$\phi_t(u) = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} H(u - \gamma_{t,x} w) f(w) dw \right] \varphi_{0,1}(x/\sqrt{t}) dx.$$

Let us denote by $h(u) = h_{t,x}(u) := \mathbf{E}[H(u - \gamma_{t,x}\zeta)]$, i.e., the function in the square brackets above. Then, h is twice continuously differentiable and there is a constant C such that for all $t \in [0, 2]$ and all x

$$\begin{aligned} |h'(u)| &\leq (1/\gamma_{t,x}) \|f'\|_{L^1} \leq C e^{C|x|} \|f'\|_{L^1}, \\ |h''(u)| &\leq (1/\gamma_{t,x}^2) \|f''\|_{L^1} \leq C e^{C|x|} \|f''\|_{L^1}. \end{aligned}$$

Since

$$\frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{C|x|} \varphi_{0,1}(x/\sqrt{t}) dx = \int_{\mathbb{R}} e^{C\sqrt{t}|x|} \varphi_{0,1}(x) dx \leq \int_{\mathbb{R}} e^{2C|x|} \varphi_{0,1}(x) dx < \infty,$$

this implies that the function $\phi_t(u)$ is twice continuously differentiable in u and the bounds (10) hold. $\square$

Remark 1. Minor changes in the above proof allows us to obtain the smoothness result for the annuity model ($c < 0$, $\xi > 0$) and for the model where the distribution F_{ξ} charges both half-axes.

4. Integro-Differential Equations

Knowing that the survival probability is a C^2 -function, the derivation of the integro-differential equation is easy and we obtain it for all variants of the model.

Proposition 2. Assume that $\Phi = (\Phi_0, \Phi_1, \dots, \Phi_{K-1}) \in C^2$. Then,

$$\frac{1}{2}\sigma_i^2 u^2 \Phi_i''(u) + (a_i u + c)\Phi_i'(u) + \mathcal{I}\Phi_i(u) + \sum_{j \in E} \lambda_{ij} \Phi_j(u), \quad i \in E, \quad (12)$$

where

$$\mathcal{I}\Phi_i(u) = \alpha \int (\Phi_i(u+x) - \Phi_i(u)) F_{\zeta}(dx) = \alpha \int \Phi_i(u+x) F_{\zeta}(dx) - \alpha \Phi_i(u).$$

Proof. Take arbitrary $h > 0$ and $\varepsilon > 0$ such that $u \in [\varepsilon, \varepsilon^{-1}] \subset (0, 1)$. Skipping, as usual, i in the notation we define the stopping time

$$\tau_h^\varepsilon := \inf \{t \geq 0: X_t \notin [\varepsilon, \varepsilon^{-1}]\} \wedge T_1 \wedge \tau_1 \wedge h.$$

The Itô formula applied to the function Φ and the semimartingale $(X, \theta) = (X^{u,i}, \theta^i)$ yields the identity

$$\Phi_{\theta_t}(X_t) = \Phi_i(u) + \Phi'_{\theta_-}(X_-) \sigma \cdot W_t + \mathcal{L}\Phi_{\theta}(X) \cdot L_t + \sum_{s \leq t} (\Phi_{\theta_s}(X_s) - \Phi_{\theta_{s-}}(X_{s-})), \quad (13)$$

where

$$\mathcal{L}\Phi_{\theta}(X) := \frac{1}{2}\sigma_{\theta}^2 X^2 \Phi''_{\theta}(X) + (a_{\theta} X + c)\Phi'_{\theta}(X), \quad \Sigma_t := \sum_{s \leq t} (\Phi_{\theta_s}(X_s) - \Phi_{\theta_{s-}}(X_{s-})).$$

Due to independence, the processes P and θ have no common jumps and

$$\begin{aligned} \Sigma_t &= \int_0^t \int_{\mathbb{R}} (\Phi_{\theta_{s-}}(X_{s-} + x) - \Phi_{\theta_{s-}}(X_{s-})) \pi(ds, dx) \\ &\quad + \int_0^t \sum_{k \in E} (\Phi_k(X_{s-}) - \Phi_{\theta_{s-}}(X_{s-})) p(ds, k). \end{aligned}$$

Now, we replace in Formula (13) t by τ_h^ε and take the expectation from both sides using the following observations:

- In virtue of the strong Markov property $\Phi_i(u) = \mathbf{E}\Phi_{\tau_h^\varepsilon}(X_{\tau_h^\varepsilon})$ (recall that $\Phi_i(u) = 0$ for $u \leq 0$).
- For any $\varepsilon > 0$, the integrands on $[0, \tau_h^\varepsilon(\omega)]$ are bounded from above, and therefore, the expectation of the stochastic integral over the Wiener process is zero.
- As $\tau_h^\varepsilon \leq \tau_1$, then we can replace θ_{s-} in the integrals by i . In addition, $\tau_h^\varepsilon = h$ when h is small enough.

According to the definition of dual predictable projections

$$\begin{aligned} \mathbf{E} \int_0^{\tau_h^\varepsilon} \int (\Phi_i(X_{s-} + x) - \Phi_i(X_{s-})) \pi(ds, dx) &= \mathbf{E} \int_0^{\tau_h^\varepsilon} \int (\Phi_i(X_{s-} + x) - \Phi_i(X_{s-})) ds \Pi(dx), \\ \mathbf{E} \int_0^{\tau_h^\varepsilon} \sum_{j \in E \setminus \{i\}} \Phi_j(X_{s-} - \Phi_i(X_{s-})) p(ds, j) &= \mathbf{E} \int_0^{\tau_h^\varepsilon} \sum_{j \in E} \Phi_j(X_{s-}) \lambda_{ij} ds. \end{aligned}$$

As a result we obtain that

$$\mathbf{E} \int_0^{\tau_h^\varepsilon} \left(\mathcal{L}\Phi_i(X_s) + \int (\Phi_i(X_s + x) - \Phi_i(X_s)) F_{\zeta}(dx) + \sum_{j \in E} (\Phi_j(X_s) - \Phi_i(X_s)) \lambda_{ij} \right) ds = 0.$$

Dividing both sides of this equality by h and letting $h \downarrow 0$ we obtain the result. $\square$

Remark 2. The obtained equation is homogeneous and holds as well for the ruin probability $\Psi(u) = 1 - \Phi(u)$.

5. Exponential Distribution and Differential Equations

Let us consider the case where $F_{\xi}(dx) = (1/\mu)e^{x/\mu}\mathbf{1}_{\{x < 0\}}dx$, that is, $-\xi$ has an exponential distribution. Then, the survival probability $\Phi \in C^2((0, \infty))$. Note that

$$\int_{-\infty}^0 \Phi_i(x+u)F_{\xi}(dx) = \frac{1}{\mu} \int_0^{\infty} \Phi_i(u-x)\mu e^{-x/\mu}dx = \frac{1}{\mu} \int_0^u \Phi_i(u-x)e^{-x/\mu}dx.$$

Taking into account that

$$\frac{d}{du} \int_0^u \Phi_i(u-x)e^{-x/\mu}dx = \Phi_i(u) - \frac{1}{\mu} \int_0^u \Phi_i(u-x)e^{-x/\mu}dx$$

and therefore,

$$\frac{d}{du} \mathcal{I}\Phi_i(u) = -\frac{1}{\mu} \mathcal{I}\Phi_i(u) - \alpha \Phi_i'(u).$$

Differentiating the integro-differential Equation (12) and adding to the resulting identity multiplied by μ Equation (12) we obtain an ODE of the third order, which can be transformed to the form

$$\Phi_i''' - p_i(u)\Phi_i'' + q_i(u)\Phi_i' + \frac{2}{\mu\sigma_i^2} \frac{1}{u^2} \sum_{j \in E} \lambda_{ij} \Phi_j'(u) - \frac{2}{\mu\sigma_i^2} \frac{1}{u^2} \sum_{j \in E} \lambda_{ij} \Phi_j(u) = 0,$$

where

$$p_i(u) := \frac{1}{\mu} - 2\left(1 + \frac{a_i}{\sigma_i^2}\right) \frac{1}{u} - \frac{2c}{\sigma_i^2} \frac{1}{u^2}, \quad q_i(u) := -\frac{2a_i}{\mu\sigma_i^2} \frac{1}{u} + (a_i - \alpha - c/\mu) \frac{2}{\sigma_i^2} \frac{1}{u^2}.$$

6. Conclusions

In this paper we consider the Cramér–Lundberg type model of an insurance company continuously investing the total of its reserve in a risky asset. It is assumed that the price dynamics of the latter is given by a stochastic volatility model describing the regime switching, e.g., due to changes in the economic environment. We show that the survival (and the ruin) probabilities are twice continuously differentiable functions provided that the density of jumps of the business process has two integrable derivatives. Previously the results on smoothness were known only for the case where the price dynamics was described by the geometric Brownian motion; see [4]. The smoothness property allows us to obtain, in a rigorous way, a system of integro-differential equations for the survival or ruin probabilities. In the case of exponentially distributed jumps one can obtain from this a system of ordinary differential equations.

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