mathematics

Article

The Maximal and Minimal Distributions of Wealth Processes in Black–Scholes Markets

Shuhui Liu

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China; shuhui.liu@polyu.edu.hk

Abstract: The Black–Scholes formula is an important formula for pricing a contingent claim in complete financial markets. This formula can be obtained under the assumption that the investor's strategy is carried out according to a self-financing criterion; hence, there arise a set of self-financing portfolios corresponding to different contingent claims. The natural questions are: If an investor invests according to self-financing portfolios in the financial market, what are the maximal and minimal distributions of the investor's wealth on some specific interval at the terminal time? Furthermore, if such distributions exist, how can the corresponding optimal portfolios be constructed? The present study applies the theory of backward stochastic differential equations in order to obtain an affirmative answer to the above questions. That is, the explicit formulations for the maximal and minimal distributions of wealth when adopting self-financing strategies would be derived, and the corresponding optimal (self-financing) portfolios would be constructed. Furthermore, this would verify the benefits of diversified portfolios in financial markets: that is, do not put all your eggs in the same basket.

Keywords: self-financing portfolio; optimal investment; maximal distribution; backward stochastic differential equation; diversified portfolio

MSC: 60G05; 60H05; 60H30; 91-10

Citation: Liu, S. The Maximal and Minimal Distributions of Wealth Processes in Black–Scholes Markets. *Mathematics* **2024**, *12*, 1503. https://doi.org/10.3390/math12101503

Academic Editor: Andrea Scozzari

Received: 11 April 2024
Revised: 6 May 2024
Accepted: 10 May 2024
Published: 11 May 2024

1. Introduction

In the realm of financial markets, the continuous trading of securities such as stocks forms the backbone of economic dynamics. This paper delves into a market comprising N securities operating within a fixed time horizon. The price trajectories of these securities are modeled by geometric Brownian motion: each is characterized by distinct drifts and volatilities. We explore the scenario of an investor investing his/her initial endowment into these N securities. The investor's portfolio, $\Pi(t) := (\pi_1(t), \cdots, \pi_N(t))$, represents the proportion of wealth invested in each stock. The notation V_t^{Π} represents the investor's wealth trajectory under the self-financing portfolio strategy $\Pi(t)$. In the context of modern portfolio theory, investors aim to balance risk and reward, with risk-averse individuals prioritizing predictability and lower risk over potentially higher but uncertain returns; see, for example, [1–4]. This preference underscores the importance of understanding the risk associated with a portfolio, particularly through the probability of the wealth process V_T^{π} falling within a specific interval.

Therefore, a natural question is: Can we obtain the maximal and minimal distributions of the wealth process V_T^{Π} on any specific interval over the portfolio set Θ. If this is possible, how can these two optimal portfolios, Π^* and Π_*, be constructed to achieve the maximal and minimal distributions, respectively? This is, for any given positive numbers $a < b$ and $0 \leq T < \infty$,

$$\mathbb{P}(V_T^{\Pi^*} \in [a,b]) = \sup_{\Pi \in \Theta} \mathbb{P}(V_T^{\Pi} \in [a,b]), \tag{1}$$

and

$$\mathbb{P}(V_T^{\Pi_*} \in [a,b]) = \inf_{\Pi \in \Theta} \mathbb{P}(V_T^{\Pi} \in [a,b]). \tag{2}$$

In the above proposed financial market model, the drift terms of the securities' price processes are not precisely known, introducing ambiguity into the market dynamics. This ambiguity reflects the real-world uncertainty that investors face when the true probabilities of future events are unclear or indeterminate. Unlike risk, which can be quantified and managed through probabilistic models, ambiguity challenges traditional decision-making frameworks and necessitates novel approaches to portfolio optimization.

The current study addresses this ambiguity by considering the range of possible drift values within known bounds $[\underline{\mu}, \overline{\mu}]$. By doing so, we aim to characterize the maximal and minimal distributions of the wealth process V_T^{Π}, which represent the best- and worst-case scenarios for an investor's wealth at time T given the uncertain drift terms. These distributions provide valuable insights for investors, particularly those who are risk-averse, as they offer a way to gauge the potential outcomes of their investment strategies in the face of ambiguous market conditions.

The study on ambiguity models dates back to Frank [5], who explains how uncertainty can create imperfect market structures. The portfolio optimization problem is studied by Hansen and Sargent [6], who model the volatility of stocks as a stochastic process such that the volatility of stocks is uncertain. Chen and Epstein [2] conceptualize the theoretical framework of ambiguity, risk and asset return with respect to a set of 'objective' probability measures. Cvitanic, Ma and Zhang [7] study the problem of computing hedging portfolios for options that may have discontinuous payoffs. Schied [8] uses risk assessment operators to solve the portfolio maximization problem. A robust mean-variance maximization problem is studied by Maccheroni, Marinacci and Ruffino [9]. Bielecki, Jin, Pliska and Zhou [10] study continuous-time mean-variance portfolio selection with bankruptcy prohibition. Jin and Zhou [11] study continuous-time portfolio selection under ambiguity, in which the appreciation rates are only known to be in a certain convex closed set, and the portfolios are allowed to be only based on historical stock prices. Bai, Ma and Xing [12] study a class of optimal dividend and investment problems with the assumption that the underlying reserve process follows the Sparre Andersen model. Hu, Jin and Zhou [13] study portfolio selection in a complete, continuous time market, in which the preference is dictated by the rank-dependent utility. Chen, Feng and Zhang [14] study sampling-strategy-driven limit theorems that generate the maximum or minimum average reward in the two-armed bandit problem.

To date, the above model has been widely studied. However, the explicit formulations of the maximal and minimal distributions remain unknown. The present study introduces a new method to investigate the above model. Specifically, based on the theory of backward stochastic differential equations (BSDEs), a confirmed answer can be obtained for the above question. That is, the explicit expression of Π^* and Π_* would be established, and the closed form of $\mathbb{P}(V_T^{\Pi^*} \in [a, b])$ and $\mathbb{P}(V_T^{\Pi_*} \in [a, b])$ would be obtained. Actually, we shall show that the maximal and minimal distributions are closely related to a BSDE that is nonlinear in z_t. Nonlinear BSDEs were initially studied by Pardoux and Peng [15]. It has been widely recognized that BSDEs provide a useful framework for formulating problems in various fields, such as financial mathematics, stochastic optimal control, and partial differential equations (PDEs). For example, El Karoui, Peng and Quenez [16] study different properties of BSDEs and their applications in finance, especially contingent claim valuation and recursive utility (independently introduced by Duffie and Epstein [17]). Pardoux and Peng [18] establish some estimates and regularity results for the solution of BSDEs and provide a Feynman–Kac representation for solutions to some nonlinear parabolic PDEs. Peng [19] obtain the general stochastic maximum principle through the theory of BSDEs. Yong [20] discusses the solvability of BSDEs with possibly unbounded coefficients and their applications in a Black–Scholes type security market with unbounded risk premium processes and/or interest rates. Chen and Epstein [21] study a central limit theorem for a sequence of random variables with a mean uncertainty, and it was revealed that the limit is defined by a BSDE, which can be interpreted as modeling an ambiguous continuous-time random walk.

Although BSDEs have been used in various problems, this method still has some limitations since the properties of z_t and the explicit solution of general nonlinear BSDEs cannot be easily established. For the z_t part, Chen, Kupperger and Wei [22] obtain an interesting comonotonic theorem of z_t for a nonlinear but special generator. Although it is difficult to obtain the explicit formulations for the solution of a general nonlinear BSDE, Chen, Liu, Qian and Xu [23] obtain explicit solutions to an interesting class of nonlinear BSDEs, which is the k-ignorance model that arose from modeling the ambiguity of asset pricing (e.g., Chen and Epstein [2]).

Motivated by these above results, the present paper uses BSDEs to study the optimal investment problem. The main ideas are as follows: First, the correlation between the maximal distribution $\sup_{\Pi\in\Theta}\mathbb{P}(V_T^{\Pi}\in[a,b])$ and the solution for a special kind of nonlinear BSDE (Theorem 1) is established. Second, through the formulation of the BSDE, the corresponding optimal portfolio is constructed (Theorem 2). Third, after obtaining the explicit solution for the derived BSDE, the maximal distribution is explicitly computed (Theorem 4). Similarly, the minimal distribution $\inf_{\Pi\in\Theta}\mathbb{P}(V_T^{\Pi}\in[a,b])$ and the corresponding optimal portfolio are similarly studied. For wider applications, a general utility function φ including the indicator function $\mathbf{1}_{[a,b]}$ is considered (Theorem 3). From the explicit formulations of the optimal strategy and the optimal distribution, it can easily be observed that diversified portfolios with two stocks would be better than portfolios with only one stock.

The present study is organized as follows. Section 2 presents the definition of maximal and minimal distributions and some basic results for the BSDEs used for the study. Section 3 presents the explicit representations of optimal portfolios Π^* and Π_*, which correspond to the maximal and minimal distributions, respectively. The explicit expressions for the maximal and minimal distributions and a general utility function case are presented in Section 4. The maximal distribution is applied to explain the benefits of diversified portfolios in Section 5.

2. Preliminaries

In this section, some notations and lemmas are provided. Let $(\Omega,\mathcal{F},\mathbb{P})$ refer to the probability space, $(B_t)_{t\geq 0}$ refer to the standard Brownian motion on this probability space, and $(\mathcal{F}_t)_{t\geq 0}$ refer to the σ-filtration generated by the Brownian motion, which is augmented by all $\mathbb{P}$-null sets $\mathcal{N}(\mathbb{P})$. That is, $\mathcal{F}_t=\sigma\{B_s;0\leq s\leq t\}\vee\mathcal{N}(\mathbb{P})$. Let $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ refer to the set of all $\mathcal{F}_T$-measurable and square-integrable random variables, $\mathcal{S}(0,T;\mathbb{R})$ refer to the set of all real-valued $\mathcal{F}_t$-adapted processes with $\mathbb{E}\left[\sup_{t\in[0,T]}|y_t|^2\right]<+\infty$, and $\mathcal{M}(0,T;\mathbb{R})$ refer to the set of all $\mathcal{F}_t$-progressively measurable real-valued processes with $\mathbb{E}\left[\int_0^T|z_t|^2dt\right]<\infty$. Throughout the study, $\mathbf{1}_A$ represents the indicator function on set A, $\mathbb{E}_{\mathbb{P}}[\cdot]$ denotes the expectation under probability measure $\mathbb{P}$, and the sign function $\mathrm{sgn}(x)$ is given by

$$\mathrm{sgn}(x)=\begin{cases}1, & x>0,\\ -1, & x\leq 0.\end{cases}$$

The definition of a maximal distribution is initially given. The minimal distribution is similarly defined.

Definition 1 (Maximal distribution)**.** *Let X^θ refer to the family of random variables over a given index set Θ. The maximal distribution of X^θ over the set Θ is denoted by the following:*

$$\sup_{\theta\in\Theta}\mathbb{P}\Big(X^\theta\in[a,b]\Big),\ \ \textit{for all } a,b\in R_+.$$

We now introduce the model of our study, which is set within a finite time horizon $0 \le T < \infty$. The price dynamics of the securities are governed by the following system of stochastic differential equations (SDEs):

$$\begin{cases} dS_i(t) = \mu_i \, S_i(t)dt + \sigma_i \, S_i(t)dB_t, \\ S_i(0) = x_i, \quad i = 1, 2, \cdots, N, \end{cases} \tag{3}$$

where μ_i represents the drift, $\sigma_i > 0$ is the volatility, x_i is the initial price, and B_t is a Brownian motion within the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A feature of our model is the ambiguity of the exact values of μ_i, with only their maximum and minimum known. For simplicity, we only consider the case $N = 2$ and $x_i = 1$ for $i = 1, 2$. This simplification does not detract from the generality of our results, which can be extended to scenarios with $N > 2$.

We explore the scenario of an investor investing his/her initial endowment into two stocks. The investor's portfolio, $\Pi(t) := (\pi(t), 1 - \pi(t))$, represents the proportion of wealth invested in each stock. The evolution of the investor's wealth, V_t^{π}, is governed by the stochastic differential equation:

$$\begin{cases} dV_t^{\Pi} = V_t^{\Pi}[\pi(t)\mu_1 + (1 - \pi(t))\mu_2]dt + V_t^{\Pi}[\pi(t)\sigma_1 + (1 - \pi(t))\sigma_2]dB_t, \quad t \in [0, T], \\ V_0^{\Pi} = 1. \end{cases} \tag{4}$$

The set of all possible self-financing portfolios, Θ, is defined as:

$$\Theta := \Big\{ \Pi(t) = (\pi(t), 1 - \pi(t)) : \pi(t) \in [\underline{\rho}, \overline{\rho}] \text{ is a predictable process} \Big\},$$

where $\underline{\rho}, \overline{\rho} \in [0, 1]$ refer to two fixed numbers that represent the constraints on the investment proportion of these two stocks.

At the end of this section, nonlinear BSDEs are briefly introduced, which were initially investigated in [15]:

$$y_t = \xi + \int_t^T g(y_s, z_s)ds - \int_t^T z_s dB_s. \tag{5}$$

Lemma 1 ([15])**.** *Assume that $g : \mathbb{R}^2 \to \mathbb{R}$ is uniformly Lipschitz continuous. Hence, for any $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and $T > 0$, the BSDE (5) has a unique pair of solution $(y, z) \in \mathcal{S}(0, T; \mathbb{R}) \times \mathcal{M}(0, T; \mathbb{R})$.*

Usually, it is difficult to obtain the closed form for the solution of the BSDE (5) when g is nonlinear. Interestingly, as shown in the following lemma, for cases $g(z) = k|z|$ and $\xi = \varphi(B_T)$, the following BSDE has a pair of explicit solutions:

$$Y_t = \varphi(B_T) + \int_t^T k|Z_s|ds - \int_t^T Z_s dB_s, \tag{6}$$

where φ satisfies the following assumption:

Hypothesis 1. *There exists some $c \in \mathbb{R}$ such that φ is symmetric on c. That is, $\varphi(c - x) = \varphi(c + x)$ for all $x \in \mathbb{R}$.*

Lemma 2 ([23])**.** *Assume that $\varphi \in C^3(\mathbb{R})$ satisfies (H.1) for some $c \in \mathbb{R}$, and $\varphi^{(i)}$ (where $i = 0, 1, 2, 3$) have, at most, polynomial growth. Then BSDE (6) has a pair of explicit solutions*

$$Y_t = H(B_t), \; Z_t = \partial_h H(B_t),$$

with H defined as follows:

(i) If $\varphi' \geq 0$ and $\varphi' \not\equiv 0$ on (c, ∞), then

$$H(h) = e^{-\frac{1}{2}k^2(T-t)} \int_{\mathbb{R}} \int_{y \geq 0} \varphi(x+h) e^{k|x-c+h|-k|c-h|-ky} P(B_{T-t} \in dx, L_{T-t}^{c-h} \in dy);$$

(ii) If $\varphi' \leq 0$ and $\varphi' \not\equiv 0$ on (c, ∞), then

$$H(h) = e^{-\frac{1}{2}k^2(T-t)} \int_{\mathbb{R}} \int_{y \geq 0} \varphi(x+h) e^{-k|x-c+h|+k|c-h|+ky} P(B_{T-t} \in dx, L_{T-t}^{c-h} \in dy),$$

where $\mathbb{P}(B_t \in dx, L_t^\ell \in dy)$ is the joint distribution of B_t and its local time L_t^ℓ with respect to ℓ and is given by

$$\begin{aligned} &\mathbb{P}(B_t \in dx, L_t^\ell \in dy) \\ &= \frac{1}{\sqrt{2\pi t^3}} (y + |x-\ell| + |\ell|) \exp\left\{ \frac{-(y+|x-\ell|+|\ell|)^2}{2t} \right\} \cdot \mathbf{1}_{\{y>0\}} dxdy \\ &\quad + \frac{1}{\sqrt{2\pi t}} \left[\exp\left\{ -\frac{x^2}{2t} \right\} - \exp\left\{ -\frac{(|x-\ell|+|\ell|)^2}{2t} \right\} \right] \cdot \mathbf{1}_{\{y=0\}} dxdy. \end{aligned} \tag{7}$$

3. Explicit Representation of Optimal Portfolios

For simplicity, in the following, we will suppress the time variable t in $\pi(t)$ when there is no confusion. This section provides the optimal portfolios $\Pi^* = (\pi^*, 1-\pi^*)$ and $\Pi_* = (\pi_*, 1-\pi_*)$ such that

$$\mathbb{P}(V_T^{\Pi^*} \in [a,b]) = \sup_{\Pi \in \Theta} \mathbb{P}(V_T^\Pi \in [a,b]) = \sup_{\Pi \in \Theta} \mathbb{P}(\log V_T^\Pi \in [\log a, \log b]),$$

and

$$\mathbb{P}(V_T^{\Pi_*} \in [a,b]) = \inf_{\Pi \in \Theta} \mathbb{P}(V_T^\Pi \in [a,b]) = \inf_{\Pi \in \Theta} \mathbb{P}(\log V_T^\Pi \in [\log a, \log b]).$$

Moreover, in the following it is assumed that $\sigma_1 = \sigma_2 = \sigma$ and $x = 1$. Then, the wealth process takes the following form:

$$\begin{cases} dV_t^\Pi = V_t^\Pi [\Pi(t)\mu_1 + (1-\Pi(t))\mu_2] dt + \sigma V_t^\Pi dB_t, \\ V_0^\Pi = 1, \quad t \in (0, T]. \end{cases} \tag{8}$$

Denote

$$\mu(t) := \pi(t)\Big(\mu_1 - \frac{1}{2}\sigma^2\Big) + (1-\pi(t))\Big(\mu_2 - \frac{1}{2}\sigma^2\Big).$$

Similarly, $\mu^*(t)$ and $\mu_*(t)$ are denoted corresponding to Π^* and Π_*, respectively. In order to study the optimal portfolios, the following result needs to be initially obtained.

Theorem 1. *Suppose that V_t^Π is the wealth process defined in (8) with $\sigma_1 = \sigma_2 = \sigma$, and $\Pi(t) = (\pi(t), 1-\pi(t))$ is the related portfolio. Assume that $\varphi(\sigma B_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Then*

(1) $\sup_{\Pi \in \Theta} \mathbb{E}\big[\varphi\big(\log V_T^\Pi\big)\big]$ is the value of the solution Y_t of the following BSDE at $t = 0$:

$$Y_t = \varphi(\sigma B_T) + \int_t^T \Big(\frac{\overline{\mu}}{\sigma} Z_s^+ - \frac{\underline{\mu}}{\sigma} Z_s^- \Big) ds - \int_t^T Z_s dB_s, \tag{9}$$

(2) $\inf_{\Pi \in \Theta} \mathbb{E}\big[\varphi\big(\log V_T^\Pi\big)\big]$ is the value of the solution y_t of the following BSDE at $t = 0$:

$$y_t = \varphi(\sigma B_T) + \int_t^T \Big(\frac{\underline{\mu}}{\sigma} z_s^+ - \frac{\overline{\mu}}{\sigma} z_s^- \Big) ds - \int_t^T z_s dB_s, \tag{10}$$

where

$$\overline{\mu} = \Big[\frac{\overline{\rho} - \underline{\rho}}{2} sgn(\mu_1 - \mu_2) + \frac{\overline{\rho} + \underline{\rho}}{2}\Big](\mu_1 - \mu_2) + \mu_2 - \frac{1}{2}\sigma^2,$$
$$\underline{\mu} = \Big[\frac{\underline{\rho} - \overline{\rho}}{2} sgn(\mu_1 - \mu_2) + \frac{\overline{\rho} + \underline{\rho}}{2}\Big](\mu_1 - \mu_2) + \mu_2 - \frac{1}{2}\sigma^2,$$

and $\bar{\rho}, \underline{\rho} \in [0,1]$ *are the upper bound and lower bound, respectively, of* $\pi(t)$*.*

Proof. Note that
$$d \log V_t^{\Pi} = \mu(t)dt + \sigma dB_t, \ \log V_0^{\Pi} = 0.$$
Let $\mathcal{H}$ be the set of $\{\mathcal{F}_t\}$-progressively measurable processes θ_s, $0 \le s \le T$ taking values in $[\underline{\mu}, \overline{\mu}]$. Then, from

$$\mu(t) = \pi(t)\Big(\mu_1 - \frac{1}{2}\sigma^2\Big) + (1 - \pi(t))\Big(\mu_2 - \frac{1}{2}\sigma^2\Big),$$

we have
$$\Pi(t) \in \Theta \Longleftrightarrow \mu(t) \in \mathcal{H}.$$
Therefore,

$$\begin{aligned}\sup_{\Pi\in\Theta} \mathbb{E}\Big[\varphi\Big(\log V_T^{\Pi}\Big)\Big] &= \sup_{\Pi\in\Theta} \mathbb{E}\Big[\varphi\Big(\sigma B_T + \int_0^T \pi(s)\Big(\mu_1 - \frac{1}{2}\sigma^2\Big) + (1 - \pi(s))\Big(\mu_2 - \frac{1}{2}\sigma^2\Big)ds\Big)\Big] \\ &= \sup_{\mu\in\mathcal{H}} \mathbb{E}\Big[\varphi\Big(\sigma B_T + \int_0^T \mu(s)ds\Big)\Big].\end{aligned} \tag{11}$$

Let (Y_t, Z_t) be the solution of BSDE (9). Define

$$a_s = \frac{\overline{\mu}}{\sigma}\mathbf{1}_{Z_s>0} + \frac{\underline{\mu}}{\sigma}\mathbf{1}_{Z_s\le 0}, \quad \text{and} \quad \widetilde{B}_s = B_s - \int_0^s a_r dr. \tag{12}$$

By Girsanov's theorem (see for example [24]), we know $\widetilde{B}_s$ is a Brownian motion under $\mathbb{Q}$, where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big(\int_0^t a_r dB_r - \frac{1}{2}\int_0^t a_r^2 dr\Big). \tag{13}$$

Therefore,

$$\begin{aligned}Y_t =& \varphi(\sigma B_T) + \int_t^T \Big(\frac{\overline{\mu}}{\sigma}Z_s^+ - \frac{\underline{\mu}}{\sigma}Z_s^-\Big)ds - \int_t^T Z_s dB_s \\ =& \varphi(\sigma B_T) - \int_t^T Z_s d\widetilde{B}_s \\ =& \varphi\Big(\sigma\widetilde{B}_T + \sigma\int_0^T a_r dr\Big) - \int_t^T Z_s d\widetilde{B}_s.\end{aligned}$$

Hence,
$$Y_0 = \mathbb{E}_{\mathbb{Q}}\Big[\varphi\Big(\sigma\widetilde{B}_T + \sigma\int_0^T a_r dr\Big)\Big] \le \sup_{\mu\in\mathcal{H}} \mathbb{E}_{\mathbb{Q}}\Big[\varphi\Big(\sigma\widetilde{B}_T + \int_0^T \mu(r)dr\Big)\Big]. \tag{14}$$

For any $\sigma\theta_s \in \mathcal{H}$, consider the following BSDE:

$$Y_t^{\theta} = \varphi(\sigma B_T) + \int_t^T \theta_s Z_s^{\theta} ds - \int_t^T Z_s^{\theta} dB_s. \tag{15}$$

Define $B_s^\theta = B_s - \int_0^s \theta_r dr$. Then B_s^θ is a Brownian motion under $\mathbb{P}^\theta$, where

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big(\int_0^t \theta_r dB_r - \frac{1}{2}\int_0^t \theta_r^2 dr\Big).$$

Thus,

$$Y_t^\theta = \varphi(\sigma B_T) + \int_t^T \theta_s Z_s^\theta ds - \int_t^T Z_s^\theta dB_s = \varphi\Big(\sigma B_T^\theta + \sigma\int_0^T \theta_r dr\Big) - \int_t^T Z_s^\theta dB_s^\theta.$$

Hence,

$$Y_0^\theta = \mathbb{E}_{\mathbb{P}^\theta}\Big[\varphi\Big(\sigma B_T^\theta + \sigma\int_0^T \theta_r dr\Big)\Big].$$

It follows from the comparison theorem of BSDE (e.g., [16]) that

$$Y_0^\theta \le Y_0.$$

Consequently,

$$\sup_{\theta\in\mathcal{H}} \mathbb{E}_{\mathbb{P}^\theta}\Big[\varphi\Big(\sigma B_T^\theta + \int_0^T \theta_r dr\Big)\Big] \le Y_0. \tag{16}$$

Note

$$\sup_{\theta\in\mathcal{H}} \mathbb{E}_{\mathbb{P}^\theta}\Big[\varphi\Big(\sigma B_T^\theta + \int_0^T \theta_r dr\Big)\Big] = \sup_{\theta\in\mathcal{H}} \mathbb{E}_{\mathbb{P}}\Big[\varphi\Big(\sigma B_T + \int_0^T \theta_r dr\Big)\Big] = \sup_{\mu\in\mathcal{H}} \mathbb{E}_{\mathbb{Q}}\Big[\varphi\Big(\sigma \widetilde{B}_T + \int_0^T \mu(r) dr\Big)\Big].$$

Combining (11), (14) and (16), we have

$$Y_0 = \sup_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^\pi)].$$

Similarly, part (2) of Theorem 1 can be proved. □

Now, we can give the main result of this section, which is about the optimal portfolios.

Theorem 2. *The optimal portfolios $\Pi^* = (\pi^*, 1-\pi^*)$ and $\Pi_* = (\pi_*, 1-\pi_*)$ defined by (1) and (2) are given as follows: For $t \in [0, T]$,*

$$\begin{aligned} \pi^*(t) =& \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{(\overline{\rho}-\underline{\rho})sgn(\mu_1-\mu_2)}{2} sgn\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) \\ =& \begin{cases} \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{\overline{\rho}-\underline{\rho}}{2} sgn\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big), & \mu_1 > \mu_2, \\ \frac{\overline{\rho}+\underline{\rho}}{2} - \frac{\overline{\rho}-\underline{\rho}}{2} sgn\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big), & \mu_1 \le \mu_2. \end{cases} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \pi_*(t) =& \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{(\underline{\rho}-\overline{\rho})sgn(\mu_1-\mu_2)}{2} sgn\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) \\ =& \begin{cases} \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{\underline{\rho}-\overline{\rho}}{2} sgn\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big), & \mu_1 > \mu_2, \\ \frac{\overline{\rho}+\underline{\rho}}{2} - \frac{\underline{\rho}-\overline{\rho}}{2} sgn\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big), & \mu_1 \le \mu_2. \end{cases} \end{aligned} \tag{18}$$

where

$$d\overline{R}_t = \Big[\frac{\overline{\mu}-\underline{\mu}}{2\sigma} sgn\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2\sigma}\Big]dt + dB_t,\ \overline{R}_0 = 0,$$

and

$$d\underline{R}_t = \Big[\frac{\underline{\mu}-\overline{\mu}}{2\sigma} sgn\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2\sigma}\Big]dt + dB_t,\ \underline{R}_0 = 0.$$

In this case,

$$\begin{cases} d\Big(\dfrac{\log V_t^{\Pi^*}}{\sigma}\Big) = \Big[\pi^*(t)\Big(\mu_1 - \dfrac{1}{2}\sigma^2\Big) + (1-\pi^*(t))\Big(\mu_2 - \dfrac{1}{2}\sigma^2\Big)\Big]dt + dB_t \\ \qquad\qquad\quad = \Big[\dfrac{\overline{\mu}-\underline{\mu}}{2\sigma} sgn\Big(-\overline{R}_t + \dfrac{\log ab}{2\sigma} - \dfrac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \dfrac{\overline{\mu}+\underline{\mu}}{2\sigma}\Big]dt + dB_t, \\ \log V_0^{\Pi^*} = 0, \end{cases}$$

and

$$\begin{cases} d\Big(\dfrac{\log V_t^{\Pi_*}}{\sigma}\Big) = \Big[\pi_*(t)\Big(\mu_1 - \dfrac{1}{2}\sigma^2\Big) + (1-\pi_*(t))\Big(\mu_2 - \dfrac{1}{2}\sigma^2\Big)\Big]dt + dB_t \\ \qquad\qquad\quad = \Big[\dfrac{\underline{\mu}-\overline{\mu}}{2\sigma} sgn\Big(-\underline{R}_t + \dfrac{\log ab}{2\sigma} - \dfrac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \dfrac{\overline{\mu}+\underline{\mu}}{2\sigma}\Big]dt + dB_t, \\ \log V_0^{\Pi_*} = 0. \end{cases}$$

That is, $e^{\sigma\overline{R}_t}$ is the wealth at time t with respect to $\Pi^* = (\pi^*, 1-\pi^*)$, and $e^{\sigma\underline{R}_t}$ is the wealth at time t with respect to $\Pi_* = (\pi_*, 1-\pi_*)$

Proof. By Theorem 1, we have

$$\sup_{\Pi\in\Theta} \mathbb{P}(V_T^{\Pi} \in [a,b]) = \sup_{\Pi\in\Theta} \mathbb{P}(\log V_T^{\Pi} \in [\log a, \log b]) = Y_0,$$

where

$$Y_t = \mathbf{1}_{[\log a, \log b]}(\sigma B_T) + \int_t^T \Big(\frac{\overline{\mu}}{\sigma}Z_s^+ - \frac{\underline{\mu}}{\sigma}Z_s^-\Big)ds - \int_t^T Z_s dB_s,$$

and $\mathbf{1}_{[\log a,\log b]}(\cdot)$ is the indicator function on $[\log a, \log b]$. Moreover,

$$Y_0 = \mathbb{E}_{\mathbb{Q}}\Big[\mathbf{1}_{[\log a,\log b]}\Big(\sigma\widetilde{B}_T + \sigma\int_0^T a_s ds\Big)\Big],$$

where a_s and $\mathbb{Q}$ are given by (12) and (13), respectively. Define $\widehat{B}_t = B_t - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}t$. We know from Girsanov's theorem that $\widehat{B}_t$ is a Brownian motion under $\widehat{\mathbb{P}}$ with

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big\{\frac{\overline{\mu}+\underline{\mu}}{2\sigma}B_t - \frac{1}{2}\Big|\frac{\overline{\mu}+\underline{\mu}}{2\sigma}\Big|^2 t\Big\},$$

and

$$\begin{aligned} Y_t =& \mathbf{1}_{[\log a,\log b]}(\sigma B_T) + \int_t^T \Big(\frac{\overline{\mu}}{\sigma}Z_s^+ - \frac{\underline{\mu}}{\sigma}Z_s^-\Big)ds - \int_t^T Z_s dB_s \\ =& \mathbf{1}_{[\frac{\log a}{\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}T, \frac{\log b}{\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}T]}(\widehat{B}_T) + \int_t^T \frac{\overline{\mu}-\underline{\mu}}{2\sigma}|Z_s|ds - \int_t^T Z_s d\widehat{B}_s. \end{aligned}$$

It follows from ([23] Corollary 6) that

$$\operatorname{sgn}(-Z_s) = sgn\Big(\widehat{B}_s - \frac{\log(ab)}{2\sigma} + \frac{\overline{\mu}+\underline{\mu}}{2\sigma}T\Big) = \operatorname{sgn}\Big(B_s - \frac{\log(ab)}{2\sigma} + \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-s)\Big).$$

Therefore,

$$Y_0 = \mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma\widetilde{B}_T + \int_0^T \Big[\frac{\overline{\mu}-\underline{\mu}}{2}\operatorname{sgn}\Big(-B_s + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-s)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2}\Big]ds\Big)\Big].$$

Define

$$d\overline{R}_t = \Big[\frac{\overline{\mu}-\underline{\mu}}{2\sigma}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2\sigma}\Big]dt + dB_t,\ \overline{R}_0 = 0.$$

We have

$$Y_0 = \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma B_T + \int_0^T \Big[\frac{\overline{\mu}-\underline{\mu}}{2}\operatorname{sgn}\Big(-\overline{R}_s + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-s)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2}\Big]ds\Big)\Big].$$

Since

$$\begin{aligned}
&\sup_{\Pi\in\Theta}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{[\log a, \log b]}(\log V_T^{\pi})] \\
&= \sup_{\mu\in\mathcal{H}}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma B_T + \int_0^T \mu(t)dt\Big)\Big] \\
&=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma B_T + \int_0^T \Big[\frac{\overline{\mu}-\underline{\mu}}{2}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2}\Big]dt\Big)\Big],
\end{aligned}$$

and $\sup_{\Pi\in\Theta}\mathbb{E}[\varphi(\log V_T^{\Pi})] = \sup_{\mu\in\mathcal{H}}\mathbb{E}\Big[\varphi\Big(\sigma B_T + \int_0^T \mu(s)ds\Big)\Big]$, from (11), we obtain that

$$\mu^*(t) = \frac{\overline{\mu}-\underline{\mu}}{2}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2}.$$

Moreover, we know

$$\mu^*(t) = \pi^*(t)(\mu_1 - \frac{1}{2}\sigma^2) + (1-\pi^*(t))(\mu_2 - \frac{1}{2}\sigma^2).$$

Thus,

$$\begin{aligned}
\pi^*(t) =& \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{\overline{\mu}-\underline{\mu}}{2(\mu_1-\mu_2)}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) \\
=& \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{(\overline{\rho}-\underline{\rho})\operatorname{sgn}(\mu_1-\mu_2)}{2}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) \\
=& \begin{cases} \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{\overline{\rho}-\underline{\rho}}{2}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big),\ \mu_1 > \mu_2, \\ \frac{\overline{\rho}+\underline{\rho}}{2} - \frac{\overline{\rho}-\underline{\rho}}{2}\operatorname{sgn}\Big(-\overline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big),\ \mu_1 \leq \mu_2. \end{cases}
\end{aligned} \tag{19}$$

Similarly, by Theorem 1, we have

$$\inf_{\Pi\in\Theta}\mathbb{P}(V_T^{\Pi}\in[a,b]) = \inf_{\Pi\in\Theta}\mathbb{P}(\log V_T^{\Pi}\in[\log a, \log b]) = y_0,$$

where

$$y_t = \mathbf{1}_{[\log a, \log b]}(\sigma B_T) + \int_t^T \Big(\frac{\underline{\mu}}{\sigma}z_s^+ - \frac{\overline{\mu}}{\sigma}z_s^-\Big)ds - \int_t^T z_s dB_s,$$

and

$$y_0 = \mathbb{E}_{\check{\mathbb{P}}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma \check{B}_T + \sigma \int_0^T \beta_s ds\Big)\Big],$$

$\check{B}_t = B_t - \int_0^t \beta_s ds$ is a Brownian motion under $\check{\mathbb{P}}$ with

$$\frac{d\check{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big(\int_0^t \beta_s dB_s - \frac{1}{2}\int_0^t \beta_s^2 ds\Big),$$

and

$$\beta_s = \frac{\underline{\mu}}{\sigma}\mathbf{1}_{z_s>0} + \frac{\overline{\mu}}{\sigma}\mathbf{1}_{z_s\le 0} = \frac{\underline{\mu}-\overline{\mu}}{2\sigma}\mathrm{sgn}(z_s) + \frac{\underline{\mu}+\overline{\mu}}{2\sigma}.$$

It follows from ([23] Corollary 6) that

$$\mathrm{sgn}(-z_s) = \mathrm{sgn}\Big(B_s - \frac{\log(ab)}{2\sigma} + \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-s)\Big).$$

Therefore,

$$\begin{aligned} y_0 =& \mathbb{E}_{\check{\mathbb{P}}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma \check{B}_T + \int_0^T \Big[\frac{\underline{\mu}-\overline{\mu}}{2}\mathrm{sgn}\Big(-B_s + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-s)\Big) + \frac{\underline{\mu}+\overline{\mu}}{2}\Big]ds\Big)\Big] \\ =& \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{[\log a, \log b]}\Big(\sigma B_T + \int_0^T \Big[\frac{\underline{\mu}-\overline{\mu}}{2}\mathrm{sgn}\Big(-\underline{R}_s + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-s)\Big) + \frac{\overline{\mu}+\underline{\mu}}{2}\Big]ds\Big)\Big]. \end{aligned}$$

Then we have

$$\mu_*(t) = \frac{\underline{\mu}-\overline{\mu}}{2}\mathrm{sgn}\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) + \frac{\underline{\mu}+\overline{\mu}}{2}.$$

From

$$\mu_*(t) = \pi_*(t)(\mu_1 - \frac{1}{2}\sigma^2) + (1-\pi_*(t))(\mu_2 - \frac{1}{2}\sigma^2),$$

we have

$$\begin{aligned} \pi_*(t) =& \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{\underline{\mu}-\overline{\mu}}{2(\mu_1-\mu_2)}\mathrm{sgn}\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) \\ =& \frac{\overline{\rho}+\underline{\rho}}{2} + \frac{(\underline{\rho}-\overline{\rho})\mathrm{sgn}(\mu_1-\mu_2)}{2}\mathrm{sgn}\Big(-\underline{R}_t + \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big) \\ =& \begin{cases} \dfrac{\overline{\rho}+\underline{\rho}}{2} + \dfrac{\underline{\rho}-\overline{\rho}}{2}\mathrm{sgn}\Big(-\underline{R}_t + \dfrac{\log(ab)}{2\sigma} - \dfrac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big), & \mu_1 > \mu_2, \\ \dfrac{\overline{\rho}+\underline{\rho}}{2} - \dfrac{\underline{\rho}-\overline{\rho}}{2}\mathrm{sgn}\Big(-\underline{R}_t + \dfrac{\log(ab)}{2\sigma} - \dfrac{\overline{\mu}+\underline{\mu}}{2\sigma}(T-t)\Big), & \mu_1 \le \mu_2. \end{cases} \end{aligned}$$

This completes the proof. □

Remark 1. *If the drifts μ_1 and μ_2 of the prices are known, based on (17) and (18), the optimal portfolios can be obtained with reference to the processes/wealth $\overline{R}_t$ and $\underline{R}_t$, respectively. For the case that μ_1 and μ_2 are unknown, the optimal portfolios cannot be applied directly. However, if $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are known while μ_1 and μ_2 are unknown, under the criterion of exploration and exploitation, the reinforcement learning technique (e.g., the ε-greedy method, ([25] Chapter 2) and [26]) and the above optimal portfolios can be combined together to construct the desired portfolios. With the estimated drifts (based on the historic data), a portfolio can be constructed to achieve the largest coverage probability on any interval $[a, b]$, for which the stock deduced by the optimal portfolios (17) and (18) with the estimated drifts is selected most of the time. However, every once in a while, such as with a small probability ε, the two stocks are chosen randomly (i.e., chosen with equal probabilities) independent of the estimated drifts for portfolios (17) and (18). Specifically, when the sign function in (17) is positive, the stock with the larger estimated drift is chosen with probability $1-\varepsilon$, and the two stocks are chosen randomly with the probability ε. Otherwise, the stock with the smaller estimated drift is chosen with* *probability $1-\varepsilon$, and the two stocks are chosen randomly with the probability ε. Similarly, when the sign function in (18) is positive, the stock with smaller estimated drift is chosen with probability $1-\varepsilon$, and the two stocks are chosen randomly with the probability ε. Otherwise, the stock with the larger estimated drift is chosen with probability $1-\varepsilon$, and the two stocks are chosen randomly with the probability ε. The algorithm with $\varepsilon = 0.1$ is presented in Appendix A.*

4. Maximal and Minimal Distributions

Next, the explicit distributions of the ambiguity portfolio model will be provided: that is, the explicit expressions of $\sup_{\Pi\in\Theta} \mathbb{P}(V_T^{\Pi} \in [a,b])$ and $\inf_{\Pi\in\Theta} \mathbb{P}(V_T^{\Pi} \in [a,b])$. In particular, the representations of $\sup_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^{\Pi})]$ and $\inf_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^{\Pi})]$ for general utility function φ are initially given. Then, the maximal and minimal distributions are obtained.

Theorem 3. *Assume that $\varphi \in C^3(\mathbb{R})$ satisfies (H.1) for some $c \in \mathbb{R}$, and $\varphi^{(i)}$ ($i = 0,1,2,3$) have, at most, polynomial growth. Set $k = \frac{\overline{\mu}-\underline{\mu}}{2\sigma}$. Then the representations of $\sup_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^{\Pi})]$ and $\inf_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^{\Pi})]$ are given as follows:*

(1) If $\varphi' \geq 0$ and $\varphi' \not\equiv 0$ on (c,∞), then

$$\sup_{\Pi\in\Theta}\mathbb{E}[\varphi(\log V_T^{\Pi})] = e^{-\frac{1}{2}k^2T} \times \left\{ \int_{\mathbb{R}}\int_{y\geq 0} \varphi\Big(\sigma x + \frac{\overline{\mu}+\underline{\mu}}{2}T\Big) \cdot \exp\{k|x-c| - k|c| - ky\}\mathbb{P}(B_T \in dx, L_T^c \in dy)\right\},$$

$$\inf_{\pi\in\Theta}\mathbb{E}[\varphi(\log V_T^{\pi})] = e^{-\frac{1}{2}k^2T} \times \left\{ \int_{\mathbb{R}}\int_{y\geq 0} \varphi\Big(\sigma x + \frac{\overline{\mu}+\underline{\mu}}{2}T\Big) \cdot \exp\{-k|x-c| + k|c| + ky\}\mathbb{P}(B_T \in dx, L_T^c \in dy)\right\},$$

where $\mathbb{P}(B_T \in dx, L_T^c \in dy)$ is given by (7).

(2) If $\varphi' \leq 0$ and $\varphi' \not\equiv 0$ on (c,∞), then

$$\sup_{\Pi\in\Theta}\mathbb{E}[\varphi(\log V_T^{\Pi})] = e^{-\frac{1}{2}k^2T} \times \left\{ \int_{\mathbb{R}}\int_{y\geq 0} \varphi\Big(\sigma x + \frac{\overline{\mu}+\underline{\mu}}{2}T\Big) \cdot \exp\{-k|x-c| + k|c| + ky\}\mathbb{P}(B_T \in dx, L_T^c \in dy)\right\},$$

$$\inf_{\Pi\in\Theta}\mathbb{E}[\varphi(\log V_T^{\Pi})] = e^{-\frac{1}{2}k^2T} \times \left\{ \int_{\mathbb{R}}\int_{y\geq 0} \varphi\Big(\sigma x + \frac{\overline{\mu}+\underline{\mu}}{2}T\Big) \cdot \exp\{k|x-c| - k|c| - ky\}\mathbb{P}(B_T \in dx, L_T^c \in dy)\right\}.$$

Proof. Let $\widetilde{\varphi}(x) = \varphi(\sigma x)$. Then $\mathbb{E}[\varphi(\log V_T^{\pi})] = \mathbb{E}\Big[\widetilde{\varphi}\Big(\frac{\log V_T^{\pi}}{\sigma}\Big)\Big]$. We will only give the proof of $\sup_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^{\Pi})]$ when $\varphi' \geq 0$ and $\varphi' \not\equiv 0$ on (c,∞) since the other case can be treated similarly. Using Theorem 1, we have

$$\sup_{\Pi\in\Theta} \mathbb{E}\Big[\widetilde{\varphi}\Big(\frac{\log V_T^{\Pi}}{\sigma}\Big)\Big] = Y_0,$$

where Y_0 is the solution Y_t of the following BSDE at $t = 0$:

$$Y_t = \widetilde{\varphi}(B_T) + \int_t^T \Big(\frac{\overline{\mu}}{\sigma} Z_s^+ - \frac{\underline{\mu}}{\sigma} Z_s^- \Big) ds - \int_t^T Z_s dB_s. \tag{20}$$

Set $\widehat{B}_s = B_s - \frac{\underline{\mu}+\overline{\mu}}{2\sigma} s$ and $\widehat{\varphi}(x) = \widetilde{\varphi}(x + \frac{\underline{\mu}+\overline{\mu}}{2\sigma} T)$. Then BSDE (20) is equivalent to the following equation:

$$Y_t = \widehat{\varphi}(\widehat{B}_T) + \int_t^T \frac{\overline{\mu} - \underline{\mu}}{2\sigma} |Z_s| ds - \int_t^T Z_s d\widehat{B}_s, \tag{21}$$

where $\widehat{B}_t$ is a Brownian motion under measure $\widehat{\mathbb{Q}}$ defined by

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\bigg\{ \int_0^t \frac{\underline{\mu}+\overline{\mu}}{2\sigma} dB_s - \frac{1}{2} \int_0^t \Big(\frac{\underline{\mu}+\overline{\mu}}{2\sigma} \Big)^2 ds \bigg\}.$$

Thus, it suffices to solve BSDE (21) on $(\Omega, \mathcal{F}, \widehat{\mathbb{Q}})$. By Lemma 2, we have

$$\sup_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^\Pi)] = Y_0 = e^{-\frac{1}{2}k^2 T} \times \bigg\{ \int_{\mathbb{R}} \int_{y\ge 0} \widehat{\varphi}(x) e^{k|x-c|-k|c|-ky} \widehat{\mathbb{Q}}(\widehat{B}_T \in dx, \widehat{L}_T^c \in dy) \bigg\},$$

where

$$\begin{aligned}
&\widehat{\mathbb{Q}}(\widehat{B}_T \in dx, \widehat{L}_T^c \in dy) = \mathbb{P}(B_T \in dx, L_T^c \in dy) \\
&= \frac{1}{\sqrt{2\pi T^3}} (y + |x-c| + |c|) \exp\bigg\{ \frac{-(y+|x-c|+|c|)^2}{2T} \bigg\} \cdot I_{\{y>0\}} dxdy \\
&\quad + \frac{1}{\sqrt{2\pi T}} \bigg[\exp\bigg\{ -\frac{x^2}{2T} \bigg\} - \exp\bigg\{ -\frac{(|x-c|+|c|)^2}{2T} \bigg\} \bigg] \cdot I_{\{y=0\}} dxdy.
\end{aligned}$$

So we obtain the expression of $\sup_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^\Pi)]$.

Similarly, applying Theorem 1, we have

$$\inf_{\Pi\in\Theta} \mathbb{E}\Big[\varphi(\log V_T^\Pi)\Big] = y_0,$$

where y_0 is the solution y_t of the following BSDE when $t = 0$:

$$y_t = \widetilde{\varphi}(B_T) + \int_t^T \Big(\frac{\underline{\mu}}{\sigma} z_s^+ - \frac{\overline{\mu}}{\sigma} z_s^- \Big) ds - \int_t^T z_s dB_s = \widehat{\varphi}(\widehat{B}_T) - \int_t^T \frac{\overline{\mu} - \underline{\mu}}{2\sigma} |z_s| ds - \int_t^T z_s d\widehat{B}_s.$$

It then follows from Lemma 2 that

$$\inf_{\Pi\in\Theta} \mathbb{E}\Big[\varphi(\log V_T^\Pi)\Big] = y_0 = e^{-\frac{1}{2}k^2 T} \times \bigg\{ \int_{\mathbb{R}} \int_{y\ge 0} \widehat{\varphi}(x) e^{-k|x-c|+k|c|+ky} \widehat{\mathbb{Q}}(\widehat{B}_T \in dx, \widehat{L}_T^c \in dy) \bigg\};$$

thus, the expression of $\inf_{\Pi\in\Theta} \mathbb{E}[\varphi(\log V_T^\Pi)]$ is obtained. □

Applying Theorem 3, the explicit formulations of the maximal and minimal distributions when $\varphi(x) = \mathbf{1}_{[a,b]}(x)$ with $0 < a < b < +\infty$ can be obtained.

Theorem 4. *Let* $k = \frac{\overline{\mu}-\underline{\mu}}{2\sigma}$ *and* $c = \frac{\log(ab)}{2\sigma} - \frac{\underline{\mu}+\overline{\mu}}{2\sigma} T$ *with* $0 < a < b < +\infty$*; then the maximal and minimal distributions are given by*

$$\sup_{\Pi\in\Theta} \mathbb{P}(V_T^\Pi \in [a,b]) = \Phi\left(-\frac{|c| - kT - \frac{\log(b/a)}{2\sigma}}{\sqrt{T}} \right) - e^{-\frac{k}{\sigma}\log(b/a)} \Phi\left(-\frac{|c| - kT + \frac{\log(b/a)}{2\sigma}}{\sqrt{T}} \right),$$

and

$$\inf_{\Pi\in\Theta}\mathbb{P}(V_T^{\Pi}\in[a,b])=\Phi\left(-\frac{|c|+kT-\frac{\log(b/a)}{2\sigma}}{\sqrt{T}}\right)-e^{\frac{k}{\sigma}\log(b/a)}\Phi\left(-\frac{|c|+kT+\frac{\log(b/a)}{2\sigma}}{\sqrt{T}}\right), \tag{22}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

Proof. First, recall that $\sup_{\Pi\in\Theta}\mathbb{E}\left[\varphi\left(\log V_T^{\Pi}\right)\right]$ is the value of the solution Y_t of the following BSDE at $t=0$:

$$\begin{aligned}Y_t=&\mathbf{1}_{[a,b]}(\sigma B_T)+\int_t^T\left(\frac{\overline{\mu}}{\sigma}Z_s^+-\frac{\underline{\mu}}{\sigma}Z_s^-\right)ds-\int_t^T Z_s dB_s\\=&\mathbf{1}_{\left[\frac{\log a}{\sigma}-\frac{\overline{\mu}+\underline{\mu}}{2\sigma}T,\frac{\log b}{\sigma}-\frac{\overline{\mu}+\underline{\mu}}{2\sigma}T\right]}(\widehat{B}_T)+\int_t^T\frac{\overline{\mu}-\underline{\mu}}{2\sigma}|Z_s|ds-\int_t^T Z_s d\widehat{B}_s,\end{aligned}\tag{23}$$

where $\widehat{B}_t=B_t-\frac{\overline{\mu}+\underline{\mu}}{2\sigma}t$ is a Brownian motion under $\widehat{\mathbb{P}}$ with

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}=\exp\left\{\frac{\overline{\mu}+\underline{\mu}}{2\sigma}B_t-\frac{1}{2}\left|\frac{\overline{\mu}+\underline{\mu}}{2\sigma}\right|^2 t\right\}.$$

For simplicity, let

$$\hat{a}=\frac{\log a}{\sigma}-\frac{\overline{\mu}+\underline{\mu}}{2\sigma}T,\quad \hat{b}=\frac{\log b}{\sigma}-\frac{\overline{\mu}+\underline{\mu}}{2\sigma}T.$$

For any $\varepsilon>0$, define

$$\varphi_\varepsilon(x):=\mathbb{E}_{\widehat{\mathbb{P}}}\left[\mathbf{1}_{[\hat{a},\hat{b}]}(x+\sqrt{\varepsilon}\xi)\right]=\int_{-\infty}^{\infty}\mathbf{1}_{[\hat{a},\hat{b}]}(v)\frac{1}{\sqrt{2\pi\varepsilon}}\exp\left[-\frac{(v-x)^2}{2\varepsilon}\right]dv,$$

where ξ is a standard normal distribution under probability measure $\widehat{\mathbb{P}}$. Then $\varphi_\varepsilon\in C^\infty(\mathbb{R})$ and $\varphi_\varepsilon(x)\to I_{[\hat{a},\hat{b}]}(x)$ as $\varepsilon\to 0$. Consider the following BSDE:

$$Y_t^\varepsilon=\varphi_\varepsilon(\widehat{B}_T)+\int_t^T k|Z_s^\varepsilon|ds-\int_t^T Z_s^\varepsilon d\widehat{B}_s.$$

By Theorem 3, we have

$$Y_t^\varepsilon=H^\varepsilon(\widehat{B}_t),$$

where

$$\begin{aligned}H^\varepsilon(h)=&e^{-\frac{1}{2}k^2(T-t)}\left\{\int_{\mathbb{R}}\int_{y\geq 0}\varphi_\varepsilon(x+h)e^{-k|x-c+h|+k|c-h|+ky}\widehat{\mathbb{P}}\left(\widehat{B}_{T-t}\in dx,\widehat{L}_{T-t}^{c-h}\in dy\right)\right\}\\=&e^{-\frac{1}{2}k^2(T-t)}\int_{\mathbb{R}}\int_{y>0}\frac{\varphi_\varepsilon(x+h)}{\sqrt{2\pi(T-t)^3}}e^{-k|x-c+h|+k|c-h|+ky}(y+|x-(c-h)|+|c-h|)\\&\exp\left\{\frac{-(y+|x-(c-h)|+|c-h|)^2}{2(T-t)}\right\}dxdy\\&+e^{-\frac{1}{2}k^2(T-t)}\int_{\mathbb{R}}\frac{\varphi_\varepsilon(x+h)}{\sqrt{2\pi(T-t)}}e^{-k|x-c+h|+k|c-h|}\\&\left[\exp\left\{-\frac{x^2}{2(T-t)}\right\}-\exp\left\{-\frac{(|x-(c-h)|+|c-h|)^2}{2(T-t)}\right\}\right]dxdy.\end{aligned}$$

Define

$$
\begin{aligned}
H(h) :=\, & e^{-\frac{1}{2}k^2(T-t)}\left\{\int_{\mathbb{R}}\int_{y\geq 0}\mathbf{1}_{[\hat{a},\hat{b}]}(x+h)e^{k|x-c+h|-k|c-h|-ky}\widehat{\mathbb{P}}\left(\widehat{B}_{T-t}\in dx,\widehat{L}^{c-h}_{T-t}\in dy\right)\right\}\\
=\, & e^{-\frac{1}{2}k^2(T-t)}\int_{\mathbb{R}}\int_{y>0}\frac{\mathbf{1}_{[\hat{a},\hat{b}]}(x+h)}{\sqrt{2\pi(T-t)^3}}e^{-k|x-c+h|+k|c-h|+ky}(y+|x-(c-h)|+|c-h|)\\
& \exp\left\{\frac{-(y+|x-(c-h)|+|c-h|)^2}{2(T-t)}\right\}dxdy\\
& + e^{-\frac{1}{2}k^2(T-t)}\int_{\mathbb{R}}\frac{\mathbf{1}_{[\hat{a},\hat{b}]}(x+h)}{\sqrt{2\pi(T-t)}}e^{-k|x-c+h|+k|c-h|}\\
& \left[\exp\left\{-\frac{x^2}{2(T-t)}\right\}-\exp\left\{-\frac{(|x-(c-h)|+|c-h|)^2}{2(T-t)}\right\}\right]dxdy.
\end{aligned}
$$

After some computations, we have

$$
H(h)=\Phi\left(-\frac{|h-c|-k(T-t)-\frac{\hat{b}-\hat{a}}{2}}{\sqrt{T-t}}\right)-e^{-k(\hat{b}-\hat{a})}\Phi\left(-\frac{|h-c|-k(T-t)+\frac{\hat{b}-\hat{a}}{2}}{\sqrt{T-t}}\right).
$$

By Lebesgue's dominated convergence theorem, we have that $H^{\varepsilon}(h)$ converges to $H(h)$ as $\varepsilon\to 0$, which means $H^{\varepsilon}(\widehat{B}_t)$ converges to $H(\widehat{B}_t)$ almost surely. Therefore, Y_t of (23) is given by

$$
Y_t=H(\widehat{B}_t)=\Phi\left(-\frac{|\widehat{B}_t-c|-k(T-t)-\frac{\hat{b}-\hat{a}}{2}}{\sqrt{T-t}}\right)-e^{-k(\hat{b}-\hat{a})}\Phi\left(-\frac{|\widehat{B}_t-c|-k(T-t)+\frac{\hat{b}-\hat{a}}{2}}{\sqrt{T-t}}\right).
$$

Finally,

$$
\begin{aligned}
&\sup_{\Pi\in\Theta}\mathbb{E}[\mathbf{1}_{[a,b]}(V_t^{\Pi})]=\sup_{\Pi\in\Theta}\mathbb{E}[\mathbf{1}_{[\log a,\log b]}(\log V_t^{\Pi})]=Y_0\\
=&\Phi\left(-\frac{|c|-kT-\frac{\hat{b}-\hat{a}}{2}}{\sqrt{T}}\right)-e^{-k(\hat{b}-\hat{a})}\Phi\left(-\frac{|c|-kT+\frac{\hat{b}-\hat{a}}{2}}{\sqrt{T}}\right).
\end{aligned}
$$

Similarly, we have (22). □

Remark 2. *It can be observed from Theorem 4 that the maximal and minimal distributions of wealth V_T^{π} are no longer* log-normal *when $\underline{\mu}\neq\overline{\mu}$. That is, if a random disturbance $\mu(t)$ is given to the Brownian motion (or the price process of the stocks), then its distribution will no longer be normal. That is, it would be a mixture of normal distributions. This is explained in the following example: If the process $(\log V_t)_{t\in[0,T]}$ follows the following SDE with some random disturbance $\mu(t)$,*

$$
d\log V_t=\mu(t)dt+dB_t,\ \log V_0=0,
$$

where $|\mu(t)|\leq\varepsilon$. Take $T=1,\varepsilon=1/2,a=-b$ and set

$$
\mathbb{F}_{B_1}(b):=\mathbb{P}(B_1\in[e^{-b},e^{b}]),\bar{\mathbb{F}}_{\log V_1}(b):=\sup_{|\mu(t)|\leq\varepsilon}\mathbb{P}(\log V_1\in[-b,b]),\underline{\mathbb{F}}_{\log V_1}(b):=\inf_{|\mu(t)|\leq\varepsilon}\mathbb{P}(\log V_1\in[-b,b]).
$$

Let $f_1(z)$ refer to the density function of B_1, and let $\bar{f}(z)$ and $\underline{f}(z)$ refer to the density functions of $\bar{\mathbb{F}}_{\log V_1}(\cdot)$ and $\underline{\mathbb{F}}_{\log V_1}(\cdot)$, respectively. Based on Theorem 4, it is not difficult to obtain

$$
\begin{cases}
\bar{\mathbb{F}}_{\log V_1}(b)=\Phi(1/2+b)-e^{-b}\cdot\Phi(1/2-b),\\
\underline{\mathbb{F}}_{\log V_1}(b)=\Phi(-1/2+b)-e^{b}\cdot\Phi(-1/2-b),
\end{cases}
$$

and consequently,

$$\begin{cases} \bar{f}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2+|z|+1/4}{2}} + 1/2 \cdot e^{-|z|} \cdot \Phi(-|z|+1/2), \\ \underline{f}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2-|z|+1/4}{2}} - 1/2 \cdot e^{|z|} \cdot \Phi(-|z|-1/2). \end{cases}$$

The differences in $\mathbb{F}_{B_1}$, $\bar{\mathbb{F}}_{\log V_1}$ and $\underline{\mathbb{F}}_{\log V_1}$ and the differences in f_1, $\bar{f}$ and $\underline{f}$ can be intuitively observed from Figure 1. This shows that the maximal and minimal distributions of V_1 are no longer log-*normal.*

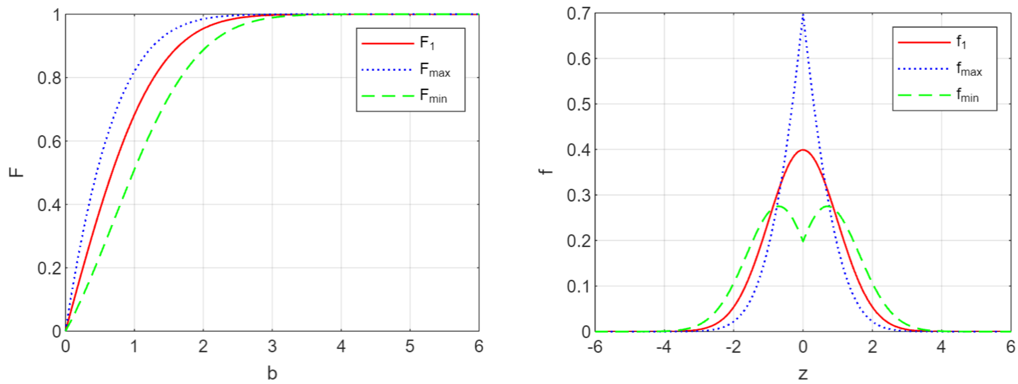

Figure 1. Differences among $\mathbb{F}_{B_1}$, $\bar{\mathbb{F}}_{\log V_1}$ and $\bar{\mathbb{F}}_{\log V_1}$ and differences in f, $\bar{f}$ and $\underline{f}$ when $a = -b$, $\varepsilon = 0.5$, $T = 1$.

5. Do Not Put All the Eggs in One Basket

'Do not put all your eggs in the same basket' is a widespread proverb that means that diversified investment is necessary in order to avoid great losses due to a single investment. On the one hand, this advice can be partly formalized by considering the volatility of the portfolio. For example, by constructing portfolios with assets that are imperfectly correlated with one another, the risk inherent in the portfolio would decline as more assets are added to the portfolio until, eventually, the volatility of the portfolio would converge to the average covariance of assets that comprise the portfolio. Therefore, diversified risks can be reduced when compared to undiversified risks. On the other hand, after obtaining the explicit formulation for the maximal distribution and the corresponding portfolio, the benefits of the diversified portfolios can be explained and the proverb from the probability framework can be formalized, as shown in the following results.

Let $\overline{\rho} = 1$ and $\underline{\rho} = 0$. Then, $\Pi_1(\cdot) \equiv (1, 0)$ and $\Pi_2(\cdot) \equiv (0, 1)$ refer to two self-financing portfolios. By applying Theorem 4, the following result can be obtained.

Proposition 1. *For $0 < a < b < +\infty$,*

$$\mathbb{P}(V_T^{\Pi^*} \in [a,b]) = \sup_{\Pi \in \Theta} \mathbb{P}(V_T^{\pi} \in [a,b]) \geq \mathbb{P}(V_T^1 \in [a,b]) \vee \mathbb{P}(V_T^2 \in [a,b]), \tag{24}$$

where $\Pi^* = (\pi^*, 1-\pi^*)$, $\pi^*(\cdot)$ is defined in (19), V^1 and V^2 are the wealth processes corresponding to portfolios $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$, respectively: that is, investing only in the first stock and only in the second stock respectively. Furthermore, let $\sigma = T = 1$, $\log b = \overline{\mu} + \delta$ and $\log a = \overline{\mu} - \delta$ for some $\delta > 0$; we have

$$\mathbb{P}(V_1^{\Pi^*} \in [a,b]) - \mathbb{P}(V_1^1 \in [a,b]) \vee \mathbb{P}(V_1^2 \in [a,b]) = (1 - e^{-(\overline{\mu}+\underline{\mu})\delta})\Phi(-\delta) > 0. \tag{25}$$

The two portfolios, $\Pi_1(\cdot) \equiv (1, 0)$ and $\Pi_2(\cdot) \equiv (0, 1)$, correspond to the cases for which all wealth is invested solely in the first and second stock, respectively. From (25), it can be observed that neither of the above portfolios is optimal in the probability framework. Instead, investing in both stocks according to $\pi^*(\cdot)$ would deduce a larger probability on any interval around the larger drift/return, thereby achieving a greater coverage probability to win a larger drift/return and reducing the risk. Therefore, a diversified portfolio with two stocks is better than a portfolio with only one stock (even when the stock has a larger drift/return). That is, the existence of a stock with a smaller drift/return does not always cause bad influences on the market. Interestingly, the combination of these two stocks would induce a larger coverage probability of wealth on any specific interval, consequently reducing the risk of the investment. Therefore, this verifies the benefits of diversified portfolios and implies the mathematical explanations for the proverb.

Remark 3. *The results for the maximal and minimal distributions can be extended to a case with more than two stocks. For example, consider that there are N $(N > 2)$ stocks in the financial market; the wealth process would follow the following SDE:*

$$\begin{cases} dV_t^\Pi = V_t^\Pi[\sum_{i=1}^N \mu_i \pi_i(t)]dt + \sigma V_t^\pi dB_t, \\ V_0^\Pi = 1, \quad t \in (0,T], \end{cases} \tag{26}$$

in which $\sum_{i=1}^N \pi_i(t) = 1$, and the set of self-financing portfolios is

$$\Theta^N := \{\Pi(t) = (\pi_1(t), \cdots, \pi_N(t)) : \pi_i(t) \in [0,1] \text{ is a predictable processes}\}.$$

Let

$$\overline{\mu} := \sup\{\mu_1 - \frac{1}{2}\sigma^2, \cdots, \mu_N - \frac{1}{2}\sigma^2\} \text{ and } \underline{\mu} := \inf\{\mu_1 - \frac{1}{2}\sigma^2, \cdots, \mu_N - \frac{1}{2}\sigma^2\}. \tag{27}$$

Then, similar to Theorem 1, it can be proved that $\sup_{\Pi \in \Theta} \mathbb{E}[\varphi(\log V_T^\Pi)]$ is equal to Y_0 of BSDE (9), with $\overline{\mu}$ and $\underline{\mu}$ given by (27). Thus, through solving BSDE (9), the maximal distributions of this case can be obtained based on Theorem 4. Furthermore, the minimal distribution can be similarly obtained.

Funding: This research was funded by Research Grants of The Hong Kong Polytechnic University under grant numbers S-ZG9U and 1-W32B.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

The algorithm is as follows:

Algorithm A1: ϵ-greedy algorithm

Input: time partition K; returns μ_1, μ_2; reward interval $[a,b]$;
Output: Wealth $\log V_1^{\pi^*}$.
1: terminal time $T = 1, t_k = \frac{k}{K}, k = 0, 1, \cdots, K$, initial condition $\log V_0^{\pi^*} = 0$, maximal drift coefficient $\overline{\mu} = \mu_1 \vee \mu_2$, minimal drift coefficient $\underline{\mu} = \mu_1 \wedge \mu_2$.
2: **for** $i = 1, 2$ **do**
3: sample means $\overline{\mu}_i(0) = 0$;
4: the number of times each state has been observed $T_i = 1$;
5: **end for**
6: **for** each $k \in [0, K]$ **do**
7: **if** k mod $10 == 0$, then
8: $j = \text{randperm}(2, 1)$;

Algorithm A1: *Cont.*

$\Delta = \frac{1}{K}\mu_j + \sigma B(\frac{1}{K})$;
$\overline{\mu}_j(t_{k+1}) = \frac{T_j - 1}{T_j}\overline{\mu}_j(t_k) + \frac{\Delta}{T_j}$;
$\overline{\mu}_i(t_{k+1}) = \overline{\mu}_i(t_k), i \neq j$;
$\log V^{\pi^*}_{t_{k+1}} = \log V^{\pi^*}_{t_k} + \Delta$;
$T_j = T_j + 1$;
else
if $\log V^{\pi^*}_{t_k} \leq \frac{\log(ab)}{2\sigma} - \frac{\overline{\mu}+\underline{\mu}}{2\sigma}(1 - t_k)$, then
find j such that $\overline{\mu}_j(t_k) = \overline{\mu}_1(t_k) \vee \overline{\mu}_2(t_k)$;
$\Delta = \frac{1}{K}\mu_j + \sigma B(\frac{1}{K})$;
$\overline{\mu}_j(t_{k+1}) = \frac{T_j - 1}{T_j}\overline{\mu}_j(t_k) + \frac{\Delta}{T_j}$;
$\overline{\mu}_i(t_{k+1}) = \overline{\mu}_i(t_k), i \neq j$;
$\log V^{\pi^*}_{t_{k+1}} = \log V^{\pi^*}_{t_k} + \Delta$;
$T_j = T_j + 1$;
else
find j such that $\overline{\mu}_j(t_k) = \overline{\mu}_1(t_k) \wedge \overline{\mu}_2(t_k)$;
$\Delta = \frac{1}{K}\mu_j + \sigma B(\frac{1}{K})$;
$\overline{\mu}_j(t_{k+1}) = \frac{T_j - 1}{T_j}\overline{\mu}_j(t_k) + \frac{\Delta}{T_j}$;
$\overline{\mu}_i(t_{k+1}) = \overline{\mu}_i(t_k), i \neq j$;
$\log V^{\pi^*}_{t_{k+1}} = \log V^{\pi^*}_{t_k} + \Delta$;
$T_j = T_j + 1$;
end if
end if
end for

References

1. Barberis, N. Investing for the Long Run when Returns Are Predictable. *J. Financ.* **2000**, *55*, 225–264. [CrossRef]
2. Chen, Z.; Epstein, L. Ambiguity, risk, and asset returns in continuous time. *Econometrica* **2002**, *70*, 1403–1443. [CrossRef]
3. Markowitz, H.M. Portfolio Selection. *J. Financ.* **1952**, *7*, 77–91.
4. Pedersen, L.H. *Efficiently Inefficient: How Smart Money Invests and Market Prices Are Determined*; Princeton University Press: Princeton, NJ, USA, 2019.
5. Frank, H.K. *Risk, Uncertainty and Profit*; BN Publishing: Hawthorne, CA, USA, 1921.
6. Hansen, L.P.; Sargent, T.J. Robust control and model uncertainty. *Am. Econ. Rev.* **2001**, *91*, 60–66. [CrossRef]
7. Cvitanic, J.; Ma, J.; Zhang, J. Efficient Computation of Hedging Portfolios for Options with Discontinuous Payoffs. *Math. Financ.* **2003**, *13*, 131–151. [CrossRef]
8. Schied, A. Risk measures and robust optimization problems. *Stoch. Model.* **2006**, *22*, 753–831. [CrossRef]
9. Maccheroni, F.; Marinacci, M.; Ruffino, D. Alpha as ambiguity: Robust mean-variance portfolio analysis. *Econometrica* **2013**, *81*, 1075–1113. [CrossRef]
10. Bielecki, T.; Jin, H.; Pliska, S.; Zhou, X. Continuous-time mean-variance portfolio selection with bankruptcy prohibition. *Math. Financ.* **2005**, *15*, 213–244. [CrossRef]
11. Jin, H.; Zhou, X. Continuous-time portfolio selection under ambiguity. *Math. Control. Relat. Fields* **2015**, *5*, 475–488. [CrossRef]
12. Bai, L.; Ma, J.; Xing, X. Optimal Dividend and Investment Problems under Sparrer-Andersen Model. *Ann. Appl. Probab.* **2017**, *27*, 3588–3632. [CrossRef]
13. Hu, Y.; Jin, H.; Zhou, X. Consistent investment of sophisticated rank-dependent utility agents in continuous time. *Math. Financ.* **2021**, *31*, 1056–1095. [CrossRef]
14. Chen, Z.; Feng, S.; Zhang, G. Strategy-Driven Limit Theorems Associated with Bandit Problems. *arXiv* **2022**, arXiv:2204.04442.
15. Pardoux, E.; Peng, S. Adapted solution of a backward stochastic differential equation. *Syst. Control. Lett.* **1990**, *14*, 55–61. [CrossRef]
16. Karoui, N.E.; Peng, S.; Quenez, M.C. Backward stochastic differential equations in finance. *Math. Financ.* **1997**, *7*, 1–71. [CrossRef]
17. Duffie, D.; Epstein, L.G. Stochastic differential utility. *Econometrica* **1992**, *60*, 353–394. [CrossRef]
18. Pardoux, E.; Peng, S. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic Partial Differential Equations and Their Applications*; Springer: Berlin/Heidelberg, Germany, 1992; Volume 176, pp. 200–217.

19. Peng, S. Backward stochastic differential equations and applications to optimal control. *Appl. Math. Optim.* **1993**, *27*, 125–144. [CrossRef]
20. Yong, J. Completeness of security markets and solvability of linear backward stochastic differential equations. *J. Math. Anal. Appl.* **2006**, *319*, 333–356. [CrossRef]
21. Chen, Z.; Epstein, L. A Central Limit Theorem for sets of probability measures. *Stoch. Process. Their Appl.* **2022**, *152*, 424–451. [CrossRef]
22. Chen, Z.; Kulperger, R.; Wei, G. A comonotonic theorem of BSDEs. *Stoch. Process. Their Appl.* **2005**, *115*, 41–54. [CrossRef]
23. Chen, Z.; Liu, S.; Qian, Z.; Xu, X. Explicit solutions for a class of nonlinear BSDEs and their nodal sets. *Probab. Uncertain. Quant. Risk* **2022**, *7*, 283–300. [CrossRef]
24. Karatzas, I.; Shreve, S. *Brownian Motion and Stochastic Calculus*; Springer: New York, NY, USA, 1988.
25. Sutton, R.S.; Barto, A.G. *Reinforcement Learning: An Introduction*; MIT Press: Cambridge, MA, USA, 1998.
26. Agrrawal, P. An automation algorithm for harvesting capital market information from the web. *Manag. Financ.* **2009**, *35*, 427–438.