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Multivariate Mittag-Leffler Solution for a Forced Fractional-Order Harmonic Oscillator

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Abstract: The harmonic oscillator is a fundamental physical–mathematical system that allows for the description of a variety of models in many fields of physics. Utilizing fractional derivatives instead of traditional derivatives enables the modeling of a more diverse array of behaviors. Furthermore, if the effect of the fractional derivative is applied to each of the terms of the differential equation, this will involve greater complexity in the description of the analytical solutions of the fractional differential equation. In this work, by using the Laplace method, the solutions to the multiple-term forced fractional harmonic oscillator are presented, described through multivariate Mittag-Leffler functions. Additionally, the cases of damped and undamped free fractional harmonic oscillators are addressed. Finally, through simulations, the effect of the fractional non-integer derivative is demonstrated, and the consistency of the result is verified when recovering the integer case.

Keywords: fractional forced oscillator; multivariate Mittag-Leffler function; fractional calculus; multivariate Laplace transform

MSC: 26A33

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1. Introduction

Fractional calculus emerged as a natural extension of the classical calculus pioneered by Leibniz and Newton. In recent years, the incorporation of non-integer-order derivatives has significantly influenced the modeling of systems with memory, characterized by power law relations [1–5]. Nonetheless, the utilization of fractional-order derivatives introduces a heightened level of complexity due to their unconventional properties. While these properties have proven invaluable in characterizing intricate dynamics of certain systems, they pose challenges in solving fractional differential equations, prompting the development and utilization of novel methodologies and tools.

Fractional oscillators represent a compelling area of study [6–8]. However, due to the complexity introduced by fractional derivatives, coupled with the diverse range of fractional operators and the singularity of their kernels, obtaining analytical solutions proves to be a non-trivial endeavor. In [9], the non-multiple-term harmonic oscillator equation focuses solely on the effect of the fractional derivative on the corresponding damping term. Here, the derivative term associated with the inertial force is set to the integer value 2, facilitating the discovery of an analytic solution through Laplace transform, which is delineated by exponential functions. In [10], a straightforward approach is introduced for formulating the fractional differential equation governing the behavior of the free harmonic oscillator, thereby ensuring a coherent physical interpretation. This work delineates two distinct scenarios of non-multiple-term fractional oscillators: one involving the application of fractional derivatives to the inertial force term, and the other involving the application of fractional derivatives to the damping factor term. The solutions are expressed in the framework of Mittag-Leffler functions. In [11], the utilization of the Weyl derivative operator provides insight into the solutions pertaining to three distinct classes of free harmonic oscillators. The first class integrates the fractional inertial force, while the second incorporates fractional order within the damping term. The third class delves into the fractional differential equation governing the non-commensurate harmonic oscillator, accounting for the impact of fractional derivatives on both the inertial force and damping. Employing Fourier transform techniques and leveraging Mittag-Leffler functions, the author provides closed-form responses (free, impulse, step, frequency, and sinusoidal) for each of these three categories of fractional oscillators.

In [12], the solution to the non-integer-order differential equation linked with the harmonic oscillator is delineated. This study addresses the multi-term issue by employing the Lorenzo–Hartley generalized G function, dependent on fractional parameters. Notably, this solution exhibits an intricate form, given its reliance on an infinite sum of convolutions; Mittag-Leffler functions are not employed for its depiction.

The multivariate Mittag-Leffler function, utilized for solving a fractional integro-differential equation [13], derives its name from its close connection to the multinomial expansion function and multinomial coefficients [14]. Numerous researchers have explored broader categories of fractional-order models, including equations with multiple terms and systems in multiple dimensions [14–16]. In [16], the operational calculus method is applied to solve fractional differential equations featuring multiple terms with various types of fractional derivatives. The study achieves a solution in a completely closed form. To accomplish this, the authors introduce multivariate Mittag-Leffler functions, which entail double (bivariate) and triple (trivariate) series. Additionally, they construct the associated fractional calculus operators.

In the present study, we conduct a comprehensive analysis on the multiple-term forced fractional harmonic oscillator by presenting its solutions in a unified and simplified manner using multivariate Mittag-Leffler functions. Section 2 introduces preliminary concepts of fractional calculus. Section 3 presents the main results, including solutions for the forced and unforced harmonic oscillators, as well as expressions for certain Laplace transforms of multivariate Mittag-Leffler functions. We also discuss the effects of fractional derivatives on the harmonic oscillator and provide corresponding graphical simulations. Finally, Section 4 offers our conclusions.

2. Preliminaries

In this section, we shall present some basic definitions. We will begin by defining the Caputo differo-integral operator.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$, $N - 1 < \gamma \leq N$, with $N \in \mathbb{Z}$. D^{γ} denotes the Caputo fractional derivative operator, as introduced below:*

$$D^{\gamma}x(t) := \begin{cases} \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} x(\tau)d\tau, & \gamma \in \mathbb{R}^- \\ x(t), & \gamma = 0 \\ D^{\gamma-\lceil\gamma\rceil} \frac{d^{\lceil\gamma\rceil}}{dt^{\lceil\gamma\rceil}}(x(t)), & \gamma \in \mathbb{R}^+ \end{cases} \tag{1}$$

To obtain an analytical solution to a fractional differential equation employing the Caputo operator, it is essential to establish the Laplace transform, [17] as stated below:

Property 1. *Let $x : [0, \infty) \to \mathbb{R}$ be a function with a Laplace transform given by $\mathcal{L}\{x\}(s) = X(s)$. Then*

$$\mathcal{L}\{D^{\gamma} x\}(s) = \begin{cases} s^{\gamma} X(s), & \gamma \in \mathbb{R}^- \\ X(s), & \gamma = 0 \\ s^{\gamma} X(s) - \sum_{k=0}^{\lceil \gamma \rceil - 1} s^{\gamma - k - 1} D^k x(0^+), & \gamma \in \mathbb{R}^+ \end{cases}$$

where D^{γ} is the Caputo operator in (1).

The Mittag-Leffler function is a generalization of the exponential function, that is, $e^z = E_{1,1}(z)$. Extensions to two or three parameters are widely recognized and extensively explored in the academic literature [18], but these still entail a single-power series in one variable.

Definition 2. *The two-parameter Mittag-Leffler function is defined as follows:*

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}$$

where $\alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $z \in \mathbb{C}$.

Another proposed generalization involves what are known as the "bivariate" and "multivariate" Mittag-Leffler functions. Unlike the conventional definition relying on a series of powers in a single variable z_1, this generalization is characterized by a series of double powers in two variables, z_1 and z_2, or even in multiple variables.

Definition 3. *The multivariate Mittag-Leffler function, referred to as such and defined in Remark 11.8 of [19], is as follows:*

$$E_{(\alpha_1,\ldots,\alpha_n),\beta}(z_1, \ldots, z_n) := \sum_{k=0}^{\infty} \sum_{L_1,\ldots,L_n \geq 0}^{L_1+\cdots+L_n=k} \binom{k}{L_1, \ldots, L_n} \frac{\prod_{j=1}^{n} z_j^{L_j}}{\Gamma\left(\beta \pm \sum_{j=1}^{n} \alpha_j L_j\right)} \tag{2}$$

where $\alpha_j, \beta \in \mathbb{C}$, $\Re(\alpha_j) > 0$, $\Re(\beta) > 0$ for $j = 1, 2, \cdots, n$, $(z_1, \ldots, z_n) \in \mathbb{C}^n$, and the multinomial coefficients are defined as

$$\binom{k}{L_1, \ldots, L_n} = \frac{k!}{L_1! \cdots L_n!}, \quad k, L_1, \ldots, L_n \in \mathbb{N}_0$$

It is easy to see that for all $\alpha_j, \beta \in \mathbb{C}$, $\Re(\alpha_j) > 0$, $\Re(\beta) > 0$, for $j = 1, 2$, the following statement is fulfilled:

$$E_{(\alpha_1,\alpha_2),\beta}(z, 0) = E_{\alpha_1,\beta}(z) \tag{3}$$

for all $z \in \mathbb{C}$ and by using the convention that $0^0 = 1$.

3. The Fractional-Order Forced Harmonic Oscillator

A damped oscillator subjected to an external force, acting on a mass from equilibrium, is described by the following second-order linear differential equation:

$$m \frac{d^2 x(t)}{dt^2} + \beta \frac{dx(t)}{dt} + \kappa x(t) = F(t)$$

where m is the mass, β is the damped coefficient, κ is the spring constant, and $F(\cdot)$ is the external force applied in the system, where the initial conditions are considered as $x(0) = 0$, $x'(0) = b_1$.

As demonstrated in [10], substituting the time derivative operator with its fractional counterpart, characterized by the introduction of the new parameter σ, ensures a physically consistent interpretation. For additional details, please refer to [20]. When considering a periodic force $F(t) = F_0 \sin(\omega t)$, we derive the fractional-order harmonic oscillator as follows:

$$\frac{m}{\sigma^{2(1-\gamma)}} \frac{d^{2\gamma} x(t)}{dt^{2\gamma}} + \frac{\beta}{\sigma^{(1-\gamma)}} \frac{d^{\gamma} x(t)}{dt^{\gamma}} + \kappa x(t) = F_0 \sin(\omega t) \tag{4}$$

Therefore, we can express the fractional-order differential equation of the harmonic oscillator in a simplified form, as illustrated below:

$$D^{2\gamma} x(t) + A D^{\gamma} x(t) + \omega_0^2 x(t) = B \sin(\omega t)$$

where $D^{\gamma} x(t)$ represents the Caputo fractional-order derivative operator of the function x of order γ with initial conditions $D^0 x(0^+) = 0$, $D^1 x(0^+) = b_1$ for $0.5 \leq \gamma \leq 1$; the term associated to the damping factor is $A = \frac{\beta \sigma^{(1-\gamma)}}{m}$, related to the retarding force between the spring and the mass $\omega_0^2 = \frac{\kappa \sigma^{2(1-\gamma)}}{m}$; and, finally, $B = \frac{F_0 \sigma^{2(1-\gamma)}}{m}$ corresponds to the driven external force. We say that the motion is (i) forced, if $B \neq 0$; (ii) unforced or free, if $B = 0$; (iii) damped, if $A > 0$; and (iv) undamped, if $A = 0$. In the case of the forced harmonic oscillator, we will distinguish between two types of forcing: constant and periodic.

The following statements provide explicit expressions for the Laplace transform of certain multivariate Mittag-Leffler functions. These expressions will be utilized in the proofs of the main theorems.

Lemma 1. *Let* $\gamma \in [0.5, 1]$, $\omega, \omega_0, A \in \mathbb{R}$, *and* $k = 0, 1$. *Then, for all* $t \geq 0$

$$\begin{aligned} &\mathcal{L}^{-1}\left\{ \frac{s^{1-k}}{(s^2 + \omega^2)(s^{2\gamma} + A s^{\gamma} + \omega_0^2)} \right\}(t) \\ &= t^{2\gamma + k} \sum_{n_1 \geq 0} \left(-\omega^2 t^2\right)^{n_1} E_{(2\gamma, \gamma), 2n_1 + 2\gamma + k + 1}\left(-(\omega_0 t^{\gamma})^2, A t^{\gamma}\right) \end{aligned} \tag{5}$$

Proof. It is widely acknowledged that for all $x \in \mathbb{R}$ such that $|x| < 1$, the following holds:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \tag{6}$$

$$\frac{1}{(1+x)^m} = \sum_{l=0}^{\infty} (-1)^l \binom{m+l-1}{m-1} x^l \tag{7}$$

where $m \in \mathbb{N}$. Therefore

$$\begin{aligned} \frac{s^{1-k}}{(s^2 + \omega^2)(s^{2\gamma} + A s^{\gamma} + \omega_0^2)} &= \frac{1}{s^{k+1}\left(1 + \frac{\omega^2}{s^2}\right)(s^{2\gamma} + A s^{\gamma})\left(1 + \frac{\omega_0^2}{s^{2\gamma} + A s^{\gamma}}\right)} \\ &= \frac{1}{s^{k+1}} \sum_{n_1 \geq 0} (-1)^{n_1} \frac{\omega^{2n_1}}{s^{2n_1}} \frac{1}{s^{2\gamma} + A s^{\gamma}} \sum_{n_2 \geq 0} (-1)^{n_2} \frac{\omega_0^{2n_2}}{(s^{2\gamma} + A s^{\gamma})^{n_2}} \\ &= \sum_{n_1, n_2 \geq 0} (-1)^{n_1 + n_2} \frac{\omega^{2n_1} \omega_0^{2n_2}}{s^{2n_1 + k + 1}} \frac{1}{(s^{2\gamma} + A s^{\gamma})^{n_2 + 1}} \\ &= \sum_{n_1, n_2 \geq 0} (-1)^{n_1 + n_2} \frac{\omega^{2n_1} \omega_0^{2n_2}}{s^{2n_1 + 2\gamma(n_2 + 1) + k + 1}} \frac{1}{\left(1 + \frac{A}{s^{\gamma}}\right)^{n_2 + 1}} \end{aligned}$$

where we have used (6) in the terms $1/(1+\omega^2/s^2)$ and $1/(1+\omega_0^2/(s^{2\gamma}+As^\gamma))$ to obtain the right-hand side of the second equality. By applying (7) in the term $1/(1+A/s^\gamma)^{n_2+1}$, we obtain

$$\begin{aligned}
&\frac{s^{1-k}}{(s^2+\omega^2)(s^{2\gamma}+As^\gamma+\omega_0^2)} \\
&= \sum_{n_1,n_2\geq 0}(-1)^{n_1+n_2}\frac{\omega^{2n_1}\omega_0^{2n_2}}{s^{2n_1+2\gamma(n_2+1)+k+1}}\sum_{n_3\geq 0}(-1)^{n_3}\binom{n_2+n_3}{n_2}\frac{A^{n_3}}{s^{\gamma n_3}} \\
&= \sum_{n_1,n_2,n_3\geq 0}^{\infty}(-1)^{n_1+n_2+n_3}\frac{\omega^{2n_1}\omega_0^{2n_2}}{s^{2n_1+2\gamma(n_2+1)+\gamma n_3+k+1}}A^{n_3}\binom{n_2+n_3}{n_2}
\end{aligned}$$

For all $s>0$ and $\gamma>-1$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\gamma+1}}\right\}(t)=\frac{t^\gamma}{\Gamma(\gamma+1)} \tag{8}$$

then

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{s^{1-k}}{(s^2+\omega^2)(s^{2\gamma}+As^\gamma+\omega_0^2)}\right\}(t) \\
&= \sum_{n_1,n_2,n_3\geq 0}^{\infty}(-1)^{n_1+n_2+n_3}\omega^{2n_1}\omega_0^{2n_2}\mathcal{L}^{-1}\left\{\frac{1}{s^{2n_1+2\gamma(n_2+1)+\gamma n_3+k+1}}\right\}(t)A^{n_3}\binom{n_2+n_3}{n_2} \\
&= \sum_{n_1,n_2,n_3\geq 0}(-1)^{n_1+n_2+n_3}\frac{t^{2n_1+2\gamma(n_2+1)+\gamma n_3+k}}{\Gamma(2n_1+2\gamma(n_2+1)+\gamma n_3+k+1)}\omega^{2n_1}\omega_0^{2n_2}A^{n_3}\binom{n_2+n_3}{n_2} \\
&= t^{2\gamma+k}\sum_{n_1\geq 0}(-\omega^2t^2)^{n_1}\sum_{n_2,n_3\geq 0}\binom{n_2+n_3}{n_2}\frac{[-(\omega_0t^\gamma)^2]^{n_2}[-At^\gamma]^{n_3}}{\Gamma(2n_1+2\gamma+k+1+2\gamma n_2+\gamma n_3)} \\
&= t^{2\gamma+k}\sum_{n_1\geq 0}(-\omega^2t^2)^{n_1}E_{(2\gamma,\gamma),2n_1+2\gamma+k+1}\Big(-(\omega_0t^\gamma)^2,-At^\gamma\Big)
\end{aligned}$$

By utilizing the definition provided in (2), we arrive at the final equality. □

Corollary 1. *Let $\gamma\in[0.5,1]$, $\omega_0, A\in\mathbb{R}$. Then, for all $t\geq 0$*

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^{2\gamma}+As^\gamma+\omega_0^2)}\right\}(t)=t^{2\gamma}E_{(2\gamma,\gamma),2\gamma+1}\Big(-(\omega_0t^\gamma)^2,-At^\gamma\Big)$$

Proof. The results are obtained by substituting $k=0$ and $\omega=0$ into (5), taking into account the convention $0^0=1$. □

Lemma 2. *Let $\gamma\in[0.5,1]$ and $\omega_0, A\in\mathbb{R}$. Then, for all $t\geq 0$*

$$\mathcal{L}^{-1}\left\{\frac{s^{2\gamma-2}}{s^{2\gamma}+As^\gamma+\omega_0^2}\right\}(t)=tE_{(2\gamma,\gamma),2}\Big(-(\omega_0t^\gamma)^2,-At^\gamma\Big)$$

Proof. We proceed similarly to the previous Lemma:

$$\begin{aligned}
\frac{s^{2\gamma-2}}{s^{2\gamma}+As^{\gamma}+\omega_0^2} &= \frac{s^{2\gamma-2}}{(s^{2\gamma}+As^{\gamma})\left(1+\frac{\omega_0^2}{s^{2\gamma}+As^{\gamma}}\right)} \\
&= \frac{s^{2\gamma-2}}{s^{2\gamma}+As^{\gamma}}\sum_{m_1\geq 0}(-1)^{m_1}\frac{\omega_0^{2m_1}}{(s^{2\gamma}+As^{\gamma})^{m_1}} \\
&= s^{2\gamma-2}\sum_{m_1\geq 0}(-1)^{m_1}\frac{\omega_0^{2m_1}}{(s^{2\gamma}+As^{\gamma})^{m_1+1}} \\
&= s^{2\gamma-2}\sum_{m_1\geq 0}(-1)^{m_1}\frac{\omega_0^{2m_1}}{s^{2\gamma(m_1+1)}}\frac{1}{\left(1+\frac{A}{s^{\gamma}}\right)^{m_1+1}} \\
&= \sum_{m_1\geq 0}(-1)^{m_1}\frac{\omega_0^{2m_1}}{s^{2\gamma m_1+2}}\sum_{m_2\geq 0}(-1)^{m_2}\binom{m_1+m_2}{m_1}\frac{A^{m_2}}{s^{\gamma m_2}} \\
&= \sum_{m_1,m_2\geq 0}^{\infty}(-1)^{m_1+m_2}\frac{\omega_0^{2m_1}}{s^{2\gamma m_1+\gamma m_2+2}}A^{m_2}\binom{m_1+m_2}{m_1}
\end{aligned}$$

where we have used (6) in the term $1/(1+\omega_0^2/(s^{2\gamma}+As^{\gamma}))$ and (7) in the term $1/(1+A/s^{\gamma})^{m_1+1}$ to obtain the second and fifth equalities, respectively.

Hence, employing (8), we find the following:

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{s^{2\gamma-2}}{s^{2\gamma}+As^{\gamma}+\omega_0^2},\right\}(t) \\
&= \sum_{m_1,m_2\geq 0}^{\infty}(-1)^{m_1+m_2}\omega_0^{2m_1}\mathcal{L}^{-1}\left\{\frac{1}{s^{2\gamma m_1+\gamma m_2+2}}\right\}(t)A^{m_2}\binom{m_1+m_2}{m_1} \\
&= \sum_{m_1,m_2\geq 0}(-1)^{m_1+m_2}\frac{t^{2\gamma m_1+\gamma m_2+1}}{\Gamma(2\gamma m_1+\gamma m_2+2)}\omega_0^{2m_1}A^{m_2}\binom{m_1+m_2}{m_1} \\
&= t\sum_{m_1,m_2\geq 0}\binom{m_1+m_2}{m_1}\frac{[-(\omega_0 t^{\gamma})^2]^{m_1}[-At^{\gamma}]^{m_2}}{\Gamma(2+2\gamma m_1+\gamma m_2)} = tE_{(2\gamma,\gamma),2}\left(-(\omega_0 t^{\gamma})^2,-At^{\gamma}\right).
\end{aligned}$$

Once more, employing the definition provided in (2), we arrive at the final equality. □

Theorem 1 (Fractional harmonic oscillator with periodic forcing)**.** *Denote a fractional-order forced harmonic oscillator as*

$$D^{2\gamma}x(t)+AD^{\gamma}x(t)+\omega_0^2 x(t) = Bp_k(\omega t) \tag{9}$$
$$D^0x(0^+)=0, D^1x(0^+)=b_1, \quad k=0,1$$

where $p_0=\cos(\cdot)$ and $p_1=\sin(\cdot)$. Then, its analytical solution expressed in terms of multivariate Mittag-Leffler functions takes the following form:

$$\begin{aligned}
x_{p_k}(t) &= B\omega^k t^{2\gamma+k}\sum_{n_1\geq 0}\left(-\omega^2t^2\right)^{n_1}E_{(2\gamma,\gamma),2n_1+2\gamma+k+1}\left(-(\omega_0t^{\gamma})^2,At^{\gamma}\right) \\
&\quad + b_1tE_{(2\gamma,\gamma),2}\left(-(\omega_0t^{\gamma})^2,-At^{\gamma}\right)
\end{aligned} \tag{10}$$

where x_{p_k} represents the solution corresponding to the periodic forcing function p_k for $k=0,1$.

Proof. By applying the Laplace transform to (9), we obtain the following:

$$s^{2\gamma}X(s) - \sum_{k=0}^{\lceil 2\gamma \rceil -1} s^{2\gamma-k-1}D^k x(0) + A(s^{\gamma}X(s)) + \omega_0^2 X(s) = \frac{B\omega^k s^{1-k}}{s^2+\omega^2}$$

By solving with respect to the variable $X(s)$, we have the following:

$$X(s) = \frac{B\omega^k s^{1-k}}{(s^2+\omega^2)(s^{2\gamma}+As^{\gamma}+\omega_0^2)} + \frac{s^{2\gamma-2}b_1}{s^{2\gamma}+As^{\gamma}+\omega_0^2} \tag{11}$$

By applying Lemmas 1 and 2 to the first and second terms of (11), respectively, and utilizing the linearity of the inverse Laplace transform, we arrive at the expression provided in (10). □

The expression provided in (10) represents the solution of a fractional harmonic oscillator subjected to an external periodic force. In Figure 1, we observe the behavior across various values of γ, with a corresponding periodic forcing $p_0 = cos(3t)$, demonstrating how the solution obtained with the Caputo derivative operator generalizes the entire case. Furthermore, the graphs illustrate that as the γ value decreases, the amplitude of the oscillation is attenuated.

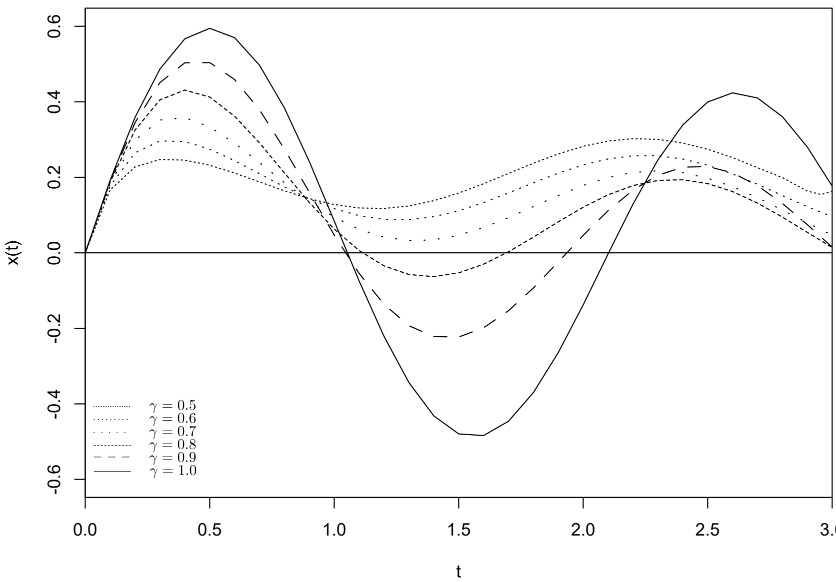

Figure 1. Solution of the fractional harmonic oscillator with periodic forcing in (10) for $p_0 = cos(3t)$ where $t \in [0,3]$, $A = 1$, $B = 1$, $b_1 = 2$, and $\omega_0 = 3$.

The forced harmonic oscillator is depicted below, particularly when subjected to constant forcing, such as that provided by the Heaviside function.

Theorem 2 (Fractional harmonic oscillator with constant forcing)**.** *Denote a fractional order forces harmonic oscillator as*

$$D^{2\gamma}x(t) + AD^{\gamma}x(t) + \omega_0^2 x(t) = BH_c(\omega t), \quad D^0x(0^+) = 0, D^1x(0^+) = b_1 \tag{12}$$

where $c > 0$ *and* $H_c(t) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{if } 0 \leq t < c \end{cases}$

Then, its analytical solution expressed in terms of multivariate Mittag-Leffler functions takes the following form:

$$x(t) = BH_c(t)(t-c)^{2\gamma}E_{(2\gamma,\gamma),2\gamma+1}\Big(-(\omega_0(t-c)^{\gamma})^2, -A(t-c)^{\gamma}\Big) \\ + b_1 t E_{(2\gamma,\gamma),2}\Big(-(\omega_0 t^{\gamma})^2, -At^{\gamma}\Big). \tag{13}$$

Proof. By applying the Laplace transform to (12) and by solving with respect to the variable s, we obtain the following:

$$X(s) = \frac{Be^{-cs}}{s(s^{2\gamma} + As^{\gamma} + \omega_0^2)} + \frac{s^{2\gamma-2}b_1}{s^{2\gamma} + As^{\gamma} + \omega_0^2}.$$

It follows from Corollary 1 and Lemma 2 that, by utilizing the definition of multivariate Mittag-Leffler functions in (2), the solution described in (13) is derived. □

In Figure 2, various values of γ are depicted for the fractional harmonic oscillator driven by a constant force defined by $H_c(\omega t)$. It can be observed that as the value of γ decreases, the oscillations are attenuated. Graphically, it is evident that the solution for $\gamma = 1$ recovers the well-known solution for the entire case.

The case of the free oscillator is derived from the general fractional harmonic oscillator, as presented below.

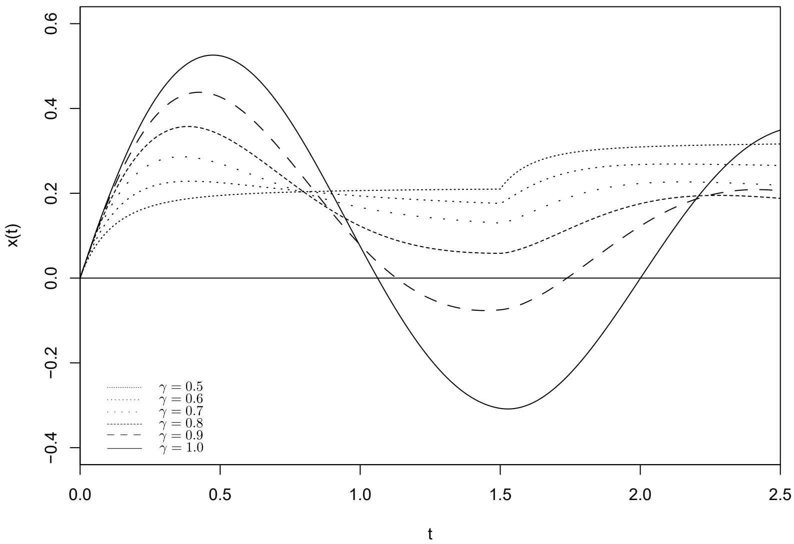

Figure 2. The solution of the fractional harmonic oscillator with constant forcing, as given in (13), evaluated for $t \in [0, 2.5]$, $A = 1$, $B = 1$, $b_1 = 2$, $c = 1.5$, and $\omega_0 = 3$.

Corollary 2 (Free undamped fractional harmonic oscillator)**.** *In the absence of an external force acting on an undamped oscillator, $A = 0$ and $B = 0$. Denote a fractional free undamped oscillator as*

$$D^{2\gamma}x(t) + \omega_0^2 x(t) = 0$$

Considering the initial conditions $D^0x(0^+) = 0$, $D^1x(0^+) = b_1$, the solution takes the following form:

$$x(t) = b_1 t E_{2\gamma,2}(-(\omega_0 t^{\gamma})^2) \tag{14}$$

Proof. From (10) and (3), the corresponding solution is as follows:

$$\begin{aligned} x(t) &= b_1 t E_{(2\gamma,\gamma),2}\Big(-(\omega_0 t^{\gamma})^2, 0\Big) \\ &= b_1 t E_{2\gamma,2}(-(\omega_0 t^{\gamma})^2) \end{aligned}$$

□

Moreover, if $\gamma = 1$, then the solution is

$$x(t) = b_1 t E_{2,2}(-(\omega_0 t)^2) = \frac{b_1}{\omega_0}\sin(\omega_0 t)$$

This is the well-known solution in the entire case, with initial conditions $x(0) = 0$ and $x'(0) = b_1$.

An interesting behavior can be observed in simple harmonic motion. Consider a mass that is suspended from a spring attached to a rigid support; this is commonly known in the entire case as a mass–spring system. Similarly, in Corollary 2, we can observe a family of oscillators of this type for different derivative values, enabling us to appreciate their behavior. Figure 3 illustrates the behavior across different values of γ, showcasing how the solution obtained with the Caputo derivative operator generalizes the entire case. Additionally, the graphs demonstrate that as the γ value decreases, the oscillations also decrease.

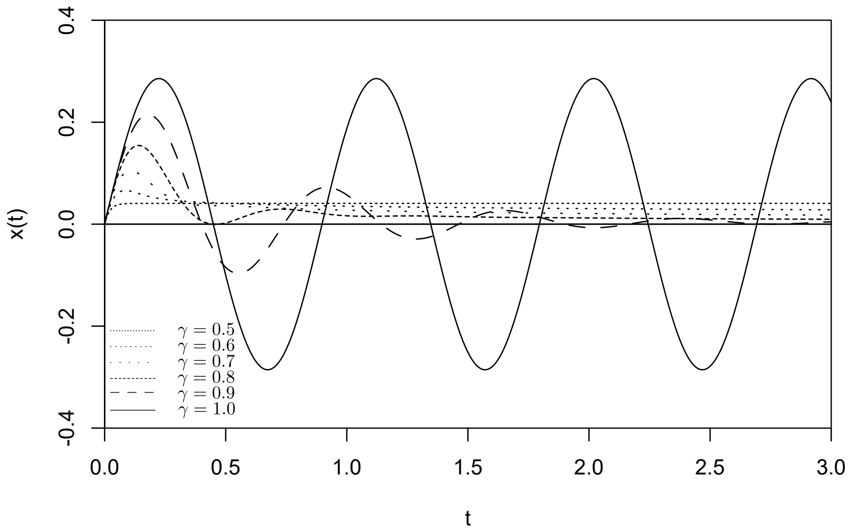

Figure 3. Free undamped fractional harmonic oscillator in (14) for $t \in [0, 3]$, $b_1 = 2$, and $\omega_0 = 7$.

Corollary 3 (Free damped fractional harmonic oscillator)**.** *As before, in the absence of an external force, that is, $B = 0$, denote the damped oscillator for $A \neq 0$ as*

$$D^{2\gamma}x(t) + AD^{\gamma}x(t) + \omega_0^2 x(t) = 0$$

Considering the initial conditions as $D^0 x(0^+) = 0$, $D^1 x(0^+) = b_1$, the solution for the fractional forced damped oscillator is as follows:

$$x(t) = b_1 t E_{(2\gamma,\gamma),2}\left(-(\omega_0 t^{\gamma})^2, -At^{\gamma}\right) \tag{15}$$

Proof. Directly from (10), we derive the solution for the fractional forced damped oscillator. □

In correspondence with damped vibrations behavior, a generalization of the fractional damped oscillator exhibits critical damping, which, in its integer-order expression, physically implies that the frictional force is sufficient to halt the object's vibration. Conversely, its fractional nature can be observed in Corollary 3. In Figure 4, the value of $\gamma = 1$ illustrates the integer-order underdamped behavior; it is evident that as the value of γ decreases, the oscillations diminish. We could even assert that for $\gamma = 0.5$, this behavior resembles that of an overdamped system.

All the MATLAB routines are available at the following address: https://www.mathworks.com/matlabcentral/fileexchange/165121-solveffharmonicosc_fun (accessed on 3 May 2024, version 1.0.0).

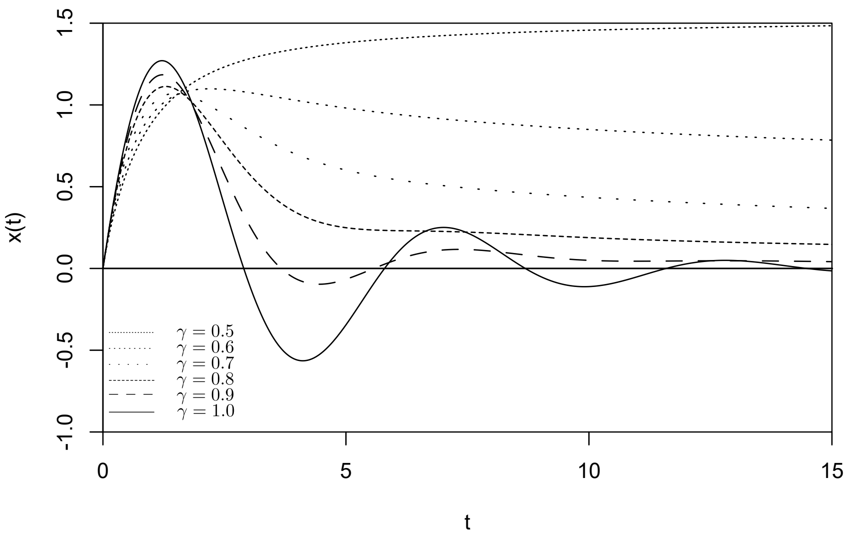

Figure 4. Fractional free damped oscillator in (15) for $t \in [0, 15]$, $A = 0.56$, $b_1 = 2$, and $\omega_0 = 1.12$.

4. Conclusions

This article presents the solution of the forced harmonic oscillator for various types of forcing, including periodic and constant forcing. The problem of fractional order is addressed by considering the incidence in all terms of the derivative. The proposed solutions provide a simplified representation of the solutions through the use of multivariate Mittag-Leffler functions. This is an improvement over the solutions proposed so far, which use infinite sums of convolutions. Additionally, during the process of describing the proposed solutions, the expressions of the inverse Laplace transforms are provided in terms of the multivariate Mittag-Leffler functions. All the solutions obtained recover the integer case, confirming the consistency of the results presented in Theorems 1 and 2.

Additionally, the free harmonic oscillator is presented for the damped and undamped case through the use of the multivariate Mittag-Leffler function, providing another way of representing the solution of this class of oscillators. We emphasize that the multivariate Mittag-Leffler functions we employ are not reported in the literature and could be highly useful for describing a wide range of physical phenomena.

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