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Article

Chebyshev–Jensen-Type Inequalities Involving χ-Products and Their Applications in Probability Theory

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Abstract: By means of the functional analysis theory, reorder method, mathematical induction and the dimension reduction method, the Chebyshev-Jensen-type inequalities involving the χ-products $\langle\cdot\rangle_\chi$ and $[\cdot]_\chi$ are established, and we proved that our main results are the generalizations of the classical Chebyshev inequalities. As applications in probability theory, the discrete with continuous probability inequalities are obtained.

Keywords: χ-product; Chebyshev inequality; Jensen inequality; countermonotone; probability density function

MSC: 26D15; 26E60

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1. Introduction

In [1], the authors points out that the classical Chebyshev's integral inequality is deeply connected with the study of positive dependence of random variables, which are monotone functions of a common random variable. The Chebyshev-type inequalities and their applications were investigated by many authors [1–13]. In [2], the authors established the Chebyshev-type inequalities involving the permanents of matrix as follows:

$$\frac{\mathrm{per}(A\odot B)}{n!}\geq\frac{\mathrm{per}A}{n!}\cdot\frac{\mathrm{per}B}{n!}\text{ and }\frac{\mathrm{per}A}{\prod_{i=1}^n\sum_{j=1}^n a_{i,j}}\leq\frac{\mathrm{per}B}{\prod_{i=1}^n\sum_{j=1}^n b_{i,j}}.$$

In [3], the authors established the following Chebyshev type inequality:

$$\frac{\langle\mathbf{a},\ \mathbf{b}\rangle}{\langle\mathbf{a}^*,\ \mathbf{b}^*\rangle}\geq\frac{\|\mathbf{a}\|_p}{\|\mathbf{a}^*\|_p}\cdot\frac{\|\mathbf{b}\|_q}{\|\mathbf{b}^*\|_q},$$

where $\mathbf{a},\mathbf{b},\mathbf{a}^*,\mathbf{b}^*\in\mathbb{R}^n_{++}$,

$$\langle\mathbf{a},\ \mathbf{b}\rangle\triangleq\sum_{i=1}^n a_ib_i\text{ and }\|\mathbf{a}\|_p\triangleq\begin{cases}\left(\sum_{i=1}^n|a_i|^p\right)^{1/p}, & 0<p<\infty\\ \max\limits_{1\leq i\leq n}\{|a_i|\}, & p=\infty\end{cases}.$$

The Jensen type inequalities and their applications were also investigated by many authors [14–24]. In [14], the authors considered that comparing two integral means for absolutely continuous functions, whose absolute value of the derivative are convex, and displayed its applications.

The probability density function [24–30] of the random variable is a basic concept in the theories of probability and statistics. In [24], the authors studied the monotonicity of the interval function

$$\mathrm{JVar}_{\phi}\varphi\left(X_{[a,b]}\right) \triangleq \frac{\int_a^b p_I \phi(\varphi)}{\int_a^b p_I} - \phi\left(\frac{\int_a^b p_I \varphi}{\int_a^b p_I}\right),$$

which involving the probability density function p_I, where $I = [a,b]$ is an interval, and displayed its applications in the analysis of variance and the higher education. In [25], the authors considered the probability density function of a stochastic HIV model with cell-to-cell infection.

This paper established the Chebyshev–Jensen-type inequalities involving the χ-products $\langle\cdot\rangle_\chi$ and $[\cdot]_\chi$, and we proved that our main results are the generalizations of the classical Chebyshev inequalities (see Corollaries 1 and 2), as well as displaying the applications of our main results in probability theory, and the discrete with continuous probability inequalities are obtained.

In Section 2, we defined the comonotone and the χ-products; in Section 3, we established the discrete Chebyshev–Jensen-type inequalities; In Section 4, we established the continuous Chebyshev–Jensen-type inequalities; in Section 5, we displayed the applications of our main results in probability theory.

The research tools of this paper include the theories of functional analysis [31–33], discrete mathematics [2,34], mean value [35,36], and probability [24–30]. The research methods of this paper are based on mathematical induction [2,36], the reorder method [2], and the dimension reduction method [36].

2. Basic Concepts and Classical Results

We will use the following hypotheses and notations throughout the paper.

$$\mathbb{N} \triangleq \{0,1,2,\ldots,j,\ldots\},\ \mathbb{N}_k^n \triangleq \{j \in \mathbb{N} : k \le j \le n\},\ \mathbb{R} \triangleq (-\infty,\infty),$$

$$\mathbb{R}_+ \triangleq [0,\infty),\ \mathbb{R}_{++} \triangleq (0,\infty),\ X \triangleq (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n,\ Y \triangleq (y_1,y_2,\ldots,y_n) \in \mathbb{R}^n,$$

$$\overline{X} \triangleq n^{-1}(x_1 + x_2 + \cdots + x_i + \cdots + x_n) \in \mathbb{R},\ X^r \triangleq (x_1^r, x_2^r, \ldots, x_i^r, \ldots, x_n^r) \in \mathbb{R}^n,$$

$$\mathbf{e} \triangleq (1,1,\ldots,1,\ldots,1) \in \mathbb{R}^n,\ f(X) \triangleq (f(x_1), f(x_2), \ldots, f(x_i), \ldots, f(x_n)) \in \mathbb{R}^n,$$

$$X^1X^2\cdots X^r\cdots X^m \triangleq \left(x_1^1x_1^2\cdots x_1^m, x_2^1x_2^2\cdots x_2^m, \ldots, x_r^1x_r^2\cdots x_r^m, \ldots, x_n^1x_n^2\cdots x_n^m\right) \in \mathbb{R}^n,$$

$$C[0,1] \triangleq \{f : [0,1] \to I : f \text{ be continuous}\},\ C_r[0,1] \triangleq \{f_r : [0,1] \to I_r : f_r \text{ be continuous}\},$$

$$\overline{f} \triangleq \int_0^1 f(t)\mathrm{d}t,\ f \in C[0,1],\ T \triangleq (t_1,t_2,\ldots,t_r,\ldots,t_m) \in \mathbb{R}^m,\ \Pi(T) \triangleq t_1t_2\cdots t_r\cdots t_m,$$

$$\chi_r(T) \triangleq \frac{\partial\chi(T)}{\partial t_r},\ \forall r \in \mathbb{N}_1^m,\ \chi_{rs}(T) \triangleq \frac{\partial^2\chi(T)}{\partial t_r \partial t_s},\ \forall r,s \in \mathbb{N}_1^m,$$

$$\mathbb{I}^m \triangleq I_1 \times I_2 \times \cdots \times I_r \times \cdots \times I_m,\ I^n = \mathbb{I}^n \Leftrightarrow I_1 = I_2 = \cdots = I_i = \cdots = I_n = I,$$

where $I, I_1, I_2, \ldots, I_r, \ldots, I_m$ are the intervals, $i \in \mathbb{N}_1^n$, $r \in \mathbb{N}_1^m$, $m \in \mathbb{N}_2^\infty$ and $n \in \mathbb{N}_2^\infty$.

Definition 1 (see [1]). *The points $X, Y \in \mathbb{R}^n$ are said to be comonotone, written as $X \backsim Y$, if*

$$\left(x_i - x_j\right)\left(y_i - y_j\right) \ge 0,\ \forall i,j \in \mathbb{N}_1^n, \tag{1}$$

and X and Y are said to be countermonotone, written as $X \backsimeq Y$, if $-X \backsim Y$; the functions $f, g : [0,1] \to \mathbb{R}$ are said to be comonotone, written as $f \backsim g$, if

$$(f(x) - f(y))(g(x) - g(y)) \ge 0,\ \forall x,y \in [0,1], \tag{2}$$

and f and g are said to be countermonotone, written as $f \backsimeq g$, if $-f \backsim g$.

Definition 2. *Let the function $\chi : \mathbb{I}^m \to \mathbb{R}$ be continuous. Then, we define the functional [31–33]*

$$\langle X^1, \ldots, X^m \rangle_\chi : I_1^n \times \cdots \times I_m^n \to \mathbb{R},\ \langle X^1, \ldots, X^m \rangle_\chi \triangleq \frac{1}{n}\sum_{i=1}^{n} \chi\left(x_i^1, \ldots, x_i^m\right) \tag{3}$$

as the first χ-product of the points $X^1, X^2, \ldots, X^m$, and the functional

$$\left[X^1, \ldots, X^m\right]_\chi : I_1^n \times \cdots \times I_m^n \to \mathbb{R},\ \left[X^1, \ldots, X^m\right]_\chi \triangleq \frac{1}{n^m}\sum_{1 \le i_1, \ldots, i_m \le n} \chi\left(x_{i_1}^1, \ldots, x_{i_m}^m\right) \tag{4}$$

as the second χ-product of the points $X^1, X^2, \ldots, X^m$.

By Definition 2, we see that the functional $\langle X^1, \ldots, X^m \rangle_\chi$ is the mean value [35,36] of the functions $\chi(x_i^1, \ldots, x_i^m)$, $\forall i \in \mathbb{N}_1^n$, and the functional $\left[X^1, \ldots, X^m\right]_\chi$ is the mean value of the functions $\chi\left(x_{i_1}^1, \ldots, x_{i_m}^m\right)$, $\forall i_1, \ldots, i_m \in \mathbb{N}_1^n$, and

$$\langle X^1, X^2, \ldots, X^m \rangle_\Pi = \overline{X^1 X^2 \cdots X^m} \tag{5}$$

with

$$\left[X^1, X^2, \ldots, X^m\right]_\Pi = \Pi\left(\overline{X^1}, \overline{X^2}, \cdots, \overline{X^m}\right) = \overline{X^1} \times \overline{X^2} \times \cdots \times \overline{X^m}. \tag{6}$$

Definition 3. *Let the function $\chi : \mathbb{I}^m \to \mathbb{R}$ be continuous. Then, we define the functional [31–33]*

$$\langle f_1, \ldots, f_m \rangle_\chi : C_1[0,1] \times \cdots \times C_m[0,1] \to \mathbb{R},\ \langle f_1, \ldots, f_m \rangle_\chi \triangleq \int_0^1 \chi(f_1, \ldots, f_m)\mathrm{d}t \tag{7}$$

as the first χ-product of the functions $f_1, f_2, \ldots, f_m$, where $f_r \triangleq f_r(t)$, $\forall r \in \mathbb{N}_1^m$, and the functional

$$[f_1, \ldots, f_m]_\chi : C_1[0,1] \times \cdots \times C_m[0,1] \to \mathbb{R},\ [f_1, \ldots, f_m]_\chi \triangleq \int_0^1 \cdots \int_0^1 \chi(f_1, \ldots, f_m)\mathrm{d}t_1 \cdots \mathrm{d}t_m \tag{8}$$

as the second χ-product of the functions $f_1, f_2, \ldots, f_m$, where $f_r \triangleq f_r(t_r)$, $\forall r \in \mathbb{N}_1^m$.

By Definition 3, we see that the functional $\langle f_1, \ldots, f_m \rangle_\chi$ is the mean value [35,36] of the function $\chi(f_1(t), \ldots, f_m(t))$, and the functional $[f_1, \ldots, f_m]_\chi$ is the mean value of the function $\chi(f_1(t_1), \ldots, f_m(t_m))$, and

$$\langle f_1, f_2, \ldots, f_m \rangle_\Pi = \overline{f_1 f_2 \cdots f_m} \tag{9}$$

with

$$[f_1, f_2, \ldots, f_m]_\Pi = \Pi\left(\overline{f_1}, \overline{f_2}, \cdots, \overline{f_m}\right) = \overline{f_1} \times \overline{f_2} \times \cdots \times \overline{f_m}. \tag{10}$$

The classical Chebyshev inequalities [1–13] can be expressed as follows.

Let $X^1, X^2 \in \mathbb{R}^n$. If $X^1 \backsim X^2$, then we have the following discrete Chebyshev inequality:

$$\frac{1}{n}\sum_{i=1}^{n} x_i^1 x_i^2 = \langle X^1, X^2 \rangle_\Pi \ge \left[X^1, X^2\right]_\Pi = \overline{X^1} \times \overline{X^2} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^1\right) \times \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right), \tag{11}$$

and

$$\langle X^1, X^2 \rangle_\Pi = \overline{X^1} \times \overline{X^2} \Leftrightarrow X^1 = x_1^1\mathbf{e} \vee X^2 = x_1^2\mathbf{e}. \tag{12}$$

Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be continuous. If $f_1 \backsim f_2$, then we have the following continuous Chebyshev inequality:

$$\int_0^1 f_1(t)f_2(t)\mathrm{d}t = \langle f_1, f_2\rangle_\Pi \geq [f_1, f_2]_\Pi = \overline{f_1} \times \overline{f_2} = \left(\int_0^1 f_1(t_1)\mathrm{d}t_1\right) \times \left(\int_0^1 f_2(t_2)\mathrm{d}t_2\right), \tag{13}$$

and

$$\langle f_1, f_2\rangle_\Pi = \overline{f_1} \times \overline{f_2} \Leftrightarrow f_1(t) \equiv f_1(0) \vee f_2(t) \equiv f_2(0). \tag{14}$$

An important hypothesis of Chebyshev inequality (11) is $X^1 \backsim X^2$, and an important hypothesis of Chebyshev inequality (13) is $f_1 \backsim f_2$. Using these methods to deal with the inequality problems is called the reorder method [2].

The classical Jensen inequalities [14–24] can be expressed as follows.

Let the function $f : I \to \mathbb{R}$ be a strictly convex function [14,21–23]. Then, for any $X \in I^n$, we have the following discrete Jensen inequality:

$$\overline{f(X)} \geq f(\overline{X}), \tag{15}$$

and

$$\overline{f(X)} = f(\overline{X}) \Leftrightarrow X = x_1\mathbf{e}. \tag{16}$$

Let the functions $g : [0, 1] \to \mathbb{R}$ and $f : g[0, 1] \to \mathbb{R}$ be continuous, where $g[0, 1]$ is the valued field of the function g, and let the function $f : g[0, 1] \to \mathbb{R}$ be a strictly convex function. Then, we have the following continuous Jensen inequality:

$$\overline{f(g)} \geq f(\overline{g}), \tag{17}$$

and

$$\overline{f(g)} = f(\overline{g}) \Leftrightarrow g(t) \equiv g(0). \tag{18}$$

This paper will generalize the Chebyshev inequalities (11) and (13), and establish the Chebyshev–Jensen-type inequalities involving the χ-products $\langle\cdot\rangle_\chi$ and $[\cdot]_\chi$.

3. Discrete Chebyshev-Jensen-Type Inequalities

Theorem 1 (Discrete Chebyshev–Jensen-type inequalities)**.** *Let the function $\chi : \mathbb{I}^m \to \mathbb{R}$ be continuous and*

$$\frac{\partial^2\chi(T)}{\partial t_r \partial t_s} > 0,\ \forall T \in \mathbb{I}^m \ \wedge\ \forall r, s \in \mathbb{N}_1^m. \tag{19}$$

If $\left(X^1, X^2, \ldots, X^m\right) \in I_1^n \times I_2^n \times \cdots \times I_m^n$ and

$$X^r \backsim X^s,\ \forall r, s \in \mathbb{N}_1^m, \tag{20}$$

then we have the following discrete Chebyshev–Jensen-type inequalities:

$$\langle X^1, X^2, \ldots, X^m\rangle_\chi \geq \left[X^1, X^2, \ldots, X^m\right]_\chi \geq \chi\left(\overline{X^1}, \overline{X^2}, \ldots, \overline{X^m}\right). \tag{21}$$

Both the equalities in (21) hold if and only if

$$X^r = x_1^r\mathbf{e},\ \forall r \in \mathbb{N}_1^m. \tag{22}$$

Lemma 1. *For the equalities*

$$\left(x_i^r - x_j^r\right)\left(x_i^s - x_j^s\right) = 0,\ \forall i, j \in \mathbb{N}_1^n \ \wedge\ \forall r, s \in \mathbb{N}_1^m \ \wedge\ r \neq s \tag{23}$$

and

$$X^s = x_1^s\mathbf{e},\ \exists k \in \mathbb{N}_1^m \ \wedge\ \forall s \in \mathbb{N}_1^m \setminus \{k\}, \tag{24}$$

then the conditions (23) and (24) are equivalent.

Proof. We first prove that (23) $\Rightarrow$ (24).

Indeed, if

$$X^s = x_1^s \mathbf{e}, \ \forall s \in \mathbb{N}_1^m, \tag{25}$$

then for any $k \in \mathbb{N}_1^m$, the equalities in (24) hold.

Assume that there exists a $k \in \mathbb{N}_1^m$ such that $X^k \neq x_1^k \mathbf{e}$. Then, there exists a $p \in \mathbb{N}_2^n$ such that $x_p^k \neq x_1^k$.

Let $i \in \mathbb{N}_1^n$. Since $x_p^k \neq x_1^k$, we have $x_i^k \neq x_p^k \ \vee \ x_i^k \neq x_1^k$. Assume that $x_i^k \neq x_1^k$. By (23), we have

$$\begin{aligned} & \left(x_i^k - x_1^k\right)(x_i^s - x_1^s) = 0, \ \forall i \in \mathbb{N}_1^n \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & x_i^s = x_1^s, \ \forall i \in \mathbb{N}_1^n \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & X^s = x_1^s \mathbf{e}, \ \exists k \in \mathbb{N}_1^m \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & (24). \end{aligned}$$

Assume that $x_i^k \neq x_p^k$. By (23), we have

$$\begin{aligned} & \left(x_i^k - x_p^k\right)\left(x_i^s - x_p^s\right) = 0, \ \forall i \in \mathbb{N}_1^n \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & x_i^s = x_p^s, \ \forall i \in \mathbb{N}_1^n \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & X^s = x_p^s \mathbf{e}, \ \exists k \in \mathbb{N}_1^m \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & X^s = x_1^s \mathbf{e}, \ \exists k \in \mathbb{N}_1^m \ \wedge \ \forall s \in \mathbb{N}_1^m \setminus \{k\} \\ \Rightarrow \ & (24). \end{aligned}$$

Next, we prove that (24) $\Rightarrow$ (23).

Indeed, since $r \neq s$, we have $r \neq k \ \vee \ s \neq k$. Assume that $r \neq k$. Then, by (24), we have $X^r = x_1^r \mathbf{e}$. Hence, (23) holds. Assume that $s \neq k$. Then, by (24), we have $X^s = x_1^s \mathbf{e}$. Hence, the equalities in (23) also hold.

In summary, the conditions (23) and (24) are equivalent. □

Lemma 2. *Let the function $\chi : \mathbb{I}^m \to \mathbb{R}$ be continuous and*

$$\frac{\partial^2 \chi(T)}{\partial t_r \partial t_s} > 0, \ \forall T \in \mathbb{I}^m \ \wedge \ \forall r, s \in \mathbb{N}_1^m \ \wedge \ r \neq s. \tag{26}$$

If $\left(X^1, X^2, \ldots, X^m\right) \in I_1^n \times I_2^n \times \cdots \times I_m^n$ and

$$X^r \backsim X^s, \ \forall r, s \in \mathbb{N}_1^m \ \wedge \ r \neq s, \tag{27}$$

then we have the following Chebyshev-type inequality:

$$\langle X^1, X^2, \ldots, X^m \rangle_\chi \geq \left[X^1, X^2, \ldots, X^m\right]_\chi. \tag{28}$$

The equality in (28) holds if and only if (24) holds.

Lemma 3. *Lemma 2 is true when $m = 2$.*

Proof. Define the auxiliary function κ as follows:

$$\kappa : I_2 \to \mathbb{R}, \ \kappa(t) \triangleq \chi\left(x_j^1, t\right) - \chi\left(x_i^1, t\right), \tag{29}$$

where $i, j \in \mathbb{N}_1^n \ \wedge \ i \neq j$. Then, by Definition 2 and (29), we have

$$
\begin{aligned}
\langle X^1, X^2\rangle_\chi - \left[X^1, X^2\right]_\chi &= \frac{1}{n}\sum_{i=1}^{n}\chi\left(x_i^1, x_i^2\right) - \frac{1}{n^2}\sum_{1\le i_1,i_2\le n}\chi\left(x_{i_1}^1, x_{i_2}^2\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\chi\left(x_i^1, x_i^2\right) - \frac{1}{n^2}\sum_{1\le i,j\le n}\chi\left(x_i^1, x_j^2\right)\\
&= \frac{1}{n^2}\left[n\sum_{i=1}^{n}\chi\left(x_i^1, x_i^2\right) - \sum_{1\le i,j\le n}\chi\left(x_i^1, x_j^2\right)\right]\\
&= \frac{1}{n^2}\left[\sum_{1\le i,j\le n}\chi\left(x_i^1, x_i^2\right) - \sum_{1\le i,j\le n}\chi\left(x_i^1, x_j^2\right)\right]\\
&= \frac{1}{n^2}\sum_{1\le i,j\le n}\left[\chi\left(x_i^1, x_i^2\right) - \chi\left(x_i^1, x_j^2\right)\right]\\
&= \frac{1}{n^2}\sum_{1\le j,i\le n}\left[\chi\left(x_j^1, x_j^2\right) - \chi\left(x_j^1, x_i^2\right)\right]\\
&= \frac{1}{2n^2}\sum_{1\le i,j\le n}\left[\chi\left(x_i^1, x_i^2\right) - \chi\left(x_i^1, x_j^2\right) + \chi\left(x_j^1, x_j^2\right) - \chi\left(x_j^1, x_i^2\right)\right]\\
&= \frac{1}{2n^2}\sum_{1\le i,j\le n}\left[\chi\left(x_j^1, x_j^2\right) - \chi\left(x_i^1, x_j^2\right) - \left(\chi\left(x_j^1, x_i^2\right) - \chi\left(x_i^1, x_i^2\right)\right)\right]\\
&= \frac{1}{2n^2}\sum_{1\le i\ne j\le n}\left[\chi\left(x_j^1, x_j^2\right) - \chi\left(x_i^1, x_j^2\right) - \left(\chi\left(x_j^1, x_i^2\right) - \chi\left(x_i^1, x_i^2\right)\right)\right]\\
&= \frac{1}{2n^2}\sum_{1\le i\ne j\le n}\left[\kappa(x_j^2) - \kappa(x_i^2)\right],
\end{aligned}
$$

i.e.,

$$\langle X^1, X^2\rangle_\chi - \left[X^1, X^2\right]_\chi = \frac{1}{2n^2}\sum_{1\le i\ne j\le n}\left[\kappa(x_j^2) - \kappa(x_i^2)\right]. \tag{30}$$

According to the Lagrange mean value theorem, there exists a

$$\xi_{i,j}^2 \in \left[\min\left\{x_i^2, x_j^2\right\}, \max\left\{x_i^2, x_j^2\right\}\right] \subseteq I_2$$

such that

$$\kappa(x_j^2) - \kappa(x_i^2) = \left(x_j^2 - x_i^2\right)\left[\frac{\mathrm{d}\kappa_{i,j}(t)}{\mathrm{d}t}\right]_{t=\xi_{i,j}^2} = \left(x_j^2 - x_i^2\right)\left(\chi_2\left(x_j^1, \xi_{i,j}^2\right) - \chi_2\left(x_i^1, \xi_{i,j}^2\right)\right), \tag{31}$$

and there exists a

$$\xi_{i,j}^1 \in \left[\min\left\{x_i^1, x_j^1\right\}, \max\left\{x_i^1, x_j^1\right\}\right] \subseteq I_1$$

such that

$$\chi_2\left(x_j^1, \xi_{i,j}^2\right) - \chi_2\left(x_i^1, \xi_{i,j}^2\right) = \left(x_j^1 - x_i^1\right)\chi_{21}\left(\xi_{i,j}^1, \xi_{i,j}^2\right). \tag{32}$$

Based on Definition 1, (26) and (27), we have

$$\left(x_j^2 - x_i^2\right)\left(x_j^1 - x_i^1\right) \ge 0 \ \wedge\ \chi_{21}\left(\xi_{i,j}^1, \xi_{i,j}^2\right) > 0,\ \forall i,j \in \mathbb{N}_1^n \ \wedge\ i \ne j. \tag{33}$$

Combining with (30), (31), (32), and (33), we obtain

$$\langle X^1, X^2\rangle_\chi - \left[X^1, X^2\right]_\chi = \frac{1}{2n^2}\sum_{1\le i\ne j\le n}\left(x_j^2 - x_i^2\right)\left(x_j^1 - x_i^1\right)\chi_{21}\left(\xi_{i,j}^1, \xi_{i,j}^2\right) \ge 0 \Rightarrow (28). \tag{34}$$

By (34), we see that the equality in (28) holds if and only if

$$\left(x_j^2-x_i^2\right)\left(x_j^1-x_i^1\right)\chi_{21}\left(\xi_{i,j}^1,\xi_{i,j}^2\right)=0,\ \forall i,j\in\mathbb{N}_1^n\ \wedge\ i\neq j. \tag{35}$$

Since

$$\chi_{21}\left(\xi_{i,j}^1,\xi_{i,j}^2\right)>0,\ \forall i,j\in\mathbb{N}_1^n\ \wedge\ i\neq j$$

and

$$i=j\Rightarrow\left(x_j^2-x_i^2\right)\left(x_j^1-x_i^1\right)=0,\ \forall j\in\mathbb{N}_1^n,$$

the equalities (35) can be rewritten as

$$\left(x_j^2-x_i^2\right)\left(x_j^1-x_i^1\right)=0,\ \forall i,j\in\mathbb{N}_1^n. \tag{36}$$

By Lemma 1, the equalities (36) can be rewritten as (24). In other words, the equality in (28) holds if and only if (24) hold. □

Lemma 4. *Under the hypotheses in Lemma 2, then, for any $i,j\in\mathbb{N}_1^n$ and any $m\in\mathbb{N}_3^\infty$, we have*

$$\chi\left(x_i^1,\ldots,x_i^{m-1},x_j^m\right)+\chi\left(x_j^1,\ldots,x_j^{m-1},x_i^m\right)\leq\chi\left(x_i^1,\ldots,x_i^m\right)+\chi\left(x_j^1,\ldots,x_j^m\right). \tag{37}$$

The equalities in (37) hold if and only if

$$\left(x_j^m-x_i^m\right)\left(x_j^k-x_i^k\right)=0,\ \forall i,j\in\mathbb{N}_1^n\ \wedge\ \forall k\in\mathbb{N}_1^m\ \wedge\ k\neq m-1. \tag{38}$$

Proof. Define an auxiliary function as follows:

$$\varphi:I_m\to\mathbb{R},\ \varphi(t)\triangleq\chi\left(x_j^1,\ldots,x_j^{m-1},t\right)-\chi\left(x_i^1,\ldots,x_i^{m-1},t\right). \tag{39}$$

First, we use the dimension reduction method [36] to prove that

$$\varphi\left(x_j^m\right)-\varphi(x_i^m)\geq 0,\ \forall i,j\in\mathbb{N}_1^n, \tag{40}$$

and the equalities in (40) hold if and only if (38) hold.

Indeed, the inequalities (40) are the equalities when $i=j$. Now, we assume that $i\neq j$.

According to the Lagrange mean value theorem, there exists a

$$\xi_{i,j}^m\in\left[\min\left\{x_i^m,x_j^m\right\},\max\left\{x_i^m,x_j^m\right\}\right]\subseteq I_m$$

such that

$$\varphi\left(x_j^m\right)-\varphi(x_i^m)=\left(x_j^m-x_i^m\right)\left[\frac{\mathrm{d}\varphi_{i,j}(t)}{\mathrm{d}t}\right]_{t=\xi_{i,j}^m},$$

i.e.,

$$\varphi\left(x_j^m\right)-\varphi(x_i^m)=\left(x_j^m-x_i^m\right)F\left(x_j^1,x_j^2,\ldots,x_j^k,\ldots,x_j^{m-1}\right), \tag{41}$$

where the function $F:\mathbb{I}^{m-1}\to\mathbb{R}$ is defined as

$$F\left(x_j^1,x_j^2,\ldots,x_j^k,\ldots,x_j^{m-1}\right)\triangleq\chi_m\left(x_j^1,\ldots,x_j^k,\ldots,x_j^{m-1},\xi_{i,j}^m\right)-\chi_m\left(x_i^1,\ldots,x_i^k,\ldots,x_i^{m-1},\xi_{i,j}^m\right),$$
$$k\in\mathbb{N}_1^{m-1}\ \wedge\ i,j\in\mathbb{N}_1^n\ \wedge\ i\neq j,$$

$x_i^1, \ldots, x_i^k, \ldots, x_i^{m-1}, \xi_{i,j}^m$ are considered to be fixed constants, and $x_j^1, x_j^2, \ldots, x_j^k, \ldots, x_j^{m-1}$ are considered to be the variables. By hypothesis (26), we have

$$\frac{\partial F\left(x_j^1, \ldots, x_j^k, \ldots, x_j^{m-1}\right)}{\partial x_j^k} = \chi_{mk}\left(x_j^1, \ldots, x_j^k, \ldots, x_j^{m-1}, \xi_{i,j}^m\right) > 0. \tag{42}$$

If $x_j^m - x_i^m = 0$, then the inequalities (40) are the equalities. Now we assume that $x_j^m - x_i^m > 0$. By the hypotheses (27) and Definition 1, we have

$$X^m \backsim X^k,\ \forall k \in \mathbb{N}_1^{m-1} \Rightarrow x_j^k - x_i^k \geq 0,\ \forall i, j \in \mathbb{N}_1^n \ \wedge\ \forall k \in \mathbb{N}_1^{m-1}. \tag{43}$$

So, based on the mathematical analysis theory, (42) and (43), for any $k \in \mathbb{N}_1^{m-1}$ and any $\left(x_j^1, \ldots, x_j^k, \ldots, x_j^{m-1}\right) \in \mathbb{I}^{m-1}$, we have

$$F\left(x_j^1, x_j^2, \ldots, x_j^k, \ldots, x_j^{m-1}\right) \geq F\left(x_j^1, x_j^2, \ldots, x_i^k, \ldots, x_j^{m-1}\right). \tag{44}$$

By (44), we have

$$\begin{aligned}
F\left(x_j^1, x_j^2, \ldots, x_j^k, \ldots, x_j^{m-1}\right) &\geq F\left(x_i^1, x_j^2, \ldots, x_j^k, \ldots, x_j^{m-1}\right) \\
&\geq F\left(x_i^1, x_i^2, \ldots, x_j^k, \ldots, x_j^{m-1}\right) \\
&\geq \cdots \\
&\geq F\left(x_i^1, x_i^2, \ldots, x_i^k, \ldots, x_i^{m-2}, x_j^{m-1}\right) \\
&\geq F\left(x_i^1, x_i^2, \ldots, x_i^k, \ldots, x_i^{m-2}, x_i^{m-1}\right) \\
&= 0.
\end{aligned}$$

Hence,

$$F\left(x_j^1, x_j^2, \ldots, x_j^k, \ldots, x_j^{m-1}\right) \geq 0. \tag{45}$$

Combining with (41), (45), and $x_j^m - x_i^m > 0$, we obtain (40).
Similarly, we can prove that (40) also holds when $x_j^m - x_i^m < 0$. Thus, (40) is proved.

According to the above proof, the equalities in (40) hold if and only if

$$x_j^m = x_i^m \ \vee\ \left(x_j^1, \ldots, x_j^k, \ldots, x_j^{m-1}\right) = \left(x_i^1, \ldots, x_i^k, \ldots, x_i^{m-1}\right),\ \forall i, j \in \mathbb{N}_1^n \Leftrightarrow (38).$$

This proves our assertion.
Next, we prove (37). By (40), we have

$$\begin{aligned}
& \chi\left(x_i^1, \ldots, x_i^m\right) + \chi\left(x_j^1, \ldots, x_j^m\right) - \left[\chi\left(x_i^1, \ldots, x_i^{m-1}, x_j^m\right) + \chi\left(x_j^1, \ldots, x_j^{m-1}, x_i^m\right)\right] \\
= {} & \chi\left(x_j^1, \ldots, x_j^m\right) - \chi\left(x_i^1, \ldots, x_i^{m-1}, x_j^m\right) - \left[\chi\left(x_j^1, \ldots, x_j^{m-1}, x_i^m\right) - \chi\left(x_i^1, \ldots, x_i^m\right)\right] \\
= {} & \varphi\left(x_j^m\right) - \varphi(x_i^m) \\
\geq {} & 0,\ \forall i, j \in \mathbb{N}_1^n \\
\Rightarrow {} & (37).
\end{aligned}$$

According to the above proof, we see that the equalities in (37) hold if and only if (38) holds. □

Now let us prove Lemma 2.

Proof. We use the mathematical induction [2,36] for m to prove Lemma 2.

(A) Let $m = 2$. According to Lemma 3, Lemma 2 is true.

(B) Suppose that Lemma 2 is true when we replace m with $m-1$ in Lemma 2, where $m \geq 3$. Now, we prove that Lemma 2 is also true as follows.

Based on the above hypothesis and Definition 2, for any $i_m \in \mathbb{N}_1^n$, we have

$$\frac{1}{n^{m-1}} \sum_{1 \leq i_1, i_2, \ldots, i_{m-1} \leq n} \chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_{m-1}}^{m-1}, x_{i_m}^m\right) \leq \frac{1}{n} \sum_{i=1}^{n} \chi\left(x_i^1, x_i^2, \ldots, x_i^{m-1}, x_{i_m}^m\right), \tag{46}$$

where $x_{i_m}^m$ is considered a constant, and the equalities in (46) hold if and only if

$$X^s = x_1^s \mathbf{e},\ \exists k \in \mathbb{N}_1^{m-1} \ \wedge\ \forall s \in \mathbb{N}_1^{m-1} \setminus \{k\}. \tag{47}$$

By Lemma 1, the equalities in (47) hold if and only if

$$\left(x_i^r - x_j^r\right)\left(x_i^s - x_j^s\right) = 0,\ \forall i, j \in \mathbb{N}_1^n \ \wedge\ \forall r, s \in \mathbb{N}_1^{m-1} \ \wedge\ r \neq s. \tag{48}$$

By Definition 2, (46) and (37) in Lemma 4, we obtain

$$\begin{aligned}
\left[X^1, X^2, \ldots, X^m\right]_\chi &= \frac{1}{n^m} \sum_{1 \leq i_1, i_2, \ldots, i_m \leq n} \chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_{m-1}}^{m-1}, x_{i_m}^m\right) \\
&= \frac{1}{n^m} \sum_{i_m=1}^{n} \sum_{1 \leq i_1, i_2, \ldots, i_{m-1} \leq n} \chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_{m-1}}^{m-1}, x_{i_m}^m\right) \\
&= \frac{1}{n} \sum_{i_m=1}^{n} \frac{1}{n^{m-1}} \sum_{1 \leq i_1, i_2, \ldots, i_{m-1} \leq n} \chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_{m-1}}^{m-1}, x_{i_m}^m\right) \\
&\leq \frac{1}{n} \sum_{i_m=1}^{n} \frac{1}{n} \sum_{i=1}^{n} \chi\left(x_i^1, x_i^2, \ldots, x_i^{m-1}, x_{i_m}^m\right) \\
&= \frac{1}{n^2} \sum_{i_m=1}^{n} \sum_{i=1}^{n} \chi\left(x_i^1, x_i^2, \ldots, x_i^{m-1}, x_{i_m}^m\right) \\
&= \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} \chi\left(x_i^1, x_i^2, \ldots, x_i^{m-1}, x_j^m\right) \\
&= \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \chi\left(x_i^1, x_i^2, \ldots, x_i^{m-1}, x_j^m\right) \\
&= \frac{1}{n^2} \sum_{1 \leq j, i \leq n} \chi\left(x_j^1, x_j^2, \ldots, x_j^{m-1}, x_i^m\right) \\
&= \frac{1}{2n^2} \sum_{1 \leq i, j \leq n} \left[\chi\left(x_i^1, x_i^2, \ldots, x_i^{m-1}, x_j^m\right) + \chi\left(x_j^1, x_j^2, \ldots, x_j^{m-1}, x_i^m\right)\right] \\
&\leq \frac{1}{2n^2} \sum_{1 \leq i, j \leq n} \left[\chi\left(x_i^1, \ldots, x_i^{m-1}, x_i^m\right) + \chi\left(x_j^1, \ldots, x_j^{m-1}, x_j^m\right)\right] \\
&= \frac{1}{2n^2} \left[\sum_{1 \leq i, j \leq n} \chi\left(x_i^1, \ldots, x_i^{m-1}, x_i^m\right) + \sum_{1 \leq i, j \leq n} \chi\left(x_j^1, \ldots, x_j^{m-1}, x_j^m\right)\right] \\
&= \frac{1}{2n^2} \left[n \sum_{i=1}^{n} \chi\left(x_i^1, \ldots, x_i^{m-1}, x_i^m\right) + n \sum_{j=1}^{n} \chi\left(x_j^1, \ldots, x_j^{m-1}, x_j^m\right)\right] \\
&= \frac{1}{n} \sum_{i=1}^{n} \chi\left(x_i^1, \ldots, x_i^{m-1}, x_i^m\right) \\
&= \langle X^1, X^2, \ldots, X^m \rangle_\chi \\
&\Rightarrow (28).
\end{aligned}$$

Hence, (28) is proved.

Based on the above proof and Lemma 4, the equality in (28) holds if and only if (38) with (48) holds. In other words, the equality in (28) holds if and only if the equalities in (23) holds. By Lemma 1, the equality in (28) holds if and only if (24) hold.

According to the principle of the mathematical induction, the proof of Lemma 2 is completed. □

Lemma 5. *Let the function $\chi : \mathbb{I}^m \to \mathbb{R}$ be continuous and*

$$\frac{\partial^2 \chi(T)}{\partial t_r^2} > 0,\ \forall T \in \mathbb{I}^m \ \wedge\ \forall r \in \mathbb{N}_1^m. \tag{49}$$

If $\left(X^1, X^2, \ldots, X^m\right) \in I_1^n \times I_2^n \times \cdots \times I_m^n$, then we have the following Jensen type inequality

$$\left[X^1, X^2, \ldots, X^m\right]_\chi \geq \chi\left(\overline{X^1}, \overline{X^2}, \ldots, \overline{X^m}\right). \tag{50}$$

The equality in (50) holds if and only if (22) holds.

Proof. By (49), we see that the function $\chi(T)$ is a strictly convex function [14,21–23] for the variable t_r, $\forall r \in \mathbb{N}_1^m$. So, according to Definitions 2 and the Jensen inequality (15), we have

$$\begin{aligned}
\left[X^1, X^2, \ldots, X^m\right]_\chi &= \frac{1}{n}\sum_{i_1=1}^{n}\frac{1}{n}\sum_{i_2=1}^{n}\cdots\frac{1}{n}\sum_{i_m=1}^{n}\chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_r}^r, \ldots, x_{i_m}^m\right)\\
&\geq \frac{1}{n}\sum_{i_1=1}^{n}\frac{1}{n}\sum_{i_2=1}^{n}\cdots\frac{1}{n}\sum_{i_{m-1}=1}^{n}\chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_{m-1}}^{m-1}, \overline{X^m}\right)\\
&\geq \frac{1}{n}\sum_{i_1=1}^{n}\frac{1}{n}\sum_{i_2=1}^{n}\cdots\frac{1}{n}\sum_{i_{m-2}=1}^{n}\chi\left(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_{m-2}}^{m-2}, \overline{X^{m-1}}, \overline{X^m}\right)\\
&\geq \cdots\\
&\geq \frac{1}{n}\sum_{i_1=1}^{n}\chi\left(x_{i_1}^1, \overline{X^2}, \ldots, \overline{X^{m-1}}, \overline{X^m}\right)\\
&\geq \chi\left(\overline{X^1}, \overline{X^2}, \ldots, \overline{X^{m-1}}, \overline{X^m}\right)\\
&\Rightarrow (50),
\end{aligned}$$

and the equality in (50) holds if and only if (22) holds by the above proof. □

Let's turn to the proof of Theorem 1.

Proof. According to the hypotheses of Theorem 1 and Lemma 2, we see that (28) holds. By the hypotheses of Theorem 1 and Lemma 5, we see that (50) holds. Combining with (28) and (50), we get the inequalities (21).

Based on the Lemmas 2 and 5, we known that both the equalities in (21) hold if and only if (22) holds.

The proof of Theorem 1 is completed. □

In Theorem 1, set $\mathbb{I}^m = \mathbb{R}_{++}^m$ and $\chi = \Pi$. Then, by (5) and (6), we have the following Corollary 1. Therefore, Theorem 1 is a generalization of the discrete Chebyshev inequality (11).

Corollary 1. *(Discrete Chebyshev type inequality) Let $X^1, X^2, \ldots, X^m \in \mathbb{R}_{++}^n$ and (27) hold. Then, we have the following discrete Chebyshev type inequality:*

$$\overline{X^1 X^2 \cdots X^m} = \langle X^1, X^2, \ldots, X^m \rangle_\Pi \geq \left[X^1, X^2, \ldots, X^m\right]_\Pi = \overline{X^1} \times \overline{X^2} \times \cdots \times \overline{X^m}, \tag{51}$$

and

$$\overline{X^1 X^2 \cdots X^m} = \overline{X^1} \times \overline{X^2} \times \cdots \times \overline{X^m} \Leftrightarrow (24).$$

4. Continuous Chebyshev–Jensen-Type Inequalities

Theorem 2. *(Continuous Chebyshev–Jensen-type inequalities) Let the function $\chi : \mathbb{I}^m \to \mathbb{R}$ be continuous and (19) hold. If $(f_1, f_2, \ldots, f_m) \in C_1[0,1] \times C_2[0,1] \times \cdots \times C_m[0,1]$ and*

$$f_r \backsim f_s,\ \forall r, s \in \mathbb{N}_1^m, \tag{52}$$

then we have the following continuous Chebyshev–Jensen-type inequalities:

$$\langle f_1, f_2, \ldots, f_m \rangle_\chi \geq [f_1, f_2, \ldots, f_m]_\chi \geq \chi\left(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\right), \tag{53}$$

and

$$\langle f_1, f_2, \ldots, f_m \rangle_\chi = [f_1, f_2, \ldots, f_m]_\chi = \chi\left(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\right) \Leftrightarrow f_r(t) \equiv f_r(0),\ \forall r \in \mathbb{N}_1^m. \tag{54}$$

Proof. Based on the theory of functional analysis [31–33], for any continuous functions $\varphi : [0,1] \to \mathbb{R}$ and $\phi : [0,1]^m \to \mathbb{R}$, we have

$$\int_0^1 \varphi(t)\mathrm{d}t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi\left(\frac{i-1}{n-1}\right) \tag{55}$$

and

$$\int_0^1 \cdots \int_0^1 \phi(t_1, \ldots, t_m)\mathrm{d}t_1 \cdots \mathrm{d}t_m = \lim_{n\to\infty} \frac{1}{n^m} \sum_{1 \leq i_1, \ldots, i_m \leq n} \phi\left(\frac{i_1 - 1}{n-1}, \ldots, \frac{i_m - 1}{n-1}\right). \tag{56}$$

Let $\left(X^1, X^2, \ldots, X^m\right) \in I_1^n \times I_2^n \times \cdots \times I_m^n$, and let

$$x_i^r = f_r\left(\frac{i-1}{n-1}\right),\ \forall i \in \mathbb{N}_1^n \ \wedge\ \forall r \in \mathbb{N}_1^m. \tag{57}$$

Then

$$x_{i_j}^r = f_r\left(\frac{i_j - 1}{n-1}\right),\ \forall i_j \in \mathbb{N}_1^n \ \wedge\ \forall r \in \mathbb{N}_1^m \ \wedge\ \forall j \in \mathbb{N}_1^m. \tag{58}$$

By Definitions 2 and 3, (55), (56), (57), and (58), we have

$$\lim_{n\to\infty} \langle X^1, X^2, \ldots, X^m \rangle_\chi = \langle f_1, f_2, \ldots, f_m \rangle_\chi, \tag{59}$$

$$\lim_{n\to\infty} \left[X^1, X^2, \ldots, X^m\right]_\chi = [f_1, f_2, \ldots, f_m]_\chi \tag{60}$$

and

$$\lim_{n\to\infty} \chi\left(\overline{X^1}, \overline{X^2}, \ldots, \overline{X^m}\right) = \chi\left(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\right). \tag{61}$$

By (52), (57), and Definition 1, (27) holds. By (27) and Theorem 1, we see that (21) holds. Combining with (59), (60), (61), and (21), we obtain

$$\begin{aligned}
\langle f_1, f_2, \ldots, f_m \rangle_\chi &= \lim_{n\to\infty} \langle X^1, X^2, \ldots, X^m \rangle_\chi \geq \lim_{n\to\infty} \left[X^1, X^2, \ldots, X^m\right]_\chi \\
&= [f_1, f_2, \ldots, f_m]_\chi \geq \lim_{n\to\infty} \chi\left(\overline{X^1}, \overline{X^2}, \ldots, \overline{X^m}\right) \\
&= \chi\left(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\right) \\
&\Rightarrow (53).
\end{aligned}$$

Base on the above proof and Theorem 1, we see that (54) holds.
This completes the proof of Theorem 2. □

In Theorem 2, set $\mathbb{I}^m = \mathbb{R}^m_{++}$ and $\chi = \Pi$. Then, by (9) and (10), we have the following Corollary 2. Therefore, Theorem 2 is a generalization of the continuous Chebyshev inequality (13).

Corollary 2. *(Continuous Chebyshev type inequality) Let the functions $f_r : [0,1] \to \mathbb{R}_{++}$ be continuous and (52) hold, $\forall r \in \mathbb{N}^m_1$. Then, we have the following continuous Chebyshev-type inequality:*

$$\overline{f_1 f_2 \dots f_m} \equiv \langle f_1, f_2, \dots, f_m \rangle_\Pi \geq [f_1, f_2, \dots, f_m]_\Pi \equiv \overline{f_1} \times \overline{f_2} \times \cdots \times \overline{f_m}, \tag{62}$$

and

$$\overline{f_1 f_2 \dots f_m} = \overline{f_1} \times \overline{f_2} \times \cdots \times \overline{f_m} \Leftrightarrow f_s(t) \equiv f_s(0),\ \exists k \in \mathbb{N}^m_1 \ \wedge\ \forall s \in \mathbb{N}^m_1 \setminus \{k\}. \tag{63}$$

5. Applications in Probability Theory

Let $\mu \triangleq (\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{I}^m$ be an m-dimensional random variable, where

$$I_r \triangleq (a_r, b_r),\ -\infty \leq a_r < b_r \leq \infty,\ \forall r \in \mathbb{N}^m_1,$$

and let its probability density function [24–30] $p : \mathbb{I}^m \to \mathbb{R}_{++}$ be continuous with $p \triangleq p(\mu)$. Then, the probability distribution function [24] of the random variable μ is

$$\chi : \mathbb{I}^m \to \mathbb{R},\ \chi(T) = \int_{a_1}^{t_1} \int_{a_2}^{t_2} \cdots \int_{a_m}^{t_m} p \mathrm{d}\mu_1 \mathrm{d}\mu_2 \cdots \mathrm{d}\mu_m, \tag{64}$$

which is also continuous, where $\chi(b_1, b_2, \dots, b_m) = 1$, and $P(R) \triangleq \chi(T) \in [0,1]$ is the probability [27–30] of the random event

$$R \triangleq: a_1 < \mu_1 \leq t_1 \wedge a_2 < \mu_2 \leq t_2 \wedge \cdots \wedge a_r < \mu_r \leq t_r \wedge \cdots \wedge a_m < \mu_m \leq t_m. \tag{65}$$

Let $\left(X^1, X^2, \dots, X^m\right) \in I^n_1 \times I^n_2 \times \cdots \times I^n_m$. Then, $P(R_i)$ and $P(R_{i_1,\dots,i_m})$ are the probabilities of the random events

$$R_i \triangleq: a_1 < \mu_1 \leq x^1_i \wedge a_2 < \mu_2 \leq x^2_i \wedge \cdots \wedge a_r < \mu_r \leq x^r_i \wedge \cdots \wedge a_m < \mu_m \leq x^m_i \tag{66}$$

and

$$R_{i_1,\dots,i_m} \triangleq: a_1 < \mu_1 \leq x^1_{i_1} \wedge a_2 < \mu_2 \leq x^2_{i_2} \wedge \cdots \wedge a_r < \mu_r \leq x^r_{i_r} \wedge \cdots \wedge a_m < \mu_m \leq x^m_{i_m}, \tag{67}$$

respectively, where $i \in \mathbb{N}^n_1$ and $i_1, \dots, i_m \in \mathbb{N}^n_1$.

Let $(f_1, \dots, f_m) \in C_1[0,1] \times \cdots \times C_m[0,1]$. Then $P(R) \triangleq P(R(t))$ and $P(R^*) \triangleq P(R^*(T))$ are the probabilities of the random events

$$R(t) \triangleq: a_1 < \mu_1 \leq f_1(t) \wedge \cdots \wedge a_r < \mu_r \leq f_r(t) \wedge \cdots \wedge a_m < \mu_m \leq f_m(t) \tag{68}$$

and

$$R^*(T) \triangleq: a_1 < \mu_1 \leq f_1(t_1) \wedge \cdots \wedge a_r < \mu_r \leq f_r(t_r) \wedge \cdots \wedge a_m < \mu_m \leq f_m(t_m), \tag{69}$$

respectively.

We first demonstrate the applications of Theorem 1 in probability theory.

Theorem 3 (Discrete probability inequalities)**.** *Let the probability density function $p : \mathbb{I}^m \to \mathbb{R}_{++}$ be continuous, and let*

$$p_r(\mu) \triangleq \frac{\partial p(\mu)}{\partial \mu_r} > 0,\ \forall \mu \in \mathbb{I}^m \ \wedge\ \forall r \in \mathbb{N}_1^m. \tag{70}$$

If $\left(X^1, X^2, \ldots, X^m\right) \in I_1^n \times I_2^n \times \cdots \times I_m^n$ and (20) holds, then we have the following discrete probability inequalities:

$$\frac{1}{n}\sum_{i=1}^{n} P(R_i) \geq \frac{1}{n^m}\sum_{1 \leq i_1, \ldots, i_m \leq n} P(R_{i_1,\ldots,i_m}) \geq \int_{a_1}^{\overline{X^1}} \cdots \int_{a_m}^{\overline{X^m}} p\mathrm{d}\mu_1 \cdots \mathrm{d}\mu_m. \tag{71}$$

Proof. Let $T \in \mathbb{I}^m$ and $r, s \in \mathbb{N}_1^m$. If $r < s$, then, based on the hypotheses of Theorem 3 and the functional analysis theory, we have

$$\frac{\partial^2 \chi(T)}{\partial t_r \partial t_s} = \int_{a_1}^{t_1} \cdots \int_{a_k}^{t_k} \cdots \int_{a_m}^{t_m} p(\mu_1, \ldots, t_r, \ldots, t_s, \ldots, \mu_m)\mathrm{d}\mu_1 \cdots \mathrm{d}\mu_k \cdots \mathrm{d}\mu_m > 0 \Rightarrow (19),$$

where $k \in \mathbb{N}_1^m \backslash \{r, s\}$. Similarly, we can prove that (19) also holds when $r > s$. Assume that $r = s$. Then, based on the functional analysis theory and (70), we have

$$\frac{\partial^2 \chi(T)}{\partial t_r \partial t_s} = \int_{a_1}^{t_1} \cdots \int_{a_k}^{t_k} \cdots \int_{a_m}^{t_m} p_r(\mu_1, \ldots, t_r, \ldots, \mu_m)\mathrm{d}\mu_1 \cdots \mathrm{d}\mu_k \cdots \mathrm{d}\mu_m > 0 \Rightarrow (19),$$

where $k \in \mathbb{N}_1^m \backslash \{r\}$. Thus, the inequalities in (19) hold.

By Definition 2 and (64), we have

$$\langle X^1, X^2, \ldots, X^m \rangle_\chi = \frac{1}{n}\sum_{i=1}^{n} P(R_i), \tag{72}$$

which is the mean value [35,36] of the probabilities $P(R_i)$, $\forall i \in \mathbb{N}_1^n$, and

$$\left[X^1, X^2, \ldots, X^m\right]_\chi = \frac{1}{n^m}\sum_{1 \leq i_1, \ldots, i_m \leq n} P(R_{i_1,\ldots,i_m}), \tag{73}$$

which is the mean value of the probabilities $P(R_{i_1,\ldots,i_m})$, $\forall i_1, \ldots, i_m \in \mathbb{N}_1^n$.

Based on the hypotheses of Theorem 3, (19), and Theorem 1, the inequalities (21) hold. By (72) and (73), we see that the inequalities (21) can be rewritten as (71). The proof of Theorem 3 is completed. □

Next, we demonstrate the applications of Theorem 2 in probability theory.

Theorem 4. *(Continuous probability inequalities) Let the probability density function $p : \mathbb{I}^m \to \mathbb{R}_{++}$ be continuous and (70) hold, and let $(f_1, \ldots, f_m) \in C_1[0,1] \times \cdots \times C_m[0,1]$ with (52) hold. If $f_1, \ldots, f_m$ are convex functions, then we have the following continuous probability inequalities:*

$$\int_0^1 P(R)\mathrm{d}t \geq \int_0^1 \cdots \int_0^1 P(R^*)\mathrm{d}t_1 \cdots \mathrm{d}t_m \geq \int_{a_1}^{f_1\left(\frac{1}{2}\right)} \cdots \int_{a_m}^{f_m\left(\frac{1}{2}\right)} p\mathrm{d}\mu_1 \cdots \mathrm{d}\mu_m. \tag{74}$$

Proof. By Definition 3 and (64), we have

$$\langle f_1, f_2, \ldots, f_m \rangle_\chi = \int_0^1 P(R)\mathrm{d}t, \tag{75}$$

which is the mean value of the probability $P(R(t))$, and

$$[f_1, f_2, \ldots, f_m]_\chi = \int_0^1 \cdots \int_0^1 P(R^*)\mathrm{d}t_1 \cdots \mathrm{d}t_m, \tag{76}$$

which is the mean value of the probability $P(R^*(T))$.

Since $f_1, \ldots, f_m$ are the convex functions [14,21–23], by Hadamard's inequality [36], we have

$$\overline{f_r} = \frac{1}{1-0}\int_0^1 f_r(t)\mathrm{d}t \geq f_r\left(\frac{0+1}{2}\right) = f_r\left(\frac{1}{2}\right), \ \forall r \in \mathbb{N}_1^m. \tag{77}$$

By the proof of Theorem 3, we see that (19) holds. Based on the hypotheses of Theorem 4, (19), and Theorem 2, the inequalities in (53) hold. By (53), (75), (76), and (77), we obtain

$$\begin{aligned}
\int_0^1 P(R)\mathrm{d}t &= \langle f_1, f_2, \ldots, f_m\rangle_\chi \\
&\geq [f_1, f_2, \ldots, f_m]_\chi \\
&= \int_0^1 \cdots \int_0^1 P(R^*)\mathrm{d}t_1 \cdots \mathrm{d}t_m \\
&\geq \chi\left(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\right) \\
&= \int_{a_1}^{\overline{f_1}} \cdots \int_{a_m}^{\overline{f_m}} p\mathrm{d}t_1 \cdots \mathrm{d}t_m \\
&\geq \int_{a_1}^{f_1\left(\frac{1}{2}\right)} \cdots \int_{a_m}^{f_m\left(\frac{1}{2}\right)} p\mathrm{d}t_1 \cdots \mathrm{d}t_m \\
&\Rightarrow (74).
\end{aligned}$$

Hence, the inequalities in (74) are proved. This completes the proof of Theorem 4. □

6. Conclusions

In this paper, we established Chebyshev–Jensen-type inequalities involving the χ-products $\langle\cdot\rangle_\chi$ and $[\cdot]_\chi$, and we proved that our main results are the generalizations of the classical Chebyshev inequalities, as well as displaying the applications of our main results in probability theory, and the discrete with continuous probability inequalities were obtained. We also demonstrated the applications of mathematical induction, the reorder method, and the dimension reduction method in establishing inequalities. The proofs of our main results are novel, concise, and interesting.

The main contributions of this article are that we extended the special function Π in the Chebyshev inequalities (11) and (13) to the general function χ, and we extended the $m = 2$ in the Chebyshev inequalities (11) and (13) to the $m \geq 2$.

Let

$$\frac{\partial^2\chi(T)}{\partial t_r \partial t_s} \equiv 0, \ \forall T \in \mathbb{I}^m \ \wedge \ \forall r, s \in \mathbb{N}_1^m. \tag{78}$$

Then, the function $\chi(T)$ is a constant. Hence, the inequalities (21) are the equalities.

If (20) does not hold, then, in general, (21) also does not hold. For example, if $m = 2$, $\chi = \Pi$ and $X^1 \backsimeq X^2$, then the inequalities in (21) are reversed.

There are a large number of functions $\chi : \mathbb{I}^m \to \mathbb{R}$ satisfying the conditions in (19). For example, we define a function as follows [2]:

$$\chi : \mathbb{R}_{++}^m \to \mathbb{R}, \ \chi(T) \triangleq \mathrm{per}\left(t_j^{\alpha_i}\right)_{m\times m} = \mathrm{per}\begin{pmatrix} t_1^{\alpha_1} & t_2^{\alpha_1} & \cdots & t_m^{\alpha_1} \\ t_1^{\alpha_2} & t_2^{\alpha_2} & \cdots & t_m^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\alpha_m} & t_2^{\alpha_m} & \cdots & t_m^{\alpha_m} \end{pmatrix}_{m\times m}, \tag{79}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in (1, \infty)^m$. Then $\chi : \mathbb{R}_{++}^m \to \mathbb{R}$ satisfies the conditions in (19).

There are a large number of probability density functions $p : \mathbb{I}^m \to \mathbb{R}$ satisfying the conditions in (70). For example, we define a probability density function as follows:

$$p : (0,1)^m \to \mathbb{R}_{++},\ p(\mu) \triangleq \frac{\operatorname{per}\left(\mu_j^{\alpha_i}\right)_{m\times m}}{\int_0^1 \int_0^1 \cdots \int_0^1 \operatorname{per}\left(\mu_j^{\alpha_i}\right)_{m\times m} \mathrm{d}\mu_1 \mathrm{d}\mu_2 \cdots \mathrm{d}\mu_m}, \tag{80}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m_{++}$. Then $p : (0,1)^m \to \mathbb{R}_{++}$ satisfies the conditions in (70).

It is worth pointing out that to find new Chebyshev-type inequalities is an important research topic and how to improve or generalize the Chebyshev-type inequalities (21) is also an important research topic. These research topics are of theoretical significance and application value in probability theory.

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