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Baire 1 Functions and the Topology of Uniform Convergence on Compacta

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Abstract: Let X be a Tychonoff topological space, $B_1(X,\mathbb{R})$ be the space of real-valued Baire 1 functions on X and τ_{UC} be the topology of uniform convergence on compacta. The main purpose of this paper is to study cardinal invariants of $(B_1(X,\mathbb{R}),\tau_{UC})$. We prove that the following conditions are equivalent: (1) $(B_1(X,\mathbb{R}),\tau_{UC})$ is metrizable; (2) $(B_1(X,\mathbb{R}),\tau_{UC})$ is completely metrizable; (3) $(B_1(X,\mathbb{R}),\tau_{UC})$ is Čech-complete; and (4) X is hemicompact. It is also proven that if X is a separable metric space with a non isolated point, then the topology of uniform convergence on compacta on $B_1(X,\mathbb{R})$ is seen to behave like a metric topology in the sense that the weight, netweight, density, Lindelof number and cellularity are all equal for this topology and they are equal to $\mathfrak{c} = |B_1(X,\mathbb{R})|$. We find further conditions on X under which these cardinal invariants coincide on $B_1(X,\mathbb{R})$.

Keywords: Baire 1 function; quasicontinuous function; topology of uniform convergence on compacta; density; weight; netweight; cellularity

MSC: 54C35; 54C08; 54C30

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1. Introduction

Let X be a topological space, $C(X,\mathbb{R})$ be the space of continuous functions on X with values in $\mathbb{R}$, $B_1(X,\mathbb{R}) = \{f \in \mathbb{R}^X : f$ is a pointwise limit of a sequence from $C(X,\mathbb{R})\}$ be the space of Baire 1 functions on X with values in $\mathbb{R}$. The research of Baire 1 functions began with Baire's paper [1] from 1899, which contains the results of their PhD thesis. Baire themselves conducted a detailed study of real Baire 1 functions on X when $X = \mathbb{R}$.

Let τ_{UC} be the topology of uniform convergence on compacta. We prove that the following conditions are equivalent: (1) $(B_1(X,\mathbb{R}),\tau_{UC})$ is metrizable; (2) $(B_1(X,\mathbb{R}),\tau_{UC})$ is completely metrizable; (3) $(B_1(X,\mathbb{R}),\tau_{UC})$ is Čech-complete; and (4) X is hemicompact.

Thus, if X is hemicompact, all cardinal invariants' cellularity, density, net weight, spread, extent, Lindeloff number, π-weight and weight coincide on $(B_1(X,\mathbb{R}),\tau_{UC})$. We find further conditions on X under which these cardinal invariants coincide on $(B_1(X,\mathbb{R}),\tau_{UC})$. If X is a separable metric space with a non isolated point, then the topology of uniform convergence on compacta on $B_1(X,\mathbb{R})$ is seen to behave like a metric topology in the sense that the cellularity, density, net weight, spread, extent, Lindeloff number, π-weight and weight are all equal for this topology and they are equal to $\mathfrak{c} = |B_1(X,\mathbb{R})|$. If X is an uncountable Polish space, then all these cardinal invariants coincide on $(B_1(X,\mathbb{R},\tau_{UC})$ and the same result holds also for the space $Q(X,\mathbb{R})$ of real-valued quasicontinuous functions.

2. Preliminaries

Denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the space of real numbers with the usual metric. Let $C(X,Y)$ be the space of all continuous functions from a topological space X into a topological space Y. The following notation is used: $B_1(X,Y) = \{f \in Y^X : f$ is a pointwise limit of a sequence from $C(X,Y)\}$ and $F_\sigma(X,Y) = \{f \in Y^X : f^{-1}(V)$ is an F_σ set for every open $V \subset Y\}$. The elements of $B_1(X,Y)$ are called functions of the first Baire class, and those of $F_\sigma(X,Y)$ are called functions of the first Borel class or F_σ measurable functions.

It is known that $B_1(X,Y) \subset F_\sigma(X,Y)$ for any topological space X and any metric space Y [2]. If $Y = \{0,1\}$, then $B_1(\mathbb{R},Y) \neq F_\sigma(\mathbb{R},Y)$. An overview of the results regarding the equality $B_1(X,Y) = F_\sigma(X,Y)$ can be found in papers [2,3].

Baire in [1] proved that if X is an interval of reals $\mathbb{R}$ and $Y = \mathbb{R}$, then $F_\sigma(X,Y) = B_1(X,Y)$. Lebesgue in [4] proved that if X is a metric space and $Y = \mathbb{R}$, then $F_\sigma(X,Y) = B_1(X,Y)$.

The following result of Laczkowich will be useful in our paper.

Proposition 1. [5] *Let X be a normal space and $Y = \mathbb{R}$; then, $F_\sigma(X,Y) = B_1(X,Y)$.*

The following Theorem was proven by Lebesgue for $Y = \mathbb{R}$.

Theorem 1. (cf. (p. 375, [6]), [7]) *Let X and Y be metric spaces. For $f\colon X \to Y$, consider the following conditions:*

(1) A function f is F_σ measurable;
(2) For each $\varepsilon > 0$, there is a cover $(X_i)_{i\in\mathbb{N}}$ of X consisting of closed sets such that $\operatorname{diam} f(X_i) \leq \varepsilon$, for all $i \in \mathbb{N}$.

Then, $(2) \Rightarrow (1)$ holds. If Y is separable, then $(1) \Rightarrow (2)$ is true.

Remark 1. *It is easy to verify that Theorem 1 works for any topological space X.*

Let X be a Hausdorff topological space. Denote by $K(X)$ the family of all nonempty compact subsets of X.

Denote by τ_{UC} the topology of uniform convergence on compact sets on $\mathbb{R}^X$. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K,\varepsilon) = \{(f,g) : \forall\, x \in K \ |f(x) - g(x)| < \varepsilon\},$$

where $K \in K(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighborhood of $f \in \mathbb{R}^X$ will be denoted by $W(f,K,\varepsilon)$, where

$$W(f,K,\varepsilon) = \{g : \forall\, x \in K \ |f(x) - g(x)| < \varepsilon\}.$$

Denote by τ_p the topology of pointwise convergence on $\mathbb{R}^X$. This topology is induced by the uniformity $\mathfrak{U}_p$ which has a base consisting of sets of the form

$$W(A,\varepsilon) = \{(f,g) : \forall\, x \in A \ |f(x) - g(x)| < \varepsilon\},$$

where A is a finite set and $\varepsilon > 0$. The general τ_p-basic neighborhood of $f \in \mathbb{R}^X$ will be denoted by $W(f,A,\varepsilon)$, where

$$W(f,A,\varepsilon) = \{g : \forall\, x \in A \ |f(x) - g(x)| < \varepsilon\}.$$

Of course, the topology τ_p of the pointwise convergence on $\mathbb{R}^X$ is just the product topology on $\mathbb{R}^X$.

3. Complete Metrizability of $(B_1(X,Y), \tau_{UC})$

It is known [8] that if X is a k-space, then the space $C(X,\mathbb{R})$ of continuous real-valued functions defined on X is a closed set in $(\mathbb{R}^X, \tau_{UC})$. However, if X is not a k-space, then $C(X,\mathbb{R})$ does not have to be closed in $(B_1(X,\mathbb{R}), \tau_{UC})$ either.

Example 1. *Let $\mathbb{N}$ be equipped with the discrete topology and $\beta\mathbb{N}$ be the Čech–Stone compactification of $\mathbb{N}$. Choose $q \in \beta\mathbb{N} \setminus \mathbb{N}$. Let $X = \mathbb{N} \cup \{q\}$ and X has the topology inherited from $\beta\mathbb{N}$. Every compact set in X is finite. X is not a k-space, since the set $\mathbb{N}$ is not closed in X and $\mathbb{N} \cap K$ is closed for every $K \in K(X)$. For every $n \in \mathbb{N}$, define $f_n : X \to \mathbb{R}$ as follows:*

$$f_n(x) = \begin{cases} 1, & x \in \{1, 2, \ldots, n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Of course, f_n is a continuous function for every $n \in \mathbb{N}$. Let $f : X \to \mathbb{R}$ be the function defined as:

$$f(x) = \begin{cases} 0, & x = q; \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to verify that f is not continuous and the sequence $\{f_n : n \in \mathbb{N}\}$ converges to f in $(\mathbb{R}^X, \tau_{UC})$. Thus, $f \in B_1(X, \mathbb{R})$ and $C(X, \mathbb{R})$ is not closed in $(B_1(X, \mathbb{R}), \tau_{UC})$.

A topological space X is hemicompact [9] if in the family of all compact subspaces of X ordered by inclusion there exists a countable cofinal subfamily. Every hemicompact space is σ-compact, but not vice versa. The space of rationals with the usual topology is a σ-compact space which is not hemicompact. A locally compact σ-compact space is hemicompact.

A Hausdorff space X is of pointwise countable type [9] if for every point $x \in X$ there exists a compact set $C \subset X$ such that $x \in C$ and $\chi(C, X) \leq \aleph_0$. Another one is the more general property of being a q-space. This is a space such that for each point there is a sequence $\{U_n : n \in \mathbb{N}\}$ of neighbourhoods of that point, so that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point [10].

Since $(B_1(X, \mathbb{R}), \tau_{UC})$ is a topological group, the equivalence $3 \Leftrightarrow 4$ in the following theorem is known [11].

Theorem 2. *Let X be a Tychonoff topological space. Then, the following are equivalent:*

1. *The uniformity $\mathfrak{U}_{UC}$ on $\mathbb{R}^X$ is induced by a metric;*
2. *The uniformity $\mathfrak{U}_{UC}$ on $B_1(X, \mathbb{R})$ is induced by a metric;*
3. *$(B_1(X, \mathbb{R}), \tau_{UC})$ is metrizable;*
4. *$(B_1(X, \mathbb{R}), \tau_{UC})$ is first countable;*
5. *$(B_1(X, \mathbb{R}), \tau_{UC})$ is of pointwise countable type;*
6. *$(B_1(X, \mathbb{R}), \tau_{UC})$ is a q-space;*
7. *X is hemicompact.*

Proof. (7) $\Rightarrow$ (1) Let $\{K_n : n \in \mathbb{N}\}$ be a countable cofinal subfamily in $K(X)$ with respect to the inclusion. The family $\{W(K, \varepsilon) : K \in K(X), \varepsilon > 0\}$ is a base of $\mathfrak{U}_{UC}$ on $\mathbb{R}^X$. Since for every $K \in K(X)$ there is $n \in \mathbb{N}$ with $K \subset K_n$, the family

$$\{W(K_n, \frac{1}{m}) : n, m \in \mathbb{N}\}$$

is a countable base of $\mathfrak{U}_{UC}$. Thus, by the metrization theorem in [8], $(\mathbb{R}^X, \mathfrak{U}_{UC})$ is metrizable.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7) Suppose that $(B_1(X, \mathbb{R}), \tau_{UC})$ is a q-space. Let f be the zero function on X. By assumption, there is a sequence $\{W(f, K_n, \varepsilon_n) : n \in \mathbb{N}\}$ such that if $f_n \in W(f, K_n, \varepsilon_n)$ for each $n \in \mathbb{N}$, then $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $(B_1(X, \mathbb{R}), \tau_{UC})$. We claim that $X = \bigcup\{K_n : n \in \mathbb{N}\}$. Suppose that there is $x \in X \setminus \bigcup\{K_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, define a function $f_n \in C(X, \mathbb{R})$ as follows:

$$f_n(z) = \begin{cases} n, & z = x; \\ 0, & z \in K_n. \end{cases}$$

Then, $f_n \in W(f, K_n, \varepsilon)$ for every $n \in \mathbb{N}$; however, $\{f_n : n \in \mathbb{N}\}$ cannot have a cluster point in $B_1(X, \mathbb{R})$.

Now, for every $n \in \mathbb{N}$, put $C_n = \cup\{K_i : i \le n\}$. We claim that $\{C_n : n \in \mathbb{N}\}$ is a cofinal family in $K(X)$ with respect to the inclusion. Suppose that this is not true. Thus, there is $K \in K(X)$ such that for each $n \in \mathbb{N}$ there is $k_n \in K \setminus C_n$. For every $n \in \mathbb{N}$, let $g_n \in C(X, \mathbb{R})$ be such that $g_n(k_n) = 1$ and $g_n(z) = 0$ for every $z \in C_n$.

Of course, for every $n \in \mathbb{N}$, $g_n \in W(f, C_n, \varepsilon_n) \subset W(f, K_n, \varepsilon_n)$. By assumption, $\{g_n : n \in \mathbb{N}\}$ has a cluster point $g \in (B_1(X, \mathbb{R}), \tau_{UC})$. Then, $g(x) = 0$ for every $x \in X$, which is a contradiction since $g_n \notin W(g, K, 1)$ for every $n \in \mathbb{N}$. □

Lemma 1. *Let X be a Tychonoff topological space. If a family $\mathcal{E}$ in $(\mathcal{B}_1(X, \mathbb{R}), \tau_{UC})$ is totally bounded, then for every compact set A in X and every $\epsilon > 0$, there is a countable family $\mathcal{B}$ of F_σ sets in X such that $\bigcup \mathcal{B} = A$ and for every $B \in \mathcal{B}$ and for every $f \in \mathcal{E}$, diam$f(B) \le \epsilon$.*

Proof. Since $\mathcal{E}$ in $(\mathcal{B}_1(X, \mathbb{R}), \tau_{UC})$ is totally bounded, there are functions $f_1, \ldots, f_n \in \mathcal{E}$ such that

$$\mathcal{E} \subset W(f_1, A, \frac{\epsilon}{3}) \cup \cdots \cup W(f_n, A, \frac{\epsilon}{3}).$$

Let $V_1, V_2, \ldots, V_m, \ldots$ be a countable open cover of $\mathbb{R}$, where the diameter of members of this cover is less than $\frac{\epsilon}{3}$. For every $i \in \{1, 2, \ldots, n\}$, $j \in \mathbb{N}$, put $B^i_j = f_i^{-1}(V_j)$. Of course, B^i_j is an F_σ set for every $i \in \{1, 2, \ldots, n\}$, $j \in \mathbb{N}$.

Let $\mathcal{F}$ be the family of all functions from $\{1, 2, \ldots, n\}$ to $\mathbb{N}$. Of course, $\mathcal{F}$ is countable. For every $g \in \mathcal{F}$, put $B_g = A \cap B^1_{g(1)} \cap \cdots \cap B^n_{g(n)}$. For every $g \in \mathcal{F}$, B_g is an F_σ set.

Now, put

$$\mathcal{B} = \{B_g : g \in \mathcal{F}, B_g \neq \emptyset\}.$$

Of course, $\bigcup \mathcal{B} = A$. Let $z \in A$; then, there is $g \in \mathcal{F}$ such that $f_i(z) \in V_{g(i)}$ and thus $z \in B^i_{g(i)}$ for every $i \in \{1, 2, \ldots, n\}$.

Finally, let $f \in \mathcal{E}$ and let $B \in \mathcal{B}$. We show that diam$f(B) \le \epsilon$. Let $p, q \in B$. There is a $g \in \mathcal{F}$ such that $B = A \cap B^1_{g(1)} \cap \cdots \cap B^n_{g(n)}$. Since $\mathcal{E}$ is a subset of $W(f_1, A, \frac{\epsilon}{3}) \cup \cdots \cup W(f_n, A, \frac{\epsilon}{3})$, there exists $i \in \{1, 2, \ldots, n\}$ such that $f \in W(f_i, A, \frac{\epsilon}{3})$. Thus, $|(f(p) - f_i(p)| < \frac{\epsilon}{3}$ and $|f(q) - f_i(q)| < \frac{\epsilon}{3}$. Because $p, q \in B^i_{g(i)}$, we have that $f_i(p) \in V_{g(i)}$ and $f_i(q) \in V_{g(i)}$. Then,

$$\begin{aligned} &|f(p) - f(q)| \le \\ \le &|f(p) - f_i(p)| + |f_i(p) - f_i(q)| + |f_i(q)) - f(q)| < \\ < &\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Proposition 2. *Let X be a Tychonoff hemicompact topological space. Then, $B_1(X, \mathbb{R})$ is a closed subset of $(\mathbb{R}^X, \tau_{UC})$.*

Proof. Let f be in the closure of $B_1(X, \mathbb{R})$ in $(\mathbb{R}^X, \tau_{UC})$. By Theorem 2, $(\mathbb{R}^X, \tau_{UC})$ is metrizable. Thus, there is a sequence $\{f_n : n \in \mathbb{N}\}$ in $B_1(X, \mathbb{R})$ which converges to f in $(\mathbb{R}^X, \tau_{UC})$. Put $\mathcal{E} = \{f_n : n \in \mathbb{N}\}$. Let $\{K_n : n \in \mathbb{N}\}$ be a countable cofinal subfamily in $K(X)$. Let $\epsilon > 0$. Apply Lemma 1 on $\mathcal{E}$ and on every K_n and ϵ. For every $n \in \mathbb{N}$, we obtain a countable family $\mathcal{B}_n$ of F_σ sets in X such that $\bigcup \mathcal{B}_n = K_n$, and for every $B \in \mathcal{B}_n$ and for every $g \in \mathcal{E}$, diam$g(B) \le \epsilon$. Put $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Then, $X = \bigcup \mathcal{B}$, $\mathcal{B}$ is a countable family of F_σ sets in X and for every $g \in \mathcal{E}$, diam$g(B) \le \epsilon$ for every $B \in \mathcal{B}$. Since $\{f_n : n \in \mathbb{N}\}$ converges to f in $(\mathbb{R}^X, \tau_{UC})$, diam$f(B) \le \epsilon$ for every $B \in \mathcal{B}$. By Remark 1, the function f is F_σ measurable. Since X is a normal space, by Proposition 1, $f \in B_1(X, \mathbb{R})$. □

Theorem 3. *Let X be a Tychonoff topological space. Then, the following are equivalent:*

1. $(B_1(X,\mathbb{R}), \tau_{UC})$ *is completely metrizable;*
2. $(B_1(X,\mathbb{R}), \tau_{UC})$ *is Čech-complete;*
3. X *is hemicompact.*

Proof. $(1) \Rightarrow (2)$ is obvious. For $(2) \Rightarrow (3)$, it is known that a Čech-complete space is a q-space. By Theorem 2 $(5) \Rightarrow (6)$, X is hemicompact.

For $(3) \Rightarrow (1)$, by Proposition 2, $B_1(X,\mathbb{R})$ is a closed set in $(\mathbb{R}^X, \tau_{UC})$. Since the uniformity $\mathfrak{U}_{UC}$ on $\mathbb{R}^X$ is complete [8], $(\mathbb{R}^X, \tau_{UC})$ is completely metrizable. Thus, $(B_1(X,\mathbb{R}), \tau_{UC})$ is completely metrizable too. □

4. Cardinal Invariants of $(\mathcal{B}_1(X,Y), \tau_{uc})$

We first recall the definitions of cardinal invariants of a topological space Z [9]. Define the weight of Z as:

$$w(Z) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base in } Z\},$$

the density of Z as:

$$d(Z) = \aleph_0 + \min\{|D| : D \text{ is a dense set in } Z\},$$

the cellularity of Z as:

$$c(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a pairwise disjoint family of nonempty open sets in } Z\},$$

and the network weight of Z as:

$$nw(Z) = \aleph_0 + \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network in } Z\}.$$

They are in general related by the inequalities

$$c(Z) \leq d(Z) \leq nw(Z) \leq w(Z).$$

The character of a point z in Z is defined as:

$$\chi(Z,z) = \aleph_0 + \min\{|\mathcal{O}| : \mathcal{O} \text{ is a base at } z\},$$

and the character of Z is defined as:

$$\chi(Z) = \sup\{\chi(Z,z) : z \in Z\}.$$

To define the π-character of Z, we first need a notion of a local π-base. If $z \in Z$, a collection $\mathcal{V}$ of nonempty open subsets of Z is called a local π-base at z provided that for each open neighborhood U of z, there exists a $V \in \mathcal{V}$ which is contained in U.

The π-character of a point z in Z is defined as:

$$\pi_\chi(Z,z) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local } \pi\text{-base at } z\},$$

and the π-character of Z is defined as:

$$\pi_\chi(Z) = \aleph_0 + \sup\{\pi_\chi(Z,z) : z \in Z\}.$$

To define the π-weight of a topological space Z, we first need a notion of a π-base. A collection $\mathcal{V}$ of nonempty open subsets of Z is called a π-base [10] provided that for each open set U in Z, there exists a $V \in \mathcal{V}$ which is contained in U. Define the π-weight of Z by:

$$\pi w(Z) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi - \text{base in } Z\}.$$

The k-cofinality of a topological space Z is defined as

$$kcof(Z) = \min\{|\beta| : \beta \text{ is a cofinal family in } K(Z)\}.$$

If $kcof(Z) = \aleph_0$, the topological space Z is hemicompact.

In what follows, let X be a Tychonoff topological space. We will consider the cardinal invariants of the space $(B_1(X,\mathbb{R}), \tau_{UC})$. Because of simplicity, the specification of the topology τ_{UC} will be omitted. Since $(B_1(X,\mathbb{R}), \tau_{UC})$ is a topological group, the character and the π-character of $B_1(X,\mathbb{R})$ and the weight and the π-weight of $B_1(X,\mathbb{R})$ coincide [11].

First we prove that the character of $(B_1(X,\mathbb{R}), \tau_{UC})$ is equal to the k-cofinality of a topological space X.

Theorem 4. *Let X be a Tychonoff topological space. Then, $\chi(B_1(X,\mathbb{R})) = kcof(X)$.*

Proof. First, we show that $kcof(X) \leq \chi(B_1(X,\mathbb{R}))$. It is easy to verify that $\chi(C(X,\mathbb{R})) \leq \chi(B_1(X,\mathbb{R}))$. It is known that $kcof(X) \leq \chi(C(X,\mathbb{R}))$ [10].

To prove that $\chi(B_1(X,\mathbb{R}) \leq kcof(X)$, let $f \in B_1(X,\mathbb{R})$ and let β be a cofinal subfamily of $K(X)$ with $|\beta| = kcof(X)$. It is easy to verify that the family $\{W(f,K,1/n) : K \in \beta, n \in \mathbb{N}\}$ is a local base at f. $\square$

For a Tychonoff space Z, define the uniform weight of Z [9]:

$$u(Z) = \aleph_0 + \min\{m : \text{there is a uniformity on } Z \text{ of weight } \leq m\}.$$

Remark 2. *Is known (see [9]) that $w(Z) = c(Z) \cdot u(Z)$, $w(Z) = e(Z) \cdot u(Z)$, where $e(Z)$ is the extent of Z defined as follows:*

$$e(Z) = \aleph_0 + \sup\{|E| : \textit{ E is a closed discrete set in } Z\}.$$

Theorem 5. *Let X be a Tychonoff topological space. Then, $u(B_1(X,\mathbb{R})) = kcof(X)$.*

Proof. Let β be a cofinal family in $K(X)$ such that $kcof(X) = |\beta|$. It is easy to verify that the family $\{W(K,1/n) : K \in \beta, n \in \mathbb{N}\}$ is a base of the uniformity $\mathfrak{U}_{UC}$. Thus, $u(B_1(X,\mathbb{R})) \leq kcof(X)$. For every Tychonoff space Z, $\chi(Z) \leq u(Z)$. Since by Theorem 4 $kcof(X) = \chi(B_1(X,\mathbb{R}))$, we have $u(B_1(X,\mathbb{R})) = kcof(X)$. $\square$

Theorem 6. *If X is locally compact, then $w(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R}))$.*

Proof. By Remark 2

$$w(B_1(X,\mathbb{R})) = c(B_1(X,\mathbb{R})) \cdot u(B_1(X,\mathbb{R})) =$$
$$= nw(B_1(X,\mathbb{R})) \cdot u(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R})) \cdot kcof(X).$$

It suffices to show that $kcof(X) \leq nw(B_1(X,\mathbb{R}))$. Because X is locally compact, it has a base $\mathcal{B}$ of relatively compact sets such that $|\mathcal{B}| = w(X)$. Then, the family of all finite unions of members of $\{\overline{B} : B \in \mathcal{B}\}$ is cofinal in $K(X)$ and has cardinality $w(X)$. So, $kcof(X) \leq w(X)$. It is known that $w(X) = nw(C(X,\mathbb{R}))$ [10]. Since $nw(C(X,\mathbb{R})) \leq nw(B_1(X,\mathbb{R}))$, we have $kcof(X) \leq nw(B_1(X,\mathbb{R}))$. $\square$

In the following lemma, the notion of the discrete cellularity introduced in [12] is used. To define the discrete cellularity of a topological space Z, we need a notion of a discrete family of subsets of Z. We say that a family $\mathcal{U}$ of subsets of a topological space Z is discrete if each point $z \in Z$ has a neighborhood that meets at most one set of the family $\mathcal{U}$.

The discrete cellularity of Z is defined as:

$$dc(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a discrete family of nonempty open sets in } Z\}.$$

Remark 3. *For every topological space Z, $dc(Z) \leq c(Z)$ and $dc(Z) \leq e(Z)$* [12].

Lemma 2. *Let X be a Tychonoff topological space which contains an infinite compact set. Then, $dc(B_1(X,\mathbb{R})) \geq \mathfrak{c}$.*

Proof. Let K be an infinite compact set in X. There is a sequence of different points $\{x_n : n \in \mathbb{N}\}$ with a cluster point $x \in K \setminus \{x_n : n \in \mathbb{N}\}$.

Let U_1, V_1 be disjont open sets such that $x_1 \in U_1$ and $x \in V_1$. There is $x_{n_2} \in \{x_n : n \in \mathbb{N}\}$ such that $x_{n_2} \in V_1$. Let U_2, V_2 be disjont open sets such that $U_2 \subset V_1$, $V_2 \subset V_1$, $x_{n_2} \in U_2$ and $x \in V_2$. Continuing this way, we can show that there is a pairwise disjoint sequence $\{U_n : n \in \mathbb{N}\}$ of open sets such that $U_n \cap K \neq \emptyset$ for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, choose a point $a_n \in U_n \cap K$. Let $2^{\mathbb{N}}$ denote the set of all functions from $\mathbb{N}$ to $\{0,1\}$. For every $\varphi \in 2^{\mathbb{N}}$, denote by $\mathbb{N}_\varphi$ the set of all $n \in \mathbb{N}$ where $\varphi(n) = 1$.

Let $\varphi \in 2^{\mathbb{N}}$ be such that $\mathbb{N}_\varphi = \{n\}$.

Let $f_{\{n\}} : X \to [0,1]$ be a continuous function defined as follows:

$$f_{\{n\}}(x) = \begin{cases} 1, & x = a_n; \\ 0, & x \in X \setminus U_n. \end{cases}$$

Let $\varphi \in 2^{\mathbb{N}}$ be such that $\mathbb{N}_\varphi = \{n_1, n_2, \ldots, n_k\}$. Define function $f_{\{n_1,n_2,\ldots,n_k\}} : X \to \mathbb{R}$ as follows:

$$f_{\{n_1,n_2,\ldots,n_k\}} = \max\{f_{\{n_1\}}, f_{\{n_2\}}, \ldots, f_{\{n_k\}}\}.$$

Then, $f_{\{n_1,n_2,\ldots,n_k\}}$ is a continuous function.

Let $\varphi \in 2^{\mathbb{N}}$ be such that $\mathbb{N}_\varphi$ is infinite and let $\mathbb{N}_\varphi = \{n_k\}_{k=1}^\infty$. For every $x \in X$, let $f_{\{n_k\}_{k=1}^\infty}(x) = \lim_{k\to\infty} f_{\{n_1,n_2,\ldots,n_k\}}(x)$.

For every $\varphi \in 2^{\mathbb{N}}$, define $B_{\mathbb{N}_\varphi} = W(f_{\mathbb{N}_\varphi}, K, 1/4)$. Let $g \in B_1(X,\mathbb{R})$. Then, $W(g,K,1/4)$ intersect at most one set of $\{B_{\mathbb{N}_\varphi} : \varphi \in 2^{\mathbb{N}}\}$. So, $\{IntB_{\mathbb{N}_\varphi} : \varphi \in 2^{\mathbb{N}}\}$ is a discrete family of open subsets of $(B_1(X,\mathbb{R}), \tau_{UC})$, where by $IntB_{\mathbb{N}_\varphi}$ we mean the interior of $B_{\mathbb{N}_\varphi}$ in $(B_1(X,\mathbb{R}), \tau_{UC})$. $\square$

For a topological space Z, define the Lindelöf degree of Z as:

$$L(Z) = \aleph_0 + \min\{\kappa : \text{every open cover of } Z \text{ has a subcover of cardinality at most } \kappa\}$$

and the spread of Z as:

$$s(Z) = \aleph_0 + \sup\{|E| : E \text{ is a discrete set in } Z\}.$$

If X is hemicompact, then by Theorem 2, $B_1(X,\mathbb{R})$ is metrizable; thus, all cardinal invariants $c, d, nw, s, e, L, \pi w, w$ coincide on $B_1(X,\mathbb{R})$. The following theorem gives other conditions on X under which the cardinal invariants coincide on $B_1(X,\mathbb{R})$.

Theorem 7. *Let X be a Tychonoff topological space which contains an infinite compact set and let $kcof(X) \leq \mathfrak{c}$. Then,*

$$\begin{aligned} &c(B_1(X,\mathbb{R})) = d(B_1(X,\mathbb{R})) = e(B_1(X,\mathbb{R})) = L(B_1(X,\mathbb{R})) = \\ &s(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R})) = \pi w(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) \geq \mathfrak{c}. \end{aligned}$$

Proof. By Remark 2 and Theorem 5, $kcof(X) \cdot e(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) = kcof(X) \cdot c(B_1(X,\mathbb{R}))$. By Lemma 2 and Remark 3, $e(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) = c(B_1(X,\mathbb{R}))$. Since other cardinal invariants are between c, w and e, the proof is finished. $\square$

Corollary 1. *Let X be a separable metric space with a non isolated point. Then,*

$$c(B_1(X,\mathbb{R})) = d(B_1(X,\mathbb{R})) = e(B_1(X,\mathbb{R})) = L(B_1(X,\mathbb{R})) =$$
$$s(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R})) = \pi w(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) = |B_1(X,\mathbb{R})| = \mathfrak{c}.$$

Proof. Let x_0 be a non isolated point in X. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of different points in X which converges to x_0. Then, the set $K = \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is an infinite compact set in X. Since X is a separable metric space $kcof(X) \leq \mathfrak{c}$. Thus, by Theorem 7

$$c(B_1(X,\mathbb{R})) = d(B_1(X,\mathbb{R})) = e(B_1(X,\mathbb{R})) = L(B_1(X,\mathbb{R})) =$$
$$s(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R})) = \pi w(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) \geq \mathfrak{c}.$$

Since $|B_1(X,\mathbb{R})| = \mathfrak{c}$ and $nw(B_1(X,\mathbb{R})) \leq |B_1(X,\mathbb{R})|$, the proof is finished. □

Corollary 2. *For every $a \in [0,1]$, put $X_a = \mathbb{R}$. Let $X = \bigoplus_{a\in[0,1]} X_a$ be the topological sum of the family $\{X_a : a \in [0,1]\}$. Then,*

$$c(B_1(X,\mathbb{R})) = d(B_1(X,\mathbb{R})) = e(B_1(X,\mathbb{R})) = L(B_1(X,\mathbb{R})) =$$
$$s(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R})) = \pi w(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) \geq \mathfrak{c}.$$

5. Comparison of Cardinal Invariants of $B_1(X,\mathbb{R})$, $Q(X,\mathbb{R})$ and $C(X,\mathbb{R})$ for an Uncountable Polish Space X

Proposition 3. *Let X be an uncountable Polish space. Then,*

$$dc(B_1(X,\mathbb{R})) = c(B_1(X,\mathbb{R})) = d(B_1(X,\mathbb{R})) = e(B_1(X,\mathbb{R})) = L(B_1(X,\mathbb{R})) =$$
$$s(B_1(X,\mathbb{R})) = nw(B_1(X,\mathbb{R})) = \pi w(B_1(X,\mathbb{R})) = w(B_1(X,\mathbb{R})) = \mathfrak{c} = |B_1(X,\mathbb{R})|.$$

Proof. X is a separable metric space. By Corollary 6.5 in [13], X contains a homeomorphic copy of the Cantor space $2^{\mathbb{N}}$. Let C be a homeomorphic copy of $2^{\mathbb{N}}$. Then, every point of C is a non isolated point. We can apply Corollary 1. □

Let X and Y be topological spaces. A function $f : X \to Y$ is quasicontinuous [14] at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$ and every open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous. A survey on quasicontinuous functions can be found in a recent monograph [15]. Denote by $Q(X,\mathbb{R})$ the space of real-valued quasicontinuous functions.

Proposition 4. *Let X be an uncountable Polish space. Then, $dc(Q(X,\mathbb{R})) = 2^{\mathfrak{c}}$.*

Proof. If X is an uncountable Polish space, then by Corollary 6.5 in [13], X contains a homeomorphic copy of the Cantor space $2^{\mathbb{N}}$. Using some ideas from [16], put

$$F = \{(x_0, x_0, x_1, x_1, x_2, x_2, \ldots, x_n, x_n, \ldots) : (x_0, x_1, x_2, \ldots, x_n, \ldots) \in 2^{\mathbb{N}}\}$$

It is easy to verify that F is a closed nowhere dense set in $2^{\mathbb{N}}$ with the cardinality $\mathfrak{c}$. Thus, X contains a compact nowhere dense set C with a cardinality $\mathfrak{c}$ homeomorphic to F. Using Lemma 6 in [17], if X is a pseudometrizable space and C is a closed nowhere dense subset of X, there is a continuous function $f : X \setminus C \to [0,1]$ such that in every neighbourhood V of $z \in C$ there are $x, y \in V \cap C^c$ with $f(x) = 0$ and $f(y) = 1$. For every subset D of C, define a function $g_D : X \to [0,1]$ as follows: $g_D(x) = f(x)$ for every $x \notin C$, $g_D(x) = 1$ for $x \in D$ and $g_D(x) = 0$ for $x \in C \setminus D$. It is easy to verify that g_D is a quasicontinuous function. For every subset D of C, define $B_D = W(g_D, C, 1/4)$. Then, $\{IntB_D : D \text{ subset of } C\}$ is a discrete family of nonempty open sets in $(Q(X,\mathbb{R}), \tau_{UC})$ with the cardinality $2^{\mathfrak{c}}$. Thus, $dc(Q(X,\mathbb{R})) = 2^{\mathfrak{c}}$. □

Proposition 5. [18] *Let X be a topological space which contains an infinite compact set and let $kcof(X) \le \mathfrak{c}$. Then,*

$$c(Q(X,\mathbb{R})) = d(Q(X,\mathbb{R})) = e(Q(X,\mathbb{R})) = L(Q(X,\mathbb{R})) = \\ s(Q(X,\mathbb{R})) = nw(Q(X,\mathbb{R})) = \pi w(Q(X,\mathbb{R})) = w(Q(X,\mathbb{R})) \ge \mathfrak{c}.$$

Theorem 8. *Let X be an uncountable Polish space. Then,*

$$dc(Q(X,\mathbb{R})) = c(Q(X,\mathbb{R})) = d(Q(X,\mathbb{R})) = e(Q(X,\mathbb{R})) = L(Q(X,\mathbb{R})) = \\ s(Q(X,\mathbb{R})) = nw(Q(X,\mathbb{R})) = \pi w(Q(X,\mathbb{R})) = w(Q(X,\mathbb{R})) = 2^{\mathfrak{c}} = |Q(X,\mathbb{R})|.$$

Proof. Use Propositions 4 and 5. □

If X is a topological space with a countable base, then, by [10], $nw(C(X,\mathbb{R})) = \aleph_0$. Thus, for an uncountable Polish space X, we have

$$dc(C(X,\mathbb{R})) = c(C(X,\mathbb{R})) = d(C(X,\mathbb{R})) = e(C(X,\mathbb{R})) = L(C(X,\mathbb{R})) = \\ s(C(X,\mathbb{R})) = nw(C(X,\mathbb{R})) = \aleph_0,$$

and by [10],

$$w(C(X,\mathbb{R})) = \pi w(C(X,\mathbb{R})) = kcof(X),\ |C(X,\mathbb{R})| = \mathfrak{c}.$$

If X is not hemicompact, then $kcof(X) > \aleph_0$; thus, the topology τ_{UC} of uniform convergence on compacta on $C(X,\mathbb{R})$ may not behave like a metric topology. The space of irrational numbers equipped with the usual Euclidean topology is an uncountable Polish space, which is not hemicompact.

6. Conclusions and Future Work

The main purpose of this paper is to study cardinal invariants of real-valued Baire 1 functions $B_1(X,\mathbb{R})$ equipped with the topology τ_{UC} of uniform convergence on compacta and compare them with the cardinal invariants of the space $C(X,\mathbb{R})$ of continuous real-valued functions equipped with the topology of uniform convergence on compacta, which were studied in [10], and also compare them with the cardinal invariants of the space $Q(X,\mathbb{R})$ of quasicontinuous real-valued functions equipped with the topology of uniform convergence on compacta, which were studied in [18]. We proved that the following conditions are equivalent: (1) $(B_1(X,\mathbb{R}),\tau_{UC})$ is metrizable; (2) $(B_1(X,\mathbb{R}),\tau_{UC})$ is completely metrizable; (3) $(B_1(X,\mathbb{R}),\tau_{UC})$ is Čech-complete; and (4) X is hemicompact.

Thus, if X is hemicompact, all cardinal invariants' cellularity, density, netweight, spread, extent, Lindeloff number, π-weight and weight coincide on $(B_1(X,\mathbb{R}),\tau_{UC})$. We found further conditions for X under which these cardinal invariants coincide on $(B_1(X,\mathbb{R}), \tau_{UC})$. If X is a separable metric space with a non isolated point, then the topology of uniform convergence on compacta on $B_1(X,\mathbb{R})$ is seen to behave like a metric topology in the sense that the cellularity, density, netweight, spread, extent, Lindeloff number, π-weight and weight are all equal for this topology and they are equal to $\mathfrak{c} = |B_1(X,\mathbb{R})|$. If X is an uncountable Polish space, then all these cardinal invariants coincide on $(B_1(X,\mathbb{R},\tau_{UC})$ and the same result holds also for the space $Q(X,\mathbb{R})$ of real-valued quasicontinuous functions.

We used the usual methods of topology and set theory in the proofs. The theory developed in this paper could be of interest to mathematicians working in fields including topology and functional analysis.

Concerning a future investigation of the space of Baire 1 functions, we plan to study cardinal invariants of $B_1(X,\mathbb{R})$ equipped with the topology of pointwise convergence and compare them with the cardinal invariants of the space $C(X,\mathbb{R})$ of continuous real-valued functions equipped with the topology of pointwise convergence, which were studied in [10].

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