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The Operational Laws of Symmetric Triangular Z-Numbers

Hui Li [1], Xuefei Liao [2,*], Zhen Li [3], Lei Pan [4], Meng Yuan [5] and Ke Qin [4]

1 School of Economics, Shanghai University, Shanghai 200444, China; youlanlihui@shu.edu.cn
2 School of Economics and Management, Zhejiang Ocean University, Zhoushan 316022, China
3 School of Logistics and Maritime Studies, Bahrain Polytechnic, Isa Town 33349, Bahrain; wesley.lee@polytechnic.bh
4 School of Management, Shanghai University, Shanghai 200444, China; pl239201@shu.edu.cn (L.P.); 2532129158@shu.edu.cn (K.Q.)
5 Qian Weichang College, Shanghai University, Shanghai 200444, China; yuanmeng@shu.edu.cn
* Correspondence: liaoxuefei@zjou.edu.cn; Tel.: +86-21-6613-4414-805

Abstract: To model fuzzy numbers with the confidence degree and better account for information uncertainty, Zadeh came up with the notion of Z-numbers, which can effectively combine the objective information of things with subjective human interpretation of perceptive information, thereby improving the human comprehension of natural language. Although many numbers are in fact Z-numbers, their higher computational complexity often prevents their recognition as such. In order to reduce computational complexity, this paper reviews the development and research direction of Z-numbers and deduces the operational rules for symmetric triangular Z-numbers. We first transform them into classical fuzzy numbers. Using linear programming, the extension principle of Zadeh, the convolution formula, and fuzzy number algorithms, we determine the operational rules for the basic operations of symmetric triangular Z-numbers, which are number-multiplication, addition, subtraction, multiplication, power, and division. Our operational rules reduce the complexity of calculation, improve computational efficiency, and effectively reduce the information difference while being applicable to other complex operations. This paper innovatively combines Z-numbers with classical fuzzy numbers in Z-number operations, and as such represents a continuation and innovation of the research on the operational laws of Z-numbers.

Keywords: Z-numbers; symmetric triangular fuzzy numbers; operational laws

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1. Introduction

In 1965, Zadeh [1] introduced the theory of fuzzy sets to effectively cope with uncertain information. The theory highlights the fuzziness and uncertainty of human thinking, reasoning, and perception of peripheral matters. It extended the characteristic function from the binary '0' or '1' relationship to the interval '0' to '1' by introducing the concept of membership degree, thereby quantitatively processing fuzzy information.

Nevertheless, relying solely on membership degree makes it difficult to accurately describe the uncertainty in practical situations. Therefore, to resolve the uncertainty of non-membership degree, researchers have made various extensions and derived batches of theories such as intuitionistic fuzzy sets [2], hesitant fuzzy sets [3], type-2 fuzzy sets [4], and interval-type intuitionistic fuzzy sets [5]. Moreover, in 2013 Masamichi and Hiroaki [6] defined the boundaries of a sequence of fuzzy sets in view of the level set of fuzzy sets and provided the boundaries, derivatives, and properties of the fuzzy set-valued mapping.

The aforementioned theories are only capable of addressing the issue of information uncertainty, and lack the ability to handle incomplete and unreliable information, which is typically only accessible in real-world situations. To this end, in 2011 Zadeh [7] introduced the notion of Z-numbers to consider the dependability of information. Compared to the traditional fuzzy sets, Z-numbers add a reliability measure to further enhance the flexibility and validity in the decision direction. Therefore, Z-numbers with fuzzy constraints are more flexible and closer to human thinking; this theory has great potential for application to the information described by probabilistic and fuzzy natural language.

From the current research outcomes on Z-numbers, we have observed four primary issues of interest. The first involves extensions and special cases of Z-number theory. Zadeh [7] initially introduced the notions of Z-information and Z^+-numbers as definitions derived from Z-numbers. Pal et al. [8] proposed Z-number-based computing with word algorithms and simulated experimental figures for evaluating demand satisfaction. Banerjee and Pal [9] introduced decision information into the structure of Z-numbers and presented the notion of Z*-numbers. Pirmuhammadi et al. [10], Peng, Wang [11], and Mondal et al. [12] proposed the concept of normal Z-numbers, hesitant uncertain linguistic Z-numbers, and linguistic hesitant Z-numbers, respectively. Tian et al. [13] introduced fuzzy ZE-numbers, while Haseli et al. [14–16] proposed a decision support model using the BCM and MARCOS methods based on fuzzy ZE-numbers. Aliev et al. [17] initiated a general method for constructing specific functions based on extension of the Z-number principle. Moreover, Massanet et al. [18] raised a new method for creating hybrid discrete Z-numbers based on discrete Z-numbers.

The second issue involves the study of various methods for sorting Z-numbers. Bakar and Gegov [19] developed a multi-layer approach to classifying Z-numbers. Aliev et al. [20] presented a method to ascertain the sorting of continuous and discrete numbers in Z-numbers. A novel Z-number ranking method which takes the weights and fuzziness degree of the prime points and the scalability of fuzzy numbers into account was extended by Jiang et al. [21]. Ezadi et al. [22] introduced sigmoidal functions and symbolic means for sorting Z-numbers.

The third issue involves studying various methods for computing Z-numbers. Aliev et al. [23] developed an approach for the direct computation of Z-numbers by combining possibility constraints with probability constraints and defined arithmetic operations for discrete Z-Numbers. Subsequently, the operations of continuous Z-numbers were further provided by Aliev et al. [24] through discretization. Aiming to reduce computational complexity and improve computational efficiency, Aliev et al. [25] presented a basic approach for developing the concept of Z-Numbers and provided examples to demonstrate the validity of their method using the Hukuhara distance. Qiu et al. [26] presented the process of computing the generalized difference for discrete and continuous Z-numbers. Shen and Wang [27] defined multidimensional Z-numbers and proved their basic operations. Kang et al. [28] presented a methodology of fuzzy set uncertainty using entropy and considering the effect of fuzzy set measure and range of fuzzy sets. Peng et al. [29] defined a series of Z-number operational laws on the basis of Archimedean t-norms and t-conorms. To balance reduced arithmetic complexity with retention of the inherent meaning of Z-numbers, Zhu et al. [30] put forward a method for approximate Z-number computation (Z-ACM) in view of kernel density estimation. Based on the idea of transformation, Kang et al. [31] proposed an improved method for converting Z-numbers into classical fuzzy numbers, greatly simplified the operations of Z-numbers with the loss of a certain amount of information, and promoted the application of Z-numbers to a degree.

The fourth issue involves research on the actual applications of Z-numbers. Zhang et al. [32] combined Z-numbers with the best–worst method and TODIM (an acronym in Portuguese referring to interactive and multi-criteria decision-making) to conduct performance evaluation for the technological service platforms. Ashraf et al. [33] and Nazari-Shirkouhi et al. [34] applied Z-numbers to supplier selection. Combining Z-numbers with DEMATEL method, Zhu et al. [35], Wang et al. [36], and Akhavein et al. [37] presented evaluation methods for the co-creative sustainable value propositions of smart product service systems, human error probability, and sustainable projects ranking. Integrating linguistic Z-numbers and the projection method, Huang et al. [38] built a new model for failure mode and effect analysis. Moreover, numerous experts have expanded the notion of Z-numbers based on hesitant fuzzy sets and used optimization models to build frameworks for solving multi-criteria decision-making, group decision-making, and three-way decision-making problems [11,39–44].

Previous research on the operational rules of Z-numbers has mainly focused on proposing a general method of computation using constraints, then used discretization to obtain the operational rules of continuous Z-numbers. This category of methods is extremely complicated, inefficient, and error-prone; for this reason, many researchers have chosen to combine Z-numbers with other methods in order to derive the operation formulas. Hence, to improve the efficiency of operation and make it easier to understand, we take symmetric triangular Z-numbers as our research object and study their operational rules, including number-multiplication, addition, subtraction, multiplication, power squares, and division.

The main contribution of our approach is as follows. First, we convert Z-numbers directly into classical fuzzy numbers using Zadeh's extension principle and the operational rules of classical fuzzy numbers for operations of Z-numbers, which does not appear in any previous related papers. Second, we use many linear correlation methods to calculate the symmetric triangular Z-numbers, which is simple in both calculation principle and process and as such can reduce the complexity of the operations. Third, we derive the formulas of the basic operations for Z-numbers, which can be directly used to simplify the complex operations involved in many realistic problems and expand the application areas of Z-numbers. Our calculation method can reduce uncertainty and prevent information loss while processing information, which can minimize information differences.

We structure the remainder of this paper as follows: Section 2 briefly introduces the related definitions and notation; Section 3 deduces the operational rules for symmetric triangular Z-numbers; finally, Section 4 draws the conclusions.

2. Preliminaries

To begin, a number of fundamental concepts are first concisely introduced.

Definition 1 (Zadeh [1])**.** *In a given domain U, the fuzzy number A can be defined as*

$$A = \{\langle t, \mu_A(t)\rangle | t \in U\},$$

where $\mu_A : U \to [0,1]$ is the membership function of A', while $\mu_A(t)$ depicts the degree of belongingness of $t \in U$ in A.

Definition 2 (Van Laarhoven and Pedrycz [45])**.** *A is a triangular fuzzy number which can be defined as (a_p, a_q, a_r); its membership function can be determined as*

$$\mu_A(t) = \begin{cases} 0, & t \in (-\infty, a_p) \\ \dfrac{t - a_p}{a_q - a_p}, & t \in (a_p, a_q) \\ \dfrac{a_r - t}{a_r - a_q}, & t \in (a_q, a_r) \\ 0, & t \in (a_r, +\infty), \end{cases}$$

where a_p and a_r are respectively the upper and lower bounds of A. When $a_r - a_q = a_q - a_p$, A is a symmetric triangular fuzzy number.

Definition 3 (Zadeh [7])**.** *A Z-number Z is an ordered fuzzy number pair, denoted as $Z = (A, B)$, where A,B could be either natural languages or numbers. Z is associated with T, which is a real-*

valued uncertain variable. Fuzzy number A represents the fuzzy constraint R(T) on the values which T can take, defined as T is A, represented as

$$R(T) : T\, is\, A \to Poss(T = t) = \mu_A(t),$$

where μ_A is the membership function of A and t is a generic value of T. When T is a random variable, the probability distribution of T represents a probabilistic restriction on T, which can be expressed as

$$R(T) : T\, is\, p,$$

where p is the probability density function of T. Under this circumstance,

$$R(T) : T\, is\, p,\ p \to Prob(t \le T \le t + dt) = p(t)dt.$$

If T is a random variable, then T is A represents a fuzzy event in R, the probability of which can be defined as

$$p = \int_R \mu_A(t) p_T(t) dt,$$

where p_T is the underlying probability density of T. Fuzzy number B is the fuzzy restriction on the reliability measure of A, expressed as

$$B = \int_R \mu_A(t) p_T(t) dt, \tag{1}$$

where p_T is not known, whereas the constraint on p_T is known, which can be presented in Figure 1.

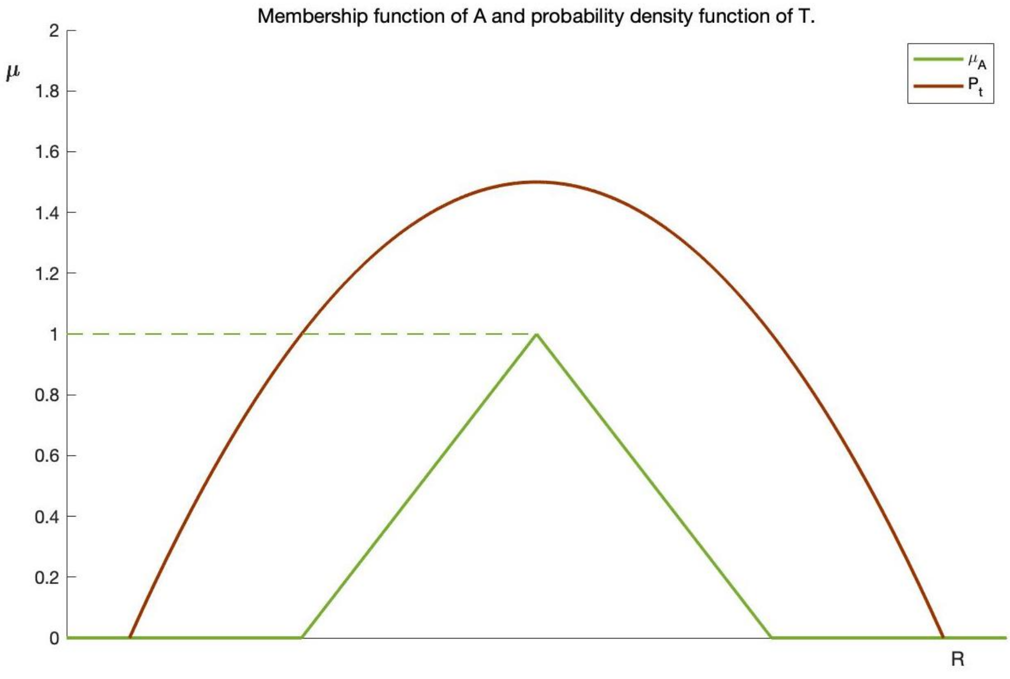

Figure 1. The membership function of A and probability density function of T.

In effect, $Z = (A, B)$ can be regarded as a restriction on T, defined as

$$Prob(T\, is\, A)\, is\, B.$$

Definition 4 (Aliev et al. [23])**.** *In a Z-number represented by $Z = (A, B)$, if the fuzzy restriction A of the real-valued indefinite variable T on the domain U is a discrete fuzzy set*

$$\mu_A : \{t_1, t_2, \cdots, t_n\} \to [0, 1], and \{t_1, t_2, \cdots, t_n\} \in R$$

and B is the reliability measure for A, which is also a discrete fuzzy set

$$\mu_B : \{b_1, b_2, \cdots, b_n\} \to [0,1], and\{b_1, b_2, \cdots, b_n\} \in [0,1],$$

then $Z = (A, B)$ is a discrete Z-number.

Definition 5 (Aliev et al. [24]). *In a Z-number represented by $Z = (A, B)$, if the fuzzy restriction A of the real-valued indefinite variable T on the domain U is a continuous fuzzy set*

$$\mu_A : U \to [0,1]$$

and B is the reliability measure for A, which is also a continuous fuzzy set, then $Z = (A, B)$ is a continuous Z-number.

Definition 6 (Zadeh [7]). *Let ζ and τ be fuzzy sets with membership functions μ and v, and let $f : \Re^2 \to \Re$ be a function; then, $f(\zeta, \tau)$ is also a fuzzy set with membership function*

$$\pi(h) = sup\{\mu(s) \wedge v(t) | h = f(s,t)\}, \tag{2}$$

where s and t are the values within the range of ζ and τ.

Definition 7 (Wang [46]). *Let ξ be a fuzzy number; its α-level sets (or α-cuts) ξ_α can be expressed as*

$$\begin{aligned} \xi_\alpha &= \left\{ t \in \Re | \mu_\xi(t) \geq \alpha \right\} \\ &= \left[min\left\{ t \in \Re | \mu_\xi(t) \geq \alpha \right\}, max\left\{ t \in \Re | \mu_\xi(t) \geq \alpha \right\} \right] = [\xi_\alpha^L, \xi_\alpha^R], \end{aligned}$$

where $\mu_\xi(t)$ is the membership function of ξ and $\Re$ is the universe of discourse. The functions ξ_α^L and ξ_α^R have the following attributes:

(a) ξ_α^L *is a monotonously growing left continuous function,*
(b) ξ_α^R *is a monotonously lessening left continuous function,*
(c) $\xi_\alpha^L \leq \xi_\alpha^R$, $\alpha \in [0,1]$.

Example 1 (Aliev et al. [24]). *Given that $A = (a_l, a_m, a_u)$ is a symmetric triangular fuzzy number, an α-cut $A_\alpha = \{t \in \Re | \mu_A(t) \geq \alpha\}$ is a closed interval:*

$$\begin{aligned} A_\alpha = [A_\alpha^L, A_\alpha^R] &= [a_l + \alpha(a_m - a_l), a_u + \alpha(a_u - a_m)] \\ &= [a_m - (1-\alpha)(a_m - a_l), a_m + (1-\alpha)(a_u - a_m)]. \end{aligned}$$

Definition 8 (Kang et al. [31]). *The basic idea of translating Z-numbers into classical fuzzy numbers is as follows. First, the reliability part B is transformed into a crisp value by defuzzification, then the weight of the crisp value is multiplied by the restriction part A, and finally, using the approximate invariance property of the fuzzy expectation, the product is converted into a commonly used classical fuzzy number.*

Step 1: *Assuming that $A = (a_k, a_l, a_m, a_n)$ is a trapezoid fuzzy number and $B = (b_l, b_m, b_n)$ is a triangular fuzzy number, B can be transformed into a crisp number by the center of gravity method with*

$$\gamma = \frac{\int t\mu_{B(t)} dt}{\int \mu_{B(t)} dt} = \frac{\int_{b_l}^{b_m} t \dfrac{t - b_l}{b_m - b_l} dt + \int_{b_m}^{b_n} t \dfrac{b_n - t}{b_n - b_m} dt}{\dfrac{1}{2}(b_n - b_l)}.$$

Thus, the gravity center of $B = (b_l, b_m, b_n)$ is computed as

$$\gamma = \frac{b_n - b_l}{2}. \tag{3}$$

Step 2: *Taking the gravity center value γ of the reliability part B as the weight of the restriction part A, the weighted Z-value can be written as*

$$Z^{\gamma} = \{(t, \mu_{A^{\gamma}}) | \mu_{A^{\gamma}}(t) = \gamma \mu_A(t), t \in [0, 1]\}. \tag{4}$$

Step 3: *Because $A = (a_k, a_l, a_m, a_n)$ is a trapezoidal fuzzy number, Z^{γ} can be calculated by*

$$Z^{\gamma} = \sqrt{\gamma} \times A = (\sqrt{\gamma} \times a_k, \sqrt{\gamma} \times a_l, \sqrt{\gamma} \times a_m, \sqrt{\gamma} \times a_n). \tag{5}$$

Remark 1. *If $A = (a_k, a_l, a_m)$ is a triangular fuzzy number, Equation (5) becomes*

$$Z^{\gamma} = \sqrt{\gamma} \times A = (\sqrt{\gamma} \times a_k, \sqrt{\gamma} \times a_l, \sqrt{\gamma} \times a_m). \tag{6}$$

Definition 9 (Kwiesielewicz [47]). *Let $A = (a_p, a_q, a_r)$, $B = (b_p, b_q, b_r)$ be two triangular fuzzy numbers, where $a_r \geq a_q \geq a_p \geq 0$ and $b_r \geq b_q \geq b_p \geq 0$. Then, their addition, difference, number-multiplication, and division can be shown as follows:*

$$A + B = [a_p + b_p, a_q + b_q, a_r + b_r], \tag{7}$$

$$A - B = [a_p - b_r, a_q - b_q, a_r - b_p], \tag{8}$$

and

$$\frac{A}{B} = [\frac{a_p}{b_r}, \frac{a_q}{b_q}, \frac{a_r}{b_p}].$$

Definition 10 (Aliev et al. [24]). *The multiplication and division of fuzzy numbers $A = (a_p, a_q, a_r)$ and $B = (b_p, b_q, b_r)$ are both fuzzy sets. The multiplication can be expressed as*

$$A \times B = \underset{\alpha \in (0,1]}{U} \alpha (A \times B)^{\alpha},$$

where the α-cut is expressed as

$$(A \times B)^{\alpha} = [min(a_1^{\alpha} \cdot b_1^{\alpha}, a_1^{\alpha} \cdot b_2^{\alpha}, a_2^{\alpha} \cdot b_1^{\alpha}, a_2^{\alpha} \cdot b_2^{\alpha}), max(a_1^{\alpha} \cdot b_1^{\alpha}, a_1^{\alpha} \cdot b_2^{\alpha}, a_2^{\alpha} \cdot b_1^{\alpha}, a_2^{\alpha} \cdot b_2^{\alpha})], \tag{9}$$

where $a_1^{\alpha} = a_p + \alpha(a_q - a_p)$, $a_2^{\alpha} = a_r + \alpha(a_r - a_q)$, $b_1^{\alpha} = b_p + \alpha(b_q - b_p)$, $b_2^{\alpha} = b_r + \alpha(b_r - b_q)$.
The division can be denoted as

$$\frac{A}{B} = \underset{\alpha \in (0,1]}{U} \alpha \left(\frac{A}{B}\right)^{\alpha},$$

where the α-cut is expressed as

$$\left(\frac{A}{B}\right)^{\alpha} = \left[min\left(\frac{a_1^{\alpha}}{b_1^{\alpha}}, \frac{a_1^{\alpha}}{b_2^{\alpha}}, \frac{a_2^{\alpha}}{b_1^{\alpha}}, \frac{a_2^{\alpha}}{b_2^{\alpha}}\right), max\left(\frac{a_1^{\alpha}}{b_1^{\alpha}}, \frac{a_1^{\alpha}}{b_2^{\alpha}}, \frac{a_2^{\alpha}}{b_1^{\alpha}}, \frac{a_2^{\alpha}}{b_2^{\alpha}}\right)\right]. \tag{10}$$

Definition 11 (Kallenberg [48]). *Suppose that (S, T) are two-dimensional continuous random variables which have probability density $f(s, t)$. Then, $R = S + T$ is still a continuous random variable with probability density*

$$f_{S+T}(r) = \int_R f(r - t, t)dt = \int_R f(s, r - s)ds. \tag{11}$$

Let the marginal probability density of (S,T) with respect to S,T be $f_S(s)$ and $f_T(t)$. If S and T are independent of each other, Equation (11) will be reduced to the convolution formula

$$f_S \circ f_T = \int_R f_S(r-t) f_T(t) dt = \int_R f_S(s) f_T(r-s) ds. \tag{12}$$

If $R = \dfrac{T}{S}$, then

$$f_S \circ f_T = \int_R |s| f_S(s) f_T(sr) ds. \tag{13}$$

If $R = ST$, then

$$f_S \circ f_T = \int_R \frac{1}{|s|} f_S(s) f_T(\frac{r}{s}) ds. \tag{14}$$

Definition 12 (Aliev et al. [24])**.** *In order to discretize fuzzy numbers, a method is presented. The idea is to split the assistance of a fuzzy number B, Supp(B), into several subintervals b_k, $k = 1, \cdots, n$. In particular, the subintervals are of the same size, i.e., the spacing is constantly $\Delta b = b_{k+1} - b_k$.*

Example 2. *Consider $B = (0.4, 0.5, 0.6)$; its support Supp(B) will be discretized into $n = 11$ points in the way shown below: $b_{j1} = 0.4$, $b_{j2} = 0.425$, $\cdots$, $b_{j11} = 0.6$. Then, the discretized fuzzy number can be attained as*

$$B = \frac{0}{0.4} + \frac{0.2}{0.42} + \frac{0.4}{0.44} + \frac{0.6}{0.46} + \frac{0.8}{0.48} + \frac{1}{0.5} + \frac{0.8}{0.52} + \frac{0.6}{0.54} + \frac{0.4}{0.56} + \frac{0.2}{0.58} + \frac{0}{0.6}.$$

In the succeeding sections, we will use the above definitions and methods to derive the operational rules of the symmetric triangular Z-numbers.

3. Operational Rules

This section introduces the operational rules for Z-numbers. The first step in the operations is all about converting Z-numbers into ordinary fuzzy numbers, as defined in Definition 8. Because the operations studied here are based on symmetric triangular Z-numbers, $A = (a_p, a_q, a_r)$, $B = (b_p, b_q, b_r)$, the value of the weight γ of the reliability part B is always as shown in Equation (3). Considering that calculating the second component of the derived Z-number requires a relatively long computation time, whereas the calculation processes are similar to each other, we only provided examples for the number-multiplication and addition operations in this section.

3.1. Number-Multiplication Formula

Theorem 1. *Let λ be a real number and let $Z = (A, B) = ((a_p, a_q, a_r), (b_p, b_q, b_r))$ be a symmetric triangular Z-number. The formula for the number-multiplication of the continuous symmetric triangular Z-number is*

$$\lambda Z = \lambda(A, B) = (\lambda A, B).$$

Proof. First, multiplying a real number $\lambda \in R$ by the base of Equation (6), we can obtain

$$\lambda Z^{\gamma} = (\lambda\sqrt{\gamma} \times a_p, \lambda\sqrt{\gamma} \times a_q, \lambda\sqrt{\gamma} \times a_r).$$

From Equation (3), $\gamma = \dfrac{b_r - b_p}{2}$. Let $\bar{Z}^{\gamma} = \lambda Z^{\gamma}$. Because the weights remain unchanged after the number-multiplication, the formula becomes

$$\bar{Z}^{\gamma} = (\lambda\sqrt{\gamma} \times a_p, \lambda\sqrt{\gamma} \times a_q, \lambda\sqrt{\gamma} \times a_r).$$

Therefore, the final likelihood measure B is unchanged, and we obtain

$$\lambda Z = \lambda(A, B) = (\lambda A, B).$$

□

Example 3. *Assume that $Z = (A, B) = ((1, 2, 3), (0.7, 0.8, 0.9))$, and calculate $3Z$.*

According to Equation (3), we have $\gamma = \frac{0.9 - 0.7}{2} = 0.1$. Accordingly, we can determine that

$$\bar{Z}^{0.1} = 3Z^{0.1} = (3\sqrt{10} \times 1, 3\sqrt{10} \times 2, 3\sqrt{10} \times 3) = (3\sqrt{10}, 6\sqrt{10}, 9\sqrt{10}).$$

Finally, we obtain

$$3Z = (A, B_{12}) = \Big((3\sqrt{10}, 6\sqrt{10}, 9\sqrt{10}), (0.7, 0.8, 0.9)\Big).$$

3.2. Addition Formula

Theorem 2. *Let $Z_1 = (A_1, B_1) = ((a_{1p}, a_{1q}, a_{1r}), (b_{1p}, b_{1q}, b_{1r}))$ and let $Z_2 = (A_2, B_2) = ((a_{2p}, a_{2q}, a_{2r}), (b_{2p}, b_{2q}, b_{2r}))$ be continuous symmetric triangular Z-numbers. Then, their sum Z_{12} can be deduced as*

$$Z_{12} = Z_1 + Z_2 = (A_{12}, B_{12}),$$

where $A_{12} = (a_1^\gamma, a_2^\gamma, a_3^\gamma) = (\sqrt{\gamma_1}a_{1p} + \sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_1}a_{1q} + \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} + \sqrt{\gamma_2}a_{2r})$, $B_{12} = \int_R \mu_{A_{12}} p_{12} dt$, $\gamma_1 = \frac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \frac{b_{2r} - b_{2p}}{2}$, $\mu_{A_{12}}$ is the membership function of A_{12}, and p_{12} is the probability density of A_{12}.

Proof. Based on Equation (6), the fuzzy transformations of Z_1 and Z_2 are

$$Z_1^\gamma = \sqrt{\gamma_1}A_1 = (\sqrt{\gamma_1}a_{1p}, \sqrt{\gamma_1}a_{1q}, \sqrt{\gamma_1}a_{1r}),$$
$$Z_2^\gamma = \sqrt{\gamma_2}A_2 = (\sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_2}a_{2r}).$$

According to Equation (7), the sum of the two can be derived as

$$Z_{12}^\gamma = Z_1^\gamma + Z_2^\gamma = (\sqrt{\gamma_1}a_{1p} + \sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_1}a_{1q} + \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} + \sqrt{\gamma_2}a_{2r}).$$

From Equation (3), it is known that $\gamma_1 = \frac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \frac{b_{2r} - b_{2p}}{2}$, where b_{1r} and b_{1p} denote the third and first possibility of Z_1, respectively, while b_{2r} and b_{2p} denote the third and first possibility of Z_2, respectively. Therefore, substitution yields

$$Z_{12}^\gamma = \left(\sqrt{\frac{b_{1r}-b_{1p}}{2}}a_{1p} + \sqrt{\frac{b_{2r}-b_{2p}}{2}}a_{2p}, \sqrt{\frac{b_{1r}-b_{1p}}{2}}a_{1q} + \sqrt{\frac{b_{2r}-b_{2p}}{2}}a_{2q}, \sqrt{\frac{b_{1r}-b_{1p}}{2}}a_{1r} + \sqrt{\frac{b_{2r}-b_{2p}}{2}}a_{2r}\right).$$

At this point, Z-numbers A and B have been transformed into simple fuzzy numbers Z_1^γ and Z_2^γ, and are transformed into Z_{12}^γ by symbolic operations; the subsequent step is to convert the simple fuzzy number Z_{12}^γ into a Z-number again.

According to Equation (4), the membership functions of Z_1 and Z_2 can be transformed into the membership functions of their corresponding fuzzy numbers as follows:

$$\mu_{A_1}^\gamma = \frac{b_{1r} - b_{1p}}{2}\mu_{A_1} = PosA_1^\gamma,$$
$$\mu_{A_2}^\gamma = \frac{b_{2r} - b_{2p}}{2}\mu_{A_2} = PosA_2^\gamma.$$

Then, according to Equation (2), we can find that

$$\mu^{\gamma}_{A_{12}}(v) = \sup_{u}(\mu^{\gamma}_{A_1}(u) \wedge \mu^{\gamma}_{A_2}(v-u)).$$

Let a_1^{γ}, a_2^{γ}, and a_3^{γ} be the three coordinate values on the horizontal axis. At this point, the range of the membership function after transformation and summation is $(0, \beta)$ instead of $(0,1)$, where β denotes the maximum of the triangular fuzzy number, i.e., the vertex of the vertical axis coordinate corresponding to a_2^{γ}. Then, β is $min\left\{\frac{b_{1r}-b_{1p}}{2}, \frac{b_{2r}-b_{2p}}{2}\right\}$, which, as it is contrary to the initial required range of $(0,1)$, should be normalized to

$$\mu_{A_{12}} = \begin{cases} \frac{t-a_1^{\gamma}}{a_2^{\gamma}-a_1^{\gamma}}, & t \in (a_1^{\gamma}, a_2^{\gamma}) \\ \frac{a_3^{\gamma}-t}{a_3^{\gamma}-a_2^{\gamma}}, & t \in (a_2^{\gamma}, a_3^{\gamma}) \\ 0, & otherwise. \end{cases} \tag{15}$$

Then, $A_{12} = (a_1^{\gamma}, a_2^{\gamma}, a_3^{\gamma}) = (\sqrt{\gamma_1}a_{1p} + \sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_1}a_{1q} + \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} + \sqrt{\gamma_2}a_{2r})$. Accordingly, we can use $\mu_{A_{12}}$ to calculate B_{12} in view of Equation (1) by

$$B_{12} = \int_R \mu_{A_{12}} p_{12} dt. \tag{16}$$

From Equation (12), we can obtain $P_{12} = P_1 \circ P_2 = \int_R p_1(u)p_2(v-u)du$, and according to Equation (1)

$$B_1 = \int_R \mu_{A_1}(t)p_1(t)dt, \quad B_2 = \int_R \mu_{A_2}(t)p_2(t)dt,$$

the values of $p_1(u)$ and $p_2(v-u)$ in calculating P_{12} can be determined.

Nevertheless, if the calculation is carried out directly, many solutions will be obtained, and they should be subject to

$$\begin{cases} \int_R p(t)dt = 1 \\ p(t) > 0 \\ \int tp(t)dt = \frac{\int t\mu_A(t)dt}{\int \mu_A(t)dt} = \frac{\int t\frac{t-a_p}{a_q-a_p}dt}{\int \frac{t-a_p}{a_q-a_p}dt}. \end{cases} \tag{17}$$

Under such conditions, p_{12} can be derived, then B_{12} can be obtained by substituting p_{12} and $\mu_{A_{12}}$ into Equation (16). Accordingly, we can obtain

$$Z_{12} = Z_1 + Z_2 = (A_{12}, B_{12}).$$

□

Example 4. *Assume that* $Z_1 = (A_1, B_1) = ((1,2,3),(0.7,0.8,0.9))$, $Z_2 = (A_2, B_2) = ((7,8,9),(0.4,0.5,0.6))$ *and calculate* $Z_{12} = Z_1 + Z_2$.

First, according to Equation (3), the values of γ_1 *and* γ_2 *can be derived as*

$$\gamma_1 = \frac{0.9-0.7}{2} = 0.1, \gamma_2 = \frac{0.6-0.4}{2} = 0.1.$$

As a result, the value of A_{12} *is calculated as follows:*

$$\begin{aligned} A_{12} &= (a_1^{\gamma}, a_2^{\gamma}, a_3^{\gamma}) = (\sqrt{\gamma_1}a_{1p} + \sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_1}a_{1q} + \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} + \sqrt{\gamma_2}a_{2r}) \\ &= (\sqrt{0.1} \times 1 + \sqrt{0.1} \times 7, \sqrt{0.1} \times 2 + \sqrt{0.1} \times 8, \sqrt{0.1} \times 3 + \sqrt{0.1} \times 9) \\ &= \left(\frac{8}{\sqrt{10}}, \sqrt{10}, \frac{12}{\sqrt{10}} \right). \end{aligned}$$

According Equation (15), the following membership function is obtained after normalization:

$$\mu_{A_{12}} = \begin{cases} \dfrac{\sqrt{10}t - 8}{2}, & t \in \left(\dfrac{8}{\sqrt{10}}, \sqrt{10} \right) \\ \dfrac{12 - \sqrt{10}t}{2}, & t \in \left(\sqrt{10}, \dfrac{12}{\sqrt{10}} \right) \\ 0, & otherwise. \end{cases}$$

The next step is to calculate P_{12}*. Considering the greater difficulty of calculating the probability density of continuous fuzzy numbers, we discretize and convert them to discrete fuzzy numbers according to Definition 12.*

First, we divide B equally into $(l-1)$ *points and define each part as* b_l*.*

For example, $B_1 = (0.7, 0.8, 0.9)$ *can be divided into 10 points:*

$$B_1 = \frac{0}{0.7} + \frac{0.2}{0.72} + \frac{0.4}{0.74} + \frac{0.6}{0.76} + \frac{0.8}{0.78} + \frac{1}{0.8} + \frac{0.8}{0.82} + \frac{0.6}{0.84} + \frac{0.4}{0.86} + \frac{0.2}{0.88} + \frac{0}{0.9}.$$

According to the discretization, it is known that

$$b_{j,l} = \sum_{i=1}^{n} \mu_{Aj}(t_{ji}) P_{j,l}(t_{ji}), \quad j = 1, 2$$

where j corresponds to Z_1 *or* Z_2*. When* $j = 12$*, it corresponds to* Z_{12}*.*

Thus, we obtain a linear programming model where $b_{j,l}$ *is a target value and the model satisfies the following constraints:*

$$\mu_{Aj}(t_{j1})P_{j,l}(t_{j1}) + \mu_{Aj}(t_{j2})P_{j,l}(t_{j2}) + \ldots + \mu_{Aj}(t_{jn})P_{j,l}(t_{jn}) \to b_{j,l}$$

$$s.t. \begin{cases} \sum P_{j,l} = 1 \\ P_{j,l} \geqslant 0 \\ \int t P_{j,l} dt = \dfrac{\int t \mu_A(t) dt}{\int \mu_A(t) dt} \end{cases}$$

Thus, we can obtain all $P_{j,l}$ *values for the lth b value; this is then continuousized and the probability density functions* $P_{1,l}$ *and* $P_{2,l}$ *are obtained by fitting. Finally, we obtain* $P_{12,l}$ *using the convolution formula* $P_{12} = P_1 \circ P_2$*, then all* P_{12} *values are obtained by iteration.*

Taking the fourth point as an example, the probability density functions after fitting are $N(2, 0.36)$ *and* $N(8, 0.76)$*, respectively; thus, the convolution formula can be used to find the probability density of the point* P_{12}*, which is equal to* $N(10, 0.83)$*.*

Substituting each copy of P_{12} *into* $B_{12,l} = \int \mu_{A_{12}} P_{12,l} dt$ *yields a series of values for* B_{12}. *Again taking the fourth point as an example, we have*

$$B_{12,4} = \int \mu_{A_{12}} P_{12,4} dt$$

$$= \int_{\frac{8}{\sqrt{10}}}^{\sqrt{10}} \frac{\sqrt{10}t - 8}{2} \cdot \frac{1}{0.83\sqrt{2\pi}} e^{-\frac{(t-10)^2}{2\cdot(0.83)^2}} dt + \int_{\sqrt{10}}^{\frac{12}{\sqrt{10}}} \frac{12 - \sqrt{10}t}{2} \cdot \frac{1}{0.83\sqrt{2\pi}} e^{-\frac{(t-10)^2}{2\cdot(0.83)^2}} dt.$$

The two endpoints and vertices are chosen to form $B_{12} = (0.62, 0.72, 0.79)$.
Finally, we obtain

$$Z_{12} = (A_{12}, B_{12}) = \left(\left(\frac{8}{\sqrt{10}}, \sqrt{10}, \frac{12}{\sqrt{10}} \right), (0.62, 0.72, 0.79) \right).$$

3.3. Subtraction Formula

The addition and subtraction operations for Z-numbers are extremely similar in thought and procedure to the addition and subtraction of ordinary numbers. To derive the subtraction expression, we transform the Z-numbers into classical fuzzy numbers first, then use the operational rules of classical fuzzy numbers to determine the expression of A_k. Finally, we apply the convolution formula to obtain the expression of B_k. The difference in the derivation process mainly lies in the fuzzy number operator formula and the convolution formula used to calculate P_{12}.

Theorem 3. *Let* $Z_1 = (A_1, B_1) = ((a_{1p}, a_{1q}, a_{1r}), (b_{1p}, b_{1q}, b_{1r}))$ *and* $Z_2 = (A_2, B_2) = ((a_{2p}, a_{2q}, a_{2r}), (b_{2p}, b_{2q}, b_{2r}))$ *be continuous symmetric triangular Z-numbers and let the difference between these two be* Z_k; *then, we have*

$$Z_k = Z_1 - Z_2 = (A_k, B_k),$$

where $A_k = (a_1^\gamma, a_2^\gamma, a_3^\gamma) = (\sqrt{\gamma_1}a_{1p} - \sqrt{\gamma_2}a_{2r}, \sqrt{\gamma_1}a_{1q} - \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} - \sqrt{\gamma_2}a_{2p})$, $B_k = \int_R \mu_{A_k} p_k dt$, $\gamma_1 = \dfrac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \dfrac{b_{2r} - b_{2p}}{2}$, μ_{A_k} *is the membership function of* A_k, *and* p_k *is the probability density of* A_k.

Proof. In views of Equation (6), the fuzzy transformations of Z_1, Z_2 are

$$Z_1^\gamma = \sqrt{\gamma_1}A_1 = (\sqrt{\gamma_1}a_{1p}, \sqrt{\gamma_1}a_{1q}, \sqrt{\gamma_1}a_{1r}),$$

$$Z_2^\gamma = \sqrt{\gamma_2}A_2 = (\sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_2}a_{2r}).$$

Then, according to Equation (8), we have

$$Z_k^\gamma = Z_1^\gamma - Z_2^\gamma = (\sqrt{\gamma_1}a_{1p} - \sqrt{\gamma_2}a_{2r}, \sqrt{\gamma_1}a_{1q} - \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} - \sqrt{\gamma_2}a_{2p}).$$

Similar to the addition process, it can be determined from Definition 6 that

$$\mu_{A_k}^\gamma(v) = \sup_u(\mu_{A_1}^\gamma(v+u) \wedge \mu_{A_2}^\gamma(u)).$$

After normalization, μ_{A_k} is expressed as Equation (15). Therefore,

$$A_k = (\sqrt{\gamma_1}a_{1p} - \sqrt{\gamma_2}a_{2r}, \sqrt{\gamma_1}a_{1q} - \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_1}a_{1r} - \sqrt{\gamma_2}a_{2p}).$$

Similar to Equation (17), P_1 and P_2 are under the three range condition restrictions. Based on Equation (12), it is easy to find that $P_k = P_1 \circ P_2 = \int_R p_1(u+v)p_2(u)du$. Then, B_{12} can be obtained by substituting p_k and μ_{A_k} into Equation (16). Hence, we can derive that

$$Z_k = Z_1 - Z_2 = (A_k, B_k).$$

□

3.4. Multiplication Formula

Theorem 4. *Let $Z_1 = (A_1, B_1) = ((a_{1p}, a_{1q}, a_{1r}), (b_{1p}, b_{1q}, b_{1r}))$ and $Z_2 = (A_2, B_2) = ((a_{2p}, a_{2q}, a_{2r}), (b_{2p}, b_{2q}, b_{2r}))$ be continuous symmetric triangular Z-numbers and let the multiplication of these two be Z^*. Then, Z^* can be expressed as*

$$Z^* = Z_1 \times Z_2 = (A^*, B^*),$$

where $A^* = (\sqrt{\gamma_1}\sqrt{\gamma_2}a_{1p}a_{2p}, \sqrt{\gamma_1}\sqrt{\gamma_2}a_{1q}a_{2q}, \sqrt{\gamma_1}\sqrt{\gamma_2}a_{1r}a_{2r})$, $B^* = \int_R \mu_{A^*} p^* dt$, $\gamma_1 = \dfrac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \dfrac{b_{2r} - b_{2p}}{2}$, and μ_{A^*} and p^* are the membership function and probability density of A^*, respectively.

Proof. Similar to the previous proof processes, we first let $Z^* = Z_1 \times Z_2$ and then transform them into classical fuzzy numbers, that is,

$$Z_1^{\gamma} = \sqrt{\gamma_1}A_1 = (\sqrt{\gamma_1}a_{1p}, \sqrt{\gamma_1}a_{1q}, \sqrt{\gamma_1}a_{1r}),$$
$$Z_2^{\gamma} = \sqrt{\gamma_2}A_2 = (\sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_2}a_{2r}),$$

where $\gamma_1 = \dfrac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \dfrac{b_{2r} - b_{2p}}{2}$.

Then, we must apply α-cuts to perform the multiplication calculation. When studying symmetric triangular fuzzy numbers, there will be a linear equation on the left and right side after α-cut processing. Next, we mark the left side to indicate the symbol as L and the right as R. The classical fuzzy number after the α-cut can be obtained as

$$Z_1^{\bar{\gamma}_1} = [Z_{1\alpha}^L, Z_{1\alpha}^R] = [\sqrt{\gamma_1}(a_{1q} - a_{1p})\alpha + \sqrt{\gamma_1}a_{1p}, \sqrt{\gamma_1}(a_{1q} - a_{1r})\alpha + \sqrt{\gamma_1}a_{1r}],$$

$$Z_2^{\bar{\gamma}_2} = [Z_{2\alpha}^L, Z_{2\alpha}^R] = [\sqrt{\gamma_2}(a_{2q} - a_{2p})\alpha + \sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_2}(a_{2q} - a_{2r})\alpha + \sqrt{\gamma_2}a_{2r}].$$

Per Equation (9), it is reasoned that

$$Z^{\bar{*}\gamma^*} = Z_1^{\bar{\gamma}_1} \times Z_2^{\bar{\gamma}_2} = [Z_{1\alpha}^L, Z_{1\alpha}^R] \times [Z_{2\alpha}^L, Z_{2\alpha}^R] = [Z_\alpha^{*L}, Z_\alpha^{*R}],$$

where

$$Z_\alpha^{*L} = min\left\{Z_{1\alpha}^L Z_{2\alpha}^L, Z_{1\alpha}^L Z_{2\alpha}^R, Z_{1\alpha}^R Z_{2\alpha}^L, Z_{1\alpha}^R Z_{2\alpha}^R\right\},$$
$$Z_\alpha^{*R} = max\left\{Z_{1\alpha}^L Z_{2\alpha}^L, Z_{1\alpha}^L Z_{2\alpha}^R, Z_{1\alpha}^R Z_{2\alpha}^L, Z_{1\alpha}^R Z_{2\alpha}^R\right\}.$$

After the modeling is completed, it is known that the ordinate corresponding to point L is less than that corresponding to point R. Therefore, after analysis, it is found that

$$Z_\alpha^{*L} = Z_{1\alpha}^L Z_{2\alpha}^L = [(a_{1q} - a_{1p})(a_{2q} - a_{2p})\alpha^2 + (a_{1p}a_{2q} + a_{2p}a_{1q} - 2a_{1p}a_{2p})\alpha + a_{1p}a_{2p}]\sqrt{\gamma_1}\sqrt{\gamma_2}, \quad (18)$$

$$Z_\alpha^{*R} = Z_{1\alpha}^R Z_{2\alpha}^R = [(a_{1q} - a_{1r})(a_{2q} - a_{2p})\alpha^2 + (a_{1r}a_{2q} + a_{2r}a_{1q} - 2a_{1r}a_{2r})\alpha + a_{1r}a_{2r}]\sqrt{\gamma_1}\sqrt{\gamma_2}. \quad (19)$$

In the next step, in order to find the membership degree μ, α needs to be calculated first.

Let $Z_\alpha^{*L} = m$, which denotes the magnitude of the length of the horizontal axis taken by the new membership degree after multiplying the two membership degrees and is an unknown number. Transforming Equation (18), we obtain

$$\alpha = \frac{-(a_{1p}\tilde{a_2} + a_{2p}\tilde{a_1}) + \sqrt{(a_{1p}\tilde{a_2} - a_{2p}\tilde{a_1})^2 + \dfrac{4\tilde{a_1}\tilde{a_2}m}{\sqrt{\gamma_1}\sqrt{\gamma_2}}}}{2\tilde{a_1}\tilde{a_2}},$$

where we discard the roots of $\alpha < 0$ and where $\tilde{a_1} = a_{1q} - a_{1p} = a_{1r} - a_{1q}$, $\tilde{a_2} = a_{2q} - a_{2p} = a_{2r} - a_{2q}$.

Similarly, letting $Z_{\alpha}^{*R} = n$, after transforming Equation (19) we have

$$\alpha = \frac{-(a_{1r}\tilde{a_2} + a_{2r}\tilde{a_1}) + \sqrt{(a_{2r}\tilde{a_1} - a_{1r}\tilde{a_2})^2 + \dfrac{4\tilde{a_1}\tilde{a_2}n}{\sqrt{\gamma_1}\sqrt{\gamma_2}}}}{2\tilde{a_1}\tilde{a_2}},$$

where we discard the roots of $\alpha > 1$ and $\tilde{a_1} = a_{1q} - a_{1p} = a_{1r} - a_{1q}$, $\tilde{a_2} = a_{2q} - a_{2p} = a_{2r} - a_{2q}$.

Due to the nature of symmetric triangular fuzzy numbers, it is obvious that

$$m + n = 2\sqrt{\gamma_1}\sqrt{\gamma_2}a_{1q}a_{2q}.$$

Therefore, the membership function of Z^* is

$$\mu_{A^*} = \begin{cases} \dfrac{-(a_{1p}\tilde{a_2}+a_{2p}\tilde{a_1})+\sqrt{(a_{1p}\tilde{a_2}-a_{2p}\tilde{a_1})^2+\dfrac{4\tilde{a_1}\tilde{a_2}m}{\sqrt{\gamma_1}\sqrt{\gamma_2}}}}{2\tilde{a_1}\tilde{a_2}}, & m \in (\sqrt{\gamma_1}\sqrt{\gamma_2}a_{1p}a_{2p}, \sqrt{\gamma_1}\sqrt{\gamma_2}a_{1q}a_{2q}) \\ \dfrac{-(a_{1r}\tilde{a_2}+a_{2r}\tilde{a_1})+\sqrt{(a_{2r}\tilde{a_1}-a_{1r}\tilde{a_2})^2+\dfrac{4\tilde{a_1}\tilde{a_2}(2\sqrt{\gamma_1}\sqrt{\gamma_2}a_{1q}a_{2q}-m)}{\sqrt{\gamma_1}\sqrt{\gamma_2}}}}{2\tilde{a_1}\tilde{a_2}}, & m \in (\sqrt{\gamma_1}\sqrt{\gamma_2}a_{1q}a_{2q}, \sqrt{\gamma_1}\sqrt{\gamma_2}a_{1r}a_{2r}) \\ 0, & otherwise. \end{cases} \tag{20}$$

As a result, $A^* = (\sqrt{\gamma_1}\sqrt{\gamma_2}a_{1p}a_{2p}, \sqrt{\gamma_1}\sqrt{\gamma_2}a_{1q}a_{2q}, \sqrt{\gamma_1}\sqrt{\gamma_2}a_{1r}a_{2r})$.

The following steps are similar to the addition and subtraction calculation processes. First calculating P^* and then applying Equation (14), we obtain

$$P^*(v) = P_1 \circ P_2 = \int_R \frac{1}{|u|} p_1(u) p_2(\frac{v}{u}) du.$$

It should be noted that a fundamental condition of the multiplicative convolution formula is that P_1 and P_2 are independent of each other. Then, B^* can be obtained by substituting p^* and μ_{A^*} into Equation (16). Thus, the conclusion is obtained:

$$Z^* = Z_1 \times Z_2 = (A^*, B^*).$$

□

3.5. Power Formula

Theorem 5. *Let λ be a real number and let $Z = (A, B) = ((a_p, a_q, a_r), (b_p, b_q, b_r))$ be a symmetric triangular Z-number. Its powers can be calculated by*

$$Z^\lambda = (A^\lambda, B^\lambda),$$

where $A^\lambda = ((\sqrt{\gamma}a_p)^\lambda, (\sqrt{\gamma}a_q)^\lambda, (\sqrt{\gamma}a_r)^\lambda)$, $B^\lambda = \int \mu_A^\lambda p(u^\lambda) du$, $\gamma_1 = \dfrac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \dfrac{b_{2r} - b_{2p}}{2}$, μ_A^λ is the membership function of A^λ, and $p(u^\lambda)$ is the probability density of A^λ.

Proof. The power operation is actually a generalization of the multiplication calculation. First, we can make $Z_1^\lambda = (A^\lambda, B^\lambda)$, which means that Z_1^λ is obtained by multiplying λ times Z_1.

When $\lambda = 1$, $Z_1^\lambda = Z_1$. When $\lambda = 2$, $Z_1^\lambda = Z_1^2 = Z_1 \times Z_1$, and according to the multiplication formula derived earlier, $Z_1^2 = (A^2, B^2)$, where $A^\lambda = A^2 = ((\sqrt{\gamma}a_p)^2, (\sqrt{\gamma}a_q)^2, (\sqrt{\gamma}a_r)^2)$. Analogously, when $\lambda = 3$, $Z_1^\lambda = Z_1^3 = Z_1 \times Z_1 \times Z_1 = (A^3, B^3)$, where $A^\lambda = ((\sqrt{\gamma}a_p)^3, (\sqrt{\gamma}a_q)^3, (\sqrt{\gamma}a_r)^3)$.

Assume that $A^n = ((\sqrt{\gamma}a_p)^n, (\sqrt{\gamma}a_q)^n, (\sqrt{\gamma}a_r)^n)$ holds when $\lambda = n$. Then, $A^{n+1} = A^n \times A = ((\sqrt{\gamma}a_p)^{n+1}, (\sqrt{\gamma}a_q)^{n+1}, (\sqrt{\gamma}a_r)^{n+1})$.

Therefore, we can find that when $\lambda \in R$, $A^\lambda = ((\sqrt{\gamma}a_p)^\lambda, (\sqrt{\gamma}a_q)^\lambda, (\sqrt{\gamma}a_r)^\lambda)$.

For B^λ, when $\lambda = 2$, $B^\lambda = B^2 = \int \mu_A^2(t)p^2(t)dt$. Using Equation (20), we can find that the membership function of A^λ is

$$\mu_A^\lambda = \begin{cases} \dfrac{\lambda\sqrt{m} - \sqrt{\gamma}a_p}{\sqrt{\gamma}\tilde{a}_1}, & m \in ((\sqrt{\gamma}a_p)^\lambda, (\sqrt{\gamma}a_q)^\lambda) \\ \dfrac{\lambda\sqrt{m} - \sqrt{\gamma}a_r}{\sqrt{\gamma}\tilde{a}_1}, & m \in ((\sqrt{\gamma}a_q)^\lambda, (\sqrt{\gamma}a_r)^\lambda) \\ 0, & otherwise. \end{cases}$$

Then, according to Equation (14), we can find

$$P(u^2) = \int_R \frac{1}{|u|} p^2(u)du.$$

The same extends to the condition when $\lambda \in R$:

$$P(u^\lambda) = \left(\int_R \frac{1}{|u|} p(u)du\right)^{\lambda-2} \left(\int_R \frac{1}{|u|} p^2(u)du\right).$$

Accordingly, by combination with $B^\lambda = \int \mu_A^\lambda p(u^\lambda)du$, we can find the value of B^λ. Finally, we obtain

$$Z^\lambda = (A^\lambda, B^\lambda).$$

□

3.6. Division Formula

Theorem 6. *Let $Z_1 = (A_1, B_1) = ((a_{1p}, a_{1q}, a_{1r}), (b_{1p}, b_{1q}, b_{1r}))$ and $Z_2 = (A_2, B_2) = ((a_{2p}, a_{2q}, a_{2r}), (b_{2p}, b_{2q}, b_{2r}))$ be continuous symmetric triangular Z-numbers and let the division formula of these two be Z_s. Then, it can be deduced that*

$$Z_s = \frac{Z_1}{Z_2} = (A_s, B_s),$$

where $A_s = \left(\dfrac{\sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_2}a_{2p}}, \dfrac{\sqrt{\gamma_1}a_{1q}}{\sqrt{\gamma_2}a_{2q}}, \dfrac{\sqrt{\gamma_1}a_{1r}}{\sqrt{\gamma_2}a_{2r}}\right)$, $B_s = \int_R \mu_{A_s} p_s dt$, $\gamma_1 = \dfrac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \dfrac{b_{2r} - b_{2p}}{2}$, μ_{A_s} is the membership function of A_s, and p_s is the probability density of A_s.

Proof. Let $Z_s = \dfrac{Z_1}{Z_2}$, where $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$. Then, we can convert Z_1 and Z_2 to fuzzy numbers as follows:

$$Z_1^{\gamma_1} = (\sqrt{\gamma_1}a_{1p}, \sqrt{\gamma_1}a_{1q}, \sqrt{\gamma_1}a_{1r}),$$
$$Z_2^{\gamma_2} = (\sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_2}a_{2q}, \sqrt{\gamma_2}a_{2r}),$$

where $\gamma_1 = \dfrac{b_{1r} - b_{1p}}{2}$, $\gamma_2 = \dfrac{b_{2r} - b_{2p}}{2}$.

For these two triangular fuzzy numbers, their α-cuts are

$$\bar{Z}_1^{\gamma_1} = [Z_{1\alpha}^L, Z_{1\alpha}^R] = [\sqrt{\gamma_1}(a_{1q} - a_{1p})\alpha + \sqrt{\gamma_1}a_{1p}, \sqrt{\gamma_1}(a_{1q} - a_{1r})\alpha + \sqrt{\gamma_1}a_{1r}],$$

$$\bar{Z}_2^{\gamma_2} = [Z_{2\alpha}^L, Z_{2\alpha}^R] = [\sqrt{\gamma_2}(a_{2q} - a_{2p})\alpha + \sqrt{\gamma_2}a_{2p}, \sqrt{\gamma_2}(a_{2q} - a_{2r})\alpha + \sqrt{\gamma_2}a_{2r}].$$

From Equation (10), the α-cut of $\bar{Z}_s^{\gamma_s} = \dfrac{\bar{Z}_1^{\gamma_1}}{\bar{Z}_2^{\gamma_2}}$ is

$$\frac{[Z_{1\alpha}^L, Z_{1\alpha}^R]}{[Z_{2\alpha}^L, Z_{2\alpha}^R]} = [Z_{s\alpha}^L, Z_{s\alpha}^R],$$

where $Z_{s\alpha}^L = min\left\{\dfrac{Z_{1\alpha}^L}{Z_{2\alpha}^L}, \dfrac{Z_{1\alpha}^L}{Z_{2\alpha}^R}, \dfrac{Z_{1\alpha}^R}{Z_{2\alpha}^L}, \dfrac{Z_{1\alpha}^R}{Z_{2\alpha}^R}\right\}$, $Z_{s\alpha}^R = max\left\{\dfrac{Z_{1\alpha}^L}{Z_{2\alpha}^L}, \dfrac{Z_{1\alpha}^L}{Z_{2\alpha}^R}, \dfrac{Z_{1\alpha}^R}{Z_{2\alpha}^L}, \dfrac{Z_{1\alpha}^R}{Z_{2\alpha}^R}\right\}$.

Similar to the proof process of multiplication, it is known that $Z^L < Z^R$. As a result,

$$Z_{s\alpha}^L = \frac{Z_{1\alpha}^L}{Z_{2\alpha}^R} = \frac{\sqrt{\gamma_1}(a_{1q} - a_{1p})\alpha + \sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_2}(a_{2q} - a_{2r})\alpha + \sqrt{\gamma_2}a_{2r}},$$

$$Z_{s\alpha}^R = \frac{Z_{1\alpha}^R}{Z_{2\alpha}^L} = \frac{\sqrt{\gamma_1}(a_{1q} - a_{1r})\alpha + \sqrt{\gamma_1}a_{1r}}{\sqrt{\gamma_2}(a_{2q} - a_{2p})\alpha + \sqrt{\gamma_2}a_{2p}}.$$

Let $Z_{s\alpha}^L = m$, which is an unknown function that represents the range of values. We discard the roots of $\alpha < 0$; hence,

$$\alpha = \frac{m\sqrt{\gamma_2}a_{2r} - \sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_1(a_{1q} - a_{1p}) - m\sqrt{\gamma_2}(a_{2q} - a_{2r})}} = \frac{m\sqrt{\gamma_2}a_{2r} - \sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_1}\tilde{a_1} - m\sqrt{\gamma_2}\tilde{a_2}},$$

where $\tilde{a_1} = a_{1q} - a_{1p} = a_{1r} - a_{1q}$, $\tilde{a_2} = a_{2q} - a_{2p} = a_{2r} - a_{2q}$.

Let $Z_{s\alpha}^R = n$, which is an unknown function that represents the range of values. We discard the roots of $\alpha > 1$; therefore,

$$\alpha = \frac{n\sqrt{\gamma_2}a_{2p} - \sqrt{\gamma_1}a_{1r}}{\sqrt{\gamma_1(a_{1q} - a_{1r}) - n\sqrt{\gamma_2}(a_{2q} - a_{2p})}} = \frac{n\sqrt{\gamma_2}a_{2p} - \sqrt{\gamma_1}a_{1r}}{\sqrt{\gamma_1}\tilde{a_1} - n\sqrt{\gamma_2}\tilde{a_2}}.$$

Due to the nature of the symmetric triangular fuzzy number, it is obvious that

$$m + n = 2\frac{\sqrt{\gamma_1}a_{1q}}{\sqrt{\gamma_2}a_{2q}}.$$

The membership function of $Z_s^{\gamma_s}$ is

$$\mu_{A_t} = \begin{cases} \dfrac{m\sqrt{\gamma_2}a_{2r} - \sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_1}\tilde{a_1} - m\sqrt{\gamma_2}\tilde{a_2}}, & m \in \left(\dfrac{\sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_2}a_{2p}}, \dfrac{\sqrt{\gamma_1}a_{1q}}{\sqrt{\gamma_2}a_{2q}}\right) \\ \dfrac{2\sqrt{\gamma_1}a_{1q}a_{2p} - \sqrt{\gamma_2}a_{2p}a_{2q}m - \sqrt{\gamma_1}a_{1r}a_{2q}}{\sqrt{\gamma_1}\tilde{a_1}a_{2q} - 2\sqrt{\gamma_1}\tilde{a_2}a_{1q} + \sqrt{\gamma_2}\tilde{a_2}a_{2q}m}, & m \in \left(\dfrac{\sqrt{\gamma_1}a_{1q}}{\sqrt{\gamma_2}a_{2q}}, \dfrac{\sqrt{\gamma_1}a_{1r}}{\sqrt{\gamma_2}a_{2r}}\right) \\ 0, & otherwise. \end{cases}$$

Then, $A_s = \left(\dfrac{\sqrt{\gamma_1}a_{1p}}{\sqrt{\gamma_2}a_{2p}}, \dfrac{\sqrt{\gamma_1}a_{1q}}{\sqrt{\gamma_2}a_{2q}}, \dfrac{\sqrt{\gamma_1}a_{1r}}{\sqrt{\gamma_2}a_{2r}}\right)$.

Next, we use Equation (13) to calculate the probability density P_s:

$$P_s(v) = P_1 \circ P_2 = \int_R |u| p_1(u) p_2(uv) du.$$

Finally, we again use Equation (16) to obtain B_t. Therefore, we have

$$Z_s = \frac{Z_1}{Z_2} = (A_s, B_s).$$

□

At this point, all of the mentioned operational rules related to Z-numbers have been proposed and proven. Compared to other computational methods, our proposed method simplifies the operations by converting them into classical fuzzy numbers and deriving the two components of the desired Z-number via the operational laws of the classical fuzzy numbers. When facing more complex Z-number operations, using these theorems of basic operations can greatly reduce computational complexity, enabling the application of Z-numbers to a wider range of fields.

4. Conclusions

Z-number proposals integrate objective natural language information and subjective human understanding, taking both the vagueness of information and the level of "trustworthiness" of fuzzy information into account. Therefore, Z-numbers provide a great deal of convenience in describing and analyzing uncertain information. Many researchers have studied the concept since its introduction. Drawn from the theoretical foundation of fuzzy sets theory and optimization methods, they have provided basic operational laws of common algebraic operations for Z-numbers. In a more straightforward manner, this paper has focused on operational laws for symmetric triangular Z-numbers. First, we transform the Z-numbers into classical triangular fuzzy numbers. After that, we employ the operational laws of the classical fuzzy numbers for reference to derive the two components of the derived Z-number. Finally, we provide the number-multiplication, addition, subtraction, multiplication, power, and division expressions for the Z-numbers. In the fields of economics, decision analysis, risk assessment, planning, and causal analysis, many real-life numbers are actually Z-numbers. In previous academic research, however, they have been simplified into other numbers due to their high computational complexity. Based on these rules of basic operations, the application prospects of Z-numbers will be greatly improved.

There are some limitations to this article. First, this paper only proposes operational laws for symmetric triangular Z-numbers, and does not apply to other types of Z-numbers; hence, it is necessary to extend this method to more general Z-numbers in further theoretical studies. Second, time and energy constraints limited our study to the multiplication, power series, and quadratic operations of Z-numbers. We have not provided definitions and operational rules for other calculations, including more complex algebraic operations, expectations, and variance, which we plan to expand upon in the future. Finally, this paper has not provided application examples to prove the applicability and scope of the proposed operational laws, which requires further investigation in future work.

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