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Article

Barrelled Weakly Köthe–Orlicz Summable Sequence Spaces

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Abstract: Let E be a Hausdorff locally convex space. We investigate the space $\Lambda_\varphi[E]$ of weakly Köthe–Orlicz summable sequences in E with respect to an Orlicz function φ and a perfect sequence space Λ. We endow $\Lambda_\varphi[E]$ with a Hausdorff locally convex topology and determine the continuous dual of the so-obtained space in terms of strongly Köthe–Orlicz summable sequences from the dual space E' of E. Next, we give necessary and sufficient conditions for $\Lambda_\varphi[E]$ to be barrelled or quasi-barrelled. This contributes to the understanding of different spaces of vector-valued sequences and their topological properties.

Keywords: summable sequences; vector-valued sequence spaces; Orlicz function; AK-space; duality; barrelled space

MSC: 46A45; 46A03; 46E30

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1. Introduction

Let E be a locally convex space. The spaces $\ell^p(E)$ and $\ell^p\{E\}$ of weakly p-summable and absolutely p-summable sequences in E, respectively, were introduced by Pietsch in [1]. The same author investigated applications of these spaces in the study of absolutely p-summing operators. In addition, he investigated the spaces $\Lambda\{E\}$ and $\Lambda(E)$ of absolutely Λ-summable and weakly Λ-summable sequences in E, respectively, where Λ is a sequence space endowed with its Köthe normal topology. Building upon Pietsch's work, Rosier [2] extended the study to the general case, wherein Λ is equipped with a general polar topology (instead of the Köthe normal topology). Rosier obtained notable results, which included a comprehensive description of the dual space of $\Lambda\{E\}_r$.

Employing the AK property, Florencio and Paúl [3] determined a representation of the elements of $\Lambda\widetilde{\otimes}_\varepsilon E$ (the completion of the injective tensor product $\Lambda \otimes_\varepsilon E$) as weakly Λ-summable sequences in E.

Later, Oubbi and Ould Sidaty extended in [4] the concept of strong summability, initially introduced by Cohen [5] for normed spaces, to the locally convex spaces. This extension allowed them to obtain a description of the continuous dual space of $\Lambda(E)_r$. Further results and properties for $\Lambda(E)$ were obtained in [6–8]. Recently, Ould Sidaty investigated in [9] the nuclearity (as a convex bornological space) of $\Lambda_b(E)$, i.e., the space of all totally Λ-summable sequences within the context defined by [10], where E represents a convex bornological space. Furthermore, Ghosh and Srivastava explored in [11] the notion of absolute Λ-summability (using an Orlicz function φ). They introduced and investigated the space $F(E,\varphi)$, consisting of all sequences $(x_n)_n$ in a Banach space E that satisfy the condition

$$\left(\varphi\left(\frac{\|x_n\|_E}{\rho}\right)\right)_n \in F$$

for some $\rho > 0$, where F denotes a normal sequence space.

It is worth noting that several kinds of sequence spaces have already been investigated in the literature. Descriptions of some of them rely on infinite Köthe matrices $(a_{i,j})_{i,j\in\mathbb{N}}$, some others rely on Cesàro operators, and others rely on different kinds of convergence or summability (see [1–4,6–20]).

Of course, Orlicz functions yield natural sequence spaces in the scalar-valued case. They are also used to construct vector-valued sequence spaces (see for example [11,17,19] and the references therein). The characterization of continuous dual or Köthe–Toeplitz dual are examples of the main issues authors are interested in (see, e.g., [21]). But first of all, a linear topology must be defined on the sequence space in consideration.

In this paper, for an Orlicz function φ and a locally convex space E, we introduce the notion of a weakly (φ, Λ)-summable sequence $(x_n)_n$ in E and examine some properties of the linear space $\Lambda_\varphi[E]$ consisting of all such sequences. Actually, weakly (φ, Λ)-summable sequences and the corresponding sequence spaces were investigated in [8] for a Banach space E. There, the author gave necessary and sufficient conditions for $\Lambda_\varphi[E]$ to be reflexive. The situation in a locally convex space is quite complicated, for the topology is no more given by a single norm but by a family of infinitely many semi-norms, which means that a bounded neighborhood of 0 may not exist there.

The outcomes of this paper extend and improve some results in the literature, especially those in [8]. We first equip $\Lambda_\varphi[E]$ with a Hausdorff locally convex topology, and then we investigate the completeness and the continuity of projections of the so-obtained locally convex space. We embed E in $\Lambda_\varphi[E]$ as a complemented subspace. In order to investigate the topological dual of $\Lambda_\varphi[E]$, we define the notion of strongly (φ, Λ)-summable sequences and the space $\Lambda_\varphi\langle E\rangle$ of all such sequences. Actually, we prove that whenever $\Lambda_\varphi[E]$ is AK, its topological dual can be given in terms of strongly summable sequences. Next, we characterize the property of barrelledness in $\Lambda_\varphi[E]$. To address this issue, we examine equicontinuous sets of the dual space of $\Lambda_\varphi[E]$. For ample information on barrelled locally convex spaces, we refer to the monograph [22].

2. Preliminaries

Throughout this paper, $\mathbb{K}$ denotes the field of real or complex numbers, $\mathbb{N}$ is the set of positive integers, and (E, τ) is a Hausdorff locally convex space over $\mathbb{K}$, for which the continuous dual is denoted by E'. If M runs over the collection $\mathcal{M}$ of all $\sigma(E', E)$-closed and equicontinuous discs of E', the topology τ is generated by the semi-norms

$$P_M(x) := \sup\{|a(x)|,\ a \in M\}, \qquad x \in E, \qquad M \in \mathcal{M}.$$

For any nonempty set X, $X^{\mathbb{N}}$ denotes the set of all sequences from X, and $X^{(\mathbb{N})}$ is the subset of $X^{\mathbb{N}}$ consisting of all sequences with finite support. If $\Omega \subset E^{\mathbb{N}}$ is a linear space, its Köthe dual, as defined in [23], is the set

$$\Omega^* := \left\{(a_n)_n \subset E' \ :\ \sum_{n=1}^{+\infty} |a_n(x_n)| < +\infty, \quad (x_n)_n \in \Omega\right\}.$$

If $t \in E$, we write te_n to mean the sequence for which the entrees are all zero, but the nth one equals t. The kth finite section of a sequence $x := (x_n)_n \in \Omega$ is defined by

$$x^{(k)} = \sum_{n=1}^{k} x_n e_n = (x_1, x_2, \ldots, x_k, 0, 0, \ldots).$$

If a topology is given on Ω, we denote by Ω_r the linear subspace of Ω consisting of those sequences x such that $x^{(k)} \in \Omega$ for all $k \in \mathbb{N}$, and $x = \lim_{k\to\infty} x^{(k)}$ in Ω.

If Λ is a normal linear subspace of $\mathbb{K}^{\mathbb{N}}$, then Λ contains the set Λ_F of all finite sections of its elements. Unless the contrary is clearly stated, it is equipped with a polar topology $\tau_{\mathcal{S}}$ defined by a topologizing family $\mathcal{S} \subset \Lambda^*$ consisting of normal closed and bounded discs with respect to the weak topology $\sigma(\Lambda^*, \Lambda)$. Such a topology is given by the semi-norms

$$P_S((\alpha_n)_n) := \sup\left\{ \sum_{n=1}^{+\infty} |\alpha_n \beta_n|, \quad (\beta_n)_n \in S \right\}, \quad (\alpha_n)_n \in \Lambda, \qquad S \in \mathcal{S}.$$

For a bounded disc A in a Hausdorff topological vector space F, F_A is the linear span of A. When no topology is specified on F_A, it is endowed with the gauge $\|.\|_A$ of A as a norm, where $\|t\|_A := \inf\{r > 0, t \in rA\}$, $t \in F_A$. We then consider without any further mention the spaces E_B, E'_M, Λ_R and Λ^*_S, where $B \subset E$, $M \in \mathcal{M}$, $S \in \mathcal{S}$, and $R \subset \Lambda$ are bounded discs, with R normal.

We refer to [23] for details concerning Köthe theory of sequence spaces and to [24] for the terminology and notations concerning the general theory of locally convex spaces.

We consider an Orlicz function φ: this is any mapping $\varphi : [0, +\infty) \to [0, +\infty]$ that is convex, vanishes at 0, and is non-constant (see [17]). The complement of φ is the function

$$\varphi^*(y) := \sup\{xy - \varphi(x), x \in [0, +\infty)\}.$$

Let us observe that φ^* is also an Orlicz function. Clearly, φ and φ^* satisfy the Young inequality; namely,

$$xy \le \varphi(x) + \varphi^*(y), \qquad x, y \ge 0.$$

The function φ is said to satisfy Δ_2 for small x (or at 0) if for each $k > 1$ there exist $R_k > 0$ and $x_k > 0$ such that $\varphi(kx) \le R_k \varphi(x)$ for all $x \in (0, x_k]$. The Orlicz sequence class associated with φ is

$$\widetilde{\ell}_\varphi = \left\{ x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} : \delta(x, \varphi) := \sum_{n=1}^{+\infty} \varphi(|x_n|) < +\infty \right\}.$$

We denote by $\widetilde{B}_\varphi$ the set $\{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} : \delta(x, \varphi) \le 1\}$.

The Orlicz sequence space associated with φ is

$$\ell_\varphi = \left\{ x := (x_n)_n \in \mathbb{K}^{\mathbb{N}} : \sum_{n\ge1} x_n y_n \text{ converges for all } y \in \widetilde{\ell}_{\varphi^*} \right\}.$$

This is a Banach space with respect to the norm

$$\begin{aligned} \|x\|_\varphi &= \sup\left\{ \left| \sum_{n\ge1} x_n y_n \right| : \delta(y, \varphi^*) \le 1 \right\} \\ &= \sup\left\{ \sum_{n\ge1} |x_n y_n| : \delta(y, \varphi^*) \le 1 \right\}. \end{aligned}$$

Like in [18], if $x \in \ell_\varphi$ and $\|x\|_\varphi \le 1$, then $x \in \widetilde{\ell}_\varphi$ and $\delta(x, \varphi) \le \|x\|_\varphi$.

3. Weakly Köthe–Orlicz Summable Sequences

In this section, we introduce the notion of weakly Köthe–Orlicz summable sequences in a locally convex space E and investigate some first properties of the linear space of all such sequences.

Definition 1. *A sequence $x = (x_n)_n \subset E$ is said to be weakly Köthe–Orlicz summable with respect to φ and Λ (for short, weakly (φ, Λ)-summable) if the sequence $(\alpha_n f(x_n))_n$ belongs to ℓ_φ for every $f \in E'$ and every $\alpha \in \Lambda^*$. The set of all such sequences is denoted by $\Lambda_\varphi[E]$.*

Since $\Lambda^* = (\Lambda^{**})^*$, we assume with no loss of generality that Λ is perfect, i.e., $\Lambda = \Lambda^{**}$.

Here are some examples of Orlicz functions and the corresponding $\Lambda_\varphi[E]$.

Example 1.

1. *Let φ be the identity map $x \mapsto x$. Then $\Lambda_\varphi[E]$ coincides with the space $\Lambda[E]$ of weakly summable sequence in E (see, e.g., [4]).*
2. *Assume $\Lambda = \ell^1$ and $E = \mathbb{K}$. Then $\Lambda_\varphi[E]$ is nothing but the classical Orlicz sequence space ℓ_φ.*
3. *Let φ be the Orlicz function defined by $\varphi(x) := +\infty$ if $x > 1$ and $\varphi(x) := 0$ if $0 \le x \le 1$. Let $\Lambda = c_0$, the space of all scalar null sequences, and let E be a Hausdorff locally convex space. We claim that $(c_0)_\varphi[E]$ is the set $c_b(E)$ of all bounded sequences in E. Indeed, since $(c_0)^{**} = \ell^\infty$, $(c_0)_\varphi[E] = \ell^\infty[E]$. Let $x \in \ell^\infty_\varphi[E]$. Then for every $f \in E'$ and $\alpha \in \ell^1 := (\ell^\infty)^*$, we have $(\alpha_n f(x_n))_n \in \ell_\varphi := \ell^\infty$. Since α is arbitrary in ℓ^1, the sequence $(f(x_n))_n$ belongs to ℓ^∞. This means that the sequence $(x_n)_n$ is weakly bounded in E, for f is arbitrary in E'. Hence, $(x_n)_n$ belongs to $c_b(E)$. The inverse inclusion $c_b(E) \subset \ell^\infty_\varphi[E]$ is trivial.*

Notice that if for every $\alpha \in \Lambda^*$ and $f \in E'$, $\psi_{\alpha,f}$ is the endomorphism of $E^{\mathbb{N}}$ defined by $\psi_{\alpha,f}((x_n)_n) = (\alpha_n f(x_n))_n$, then

$$\Lambda_\varphi[E] = \bigcap \{\psi^{-1}_{\alpha,f}(\ell_\varphi),\ \alpha \in \Lambda^*,\ f \in E'\}.$$

This shows that $\Lambda_\varphi[E]$ is a linear space.

Lemma 1. *For every $x = (x_n)_n \in \Lambda_\varphi[E]$ and $S \in \mathcal{S}$, the set A^φ_S below is bounded in E.*

$$A^\varphi_S = \left\{ \sum_{n=1}^{p} \alpha_n y_n x_n : \alpha \in S,\ y \in \widetilde{B}_{\varphi^*},\ p \in \mathbb{N} \right\}.$$

Therefore, for every $S \in \mathcal{S}$ and $M \in \mathcal{M}$, a semi-norm $\varepsilon^\varphi_{S,M}$ is defined on $\Lambda_\varphi[E]$, where

$$\varepsilon^\varphi_{S,M}(x) := \sup_{\alpha \in S, f \in M} \|(\alpha_n f(x_n))_n\|_\varphi, \quad x = (x_n)_n \in \Lambda_\varphi[E].$$

Proof. Let $x = (x_n)_n \in \Lambda_\varphi[E]$, $\alpha \in S$, $y \in \widetilde{B}_{\varphi^*}$, $p \in \mathbb{N}$, and $f \in E'$ be given. Then

$$\left| f\left(\sum_{n=1}^{p} \alpha_n y_n x_n \right) \right| = \left| \sum_{n=1}^{p} \alpha_n y_n f(x_n) \right| \le \|(\alpha_n f(x_n))_n\|_\varphi.$$

Define a linear mapping $g_f : \Lambda^*_S \longrightarrow \ell_\varphi$ by $g_f(\beta) = (\beta_n f(x_n))$. Since Λ^*_S is a Banach space ([4], Lemma 3), g_f is continuous by the closed graph theorem. Therefore, it is bounded on S by the norm $\|g_f\|$ of g_f. This is

$$\left| f\left(\sum_{n=1}^{p} \alpha_n y_n x_n \right) \right| \le \|(\alpha_n f(x_n))_n\|_\varphi \le \|g_f\|.$$

Since f was arbitrary in E', A^φ_S is weakly bounded and is then also bounded in E. The remainder is trivial. □

We denote by $\varepsilon^{\varphi}_{\mathcal{S},\mathcal{M}}$ the locally convex topology defined on $\Lambda_{\varphi}[E]$ by the family $\left(\varepsilon^{\varphi}_{S,M}\right)_{\substack{S\in\mathcal{S},\\ M\in\mathcal{M}}}$ of semi-norms.

Example 2.

1. *If φ is the identity of $\mathbb{R}_{+}$, the topology $\varepsilon^{\varphi}_{\mathcal{S},\mathcal{M}}$ of $\Lambda_{\varphi}[E]$ is nothing but the topology $\varepsilon_{\mathcal{S},\mathcal{M}}$ given in [4].*
2. *In case $\Lambda = \ell^{1}$ and $E = \mathbb{K}$, the topology $\varepsilon^{\varphi}_{\mathcal{S},\mathcal{M}}$ coincides with the norm topology of ℓ_{φ}.*
3. *When φ is the Orlicz function in (3) of Example 1, $\varepsilon^{\varphi}_{\mathcal{S},\mathcal{M}}$ is given by the semi-norms*

$$\varepsilon_{M}(x) := \sup_{f\in M} \|(f(x_n))_n\|_{\infty}, \quad x \in c_b(E),\ M \in \mathcal{M}.$$

Lemma 2. *The topology $\varepsilon^{\varphi}_{\mathcal{S},\mathcal{M}}$ is Hausdorff. Moreover:*

1. *For every $n \in \mathbb{N}$, the projection $\mathcal{I}_n : x := (x_k)_k \mapsto x_n$ is a continuous mapping from $\Lambda_{\varphi}[E]$ into E;*
2. *$\Lambda_{\varphi}[E]_r$ is a closed subspace of $\Lambda_{\varphi}[E]$.*

Proof. It is easily seen that $\varepsilon^{\varphi}_{\mathcal{S},\mathcal{M}}$ is Hausdorff. To show this:

1. Fix $n \in \mathbb{N}$, $M \in \mathcal{M}$ and choose $S \in \mathcal{S}$ such that $e_n \in S$. For all $x = (x_n)_n \in \Lambda_{\varphi}[E]$, we have

$$\begin{aligned} P_M(\mathcal{I}_n(x)) = P_M(x_n) &= \frac{1}{\|e_n\|_{\varphi}} \|P_M(x_n)e_n\|_{\varphi} \\ &\leq \frac{1}{\|e_n\|_{\varphi}} \varepsilon^{\varphi}_{S,M}(x). \end{aligned}$$

Then $\mathcal{I}_n$ is continuous.

2. Let $x \in \overline{\Lambda_{\varphi}[E]_r}$. Then for all $\varepsilon > 0$, $M \in \mathcal{M}$, and $S \in \mathcal{S}$, there is $y \in \Lambda_{\varphi}[E]_r$ such that $\varepsilon^{\varphi}_{S,M}(x-y) \leq \dfrac{\varepsilon}{3}$. Since $y \in \Lambda_{\varphi}[E]_r$, there is $n_0 \in \mathbb{N}$ such that for all $i \geq n_0$, $\varepsilon^{\varphi}_{S,M}(y^{(i)} - y) \leq \dfrac{\varepsilon}{3}$. So for all $i \geq n_0$:

$$\begin{aligned} \varepsilon^{\varphi}_{S,M}(x^{(i)} - x) &\leq \varepsilon^{\varphi}_{S,M}(x^{(i)} - y^{(i)}) + \varepsilon^{\varphi}_{S,M}(y^{(i)} - y) + \varepsilon^{\varphi}_{S,M}(x - y) \\ &\leq \varepsilon^{\varphi}_{S,M}((x-y)^{(i)}) + \varepsilon^{\varphi}_{S,M}(y^{(i)} - y) + \varepsilon^{\varphi}_{S,M}(x-y) \\ &\leq 2\varepsilon^{\varphi}_{S,M}(x-y) + \varepsilon^{\varphi}_{S,M}(y^{(i)} - y) \quad \leq \varepsilon. \end{aligned}$$

Then $\Lambda_{\varphi}[E]_r$ is closed.

□

Remark 1. *According to the proof above, for every $S \in \mathcal{S}$, the set $\{\mathcal{I}_n, e_n \in S\}$ is even equicontinuous. In particular, if Λ is a normed space so that $\|e_n\|_{\Lambda^*} \leq 1$ for every n, then $\{\mathcal{I}_n, n \in \mathbb{N}\}$ is equicontinuous and is then also equibounded. An instance where this occurs is $\Lambda = \ell^{p}$.*

The following lemma shows that not only is E (identified with) a subspace of $\Lambda_{\varphi}[E]$, but it is also complemented in it.

Lemma 3. *The space E is complemented in both spaces $\Lambda_{\varphi}[E]$ and $\Lambda_{\varphi}[E]_r$.*

Proof. Set $[E] := \{te_1 : t \in E\}$ and consider the mapping $p : \Lambda_{\varphi}[E] \to [E]$ defined for all $(x_n)_n \in \Lambda_{\varphi}[E]$ by $p((x_n)_n) = x_1 e_1$. This is a projection, and since

$$\varepsilon^{\varphi}_{S,M}(p((x_n)_n)) \leq \varepsilon^{\varphi}_{S,M}((x_n)_n), \qquad (x_n)_n \in \Lambda_{\varphi}[E], \qquad (S,M) \in \mathcal{S} \times \mathcal{M},$$

p is a continuous. Therefore, $[E]$ is complemented in $\Lambda_\varphi[E]$. Now, the mapping $\phi : t \mapsto te_1$ is a bicontinuous linear isomorphism from E into $[E]$ because for all $t \in E$ and all $(S, M) \in \mathcal{S} \times \mathcal{M}$,

$$\varepsilon^\varphi_{S,M}(te_1) = \|e_1\|_\varphi P_S(e_1)P_M(t).$$

Identifying E and $[E]$, E is complemented in $\Lambda_\varphi[E]$.

The same proof also works for $\Lambda_\varphi[E]_r$. □

The following theorem shows when $\Lambda_\varphi[E]$ is complete or sequentially complete.

Theorem 1. *The space $\Lambda_\varphi[E]$ is (sequentially) complete if and only if E is (sequentially) complete.*

Proof. This necessity is derived from Lemma 3. As to the sufficiency, assume E is complete, and let $(x^i)_{i\in I}$ be a Cauchy net in $\Lambda_\varphi[E]$, with $(I, \leq)$ being an upwardly directed ordered set. The continuity of the projection $\mathcal{I}_n$ implies that $(x^i_n)_i$ is a Cauchy net in E for all n. Hence, it converges to some $x_n \in E$.

We claim that $x := (x_n)_n$ belongs to $\Lambda_\varphi[E]$. For every $S \in \mathcal{S}$, $M \in \mathcal{M}$, and $\varepsilon > 0$, choose $k \in I$ such that for all $i, j > k$, $\varepsilon^\varphi_{S,M}(x^i - x^j) < \varepsilon$. Then, by normality of ℓ_φ, for every $\alpha \in S$, $f \in M$, and $i, j > k$, one has

$$\left\| \left(\alpha_n f(x^i_n) - \alpha_n f(x^j_n) \right)_n \right\|_\varphi \leq \varepsilon^\varphi_{S,M}(x^i - x^j) < \varepsilon.$$

Therefore, $(\alpha_n f(x^i_n))_i$ is a Cauchy sequence in the Banach space ℓ_φ for all $n \in \mathbb{N}$. Let $\gamma := (\gamma_n)_n$ be its limit in ℓ_φ. Then for every $n \in \mathbb{N}$, we have

$$\alpha_n f(x_n) = \alpha_n f(\lim_i x^i_n) = \lim_i \alpha_n f(x^i_n) = \gamma_n.$$

But for $i, j \geq k$, $\alpha \in \mathcal{S}$, and $N \in \mathbb{N}$, we have

$$\sup_{\delta(y,\varphi^*)\leqslant 1} \sum_{n=1}^{N} \left| y_n \alpha_n f(x^i_n - x^j_n) \right| \leq \left\| \left(\alpha_n f(x^i_n - x^j_n) \right)_n \right\|_\varphi \leq \varepsilon^\varphi_{S,M}(x^i - x^j) < \varepsilon.$$

Passing to the limit on j, we get for all $N \geq n_0$

$$\sup_{\delta(y,\varphi^*)\leqslant 1} \sum_{n=1}^{N} \left| y_n \alpha_n f(x^i_n - x_n) \right| \leq \varepsilon,$$

and then $\varepsilon^\varphi_{S,M}(x^i - x) \leq \varepsilon$ for every $i \geq k$. This shows at once that x belongs to $\Lambda_\varphi[E]$ and that $(x^i)_{i\in I}$ converges to x in $\Lambda_\varphi[E]$.

With a similar proof, one shows that $\Lambda_\varphi[E]$ is sequentially complete if and only if E is sequentially complete. □

Lemma 3 and Theorem 1 show that the three spaces E, $\Lambda_\varphi[E]$, and $\Lambda_\varphi[E]_r$ are simultaneously complete or simultaneously not complete.

Proposition 1. *If E is fast-barrelled, then*

$$\Lambda_\varphi\left[E'_\beta\right] = \{a = (a_n)_n \subset E' : (\alpha_n a_n(x))_n \in \ell_\varphi,\ x \in E,\ \alpha \in \Lambda^*\}.$$

Moreover, the topology of $\Lambda_\varphi\left[E'_\beta\right]$ is given by the semi-norms

$$\varepsilon^\varphi_{S,B}(a) = \sup_{\alpha\in S, x\in B} \|(\alpha_n a_n(x))_n\|_\varphi,$$

where S runs over $\mathcal{S}$, and B runs over the collection $\mathcal{B}$ of all closed and bounded discs in E.

Proof. If
$$\Delta := \{a = (a_n)_n \subset E' : (\alpha_n a_n(x))_n \in \ell_\varphi,\ x \in E,\ \alpha \in \Lambda^*\},$$
then clearly, $\Lambda_\varphi\left[E'_\beta\right] \subset \Delta$.

Conversely, consider $a := (a_n)_n \in \Delta$, $f \in (E'_\beta)'$, $y \in \widetilde{B}_{\varphi^*}$, and $\beta \in \Lambda^*$. Choose $x \in E$. Then
$$\left|\sum_{n=1}^{p} y_n\beta_n a_n(x)\right| \leq \sum_{n=1}^{+\infty}|y_n\beta_n a_n(x)| < +\infty, \qquad p \in \mathbb{N}.$$
Therefore,
$$A := \left\{\sum_{n=1}^{p} y_n\beta_n a_n, p \in \mathbb{N}\right\}$$
is $\sigma(E', E)$-bounded. Since E is fast-barrelled, A is bounded in E'_β. Hence, there is some $K > 0$ such that
$$\sum_{n=1}^{+\infty}|y_n\beta_n f(a_n)| \leq K.$$
Consequently,
$$a \in \Lambda_\varphi\left[E'_\beta\right].$$

Now, let M be a closed equicontinuous disc in $(E'_\beta)'$. Then the polar M° of M is a 0-neighborhood in E'_β. If B is the polar in E of M°, then B is a closed bounded disc in E such that
$$M = M^{\circ\circ} \subset B^{\circ\circ} = \overline{B}^{\sigma(E'',E')}.$$
Then for every $a \in E'$, we have
$$\sup_{f\in M}|f(a)| \leq \sup_{x\in B^{\circ\circ}}|a(x)| \leq \sup_{x\in B}|a(x)|.$$

In particular, for $a = \sum_{n=1}^{p}\alpha_n y_n a_n \in E'$ with $y \in \widetilde{B}_{\varphi^*}$, $\alpha \in S$ and $a \in \Lambda_\varphi\left[E'_\beta\right]$, we have
$$\sup_{f\in M}\left|\sum_{n=1}^{p}\alpha_n y_n f(a_n)\right| \leq \sup_{x\in B^{\circ\circ}}\left|\sum_{n=1}^{p}\alpha_n y_n a_n(x)\right| \leq \sup_{x\in B}\left|\sum_{n=1}^{p}\alpha_n y_n a_n(x)\right|.$$
Passing to the supremum on p, first on $y \in \widetilde{B}_{\varphi^*}$ and then on $\alpha \in S$, we get
$$\varepsilon^{\varphi}_{S,M}(a) \leq \varepsilon^{\varphi}_{S,B}(a),$$
which completes the proof. □

4. Continuous Dual Space of $\Lambda_\varphi[E]$

In the literature, several kinds of duals are considered when dealing with sequence spaces: mainly the Köthe-dual or the α-dual, the β-dual, the Köthe–Toeplitz dual, the algebraic dual and, whenever the sequence space is equipped with a linear topology, the continuous dual (see [4,8,21]). In order to determine the continuous dual space of $\Lambda_\varphi[E]$, we introduce the notion of strongly Köthe–Orlicz summable sequences.

Definition 2. *A sequence $x = (x_n) \subset E$ is said to be strongly Köthe–Orlicz summable with respect to φ and Λ (for short, strongly (φ, Λ)-summable), if for every $M \in \mathcal{M}$ and every $a = (a_n)_n \in (\Lambda^*)_{\varphi^*}[E'_M]$, the sequence $(a_n(x_n))_n$ belongs to ℓ^1.*

The set of all strongly (φ, Λ)-summable sequences is denoted by $\Lambda_\varphi\langle E\rangle$.

Proposition 2. *Let $S \in \mathcal{S}$ and $M \in \mathcal{M}$. Then:*

1. *The space $(\Lambda_S^*)_{\varphi^*}[E'_M]$ is a Banach space for the norm $\varepsilon^{\varphi^*}_{S^\circ,M^\circ}$ defined by*

$$\varepsilon^{\varphi^*}_{S^\circ,M^\circ}(a) := \sup_{f\in M^\circ,\ \alpha\in S^\circ} \|(\alpha_n f(a_n))_n\|_{\varphi^*}, \quad a := (a_n)_n \in (\Lambda_S^*)_{\varphi^*}[E'_M],$$

with S° being the polar of S in Λ. Moreover, the projections $(a_n)_n \mapsto a_n$ are continuous.

2. *The mapping $\sigma^\varphi_{S,M}$ is a semi-norm on $\Lambda_\varphi\langle E\rangle$, where for all $x \in \Lambda_\varphi\langle E\rangle$,*

$$\sigma^\varphi_{S,M}(x) = \sup\left\{ \sum_{n=1}^{+\infty} |a_n(x_n)|; a = (a_n)_n \in (\Lambda_S^*)_{\varphi^*}\left[E'_M\right], \varepsilon^{\varphi^*}_{S^\circ,M^\circ}(a) \le 1 \right\}.$$

Proof. 1. If $\mathcal{S}' := \{rS', r \ge 0\}$, where S' denotes the $\sigma((\Lambda_S^*)^*, \Lambda_S^*)$-closure of S° in $(\Lambda_S^*)^*$, then the norm topology of Λ_S^* is nothing but the $\mathcal{S}'$-topology. Therefore, by Theorem 1, $(\Lambda_S^*)_{\varphi^*}[E'_M]$ is the Banach space. Moreover, by Lemma 2, the projections are continuous.

2. It suffices to show that $\sigma^\varphi_{S,M}(x)$ is finite for every $x \in \Lambda_\varphi\langle E\rangle$. Fix then such an x and define a linear mapping T_x from $(\Lambda_S^*)_{\varphi^*}\left[E'_M\right]$ into ℓ^1 by $T_x((a_n)_n) = (a_n(x_n))_n$. Suppose that $(a^i)_i \in (\Lambda_S^*)_{\varphi^*}\left[E'_M\right]$ converges to $a := (a_n)_n$ and $(T_x(a^i))_i$ converges in ℓ^1 to $(\gamma_n)_n$. By continuity of the projections, $(a^i_n)_i$ converges in E'_M to some a_n for every $n \in \mathbb{N}$. Then $(a^i_n(x_n))_i$ converges to $a_n(x_n)$ as well. It follows that $(a_n(x_n))_n = (\gamma_n)_n$: hence, the closedness of the graph of T_x. Therefore, T_x is continuous and is then bounded on the unit ball of $(\Lambda_S^*)_{\varphi^*}\left[E'_M\right]$. This yields $\sigma^\varphi_{S,M}(x) < +\infty$. □

The following lemma can be shown using a standard argument. Its proof is thus omitted.

Lemma 4. *If $\gamma := (\gamma_n)_n \in c_0$, then $\gamma x = (\gamma_n x_n)_n \in \Lambda_\varphi[E]_r$ for every $x = (x_n)_n \in \Lambda_\varphi[E]$.*

For a continuous linear functional F on $\Lambda_\varphi[E]$ (or on $\Lambda_\varphi[E]_r$), let $F_n(t) := F(te_n)$ for $n \in \mathbb{N}$ and $t \in E$. The following lemma shows that in some sense, the topological dual space of $\Lambda_\varphi[E]_r$ is contained in $\left(\Lambda_\varphi[E]\right)^*$.

Lemma 5. *Let F be a continuous linear functional on $\Lambda_\varphi[E]$. Then:*

1. *There exists $M \in \mathcal{M}$ such that $(F_n)_n \in E'_M$.*
2. *The sequence $(F_n)_n$ belongs to $\left(\Lambda_\varphi[E]\right)^*$.*

If, in addition, the family $\{e_n, n \in \mathbb{N}\}$ is $\tau_{\mathcal{S}}$-bounded, then $(F_n)_n$ is equicontinuous.

Proof. By continuity of F, for every $x \in \Lambda_\varphi[E]_r$, we have

$$F(x) = F\left(\sum_{n\ge1} x_n e_n\right) = \sum_{n\ge1} F(x_n e_n) = \sum_{n\ge1} F_n(x_n).$$

Moreover, there exist $S \in \mathcal{S}$ and $M \in \mathcal{M}$ such that $|F(x)| \le \varepsilon^\varphi_{S,M}(x)$ for all $x \in \Lambda_\varphi\{E\}$. Fix $n \in \mathbb{N}$ and $t \in E$. We have

$$\left|F_n(t)\right| = \left|F(te_n)\right| \le \varepsilon^\varphi_{S,M}(te_n) = \|e_n\|_\varphi P_S(e_n) P_M(t). \tag{1}$$

It follows that F_n belongs to E'_M and thus Condition 1 is proved.

For Condition 2, let $x \in \Lambda_\varphi[E]$ be arbitrary. For all $\gamma \in c_0$, $\gamma x \in \Lambda_\varphi[E]_r$. Choose a scalar sequence $\lambda = (\lambda_n)_n$ such that $|\lambda_n| = 1$ and $|\gamma_n F_n(x_n)| = \lambda_n \gamma_n F_n(x_n)$ for all $n \in \mathbb{N}$. Since $\gamma\lambda x \in \Lambda_\varphi[E]_r$, we have

$$\sum_{n\geq 1} |\gamma_n F_n(x_n)| = \sum_{n\geq 1} \gamma_n \lambda_n F_n(x_n) = \sum_{n\geq 1} F_n(\gamma_n \lambda_n x_n) = F(\lambda\gamma x) < +\infty.$$

As $\gamma \in c_0$ was arbitrary, this shows that

$$\sum_{n\geq 1} |F_n(x_n)| < +\infty.$$

Hence, $(F_n)_n \in \left(\Lambda_\varphi[E]\right)^*$.

Now, if in addition, the family $\{e_n, n \in \mathbb{N}\}$ is $\tau_{\mathcal{S}}$-bounded, choose $s > 0$ such that for every $n \in \mathbb{N}$, $P_S(e_n) \leq s$, $\|e_n\|_\varphi \leq s$. We then get

$$|F_n(t)| \leq \|e_n\|_\varphi P_M(t) P_S(e_n) \leq s^2 P_M(t).$$

Therefore, $(F_n)_n$ is equicontinuous. $\square$

Now, we give a better description of continuous functionals on $\Lambda_\varphi[E]$.

Theorem 2. *If F is a continuous functional on $\Lambda_\varphi[E]$, then there exist $M \in \mathcal{M}$ and $S \in \mathcal{S}$ such that the sequence $(F_n)_n$ is strongly (φ^*, Λ_S^*)-summable in E'_M, i.e., $(F_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M \rangle$.*

Proof. Let $S \in \mathcal{S}$ and $M \in \mathcal{M}$ be such that

$$|F(x)| \leq \varepsilon^\varphi_{S,M}(x), \qquad x = (x_n)_n \in \Lambda_\varphi[E].$$

By Lemma 5, $(F_n)_n \subset E'_M$. Now, fix $(f_n)_n \in (\Lambda_S^*)^*_\varphi\left[(E'_M)'\right]$. We claim that $\left(f_n(F_n)\right)_n$ belongs to ℓ^1. Indeed, take an arbitrary $k \in \mathbb{N}$ and $\delta > 0$, and denote by X the completion of the normed space $(E/M^\perp, \overline{P_M})$ and by B_k the linear span of $\{F_1, F_2, \ldots, F_k\}$. Here, $M^\perp$ is the annihilator of M in E', and as usual,

$$\overline{P_M}(x + M^\perp) := P_M(x).$$

Since E'_M is isometrically isomorphic to $(E/M^\perp)' = X'$, we have $B_k \subset X'$. But

$$(f_n)_n \in (\Lambda_S^*)^*_\varphi\left[(E'_M)'\right],$$

hence

$$(f_n)_n \subset (E'_M)' = X''.$$

Let A_k be the linear span of $\{f_1, f_2, \ldots, f_k\}$. By the principle of local reflexivity, there exists a continuous operator $T_k : A_k \longrightarrow X$ such that:

1. $\|T_k\| \leq 1 + \delta$ with $\|T_k\| = \sup\limits_{f \in M^\circ} \|T_k(f)\|_X$;
2. $F_n(T_k f_n) = f_n(F_n), \qquad n \in \{1, 2, \ldots, k\}$.

Since $E/M^\perp$ is dense in X, for any

$$0 < \delta_n \leq \frac{\delta}{k(1 + \|e_n\|_\varphi P_S(e_n))},$$

there is $x_n \in E$ such that:

$$\overline{P_M}(x_n + M^\perp - T_k f_n) \leq \delta_n.$$

Next, (1) implies that $\|F_n\|_M \le \|e_n\|_\varphi P_S(e_n)$. Therefore, as F_n is continuous,

$$\begin{aligned}\left|F_n(x_n + M^\perp - T_k f_n)\right| &\le \|F_n\|_M P_M(x_n - T_k f_n)\\ &\le \|e_n\|_\varphi P_S(e_n)\frac{\delta}{k(1+\|e_n\|_\varphi P_S(e_n))}\\ &\le \frac{\delta}{k}.\end{aligned}$$

Choose λ_n in the unit complex circle so that $|F(x_n e_n)| = \lambda_n F(x_n e_n)$. Then

$$\begin{aligned}\sum_{n=1}^{k}|f_n(F_n)| &= \sum_{n=1}^{k}|F_n(T_k f_n)|\\ &\le \sum_{n=1}^{k}\left|F_n(x_n + M^\perp - T_k f_n)\right| + \left|F\left(\sum_{n=1}^{k}\lambda_n x_n e_n\right)\right|\\ &\le \delta + \varepsilon^{\varphi}_{S,M}((x_1, x_2, \ldots, x_k, 0, \ldots))\\ &= \delta + \sup\left\{\left|\sum_{n=1}^{k} y_n \alpha_n a(x_n)\right| : (\alpha_n)_n \in S, a \in M, y \in \widetilde{B}_{\varphi^*}\right\}.\end{aligned}$$

But for every $(\alpha_n)_n \in S$, $y \in \widetilde{B}_{\varphi^*}$, and $a \in M$,

$$\begin{aligned}\left|\sum_{n=1}^{k} y_n\alpha_n a(x_n)\right| &\le \left|\sum_{n=1}^{k} y_n\alpha_n a(x_n + M^\perp - T_k f_n)\right| + \left|\sum_{n=1}^{k} y_n\alpha_n a(T_k f_n)\right|\\ &\le \sum_{n=1}^{k}\left|y_n\alpha_n\right|\left|a(x_n + M^\perp - T_k f_n)\right| + \left|a\left(T_k\left(\sum_{n=1}^{k} y_n\alpha_n f_n\right)\right)\right|\\ &\le \sum_{n=1}^{k}|y_n\alpha_n|\|a\|_M\delta_n + \|a\|_M\|T_k\|\sup_{x'\in M}\left\{\left|\sum_{n=1}^{k} y_n\alpha_n f_n(x')\right|\right\}\\ &\le \delta + (1+\delta)\varepsilon^{\varphi}_{S,M}((f_n)_n).\end{aligned}$$

Consequently,

$$\sum_{n=1}^{k}|f_n(F_n)| \le 2\delta + (1+\delta)\varepsilon^{\varphi}_{S,M}((f_n)_n), \quad k \in \mathbb{N}.$$

Hence, $(f_n(F_n))_n$ belongs to ℓ^1. □

Remark 2. *Since in the proof of Theorem 2, δ is arbitrary, it follows that*

$$\sum_{n=1}^{+\infty}\left|f_n(F_n)\right| \le \varepsilon^{\varphi}_{S,M}((f_n)_n).$$

Using the Hahn–Banach theorem, we get:

Corollary 1. *If F is a continuous functional on $\Lambda_\varphi[E]_r$, then there exist $M \in \mathcal{M}$ and $S \in \mathcal{S}$ such that $(F_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle$.*

The following proposition is interesting on its own.

Proposition 3. *Let $S \in \mathcal{S}$ and $M \in \mathcal{M}$. If $(a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle$, then $(\|y_n a_n\|_M)_n \in \Lambda_S^*$ for every $y \in \widetilde{B}_{\varphi^*}$.*

Proof. Fix $(a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle$ and $y \in \widetilde{B}_{\varphi^*}$, and let $(\alpha_n)_n \in \Lambda$ and $\varepsilon > 0$ be given. We have

$$\|y_n\alpha_n a_n\|_M = \sup_{t\in M^\circ} |y_n\alpha_n a_n(t)|, \quad n \in \mathbb{N}.$$

Hence, for every $n \in \mathbb{N}$, there is $t_n \in M^\circ \subset E$ such that

$$\|y_n\alpha_n a_n\|_M \leq \frac{\varepsilon}{2^n} + |y_n\alpha_n a_n(t_n)|.$$

Fix $n \in \mathbb{N}$ and $a \in E'_M$ and define $f_n(a) := \alpha_n a(t_n)$. Then

$$|f_n(a)| = |\alpha_n a(t_n)| \leq \|a\|_M P_M(t_n)|\alpha_n| \leq \|a\|_M|\alpha_n|.$$

Since $a \in E'_M$, there is $\mu > 0$ such that $a \in \mu M$. Therefore, $|y_n f_n(a)| \leq \mu\|y\|_\infty|\alpha_n|$, and as Λ is normal, $(y_n f_n(a))_n \in \Lambda$. Hence, $(y_n f_n(a))_n \in (\Lambda_S^*)^*$ for $\Lambda \subset (\Lambda_S^*)^*$. Using Proposition 1, we come to

$$(f_n)_n \in (\Lambda_S^*)_\varphi^*\left[(E'_M)'\right].$$

Further, since $(a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle$, the series

$$\sum_{n=1}^{+\infty} f_n(a_n) = \sum_{n=1}^{+\infty} \alpha_n a_n(t_n)$$

is absolutely convergent. As

$$\sum_{n=1}^{+\infty} \|y_n\alpha_n a_n\|_M \leq \varepsilon + \sum_{n=1}^{+\infty} |y_n\alpha_n a_n(t_n)| \leq \varepsilon + \|y\|_\infty \sum_{n=1}^{+\infty} |f_n(a_n)|,$$

the series

$$\sum_{n=1}^{+\infty} |\alpha_n|\|y_n a_n\|_M$$

is convergent. Hence, $(\|y_n a_n\|_M)_n \in \Lambda^*$ because α was arbitrary in Λ.

Now, if $(\alpha_n)_n \in S^\circ \subset \Lambda$, by Remark 2, we have:

$$\sum_{n=1}^{+\infty} |y_n\alpha_n a_n(t_n)| \leq \|y\|_\infty \sum_{n=1}^{+\infty} |f_n(a_n)| \leq \|y\|_\infty \varepsilon_{S,M}^{\varphi}((f_n)_n).$$

But

$$\begin{aligned}
\varepsilon_{S,M}^{\varphi}((f_n)_n) &= \sup\left\{\sum_{n=1}^{+\infty} |z_n\beta_n f_n(a)| : (\beta_n)_n \in S, a \in M, z \in \widetilde{B}_{\varphi^*}\right\} \\
&\leq \sup\left\{\|a\|_M t_{\varphi^*} \sum_{n=1}^{+\infty} |\beta_n\alpha_n| : (\beta_n)_n \in S, a \in M\right\} \\
&\leq t_{\varphi^*},
\end{aligned}$$

where $t_{\varphi^*} := \sup\left\{t \in [0;+\infty), \varphi^*(t) \leq 1\right\}$. Consequently,

$$\sum_{n=1}^{+\infty} |y_n\alpha_n a_n(t_n)| \leq \|y\|_\infty t_{\varphi^*},$$

whereby

$$\sum_{n=1}^{+\infty} \|y_n\alpha_n a_n\|_M \leq \|y\|_\infty t_{\varphi^*} + \varepsilon.$$

This means that

$$(\|y_n a_n\|_M)_n \in (\|y\|_\infty t_{\varphi^*} + \varepsilon)S^{\circ\circ} = (\|y\|_\infty t_{\varphi^*} + \varepsilon)S;$$

hence $(\|y_n a_n\|_M)_n \in \Lambda_S^*$. $\square$

Proposition 4. *For every $S \in \mathcal{S}$, $M \in \mathcal{M}$ and $a = (a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M \rangle$, the mapping*

$$f_a : x \longmapsto \sum_{n=1}^{+\infty} a_n(x_n)$$

defines a continuous linear functional on $\Lambda_\varphi[E]$.

Proof. Fix an arbitrary $S \in \mathcal{S}$, $M \in \mathcal{M}$, and $a = (a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M \rangle$, and for every $t \in E$, denote by $\widehat{t}$ the continuous linear map on E'_M defined by $\widehat{t}(f) := f(t)$. Next, for $x = (x_n)_n \in \Lambda_\varphi[E]$, $u \in E'_M \subset E'$, and $y \in \widetilde{B}_\varphi$, we have

$$(y_n \widehat{x}_n(u))_n = (y_n u(x_n))_n \in \Lambda \subset (\Lambda_S^*)^*.$$

So using Proposition 1, we get

$$(\widehat{x}_n)_n \in (\Lambda_S^*)_\varphi^* \left[(E'_M)'_\beta \right].$$

Consequently,

$$\sum_{n=1}^{+\infty} a_n(x_n) = \sum_{n=1}^{+\infty} \widehat{x}_n(a_n)$$

is convergent, and therefore, f_a is well-defined.

Further, observe also that the mapping $\psi_a : (\Lambda_S^*)_\varphi^* \left[(E'_M)' \right] \longrightarrow \ell^1$, given by

$$(f_n)_n \longmapsto \psi_a((f_n)_n) = (f_n(a_n))_n,$$

is well-defined.

In fact, let $(f_n)_n \in (\Lambda_S^*)_\varphi^* \left[(E'_M)' \right]$ be given. Since $a = (a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M \rangle$, the series

$$\sum_{n=1}^{+\infty} f_n(a_n)$$

is absolutely convergent; hence, $(f_n(a_n))_n \in \ell^1$.

Since $(\Lambda_S^*)^*$ is perfect and $(E'_M)'$ is a Banach space, $((\Lambda_S^*)_\varphi^* \left[(E'_M)' \right], \varepsilon_{S,M}^\varphi)$ is also a Banach space. Further, assume that $((f_n)_n^i)_i$ is a null sequence in $(\Lambda_S^*)_\varphi^* \left[(E'_M)' \right]$ such that $(\psi_a((f_n)_n^i))_i$ converges in ℓ^1 to $(\alpha_n)_n$. As the projections $(f_n)_n \mapsto f_n$ are continuous, $(f_n^i)_i$ converges in $(E'_M)'$ to 0 for all $n \in \mathbb{N}$. Hence, the sequence $(\psi_a((f_n^i)_n))_i = ((f_n^i(a_n))_n)_i$ converges to 0, whereby $\alpha_n = 0$ for every n. By the closed graph theorem, φ_a is continuous. Therefore, there is $K > 0$ such that for every $(f_n)_n \in (\Lambda_S^*)_\varphi^* \left[(E'_M)' \right]$, we have the inequality

$$\|\psi_a((f_n)_n)\|_1 \le K \varepsilon_{S,M}^\varphi((f_n)_n),$$

which means that

$$\sum_{n=1}^{+\infty} |f_n(a_n)| \le K \varepsilon_{S,M}^\varphi((f_n)_n).$$

But $(\widehat{x}_n)_n \in (\Lambda_S^*)_\varphi^* \left[(E'_M)' \right]$; hence,

$$|f_a(x)| = \left| \sum_{n=1}^{+\infty} \widehat{x}_n(a_n) \right| \le K \varepsilon_{S,M}^\varphi((\widehat{x}_n)_n) \le K \varepsilon_{S,M}^\varphi(x).$$

Consequently, f_a is continuous. □

Theorem 3. *The following equality is valid:*

$$(\Lambda_\varphi[E])_r)' = \bigcup_{S\in\mathcal{S},M\in\mathcal{M}} (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle.$$

Proof. By Proposition 4, for every $S \in \mathcal{S}$, $M \in \mathcal{M}$, and $a := (a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle$, we have $f_a \in (\Lambda_\varphi[E]_r)'$. Therefore, the function

$$\phi : \bigcup_{S\in\mathcal{S},M\in\mathcal{M}} (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle \longrightarrow (\Lambda_\varphi[E]_r)'$$

given by

$$a \longmapsto f_a,$$

is well-defined and linear. Clearly, ϕ is injective.

Moreover, observe that if $F \in (\Lambda_\varphi[E]_r)'$, then Corollary 1 implies that there exist $S \in \mathcal{S}$, $M \in \mathcal{M}$ such that the sequence $a := (F_n)_n$ belongs to $(\Lambda^*)_{\varphi^*}\langle E'_M\rangle$. Next, for each $x \in \Lambda_\varphi[E]_r$, by the continuity of F, we have

$$\begin{aligned} F(x) &= \lim_k F(x^{(k)}) = \lim_k \sum_{n=1}^{k} F(x_n e_n) \\ &= \sum_{n=1}^{+\infty} F_n(x_n) = f_a(x). \end{aligned}$$

This means that ϕ is also surjective. Consequently ϕ is an isomorphism. □

In the following, we describe a fundamental base of equicontinuous subsets of $(\Lambda_\varphi[E]_r)'$. In order to establish it, let us denote for $S \in \mathcal{S}$ and $M \in \mathcal{M}$:

$$K_{S,M}^{\varphi} = \left\{(f_n)_n \in \Lambda_\varphi[(E'_M)'] : (y_n f_n(a))_n \in S^\circ, \quad a \in M,\ y \in \widetilde{B}_{\varphi^*}\right\}.$$

Theorem 4. *The family of sets of the form*

$$S_\varphi\langle M\rangle = \left\{(a_n)_n \in (\Lambda_S^*)_{\varphi^*}\langle E'_M\rangle : \sum_{n=1}^{+\infty} |f_n(a_n)| \le 1, \quad (f_n)_n \in K_{S,M}^{\varphi}\right\},$$

with S running over $\mathcal{S}$ and M over $\mathcal{M}$ yields a fundamental system of equicontinuous subsets of $(\Lambda_\varphi[E]_r)'$.

Proof. Let us first show that $S_\varphi\langle M\rangle$ is equicontinuous. If $x = (x_n)_n \in \Lambda_\varphi[E]$ is such that $\varepsilon_{S,M}^{\varphi}(x) \le 1$, then, as in the proof of Proposition 4, one has

$$\sum_{n=1}^{+\infty} |y_n \alpha_n \widehat{x}_n(u)| = \sum_{n=1}^{+\infty} |y_n \alpha_n u(x_n)| \le \varepsilon_{S,M}^{\varphi}(x) \le 1$$

for all $y \in \widetilde{B}_{\varphi^*}$, $u \in M$ and $\alpha \in S$. Hence,

$$(y_n \widehat{x}_n(u))_n \in S^\circ.$$

Therefore, $(\widehat{x}_n)_n \in K_{S,M}^{\varphi}$. Moreover, if $a = (a_n)_n \in S_\varphi\langle M\rangle$, then

$$\left|\sum_{n=1}^{+\infty} \widehat{x}_n(a_n)\right| = \left|\sum_{n=1}^{+\infty} a_n(x_n)\right| \le 1.$$

Consequently, $S_\varphi\langle M\rangle$ is equicontinuous.

Now, if $\mathbb{H} \subset (\Lambda_\varphi[E]_r)'$ is equicontinuous, then there are $S \in \mathcal{S}$ and $M \in \mathcal{M}$ such that:

$$\left|\sum_{n=1}^{+\infty} a_n(x_n)\right| \leq \varepsilon^{\varphi}_{S,M}(x)$$

for all $x = (x_n)_n \in \Lambda_\varphi[E]_r$ and $a = (a_n)_n \in \mathbb{H}$. Let $f = (f_n)_n \in K^{\varphi}_{S,M}$. Then $\varepsilon^{\varphi}_{S,M}(f) \leq 1$, and by Remark 2, we have:

$$\sum_{n=1}^{+\infty} |f_n(a_n)| \leq \varepsilon^{\varphi}_{S,M}(f) \leq 1.$$

Consequently, $\mathbb{H} \subset S_\varphi\langle M\rangle$. $\square$

Let us consider the collections:

$$\begin{aligned}
\mathcal{B}' &:= \{B' \subset E' : B' \text{ is a closed weak*-bounded disc}\},\\
\mathcal{R} &:= \{R \subset \Lambda : R \text{ is a closed bounded and normal disc}\},\\
\mathcal{R}' &:= \{R' \subset \Lambda^* : R' \text{ is a closed weak*-bounded and normal disc}\},
\end{aligned}$$

and for every $R' \in \mathcal{R}'$ and $B' \in \mathcal{B}'$, the sets:

$$\begin{aligned}
K_{R',B'} &:= \left\{(f_n)_n \in \Lambda_\varphi[(E'_{B'})'] : (y_n f_n(a))_n \in (R')^\circ, \quad a \in B',\ y \in \widetilde{B}_{\varphi^*}\right\},\\
R'_\varphi\langle B'\rangle &:= \left\{(a_n)_n \in (\Lambda_\varphi[E]_r)' : \sum_{n=1}^{+\infty} |f_n(a_n)| \leq 1, \quad (f_n)_n \in K_{R',B'}\right\}.
\end{aligned}$$

The following theorem gives a necessary and sufficient condition for the space $\Lambda_\varphi[E]_r$ to be barrelled or quasi-barrelled.

Theorem 5. *Assume that Λ is barrelled (quasi-barrelled). Then $\Lambda_\varphi[E]_r$ is barrelled (resp. quasi-barrelled) if and only if the following two conditions are satisfied:*

(i) E is barrelled (resp. quasi-barrelled).

(ii) For each weak bounded (resp. strongly bounded) subset $\mathbb{B}$ of $(\Lambda_\varphi[E]_r)'$, there exist $B' \in \mathcal{B}'$ and $R' \in \mathcal{R}'$ such that $\mathbb{B} \subset R'_\varphi\langle B'\rangle$.*

Proof. Let T be a barrel (resp. bornivorous barrel) in $\Lambda_\varphi[E]_r$. Then T° is a weakly bounded (resp. strongly bounded) subset of $(\Lambda_\varphi[E]_r)'$. By (ii), there exists $R' \in \mathcal{R}'$ and $B' \in \mathcal{B}'$ such that $T^\circ \subset R'_\varphi\langle B'\rangle$. Since E is barrelled (resp. quasi-barrelled), B' is equicontinuous. Hence, it is contained in some $M \in \mathcal{M}$.

Similarly, since Λ is barrelled (resp. quasi-barrelled), there exists $S \in \mathcal{S}$ such that $R' \subset S$. Hence, $T^\circ \subset R'_\varphi\langle B'\rangle \subset S_\varphi\langle M\rangle$. Therefore, T° is equicontinuous and consequently T is a neighborhood of 0 in $\Lambda_\varphi[E]_r$.

Now, assume that $\Lambda_\varphi[E]_r$ is barrelled. By Lemma 3, E is complemented in $\Lambda_\varphi[E]_r$. Therefore, E is a barrelled (resp. quasi-barrelled) space, whereby (i) is satisfied. Moreover, Let $\mathbb{B}$ be a weakly bounded (resp. strongly bounded) subset of $(\Lambda_\varphi[E]_r)'$. Then $\mathbb{B}$ is an equicontinuous subset of $(\Lambda_\varphi[E]_r)'$. By Theorem 4, there exist $S \in \mathcal{S}$ and $M \in \mathcal{M}$ such that $\mathbb{B} \subset S_\varphi\langle M\rangle$. Hence, (ii) is satisfied, too. $\square$

Example 3.

1. *If φ is the identity of $\mathbb{R}_+$, the continuous dual of $\Lambda_\varphi[E]_r$ is as given in [4].*
2. *In case $\Lambda = \ell^1$ and $E = \mathbb{K}$, the continuous dual of $\Lambda_\varphi[E]_r$ is ℓ_{φ^*}.*
3. *When φ is the Orlicz function in (3) of Example 1, the continuous dual of $(c_0)_\varphi[E]_r := c_b(E)_r$ is $\bigcup_{M\in\mathcal{M}} \ell^1\langle E'_M\rangle$.*

In order to give further examples as applications of our results, we determine the duals of some concrete sequence spaces and characterize the barrelledness therein. For this, let $p \geq 1$ be a real number and q its conjugate (i.e., $\frac{1}{p} + \frac{1}{q} = 1$ if $p \neq 1$, and $q = +\infty$ if $p = 1$) and let $(E, \|.\|_E)$ be a normed space. Then the topology of $\ell^q_{\varphi^*}\langle E' \rangle$ is defined by the single norm $\sigma^{\varphi^*}_{R',B'}$; it is also denoted by $\sigma^{\varphi^*}_{q,E'}$. Here, R' and B' are the closed unit bulls of ℓ^q and E', respectively.

We have the following proposition:

Proposition 5. *The topological dual of $\ell^p_\varphi[E]_r$ is $\ell^q_{\varphi^*}\langle E' \rangle$. Moreover, $\ell^p_\varphi[E]_r$ is barrelled if and only if E is barrelled.*

Proof. The first assertion results immediately from Theorem 3.

For the second one, notice that since ℓ^p is a Banach space, it is barrelled. As $\ell^q_{\varphi^*}\langle E' \rangle$ is a Banach space, it is sufficient to show that if E is barrelled, then the unit ball $\mathbb{B}$ of $\ell^q_{\varphi^*}\langle E' \rangle$ is contained in $R'_\varphi\langle B' \rangle$, where R' and B' are the unit balls of ℓ^q and E', respectively.

So choose an arbitrary $(a_n)_n \in \mathbb{B}$ and $(f_n)_n \in K_{R',B'}$. Then $(y_n f_n(b)) \in (R')^\circ$ for every $b \in B'$ and every $y \in \widetilde{B}_{\varphi^*}$, whereby

$$\sup_{\alpha \in R', b \in B'} \sup_{\delta(y,\varphi^*) \leqslant 1} \sum_{n \geq 1} |y_n \alpha_n f_n(b)| \leq 1.$$

This shows that

$$\varepsilon^\varphi_{p,E''}(f) := \varepsilon^\varphi_{R',B'}(f) \leq 1.$$

Hence,

$$\sum_{n \geq 1} |f_n(a_n)| \leq \sigma^{\varphi^*}_{q,E'}((a_n)_n) \leq 1,$$

and consequently, $(a_n)_n \in R'_\varphi\langle B' \rangle$. □

In the special case where φ is the identity $x \mapsto x$, the space $\ell^p_\varphi[E]_r$ is nothing but the space $\ell^p[E]$ introduced by H. Apiola [13]. We then obtain a characterization of barrelledness in such spaces.

Corollary 2. *$\ell^p[E]_r$ is barrelled if and only if E is barrelled.*

5. Conclusions and Future Work

We introduce the notions of weakly (resp. strongly) (φ, Λ)-summable sequences in a locally convex space E and investigate topological properties of the linear space $\Lambda_\varphi[E]$ of all such sequences endowed with the topology induced by an appropriate family of semi-norms. We obtain that E is embedded as a complemented subspace in $\Lambda_\varphi[E]$. Whenever $\Lambda_\varphi[E]$ has the property AK, we characterize its continuous dual in terms of strongly (φ, Λ)-summable sequences in E', which is the continuous dual of E. We further provide necessary and sufficient conditions under which $\Lambda_\varphi[E]$ is barrelled or quasi-barrelled. To illustrate the proposed results, we have included as applications concrete examples of such spaces (see Proposition 5 and Corollary 2). The outcomes of our paper extend and improve known results: in particular, of [8]. Our work paves the way for further investigations of these sequence spaces: namely, for studying reflexivity and distinguishedness.

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