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Curvature Pinching Problems for Compact Pseudo-Umbilical PMC Submanifolds in $\mathbb{S}^m(c) \times \mathbb{R}$

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Abstract: Let $\mathbb{S}^m(c)$ denote a sphere with a positive constant curvature c and M^n $(n \geq 3)$ be an n-dimensional compact pseudo-umbilical submanifold in a Riemannian product space $\mathbb{S}^m(c) \times \mathbb{R}$ with a nonzero parallel mean curvature vector (PMC), where $\mathbb{R}$ is a Euclidean line. In this paper, we prove a sequence of pinching theorems with respect to the Ricci, sectional and scalar curvatures of M^n, which allow us to generalize some classical curvature pinching results in spheres.

Keywords: sectional curvature; Ricci curvature; parallel mean curvature vector; pseudo-umbilical; product space; pinching theorems

MSC: 53C40; 53C24

1. Introduction

Curvature pinching problems, as one of the most important topics of differential geometry, have recently attracted significant research interest. Generally speaking, a typical conclusion is that a general manifold M retains particular geometric and topological properties under the given assumptions for a sectional, Ricci or scalar curvature if it is bounded between suitable constants (e.g., [1–5]).

Let M^n be an n-dimensional compact manifold which is minimally immersed in an $(n+p)$ sphere $\mathbb{S}^{n+p}(c)$. The scalar curvature pinching problem of minimal submanifolds of spheres was initiated in 1968 in [6], in which Simons established the famous Simons integral inequality:

$$\int_M S\Big\{(2-\frac{1}{p})S - nc\Big\}dV \geq 0, \tag{1}$$

where S denotes the square of the second fundamental form and dV is the volume form of M^n. It follows from Equation (1) that if $0 \leq S \leq C_1 = nc/(2-\frac{1}{p})$, then either $S = 0$ and M^n is a totally geodesic submanifold, or $S = C_1$. Soon after, Lawson [7] and Chern, do Carmo and Kobayashi [8] proved that if $S \equiv C_1$, then either $p = 1$ and M^n is a Clifford minimal hypersurface $M_{k,n-k} = \mathbb{S}^k(\frac{nc}{k}) \times \mathbb{S}^{n-k}(\frac{nc}{n-k})$ in $\mathbb{S}^{n+1}(c)$ with $1 \leq k \leq n-1$, or $n = p = 2$, and M^n is a Veronese surface M^2 in $\mathbb{S}^4(c)$. Later, Li and Li [9] improved the first pinching constant C_1 to $\frac{2nc}{3}$ for cases where $p \geq 2$ (see also Chen and Xu's result [10] with another method). Xu [11] obtained a sharp generalization of the above theorems for the compact PMC submanifolds of a sphere.

Instead of the scalar curvature assumption, Yau improved Simons' inequality under the sectional curvature assumption (see Theorem 15 in [12]). In fact, it was proven in [12] that if M^n is a compact minimal submanifold in $\mathbb{S}^{n+p}(c)$ with the sectional curvature of M^n not being less than $\frac{(p-1)c}{2p-1}$, then M^n is a totally geodesic sphere $\mathbb{S}^n(c)$, a Clifford minimal hypersurface $M_{k,n-k}$ in $\mathbb{S}^{n+1}(c)$ with $1 \leq k \leq n-1$ or a Veronese surface M^2 in $\mathbb{S}^4(c)$. Later on, Itoh proved in [13] that if M^n is a compact minimal submanifold in $\mathbb{S}^{n+p}(c)$, and the sectional curvature of M^n is not less than $\frac{nc}{2(n+1)}$, then M^n is either a totally geodesic sphere

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$\mathbb{S}^n(c)$ or a Veronese manifold of a constant sectional curvature $\frac{nc}{2(n+1)}$ with the co-dimension $p = \frac{1}{2}(n-1)(n+2)$. In 2005, Xu and Han [14] established a geometric rigidity theorem for submanifolds with parallel mean curvatures in a space form, which generalizes Yau and Itoh's pinching theorems.

In 1979, Ejiri [15] considered the question of classifying all compact minimal n-dimensional submanifolds of a sphere under some restrictions on the Ricci curvature. Actually, the author proved in [15] that if $M^n(n \geq 4)$ is an n-dimensional simply connected compact minimal submanifold immersed in $\mathbb{S}^{n+p}(c)$, the immersion is full, and the Ricci curvature of M^n is not less than $(n-2)c$, then M^n is a totally geodesic sphere $\mathbb{S}^n(c)$, a Clifford minimal hypersurface $M_{k,k}$ in $\mathbb{S}^{n+1}(c)$ with $n = 2k$ or a two-dimensional complex projective space $\mathbb{C}P^2(\frac{4}{3}c)$ with a holomorphic sectional curvature $\frac{4}{3}c$ in $\mathbb{S}^7(c)$. Moreover, Shen [16] and Li [17] settled Ejiri's type of pinching problem for the case where $n = 3$. Xu and Tian [18] showed further that Ejiri's pinching theorem still holds for $n \geq 3$ by removing the simply connected condition. In the case of a nonzero parallel mean curvature vector, Sun [19] initially established an Ejiri-type theorem. He proved in 1987 that if M^n is a compact PMC submanifold in $\mathbb{S}^{n+p}(c)$ with $n \geq 4$, and under the additional assumption that the Ricci curvature of M^n is not less than $n(n-2)(c+H^2)/(n-1)$, then M^n is a totally umbilic sphere. By replacing Sun's pinching constant with the optimal possible constant $(n-2)(c+H^2)$, He and Luo [20] and Xu et al. [21,22] further generalized Ejiri's pinching theorem to compact PMC submanifolds in a space form.

Pinching phenomena in submanifolds also arose in the Riemannian product spaces. In 2011, Chen and Cui [23] obtained a pinching theorem concerning the squared length of the second fundamental form or the scalar curvature for compact minimal submanifolds in a kind of product space $\mathbb{S}^m(1) \times \mathbb{R}$. By replacing the real line $\mathbb{R}$ with a general Euclidean space with an arbitrary dimension, the second author [24] further extended Chen and Cui's rigidity theorem to the compact minimal submanifolds in a generalized cylinder $\mathbb{S}^{n_1}(c) \times \mathbb{R}^{n_2}$. After deriving a Simons-type formula on the squared length of the second fundamental form, Chen-Chen-Li [25] investigated the pinching problems of compact minimal submanifolds in $\mathbb{S}^m(1) \times \mathbb{R}$. Actually, they obtained a series of pinching theorems involving the scalar, Ricci and sectional curvatures, which generalize the rigidity theorems due to the works of Li and Li [9], Ejiri, Shen and Li [15–17] and Itoh [13]. Later on, the authors of [26] considered a compact minimal submanifold M^n in $\mathbb{S}^{n_1}(c) \times \mathbb{R}^{n_2}$ and obtained several rigidity results depending on the Ricci curvature, the squared length and the squared maximum norm of the second fundamental form on M^n. Some other types of pinching theorems in general product spaces seem to be of interest and worth further discussion (refer to [27–30]).

Inspired by the previous results, the aim of this paper is to investigate the geometric rigidity of compact submanifolds in a kind of Riemannian product space with a parallel mean curvature vector (PMC). We extend some pinching results for the minimal submanifolds to the PMC submanifolds in $\mathbb{S}^m(c) \times \mathbb{R}$. The outline of this paper is as follows. In Section 2, we briefly recall some basic notations, concepts and general properties of compact pseudo-umbilical PMC submanifolds in $\mathbb{S}^m(c) \times \mathbb{R}$. In Section 3, we summarize several relevant inequalities, which is crucial for obtaining our curvature pinching theorems. In Sections 4–6, we give the proofs of our pinching theorems (For details see Theorems 1, 2 and 4) for the compact pseudo-umbilical PMC submanifolds in $\mathbb{S}^m(c) \times \mathbb{R}$ in terms of the Ricci, sectional and scalar curvatures of the submanifolds, respectively.

2. Preliminaries

Suppose that $\widetilde{M} := \mathbb{S}^m(c) \times \mathbb{R}$ is an $(m+1)$-dimensional Riemannian product space, M^n is a manifold of a dimension $n(\geq 3)$ and $\varphi : M^n \longrightarrow \widetilde{M}$ is an isometric immersion with the co-dimension $p = m+1-n$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M^n

and $\widetilde{M}$, respectively. Throughout this paper, we agree on the following ranges of indices and the Einstein summation convention to be used unless otherwise stated:

$$1 \leq A, B, C, \cdots \leq m+1; \quad 1 \leq i, j, k, \cdots \leq n; \quad n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p.$$

We can choose $\{e_i\}$ to be a local orthonormal basis of the tangent bundle TM with a dual basis $\{\omega_i\}$ and $\{e_\alpha\}$ to be a local orthonormal basis of the normal bundle $T^\perp M$ with a dual basis $\{\omega_\alpha\}$. Let $\{\omega_{AB}\}$ be the connection 1-forms associated with $\{\omega_A\}$. Since $\omega_\alpha = 0$ when restricted on M, it follows immediately that

$$0 = d\omega_\alpha = -\omega_{i\alpha} \wedge \omega_i.$$

Under Cartan's lemma, we can write

$$\omega_{i\alpha} = h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2}$$

We denote by $A_\alpha = (h^\alpha_{ij})$, $h = h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha$ and $\nu = H^\alpha e_\alpha = \frac{1}{n}h^\alpha_{ii}e_\alpha$ the shape operator with respect to e_α, the second fundamental form and the mean curvature vector field, respectively. The length of ν, denoted by H, is defined to be the mean curvature of M^n. Moreover, M^n is defined to be a submanifold with a parallel mean curvature vector if ν is parallel on the normal bundle. M^n is said to be a minimal submanifold if ν vanishes.

The first and second covariant derivatives of h^α_{ij} are defined as

$$\nabla h^\alpha_{ij} = dh^\alpha_{ij} + h^\alpha_{mj}\omega_{mi} + h^\alpha_{im}\omega_{mj} + h^\beta_{ij}\omega_{\beta\alpha} := h^\alpha_{ijk}\omega_k, \tag{3}$$

$$\nabla h^\alpha_{ijk} = dh^\alpha_{ijk} + h^\alpha_{mjk}\omega_{mi} + h^\alpha_{imk}\omega_{mj} + h^\alpha_{ijm}\omega_{mk} + h^\beta_{ijk}\omega_{\beta\alpha} := h^\alpha_{ijkl}\omega_l. \tag{4}$$

We denote by t the Cartesian coordinate on the Euclidean line $\mathbb{R}$ and $\partial_t := \frac{\partial}{\partial t}$ the unit vector field tangent to $\mathbb{R}$. By decomposing ∂_t into its tangential and normal components, we see that

$$\partial_t = T + \eta = \sum_i T_i e_i + \sum_\alpha \eta_\alpha e_\alpha, \tag{5}$$

which satisfies $|T|^2 = \sum_i T_i^2$, $|\eta|^2 = \sum_\alpha \eta_\alpha^2$ and $|T|^2 + |\eta|^2 = 1$. Moreover, the covariant derivatives of T_i and η_α are defined to be

$$\nabla T_i = dT_i + T_j\omega_{ji} := T_{i,j}\omega_j, \tag{6}$$

$$\nabla \eta_\alpha = d\eta_\alpha + \eta_\beta\omega_{\beta\alpha} := \eta_{\alpha,j}\omega_j. \tag{7}$$

Since $\widetilde{\nabla}$ is a Euclidean connection when restricted to $\mathbb{R}$, it follows that ∂_t is parallel in $\widetilde{M}$. Taking the covariant derivative of each side of Equation (5) and using Equations (6) and (7) gives

$$T_{i,j} = \sum_\beta h^\beta_{ij}\eta_\beta, \tag{8}$$

$$\eta_{\alpha,i} = -\sum_k h^\alpha_{ik}T_k. \tag{9}$$

Let $T\widetilde{M}$ denote the tangent bundle of $\widetilde{M}$. Recall that the curvature tensor of $\widetilde{M}$ is the map $\widetilde{R} : T\widetilde{M} \times T\widetilde{M} \times T\widetilde{M} \longrightarrow T\widetilde{M}$, defined as

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z, \tag{10}$$

and it is written in terms of the orthonormal basis $\{e_A\}$ as

$$\widetilde{R}(e_C, e_D)e_B = \sum_A \widetilde{R}^A_{BCD}e_A.$$

Moreover, the Riemannian curvature tensor $\widetilde{R}_{ABCD}$ of $\widetilde{M}$ is the covariant 4-tensor field obtained from the (1,3)-type curvature tensor field $\widetilde{R}^{A}_{BCD}$ of $\widetilde{M}$ by lowering the first index. Here, we continue to use the same symbol $\widetilde{R}$ for the Riemannian curvature tensor when there is no risk of confusion. With this definition, we write

$$\widetilde{R}(e_A, e_B)e_C = \sum_D \widetilde{R}_{DCAB} e_D,$$

and by utilizing the symmetries of the Riemannian curvature tensor, we obtain

$$\langle \widetilde{R}(e_A, e_B)e_C, e_D \rangle = \widetilde{R}_{DCAB} = \widetilde{R}_{ABDC}. \tag{11}$$

It is worth pointing out that we can define the Riemannian curvature tensor R and the normal curvature tensor $R^{\perp}$ of M^n. It is convenient to write

$$\langle R(e_i, e_j)e_k, e_l \rangle = R_{lkij} = R_{ijlk}, \quad \langle R^{\perp}(e_i, e_j)e_\alpha, e_\beta \rangle = R^{\perp}_{\beta\alpha ij}. \tag{12}$$

As we know, the standard Gauss, Codazzi and Ricci equations are given by these respective equations:

$$R_{ijkl} = \widetilde{R}_{ijkl} + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \tag{13}$$

$$-\widetilde{R}_{\alpha ijk} = h^\alpha_{ijk} - h^\alpha_{ikj}, \tag{14}$$

$$R^{\perp}_{\alpha\beta kl} = \widetilde{R}_{\alpha\beta kl} + \sum_k (h^\alpha_{ik} h^\beta_{jk} - h^\beta_{ik} h^\alpha_{jk}). \tag{15}$$

Moreover, the Ricci identity is written as

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = h^\alpha_{mj} R_{mikl} + h^\alpha_{im} R_{mjkl} + h^\beta_{ij} R^{\perp}_{\beta\alpha kl}. \tag{16}$$

Let $\pi : \widetilde{M} = \mathbb{S}^m(c) \times \mathbb{R} \longrightarrow \mathbb{S}^m(c)$ denote the natural projection map. Since e_A projects to the vector field $\pi_*(e_A)$ on the tangent bundle $T\mathbb{S}^m(c)$, then

$$\pi_*(e_A) = e_A - \langle e_A, \partial_t \rangle \partial_t. \tag{17}$$

By taking into account Equations (10), (11) and (17), it follows that

$$\begin{aligned} \widetilde{R}_{ABCD} = c \cdot \Big\{ & \langle \pi_*(e_A), \pi_*(e_C) \rangle \langle \pi_*(e_B), \pi_*(e_D) \rangle \\ & - \langle \pi_*(e_A), \pi_*(e_D) \rangle \langle \pi_*(e_B), \pi_*(e_C) \rangle \Big\}. \end{aligned} \tag{18}$$

Instituting Equation (18) into Equations (13)–(15) leads to the following result.

Proposition 1 (c.f. [25]). *Let M^n be an n-dimensional immersed submanifold in an $(m+1)$-dimensional Riemannian product space $\widetilde{M} := \mathbb{S}^m(c) \times \mathbb{R}$. Then, the Gauss, Codazzi and Ricci equations are given by these respective equations:*

$$\begin{aligned} R_{ijkl} = & c\{\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + T_j T_k \delta_{il} + T_i T_l \delta_{jk} - T_i T_k \delta_{jl} - T_j T_l \delta_{ik}\} \\ & + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}), \end{aligned} \tag{19}$$

$$-\widetilde{R}_{\alpha ijk} = h^\alpha_{ijk} - h^\alpha_{ikj} = c\eta_\alpha (T_j \delta_{ik} - T_k \delta_{ij}), \tag{20}$$

$$R^{\perp}_{\alpha\beta ij} = \sum_k (h^\alpha_{ik} h^\beta_{kj} - h^\beta_{ik} h^\alpha_{kj}). \tag{21}$$

Using Equation (20) gives the following proposition derived in Proposition 1 of [31]:

Proposition 2 ((c.f. [31]). *Let* $g^{\alpha}_{ijk} = \frac{1}{3}\left(h^{\alpha}_{ijk} + h^{\alpha}_{jki} + h^{\alpha}_{kij}\right)$. *Then, we have*

$$\sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 = \sum_{\alpha,i,j,k} (g^{\alpha}_{ijk})^2 + \frac{2}{3}(n-1)c^2|T|^2|\eta|^2. \tag{22}$$

Remark 1. *If* $h^{\alpha}_{ijk} = 0$ *for any* α, i, j, k *in Equation* (22), *then* $|T| = 0$ *or* $|\eta| = 0$ *(i.e.,* M^n *is either contained in a slice* $\mathbb{S}^m(c)$, *or* ∂_t *is tangent to* $\mathbb{R}$ *everywhere). In particular, suppose* M^n *is a compact submanifold of* $\mathbb{S}^m(c) \times \mathbb{R}$. *It is easy to see that* $|T| = 0$, *and thus* M^n *lies in* $\mathbb{S}^m(c)$.

The Ricci curvature is a symmetric 2-tensor field, and we find with Equation (19) that the Ricci curvature of M^n is given by

$$R_{ik} = c\left\{(n-1-|T|^2)\delta_{ik} - (n-2)T_iT_k\right\} + n\sum_{\alpha} H^{\alpha}h^{\alpha}_{ik} - \sum_{\alpha,j} h^{\alpha}_{ij}h^{\alpha}_{jk}. \tag{23}$$

In particular, using $k = i$ in Equation (23) gives

$$R_{ii} = c\left\{n-1-|T|^2-(n-2)T_i^2\right\} + n\sum_{\alpha} H^{\alpha}h^{\alpha}_{ii} - \sum_{\alpha,j}\left(h^{\alpha}_{ij}\right)^2. \tag{24}$$

In this paper, we assume that M^n has nonzero parallel mean curvature vector (PMC), and thus we can choose $e_{n+1} = \nu/|\nu|$ such that H is a constant, and for $\alpha \neq n+1$, we have

$$H^{\alpha} = \frac{1}{n}\mathtt{Tr}A_{\alpha} = 0, \tag{25}$$

$$H^{n+1} = \frac{1}{n}\mathtt{Tr}A_{n+1} = H, \tag{26}$$

$$R^{\perp}_{n+1,\alpha ij} = 0. \tag{27}$$

The detailed proofs are given in [12].

Remark 2. *Suppose that* M^n *is a complete pseudo-umbilical PMC hypersurface in a product space* $\mathbb{S}^n(c) \times \mathbb{R}$. *It follows that* $h^{n+1}_{ijk} = 0$ *for any* $1 \leq i, j, k \leq n$. *According to Remark 1, it may be concluded that* M^n *is either a sphere* $\mathbb{S}^n(c)$ *or a product space* $\mathbb{S}^{n-1}(c + H^2) \times \mathbb{R}$ *of an* $(n-1)$*-dimensional sphere of a constant sectional curvature* $c + H^2$ *and a real line* $\mathbb{R}$. *In the remainder of this paper, we require* $p \geq 2$.

Let S_I be a function given by

$$S_I = \sum_{\alpha \neq n+1} \mathtt{Tr}(A^2_{\alpha}).$$

We intend to compute the Laplacian of S_I. In fact, using the Codazzi equation (Equation (14)) and Ricci identity (Equation (16)), we deduce that

$$\begin{aligned}\frac{1}{2}\Delta S_I =& \frac{1}{2}\Delta\Big(\sum_{\alpha\neq n+1}\sum_{i,j}(h^{\alpha}_{ij})^2\Big)\\ =& \sum_{\alpha\neq n+1}\sum_{i,j,k}(h^{\alpha}_{ijk})^2 + \sum_{\alpha\neq n+1}\sum_{i,j,k} h^{\alpha}_{ij}h^{\alpha}_{ijkk}\\ =& I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{28}$$

where

$$I_1 = \sum_{\alpha\neq n+1}\sum_{i,j,k}(h^{\alpha}_{ijk})^2,$$

$$I_2 = -\sum_{\alpha\neq n+1}\sum_{i,j,k} h^\alpha_{ij}(\widetilde{R}_{\alpha kikj} + \widetilde{R}_{\alpha ijkk}),$$
$$I_3 = \sum_{\alpha\neq n+1}\sum_{i,j,k,\beta} h^\alpha_{ij}h^\beta_{ki}R^\perp_{\beta\alpha jk},$$
$$I_4 = \sum_{\alpha\neq n+1}\sum_{i,j,k,m} h^\alpha_{ij}(h^\alpha_{mi}R_{mkjk} + h^\alpha_{km}R_{mijk}).$$

Recall that M^n is said to be pseudo-umbilical if the shape operator A_ν in the direction of the mean curvature vector field ν is proportional to the identity map id (i.e., $A_\nu = \lambda \cdot id$ for some function λ). In particular, M^n is called totally umbilical if its second fundamental form h and its mean curvature vector field ν satisfy

$$h(X,Y) = \langle X,Y\rangle\nu, \quad X,Y \in T_pM^n,\ p \in M^n. \tag{29}$$

We now assume M^n to be pseudo-umbilical. Equation (29) means that we can choose a suitable local orthonormal basis $\{e_i\}$ such that

$$h^{n+1}_{ij} = H\delta_{ij}, \tag{30}$$

which gives

$$h^{n+1}_{ijk} = H_{,k}\delta_{ij} = 0. \tag{31}$$

When tracing Equation (23) on i and k, the scalar curvature $\mathcal{R}$ of M^n takes the form

$$\mathcal{R} = n(n-1)(c+H^2) - 2(n-1)c|T|^2 - S_I. \tag{32}$$

The fundamental $(p\times p)$-matrix $(S_{\alpha\beta})$, defined by $S_{\alpha\beta} := \mathtt{Tr}(A_\alpha A_\beta)$, is symmetric. We thus find that for each $\alpha \neq n+1$, we have

$$S_{n+1,\alpha} = \mathtt{Tr}(A_{n+1}A_\alpha) = \sum_{i,j} H\delta_{ij}h^\alpha_{ij} = H\sum_i h^\alpha_{ii} = 0. \tag{33}$$

We fix $e_{n+1} = \nu/|\nu|$ and choose a suitable orthogonal basis $\{e_\alpha\}_{\alpha=n+2}^{n+p}$ such that the $(p\times p)$ matrix $(S_{\alpha\beta})$ is diagonal; in other words, we have

$$S_{\alpha\beta} = \begin{cases} S_\alpha, & \alpha = \beta; \\ 0, & \alpha \neq \beta. \end{cases}$$

By the definition of S_I, we conclude that

$$S_I = \sum_{\alpha\neq n+1} S_\alpha,$$

and therefore

$$\sum_{\alpha\neq n+1} S^2_\alpha \leq S^2_I. \tag{34}$$

According to Equations (19)–(21), (25)–(27) and (33), one computes

$$I_2 = nc\sum_{i,j}\Big(\sum_{\alpha\neq n+1} h^\alpha_{ij}\eta_\alpha\Big)^2 - nc\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2, \tag{35}$$
$$I_3 = -\sum_{\alpha,\beta\neq n+1}\Big[\mathtt{Tr}(A^2_\alpha A^2_\beta) - \mathtt{Tr}(A_\alpha A_\beta)^2\Big], \tag{36}$$
$$I_4 = \Big[n(c+H^2) - c|T|^2\Big]S_I - nc\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2 - \sum_{\alpha,\beta\neq n+1}\Big[\mathtt{Tr}(A^2_\alpha A^2_\beta) - \mathtt{Tr}(A_\alpha A_\beta)^2\Big] - \sum_{\alpha\neq n+1} S^2_\alpha. \tag{37}$$

Remark 3. *It should be emphasized that the formulas mentioned above in this section are closely associated with the similar computations obtained in the proof of Equation (4.3) in [30], where the authors used the method of an invariant operator.*

3. Several Key Estimates

For future purposes, in this section, we give several key inequalities which will play crucial roles in the proof of our rigidity theorems. At first, Lemma 1 is immediate from the Cauchy–Schwarz inequality.

Lemma 1. *The expression* $\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2$ *satisfies the following upper bound estimate:*

$$\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2=\sum_{\alpha\neq n+1}\sum_i\Big(\sum_j h_{ij}^\alpha T_j\Big)^2\leq|T|^2S_I. \tag{38}$$

Lemma 2 (c.f. [12])**.** *Let* K_0 *stand for the infimum of the sectional curvature of* M *at each* $x\in M$*. Then, we have a lower bound estimate of* I_4*:*

$$I_4=\sum_{\alpha\neq n+1}\sum_{i,j,k,m}h_{ij}^\alpha(h_{mi}^\alpha R_{mkjk}+h_{km}^\alpha R_{mijk})\geq nK_0S_I. \tag{39}$$

Proof. We fix each $\alpha\neq n+1$ and let $\lambda_1^\alpha,\lambda_2^\alpha,\cdots,\lambda_n^\alpha$ be the eigenvalues of A_α. The matrix (h_{ij}^α) can be diagonalized for a suitable choice of an orthonormal basis $\{\widetilde{e}_i\}$ at each point $x\in M$. It can easily be checked that

$$\begin{aligned}
&\sum_{i,j,k,m}h_{ij}^\alpha(h_{mi}^\alpha R_{mkjk}+h_{km}^\alpha R_{mijk})\\
=&\sum_{i,k}\lambda_i^\alpha(\lambda_i^\alpha R_{ikik}+\lambda_k^\alpha R_{kiik})\\
=&\frac{1}{2}\sum_{i,k}(\lambda_i^\alpha-\lambda_k^\alpha)^2R_{ikik}\\
\geq&\frac{1}{2}\sum_{i,k}(\lambda_i^\alpha-\lambda_k^\alpha)^2K_0\\
=&nK_0\sum_i(\lambda_i^\alpha)^2-K_0\sum_i\lambda_i^\alpha\sum_k\lambda_k^\alpha\\
=&nK_0\sum_{i,k}(h_{ik}^\alpha)^2.
\end{aligned} \tag{40}$$

By taking the sum over the index α in Equation (40), not including $n+1$, we can obtain the inequality in Equation (39). In fact, the above inequality (Equation (40)) is originally due to Yau's estimate (Equation (10.9)) in [12]. □

We end this section by recalling a well-known inequality from the results due to [9].

Lemma 3 (c.f. [9])**.** *We abbreviate* $\mathtt{Tr}(A_\alpha^2A_\beta^2)-\mathtt{Tr}(A_\alpha A_\beta)^2$ *to* $N_{\alpha\beta}$ *for* $\alpha,\beta\neq n+1$*. Furthermore, we set* $N_I=\sum_{\alpha,\beta\neq n+1}N_{\alpha\beta}$ *and* $L_I=\sum_{\alpha\neq n+1}S_\alpha^2$*. Then, we have*

$$2N_I+L_I\leq\Big[1+\frac{1}{2}\mathrm{sgn}(p-2)\Big]S_I^2, \tag{41}$$

where $\operatorname{sgn}(\cdot)$ *stands for the sign function. The equality holds if and only if at most two of the shape operators* $\{A_\alpha\}$ *are nonzero for* $n+2 \le \alpha \le n+p$. *And if we assume that* $A_{n+2} \neq 0$, $A_{n+3} \neq 0$ *and* $A_\alpha = 0$ *for any* $n+4 \le \alpha \le n+p$, *then there exists an orthogonal matrix* U *such that*

$$U^\top A_{n+2} U = \lambda \begin{pmatrix} J_{2\times 2} & (0)_{2\times r} \\ (0)_{r\times 2} & (0)_{r\times r} \end{pmatrix}, \quad U^\top A_{n+3} U = \lambda \begin{pmatrix} K_{2\times 2} & (0)_{2\times r} \\ (0)_{r\times 2} & (0)_{r\times r} \end{pmatrix},$$

where $r = n-2$, λ *is a nonzero constant and*

$$J_{2\times 2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4. Ricci Curvature Pinching Theorem

In this section, we consider the Ricci curvature pinching problem. Assume that Q_0 denotes the infimum of the Ricci curvature of M at every point $x \in M$. From Equations (24)–(26) and (30), one has

$$\sum_{\alpha \neq n+1} \sum_j \left(h_{ij}^\alpha\right)^2 \le (n-1)(c+H^2) - c|T|^2 - (n-2)cT_i^2 - Q_0. \tag{42}$$

Taking the sum over i in Equation (42) gives

$$S_I \le n(n-1)(c+H^2) - 2(n-1)c|T|^2 - nQ_0. \tag{43}$$

Since $(\lambda_i^\alpha - \lambda_k^\alpha)^2 \le 2\left[(\lambda_i^\alpha)^2 + (\lambda_k^\alpha)^2\right]$, where the equality holds if and only if $\lambda_i^\alpha = -\lambda_k^\alpha$, we obtain

$$\begin{aligned} N_{\alpha\beta} &= \sum_{i,k} (\lambda_i^\alpha \lambda_i^\alpha h_{ik}^\beta h_{ki}^\beta - \lambda_i^\alpha h_{ik}^\beta \lambda_k^\alpha h_{ki}^\beta) \\ &= \frac{1}{2} \sum_{\substack{i,k \\ i\neq k}} (\lambda_i^\alpha - \lambda_k^\alpha)^2 (h_{ik}^\beta)^2 \\ &\le \sum_{\substack{i,k \\ i\neq k}} \left[(\lambda_i^\alpha)^2 + (\lambda_k^\alpha)^2\right] (h_{ik}^\beta)^2. \end{aligned} \tag{44}$$

This implies in particular that

$$N_{\alpha\beta} \le \sum_{i,k} \left[(\lambda_i^\alpha)^2 + (\lambda_k^\alpha)^2\right] (h_{ik}^\beta)^2 = 2 \sum_i \left((\lambda_i^\alpha)^2 \sum_k (h_{ik}^\beta)^2 \right). \tag{45}$$

Combining Equations (45) and (42) gives

$$\begin{aligned} N_I = \sum_{\alpha,\beta \neq n+1} N_{\alpha\beta} &\le 2 \sum_{\alpha \neq n+1} \sum_i \left((\lambda_i^\alpha)^2 \sum_{\beta \neq n+1} \sum_j (h_{ij}^\beta)^2 \right) \\ &\le 2 \sum_{\alpha \neq n+1} \sum_i (\lambda_i^\alpha)^2 \left((n-1)(c+H^2) - c|T|^2 - (n-2)cT_i^2 - Q_0 \right) \\ &= 2\left((n-1)(c+H^2) - c|T|^2 - Q_0 \right) S_I - 2(n-2)c \sum_{\alpha \neq n+1} \sum_i T_i^2 (\lambda_i^\alpha)^2 \\ &\le 2\left((n-1)(c+H^2) - c|T|^2 - Q_0 \right) S_I. \end{aligned} \tag{46}$$

On the other hand, with Equation (44), one obtains

$$N_{\alpha\beta} \le \sum_{\substack{i,k \\ i\neq k}} \left[(\lambda_i^\alpha)^2 + (\lambda_k^\alpha)^2\right] (h_{ik}^\beta)^2 \le \sum_j (\lambda_j^\alpha)^2 \sum_{i,k} (h_{ik}^\beta)^2 = S_\alpha S_\beta,$$

from which we have

$$N_I = \sum_{\alpha,\beta\neq n+1} N_{\alpha\beta} \leq \sum_{\alpha,\beta\neq n+1} S_\alpha S_\beta = S_I^2 - \sum_{\alpha\neq n+1} S_\alpha^2 = S_I^2 - L_I. \tag{47}$$

With these preparations, we shall now prove the following pinching theorem for the Ricci curvature:

Theorem 1. *Let M^n be a compact pseudo-umbilical PMC submanifold immersed in a product space $\mathbb{S}^m(c)\times\mathbb{R}$ $(n\geq 3, p\geq 2)$. If the infimum of the Ricci curvature of M^n satisfies $Q_0 \geq D_1$, where D_1 is defined as Equation (51), then $|T| = 0$, and M^n is a totally umbilical sphere $\mathbb{S}^n(c+H^2)$ in a slice $\mathbb{S}^m(c)$.*

Proof. According to Equations (35)–(37), a direct computation shows that

$$\begin{aligned}\frac{1}{2}\Delta S_I =& I_1 + I_2 + I_3 + I_4\\ =& I_1 + \left[n(c+H^2) - c|T|^2\right]S_I + nc\sum_{i,j}\Big(\sum_{\alpha\neq n+1} h_{ij}^\alpha \eta_\alpha\Big)^2\\ & - 2nc\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2 - 2N_I - L_I\\ \geq& I_1 + \left[n(c+H^2) - c|T|^2\right]S_I - 2nc\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2 - N_I - (N_I + L_I).\end{aligned} \tag{48}$$

By applying Equations (38) and (43)–(48), it is straightforward to show that

$$\begin{aligned}\frac{1}{2}\Delta S_I \geq& I_1 + \left[n(c+H^2) - c|T|^2\right]S_I - 2nc|T|^2 S_I\\ & - 2\Big((n-1)(c+H^2) - c|T|^2 - Q_0\Big)S_I - S_I^2\\ \geq& I_1 + \left[n(c+H^2) - c|T|^2\right]S_I - 2nc|T|^2 S_I\\ & - 2\Big((n-1)(c+H^2) - c|T|^2 - Q_0\Big)S_I\\ & - \Big(n(n-1)(c+H^2) - 2(n-1)c|T|^2 - nQ_0\Big)S_I\\ =& I_1 + (n+2)\Big(Q_0 - \frac{n^2-2}{n+2}(c+H^2) - \frac{1}{n+2}c|T|^2\Big)S_I.\end{aligned} \tag{49}$$

On the other hand, applying Equations (38), (41) and (43) to Equation (48) gives rise to

$$\begin{aligned}\frac{1}{2}\Delta S_I \geq& I_1 + \left[n(c+H^2) - c|T|^2\right]S_I - 2nc|T|^2 S_I - \Big[1+\frac{1}{2}\mathrm{sgn}(p-2)\Big]S_I^2\\ \geq& I_1 + \left[n(c+H^2) - (2n+1)c|T|^2\right]S_I\\ & - \Big[1+\frac{1}{2}\mathrm{sgn}(p-2)\Big]\Big(n(n-1)(c+H^2) - 2(n-1)c|T|^2 - nQ_0\Big)S_I\\ =& I_1 + n\Big[1+\frac{1}{2}\mathrm{sgn}(p-2)\Big]\Big[Q_0 - \frac{2(n-2)+(n-1)\mathrm{sgn}(p-2)}{2+\mathrm{sgn}(p-2)}(c+H^2)\\ & - \frac{6-2(n-1)\mathrm{sgn}(p-2)}{n\left[2+\mathrm{sgn}(p-2)\right]}c|T|^2\Big]S_I.\end{aligned} \tag{50}$$

We set

$$\begin{aligned}D_1 = \min\Big\{& \frac{n^2-2}{n+2}(c+H^2) + \frac{1}{n+2}c|T|^2,\\ & \frac{2(n-2)+(n-1)\mathrm{sgn}(p-2)}{2+\mathrm{sgn}(p-2)}(c+H^2)\end{aligned}$$

$$+\frac{6-2(n-1)\mathrm{sgn}(p-2)}{n\big[2+\mathrm{sgn}(p-2)\big]}c|T|^2\Big\}. \tag{51}$$

Provided that $Q_0 \geq D_1$, taking integration over M^n on Equations (49) and (50), and using the compactness of M^n, we are led to the conclusion that $\Delta S_I = 0$ and

$$I_1 = \sum_{\alpha\neq n+1}\sum_{i,j,k}(h^\alpha_{ijk})^2 = 0. \tag{52}$$

Taking into consideration Equations (31) and (52), we find $h^\alpha_{ijk} = 0$ for any α, i, j, k. It follows from Remark 1 that $|T| = 0$, and M^n is a compact pseudo-umbilical PMC submanifold in $\mathbb{S}^m(c)$. Under Theorem 1 in [32] (see also [33]), we find that M^n is either a sphere $\mathbb{S}^{m-1}(c+H^2)$ or a minimal submanifold immersed in $\mathbb{S}^{m-1}(c+H^2)$ with the co-dimension $\widetilde{p} = p-2$. Moreover, we see that

$$S_I = \sum_{\alpha\neq n+1}\sum_{i,j}(h^\alpha_{ij})^2 = 0,$$

or

$$Q_0 = D_1 = \min\Big\{\frac{n^2-2}{n+2}(c+H^2), \frac{2(n-2)+(n-1)\mathrm{sgn}(p-2)}{2+\mathrm{sgn}(p-2)}(c+H^2)\Big\}.$$

If $S_I = 0$ or $p = 2$, then M^n is a totally umbilical sphere $S^n(c+H^2)$ in $\mathbb{S}^m(c)$.

If $S_I > 0$ and $p \geq 3$, then the Ricci curvature of M^n is identically equal to D_1. Now, we also observe that

$$\begin{aligned} D_1 =& \min\Big\{\frac{n^2-2}{n+2}(c+H^2), \frac{3n-5}{3}(c+H^2)\Big\} \\ =& \begin{cases} \frac{4}{3}(c+H^2), & \text{if } n=3; \\ \frac{n^2-2}{n+2}(c+H^2), & \text{if } n\geq 4. \end{cases} \end{aligned}$$

It can easily be seen that the Ricci curvature satisfies

$$R_{ik} = D_1 > (n-2)(c+H^2)$$

for any $n \geq 3$. This, combined with the results of Ejiri [15] and Li [17], implies that M^n must be a totally geodesic sphere $S^n(c+H^2)$ in $\mathbb{S}^{m-1}(c+H^2)$, which contradicts the assumption that $S_I > 0$. We have thus proven Theorem 1. □

5. Sectional Curvature Pinching Theorem

Now, we shall state and prove the following sectional curvature pinching theorem:

Theorem 2. *Let M^n be a compact pseudo-umbilical PMC submanifold immersed in a product space $\mathbb{S}^m(c)\times\mathbb{R}$ $(n\geq 3, p\geq 2)$. If the infimum of the sectional curvature of M^n satisfies $K_0 \geq D_2$, where D_2 is defined as Equation (60), then $|T| = 0$ (i.e., M^n lies in a slice $\mathbb{S}^m(c)$). Furthermore, M^n is locally either a totally umbilical sphere $\mathbb{S}^n(c+H^2)$ in $\mathbb{S}^m(c)$ or a Veronese manifold with a constant sectional curvature $\frac{n}{2(n+1)}(c+H^2)$.*

Proof. By substituting Equations (35)–(37) into Equation (28), a direct computation shows that

$$\begin{aligned} \frac{1}{2}\Delta S_I =& I_1 + I_2 + I_3 + (1+\frac{n}{n+2})I_4 - \frac{n}{n+2}I_4 \\ =& I_1 + \frac{2(n+1)}{n+2}I_4 - \frac{n}{n+2}\big[n(c+H^2) - c|T|^2\big]S_I + nc\sum_{i,j}\Big(\sum_{\alpha\neq n+1}h^\alpha_{ij}\eta_\alpha\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&-\frac{2nc}{n+2}\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2+\frac{1}{n+2}(nL_I-2N_I)\\
\geq& I_1+\frac{2(n+1)}{n+2}I_4-\frac{n}{n+2}\big[n(c+H^2)-c|T|^2\big]S_I\\
&-\frac{2nc}{n+2}\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2+\frac{1}{n+2}(nL_I-2N_I).
\end{aligned}\tag{53}
$$

In light of Equations (38) and (39), we can derive through Equation (53) that

$$
\begin{aligned}
\frac{1}{2}\Delta S_I \geq& I_1+\frac{2n(n+1)}{n+2}K_0S_I-\frac{n}{n+2}\big[n(c+H^2)-c|T|^2\big]S_I\\
&-\frac{2nc}{n+2}|T|^2S_I+\frac{1}{n+2}(nL_I-2N_I)\\
\geq& I_1+\frac{2n(n+1)}{n+2}\Big(K_0-\frac{n}{2(n+1)}(c+H^2)-\frac{c|T|^2}{2(n+1)}\Big)S_I,
\end{aligned}\tag{54}
$$

where the last inequality is followed by

$$N_I\leq\frac{n}{2}L_I,$$

which is taken from T. Itoh (see [34], Proposition 1).

Likewise, by substituting Equations (35)–(37) into Equation (28) again, we have

$$
\begin{aligned}
\frac{1}{2}\Delta S_I =& I_1+I_2+I_3+(1+\frac{p-2}{p-1})I_4-\frac{p-2}{p-1}I_4\\
=& I_1+\frac{2p-3}{p-1}I_4-\frac{p-2}{p-1}\big[n(c+H^2)-c|T|^2\big]S_I+nc\sum_{i,j}\Big(\sum_{\alpha\neq n+1}h^\alpha_{ij}\eta_\alpha\Big)^2\\
&-\frac{nc}{p-1}\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2+\frac{1}{p-1}\Big((p-2)L_I-N_I\Big)\\
\geq& I_1+\frac{2p-3}{p-1}I_4-\frac{p-2}{p-1}\big[n(c+H^2)-c|T|^2\big]S_I\\
&-\frac{nc}{p-1}\sum_{\alpha\neq n+1}|\nabla\eta_\alpha|^2+\frac{1}{p-1}\Big((p-2)L_I-N_I\Big).
\end{aligned}\tag{55}
$$

Applying Equations (38)–(39) to Equation (55) implies that

$$
\begin{aligned}
\frac{1}{2}\Delta S_I \geq& I_1+\frac{2p-3}{p-1}nK_0S_I-\frac{p-2}{p-1}\big[n(c+H^2)-c|T|^2\big]S_I\\
&-\frac{nc}{p-1}|T|^2S_I+\frac{1}{p-1}\Big((p-2)L_I-N_I\Big)\\
=& I_1+\frac{2p-3}{p-1}n\Big(K_0-\frac{p-2}{2p-3}(c+H^2)-\frac{n+2-p}{n(2p-3)}c|T|^2\Big)S_I\\
&+\frac{1}{p-1}\Big((p-2)L_I-N_I\Big).
\end{aligned}\tag{56}
$$

With the Cauchy–Schwarz inequality, we obtain

$$S_I^2=\Big(\sum_{\alpha\neq n+1}1\cdot S_\alpha\Big)^2\leq\sum_{\alpha\neq n+1}1^2\cdot\sum_{\alpha\neq n+1}S_\alpha^2=(p-1)\sum_{\alpha\neq n+1}S_\alpha^2=(p-1)L_I,\tag{57}$$

where the equality holds if and only if one of $\{S_\alpha\}_{\alpha=n+2}^{n+p}$ is nonzero or

$$S_{n+2}=S_{n+3}=\cdots=S_{n+p}.$$

Also, it follows from Equations (47) and (57) that

$$(p-2)L_I - N_I \geq (p-2)L_I - (S_I^2 - L_I) = (p-1)L_I - S_I^2 \geq 0. \tag{58}$$

Applying Equation (58) to Equation (56) shows that

$$\frac{1}{2}\Delta S_I \geq I_1 + \frac{2p-3}{p-1}n\Big(K_0 - \frac{p-2}{2p-3}(c+H^2) - \frac{n+2-p}{n(2p-3)}c|T|^2\Big)S_I. \tag{59}$$

We set

$$\begin{aligned} D_2 = \min\Big\{&\frac{n}{2(n+1)}(c+H^2) + \frac{c|T|^2}{2(n+1)}, \\ &\frac{p-2}{2p-3}(c+H^2) + \frac{n+2-p}{n(2p-3)}c|T|^2\Big\}. \end{aligned} \tag{60}$$

Provided that $K_0 \geq D_2$, by integrating Equations (54) and (59) and using the compactness of M^n, we therefore conclude that $\Delta S_I = 0$ and

$$I_1 = \sum_{\alpha \neq n+1}\sum_{i,j,k}(h_{ijk}^{\alpha})^2 = 0.$$

An argument similar to that used in proof of Theorem 1 shows that $|T| = 0$, and M^n is either a sphere $\mathbb{S}^{m-1}(c+H^2)$ or a minimal submanifold immersed in $\mathbb{S}^{m-1}(c+H^2)$ with the co-dimension $\widetilde{p} = p-2$. Moreover, we see that

$$S_I = \sum_{\alpha \neq n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 = 0,$$

or

$$K_0 = D_2 = \min\Big\{\frac{n}{2(n+1)}(c+H^2), \frac{\widetilde{p}}{2\widetilde{p}+1}(c+H^2)\Big\}.$$

For the former case, M^n is a totally umbilical sphere $S^n(c+H^2)$ in $\mathbb{S}^m(c)$. Hereafter, we consider the latter case, where the sectional curvature R_{ikik} is identically equal to D_2.

Case A. If $2\widetilde{p} \geq n$, then

$$D_2 = \frac{n}{2(n+1)}(c+H^2),$$

Therefore, M^n is a Veronese manifold with a constant sectional curvature $\frac{n}{2(n+1)}(c+H^2)$ and $\widetilde{p} = \frac{1}{2}(n-1)(n+2)$ (see Example 2 in [8] and Main Theorem in [13]). Here, we assume that the immersion

$$\varphi_1 : M^n \longrightarrow \mathbb{S}^{m-1}(c+H^2)$$

is full. In particular, for $n = \widetilde{p} = 2$, we recover the Veronese surface.

Case B. If $2\widetilde{p} < n$, then

$$D_2 = \frac{\widetilde{p}}{2\widetilde{p}+1}(c+H^2),$$

Therefore, the equality holds in Equation (58), and so does Equation (57).

Suppose that $S_{n+2} = S_{n+3} = \cdots = S_{n+p} = 0$. It is easy to see that M^n is a totally geodesic sphere $S^n(c+H^2)$ in $\mathbb{S}^{m-1}(c+H^2)$.

Suppose that $S_{n+2} = S_{n+3} = \cdots = S_{n+p} \neq 0$. Note that M^n is contained in a slice $\mathbb{S}^m(c)$, and we thus find that $\partial_t = \eta$ is perpendicular to the mean curvature vector field ν. We can certainly assume that $e_{n+2} = \eta$. Since the equality holds in Equation (53), we obtain $(h_{ij}^{n+2})^2 = 0$ for arbitrary $1 \leq i, j \leq n$, namely $S_{n+2} = 0$, which is a contradiction.

We now turn our attention to the case where $S_{n+2} \neq 0$ and $S_{n+3} = \cdots = S_{n+p} = 0$. Therefore, we assume that M^n is a minimal immersed hypersurface of $\mathbb{S}^{n+1}(c+H^2)$ with the shape operator $\widetilde{A} = diag(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \cdots, \widetilde{\lambda}_n)$. According to the Gauss equation, it follows that

$$\frac{\widetilde{p}}{2\widetilde{p}+1}(c+H^2) = (c+H^2) + \widetilde{\lambda}_i\widetilde{\lambda}_k \tag{61}$$

for any $i \neq k$. Hence, $\widetilde{\lambda}_1 = \widetilde{\lambda}_2 = \cdots = \widetilde{\lambda}_n$, which is due to the fact that $n \geq 3$. From Equation (61), one easily finds that for any i,

$$\widetilde{\lambda}_i^2 = -\frac{\widetilde{p}+1}{2\widetilde{p}+1}(c+H^2) < 0,$$

which is impossible. This finishes the proof of Theorem 2. □

6. Scalar Curvature Pinching Theorem

Upon substituting Equations (38) and (41) into Equation (48), one sees immediately that

$$\begin{aligned}\frac{1}{2}\Delta S_I \geq & I_1 + \left[n(c+H^2) - c|T|^2\right]S_I - 2nc|T|^2S_I - \left[1+\frac{1}{2}\mathrm{sgn}(p-2)\right]S_I^2 \\ = & I_1 + \left\{n(c+H^2) - (2n+1)c|T|^2 - \left[1+\frac{1}{2}\mathrm{sgn}(p-2)\right]S_I\right\}S_I. \end{aligned} \tag{62}$$

By the same argument as that in [8,9], we can now establish the desired pinching theorem for S_I.

Theorem 3. *Let M^n be a compact pseudo-umbilical PMC submanifold immersed in a product space $\mathbb{S}^m(c) \times \mathbb{R}$ $(n \geq 3, p \geq 2)$. If $S_I \leq D_3$ everywhere on M^n, then $|T| = 0$ (i.e., M^n lies in a slice $\mathbb{S}^m(c)$). Here, D_3 is given by*

$$D_3 = \frac{1}{1+\frac{1}{2}\mathrm{sgn}(p-2)}\{n(c+H^2) - (2n+1)c|T|^2\}.$$

Furthermore, one of the following assertions holds:

(i) *M^n is a sphere $\mathbb{S}^n(c+H^2)$;*

(ii) *M^n is a Clifford minimal hypersurface $M_{k,m-2-k}$ contained in $\mathbb{S}^{m-1}(c+H^2)$ with $1 \leq k \leq m-3$, where the Clifford minimal hypersurface $M_{k,m-2-k}$ is given by*

$$M_{k,m-2-k} = S^k\Big(\frac{(m-2)(c+H^2)}{k}\Big) \times S^{m-2-k}\Big(\frac{(m-2)(c+H^2)}{m-2-k}\Big);$$

(iii) *M^n is a Veronese surface M^2 in $\mathbb{S}^4(c+H^2)$.*

In light of Equation (32), we point out that if $S_I \leq D_3$, then $\mathcal{R} \geq D_4$, where

$$\begin{aligned} D_4 = & n(n-1)(c+H^2) - 2(n-1)c|T|^2 - D_3 \\ = & \frac{2(n-2)+(n-1)\mathrm{sgn}(p-2)}{2+\mathrm{sgn}(p-2)}n(c+H^2) \\ & + \frac{6-2(n-1)\mathrm{sgn}(p-2)}{2+\mathrm{sgn}(p-2)}c|T|^2. \end{aligned}$$

Consequently, Theorem 3 can be formulated equivalently in terms of the scalar curvature $\mathcal{R}$ of M^n as follows:

Theorem 4. *Let M^n be a compact pseudo-umbilical PMC submanifold immersed in a product space $\mathbb{S}^m(c) \times \mathbb{R}$ $(n \geq 3, p \geq 2)$. If $\mathcal{R} \geq D_4$ everywhere on M^n, then $|T| = 0$, and M^n is a sphere $\mathbb{S}^n(c + H^2)$, a Clifford minimal hypersurface $M_{k,m-2-k}$ contained in $\mathbb{S}^{m-1}(c + H^2)$ or a Veronese surface M^2 in $\mathbb{S}^4(c + H^2)$.*

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