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Multiplicity Results of Solutions to the Double Phase Problems of Schrödinger–Kirchhoff Type with Concave–Convex Nonlinearities

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Abstract: The present paper is devoted to establishing several existence results for infinitely many solutions to Schrödinger–Kirchhoff-type double phase problems with concave–convex nonlinearities. The first aim is to demonstrate the existence of a sequence of infinitely many large-energy solutions by applying the fountain theorem as the main tool. The second aim is to obtain that our problem admits a sequence of infinitely many small-energy solutions. To obtain these results, we utilize the dual fountain theorem. In addition, we prove the existence of a sequence of infinitely many weak solutions converging to 0 in L^∞-space. To derive this result, we exploit the dual fountain theorem and the modified functional method.

Keywords: Kirchhoff function; double phase problems; Musielak–Orlicz–Sobolev spaces; multiple solutions; variational methods

MSC: 35B38; 35D30; 35J10; 35J20; 35J62

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1. Introduction

In this paper, we demonstrate the existence of multiple solutions for the following double phase problem in $\mathbb{R}^N$:

$$-M\left(\int_{\mathbb{R}^N}\frac{1}{p}|\nabla w|^p+\frac{\nu(y)}{q}|\nabla w|^q\,dy\right)\operatorname{div}(|\nabla w|^{p-2}\nabla w+\nu(y)|\nabla w|^{q-2}\nabla w)$$
$$+\mathfrak{V}(y)(|w|^{p-2}w+\nu(y)|w|^{q-2}w)=\sigma(y)|w|^{r-2}w+\theta g(y,w)\quad\text{in }\mathbb{R}^N,\quad(1)$$

where $N\geq 2$, $1<p<q<N$, $1<r<p$, θ is a positive real parameter, $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function,

$$\frac{q}{p}\leq 1+\frac{1}{N},\quad \nu:\mathbb{R}^N\to[0,\infty)\text{ is Lipschitz continuous,}$$

and $\mathfrak{V}:\mathbb{R}^N\to(0,\infty)$ is a potential function satisfying

(V) $\mathfrak{V}\in C(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{y\in\mathbb{R}^N}\mathfrak{V}(y)>0$, and $\operatorname{meas}\{y\in\mathbb{R}^N:\mathfrak{V}(y)\leq\mathcal{V}_0\}<+\infty$, for all $\mathcal{V}_0\in\mathbb{R}$.

Furthermore, let us assume that a Kirchhoff function $M:\mathbb{R}_0^+\to\mathbb{R}^+$ satisfies the following conditions:

(M1) $M\in C(\mathbb{R}^+)$ fulfills $\inf_{\zeta\in\mathbb{R}^+}M(\zeta)\geq\kappa_0>0$, where κ_0 is a constant;

(M2) There exists a constant $\vartheta\geq 1$ such that $\vartheta\mathcal{M}(\zeta)=\vartheta\int_0^\zeta M(\tau)d\tau\geq M(\zeta)\zeta$ for $\zeta\geq 0$.

The double phase operator, which is the natural generalization of the p-Laplace operator, has been studied extensively by many researchers. The research interest in differential equations and variational problems with double phase operators can be regarded as a key factor in diverse fields of mathematical physics, such as strongly anisotropic materials, the Lavrentiev phenomenon, plasma physics, biophysics, chemical reactions, etc.; for more information, see [1,2]. In relation to regularity theory for double phase functionals, there is a series of remarkable papers by Mingione et al. [3–8]. Eigenvalue problems for a class of double phase variational integrals driven by Dirichlet double phase operators have been dealt with [9]. A study on a remarkable existence result of solutions to quasilinear equations involving a general variable exponent elliptic operator was investigated in the recent work by Zhang and Radulescu [10]. Recently, the authors in [11] provided a new class of double phase operators with variable exponents. As its application, they gave the existence and uniqueness results for quasilinear elliptic equations with a convection term. Other existence results for double phase problems can be found in the papers [12,13].

The study of elliptic problems with the non-local Kirchhoff term was initially introduced by Kirchhoff [14] in order to study an extension of the classical d'Alembert's wave equation by taking into account the changes to the lengths of strings during vibration. The variational problems of the Kirchhoff type have had influence in various applications in physics and have been intensively investigated by many researchers in recent years; for examples, see [15–28] and the references therein. A detailed discussion about the physical implications based on the fractional Kirchhoff model was initially suggested by the work of Fiscella and Valdinoci [20]. They derived the existence of non-trivial solutions by taking advantage of the mountain-pass theorem and a truncation argument on a non-local Kirchhoff term. In particular, the conditions imposed on the non-degenerated Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ were that M is an increasing and continuous function with (M1); also, see [24] and references therein. However, this increasing condition eliminated the case that is not monotone; for example,

$$M(\zeta) = (1+\zeta)^k + (1+\zeta)^{-1} \text{ with } 0 < k < 1$$

for all $\zeta \in \mathbb{R}_0^+$. In this regard, the existence of multiple solutions to a class of Schrödinger–Kirchhoff-type equations involving the fractional p-Laplacian was provided by reference [25] when the Kirchhoff function M is continuous and satisfies (M1) and the condition:

(M3) For $0 < s < 1$, there is $\vartheta \in [1, \frac{N}{N-sp})$ such that $\vartheta \mathcal{M}(\zeta) \geq M(\zeta)\zeta$ for any $\zeta \geq 0$.

We also referred to [15,16,25–29] for recent results.

Recently, the authors of [22] studied the existence result of a positive ground-state solution for an elliptic problem of the Kirchhoff type with critical exponential growth under the following condition:

(M4) There exists $\vartheta > 1$ such that $\frac{M(\zeta)}{\zeta^{\vartheta-1}}$ is non-increasing for $\zeta > 0$.

From this condition and direct computation, we immediately recognize that $\vartheta\mathcal{M}(\zeta) - M(\zeta)\zeta$ is non-decreasing for all $\zeta \geq 0$, and thus, this implies the condition (M2). A typical model for the Kirchhoff function M satisfying (M2) is given by $M(\zeta) = 1 + a\zeta^{\vartheta}$, with $a \geq 0$ for all $\zeta \geq 0$. Hence, the condition (M2) includes this classical example as well as cases that are not monotone. Under this condition, the authors of [18] obtained multiplicity results for certain classes of double phase problems of the Kirchhoff type with nonlinear boundary conditions; also, see [19] for the Dirichlet boundary condition. For these reasons, the nonlinear elliptic equations with a Kirchhoff coefficient satisfying (M2) have been comprehensively investigated by many researchers in recent years [15,17–19,21,25,27,28].

The main aim of the present paper is to provide several multiplicity results of solutions for Schrödinger–Kirchhoff-type problems involving a double phase operator for the combined effect of concave–convex nonlinearities. In this paper, we first discuss that Problem (1) has infinitely many large-energy solutions. Second, we demonstrate the existence of a sequence of infinitely many small-energy solutions. Finally, we provide the existence of a sequence of infinitely many weak solutions converging to 0 in L^∞-space. To derive such results, we exploit the fountain theorem, the dual fountain theorem, and the modified functional method as the main tools. The present paper is motivated by recent work in [30,31]. Moreover, the authors of [30] obtained multiplicity results to the double phase problem as follows:

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \nu(y)|\nabla u|^{q-2}\nabla u) + \mathfrak{V}(y)(|u|^{p-2}u + \nu(y)|u|^{q-2}u)$$
$$= \lambda\sigma(y)|w|^{r-2}w + g(y,u) \text{ in } \mathbb{R}^N,$$

where $\mathfrak{V} : \mathbb{R}^N \to (0,\infty)$ is a potential function satisfying (V) and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfills the Carathéodory condition. In particular, in the work [30], the authors obtained the existence of a sequence of small-energy solutions under specific conditions of the nonlinear term that were different from those in previous studies [23,32–37]. More precisely, in view of [32–35], the conditions of the nonlinear term g near zero as well as at infinity were decisive for proving the hypotheses in the dual-fountain theorem. However, the authors also ensured the hypotheses when the behavior at infinity was not assumed, and the condition near zero—namely, $g(y,\zeta) = o(|\zeta|^{p-2}\zeta)$ as $|\zeta| \to 0$ uniformly for all $y \in \mathbb{R}^N$—was replaced by (G4), which is discussed in Section 2. Although this study is inspired by [30,31], the presence of the non-local Kirchhoff coefficient M required more complicated analyses that had to be performed meticulously. In particular, one of the key ingredients to obtain this multiplicity result in [30,31] is that the potential function $V \in C(\mathbb{R}^N, (0,\infty))$ is coercive: that is, $\lim_{|x|\to\infty} V(x) = +\infty$, which is crucial to guarantee the compactness condition of the Palais–Smale type. However, in order to prove this condition, we employ a weaker condition (V) than the coercivity of the function V. Therefore, in this study, we develop a multiplicity result for double phase problems of the Kirchhoff type under various conditions on the convex term g.

Our multiplicity result of infinitely many small-energy solutions converging to 0 in L^∞-space is motivated by [38–42]. However, in contrast to [38,41,42], we utilize the dual-fountain theorem instead of the global variational formulation in [43]. This multiplicity result yielding small-energy solutions for variational elliptic equations based on the dual fountain theorem does not guarantee the boundedness of the solutions. For this reason, the authors of [39,40] combined the modified functional method with the dual-fountain theorem in order to demonstrate the existence of multiple small-energy solutions converging to zero in L^∞-space. In this direction, our final result is based on recent research [39,40]. However, our approach differs from [40] when validating a condition in the dual fountain theorem, as shown in the Section 4. Furthermore, we have to carry out more complicated analyses than those in [39]: not only because our problem has the Kirchhoff coefficient M but also because the given domain is the whole space $\mathbb{R}^N$.

The outline of this paper is as follows. We present necessary preliminary knowledge of function spaces for the present paper. Next, we provide the variational framework related to problem (1), and then we establish various existence results of infinitely many nontrivial solutions to the Kirchhoff-type double phase equations with concave–convex-type nonlinearities under certain conditions on g.

2. Preliminaries

In this section, we briefly discuss the definitions and the essential properties of Musielak–Orlicz–Sobolev space. For more in-depth examinations of these spaces, we refer to [9,44–46].

The functions $\mathcal{H} : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ and $\mathcal{H}_{\mathfrak{V}} : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ are defined as follows:

$$\mathcal{H}(y,\zeta) := \zeta^p + \nu(y)\zeta^q, \quad \mathcal{H}_{\mathfrak{V}}(y,\zeta) := \mathfrak{V}(y)(\zeta^p + \nu(y)\zeta^q) \tag{2}$$

For almost all $y \in \mathbb{R}^N$ and for any $\zeta \in [0,\infty)$ with $1 < p < q$,

$$\frac{q}{p} \le 1 + \frac{1}{N}, \quad \nu : \mathbb{R}^N \to [0,\infty) \text{ is Lipschitz continuous,}$$

and $\mathfrak{V} : \mathbb{R}^N \to \mathbb{R}$ is a function satisfying (V).

We define the Musielak–Orlicz space $L^{\mathcal{H}}(\mathbb{R}^N)$ as

$$L^{\mathcal{H}}(\mathbb{R}^N) := \Big\{ v : \mathbb{R}^N \to \mathbb{R} \text{ is measurable} : \ \varsigma_{\mathcal{H}}(v) < \infty \Big\},$$

induced by the Luxemburg norm

$$\|v\|_{\mathcal{H}} := \inf\Big\{ \lambda > 0 : \ \varsigma_{\mathcal{H}}\Big(y, \Big|\frac{v}{\lambda}\Big|\Big) \le 1 \Big\},$$

where $\varsigma_{\mathcal{H}}$ denotes the $\mathcal{H}$-modular function with

$$\varsigma_{\mathcal{H}}(v) := \int_{\mathbb{R}^N} \mathcal{H}(y, |v|) dy. \tag{3}$$

If we replace $\mathcal{H}$ with $\mathcal{H}_{\mathfrak{V}}$, we obtain the definition of the Musielak–Orlicz space $(L_{\mathcal{H}_{\mathfrak{V}}}(\mathbb{R}^N), \| \cdot \|_{\mathcal{H}_{\mathfrak{V}}})$, i.e.,

$$L_{\mathcal{H}_{\mathfrak{V}}}(\mathbb{R}^N) := \Big\{ v : \mathbb{R}^N \to \mathbb{R} \text{ is measurable} : \ \varsigma_{\mathfrak{V}}^{\mathcal{H}}(v) < \infty \Big\},$$

induced by the Luxemburg norm

$$\|v\|_{\mathcal{H}_{\mathfrak{V}}} := \inf\Big\{ \lambda > 0 : \ \varsigma_{\mathcal{V}}^{\mathcal{H}}\Big(y, \Big|\frac{v}{\lambda}\Big|\Big) \le 1 \Big\},$$

where $\varsigma_{\mathfrak{V}}^{\mathcal{H}}$ denotes the $\mathcal{H}_{\mathfrak{V}}$-modular function as

$$\varsigma_{\mathfrak{V}}^{\mathcal{H}}(v) := \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |v|) dy. \tag{4}$$

According to [45,47], the spaces $L^{\mathcal{H}}(\mathbb{R}^N)$ and $L_{\mathcal{H}_{\mathfrak{V}}}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Lemma 1 ([47])**.** *For $\varsigma_{\mathfrak{V}}^{\mathcal{H}}(v)$ given in (4) and $v \in L_{\mathcal{H}_{\mathfrak{V}}}(\mathbb{R}^N)$, we have:*
(i) for $v \ne 0, \|v\|_{\mathcal{H}_{\mathfrak{V}}} = \lambda$ iff $\varsigma_{\mathfrak{V}}^{\mathcal{H}}(\frac{v}{\lambda}) = 1$;
(ii) $\|v\|_{\mathcal{H}_{\mathfrak{V}}} < 1 (= 1; > 1)$ iff $\varsigma_{\mathfrak{V}}^{\mathcal{H}}(v) < 1 (= 1; > 1)$;
(iii) if $\|v\|_{\mathcal{H}_{\mathfrak{V}}} > 1$, then $\|v\|_{\mathcal{H}_{\mathfrak{V}}}^p \le \varsigma_{\mathfrak{V}}^{\mathcal{H}}(v) \le \|v\|_{\mathcal{H}_{\mathfrak{V}}}^q$;
(iv) if $\|v\|_{\mathcal{H}_{\mathfrak{V}}} < 1$, then $\|v\|_{\mathcal{H}_{\mathfrak{V}}}^q \le \varsigma_{\mathfrak{V}}^{\mathcal{H}}(v) \le \|v\|_{\mathcal{H}_{\mathfrak{V}}}^p$.
Furthermore, analogous results hold for $\varsigma_{\mathcal{H}}(u)$, given in (3), and $\| \cdot \|_{\mathcal{H}}$.

The weighted Musielak–Orlicz–Sobolev space $W_{\mathfrak{V}}^{1,\mathcal{H}}(\mathbb{R}^N)$ is defined by

$$W_{\mathfrak{V}}^{1,\mathcal{H}}(\mathbb{R}^N) = \{ v \in L_{\mathcal{H}_{\mathfrak{V}}}(\mathbb{R}^N) : |\nabla v| \in L^{\mathcal{H}}(\mathbb{R}^N) \}.$$

Then, it is provided with the following norm:

$$\|v\| = \|\nabla v\|_{\mathcal{H}} + \|v\|_{\mathcal{H}_{\mathfrak{V}}}.$$

Note that $W_{\mathfrak{V}}^{1,\mathcal{H}}(\mathbb{R}^N)$ is a separable reflexive Banach space [45]. In the following calculations, the notation $E \hookrightarrow F$ indicates that space E is *continuously* embedded into space F, while $E \hookrightarrow\hookrightarrow F$ denotes that E is *compactly* embedded into F.

According to Lemma 1, we obtain the following results:

Lemma 2 ([47])**.** *The following embeddings hold:*
(i) $L_{\mathcal{H}_{\mathfrak{V}}}(\mathbb{R}^N) \hookrightarrow L^{\mathcal{H}}(\mathbb{R}^N)$;
(ii) $W_{\mathfrak{V}}^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^{\tau}(\mathbb{R}^N)$ for $\tau \in [p, p^*]$;

(iii) $W^{1,\mathcal{H}}_{\mathfrak{V}}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^\tau(\mathbb{R}^N)$ *for* $\tau \in [p, p^*)$.

Lemma 3 ([47]). *Let*

$$A(v) := \int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla v|)dy + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |v|)dy.$$

Then, the following properties hold:

(i) $A(v) \le \|v\|^p + \|v\|^q$ *for all* $v \in W^{1,\mathcal{H}}_{\mathfrak{V}}(\mathbb{R}^N)$;
(ii) *If* $\|v\| \le 1$, *then* $2^{1-q}\|v\|^q \le A(v) \le \|v\|^p$;
(iii) *If* $\|v\| \ge 1$, *then* $2^{-p}\|v\|^p \le A(v) \le 2\|v\|^q$.

Let us define the functional $\Phi : \mathfrak{E} := W^{1,\mathcal{H}}_{\mathfrak{V}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi(w) = \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w|)\, dy,$$

where the functions $\mathcal{H}_{p,q} : \mathbb{R}^N \times [0, \infty) \to [0, \infty)$ and $\mathcal{H}_{\mathfrak{V},p,q} : \mathbb{R}^N \times [0, \infty) \to [0, \infty)$ are defined as

$$\mathcal{H}_{p,q}(y, \zeta) := \frac{1}{p}\zeta^p + \frac{\nu(y)}{q}\zeta^q \quad \text{and} \quad \mathcal{H}_{\mathfrak{V},p,q}(y, \zeta) := \mathfrak{V}(y)\left(\frac{1}{p}\zeta^p + \frac{\nu(y)}{q}\zeta^q\right).$$

Then, it is standard to check that $\Phi \in C^1(\mathfrak{E}, \mathbb{R})$, and its Fréchet derivative $\Phi' : \mathfrak{E} \to \mathfrak{E}^*$ is defined as follows:

$$\begin{aligned}\left\langle \Phi'(w), v\right\rangle =& M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy\right) \int_{\mathbb{R}^N} (|\nabla w|^{p-2}\nabla w \cdot \nabla v + \nu(y)|\nabla w|^{q-2}\nabla w \cdot \nabla v)\, dy \\ &+ \int_{\mathbb{R}^N} \mathfrak{V}(y)(|w|^{p-2}wv + \nu(y)|w|^{q-2}wv)\, dy\end{aligned}$$

for all $w, v \in \mathfrak{E}$, where $\mathfrak{E}^*$ denotes the dual space of $\mathfrak{E}$, and $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{E}$ and $\mathfrak{E}^*$.

Throughout this paper, the Kirchhoff function M satisfies the conditions (M1)–(M2), and the potential $\mathfrak{V}$ fulfills the condition (V).

Definition 1. *We say that* $w \in \mathfrak{E}$ *is a weak solution for Problem* (1) *if*

$$\begin{aligned}&M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy\right) \int_{\mathbb{R}^N} (|\nabla w|^{p-2}\nabla w \cdot \nabla u + \nu(y)|\nabla w|^{q-2}\nabla w \cdot \nabla u)\, dy \\ &+ \int_{\mathbb{R}^N} \mathfrak{V}(y)(|w|^{p-2}wu + \nu(y)|w|^{q-2}wu)\, dy = \int_{\mathbb{R}^N} \sigma(y)|w|^{r-2}wu\, dy + \theta\int_{\mathbb{R}^N} g(y, w)u\, dy\end{aligned}$$

for any $u \in \mathfrak{E}$.

We assume the following:

(B1) $1 < r < p < q < \ell < p^*$;
(B2) $0 \le \sigma \in L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with meas $\{y \in \mathbb{R}^N : \sigma(y) \ne 0\} > 0$ for any γ_0 with $p < \gamma_0 < p^*$;
(G1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, and there is an $s \in [p, p^*)$, $0 \le \rho_1 \in L^{s'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and a positive constant ρ_2 such that

$$|g(y, \zeta)| \le \rho_1(y) + \rho_2|\zeta|^{\ell-1}$$

for all $\zeta \in \mathbb{R}$ and for almost all $y \in \mathbb{R}^N$;

(G2) There exist $\mu > \vartheta q$ and $\mathfrak{M}_0 > 0$ such that

$$g(y,\zeta)\zeta - \mu G(y,\zeta) \geq 0$$

for all $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$ with $|\zeta| \geq \mathfrak{M}_0$ where $G(y,\zeta) = \int_0^\zeta g(y,s)\,ds$;

(G3) There exist $\mu > \vartheta q$, $\varsigma \geq 0$, and $\mathfrak{M}_1 > 0$ such that

$$g(y,\zeta)\zeta - \mu G(y,\zeta) \geq -\varsigma|\zeta|^p$$

for all $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$ with $|\zeta| \geq \mathfrak{M}_1$;

(G4) There exist $\mathfrak{M}_2 > 0$, $1 < d < p$, $\tau > 1$ with $p \leq \tau' d \leq p^*$, and a positive function $\xi \in L^\tau(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\liminf_{|\zeta|\to 0} \frac{g(y,\zeta)}{\xi(y)|\zeta|^{d-2}\zeta} \geq \mathfrak{M}_2$$

uniformly for almost all $y \in \mathbb{R}^N$.

Remark 1. *It is clear that the condition* (G3) *is weaker than* (G2), *which was initially provided by [48]. If we consider the function*

$$g(y,\zeta) = \rho(y)\left(\xi(y)|\zeta|^{d-2}\zeta + |\zeta|^{p-2}\zeta + \frac{2}{p}\sin\zeta\right)$$

with its primitive function

$$G(y,\zeta) = \rho(y)\left(\frac{\xi(y)}{d}|\zeta|^d + \frac{1}{p}|\zeta|^p - \frac{2}{p}\cos\zeta + \frac{2}{p}\right),$$

where $\rho \in C(\mathbb{R}^N,\mathbb{R})$ *with* $0 < \inf_{y\in\mathbb{R}^N}\rho(y) \leq \sup_{y\in\mathbb{R}^N}\rho(y) < \infty$, *and* d, ξ *are given in* (G4), *then it is obvious that this example satisfies the condition* (G3) *but not* (G2). *However, the conditions* (G1) *and* (G4) *are also satisfied.*

Let us define the functional $\Psi_\theta : \mathfrak{E} \to \mathbb{R}$ as

$$\Psi_\theta(w) = \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\,dy + \theta\int_{\mathbb{R}^N} G(y,w)\,dy.$$

Then, it is easy to show that $\Psi_\theta \in C^1(\mathfrak{E},\mathbb{R})$, and its Fréchet derivative is

$$\left\langle \Psi_\theta'(w), z\right\rangle = \int_{\mathbb{R}^N} \sigma(y)|w|^{r-2}wz\,dy + \theta\int_{\mathbb{R}^N} g(y,w)z\,dy$$

for any $w,z \in \mathfrak{E}$ [47]. Next, we define the functional $\mathcal{E}_\theta : \mathfrak{E} \to \mathbb{R}$ by

$$\mathcal{E}_\theta(w) = \Phi(w) - \Psi_\theta(w).$$

Then, it follows that the functional $\mathcal{E}_\theta \in C^1(\mathfrak{E},\mathbb{R})$ and its Fréchet derivative is:

$$\left\langle \mathcal{E}_\theta'(w), z\right\rangle = \left\langle \Phi'(w), z\right\rangle - \left\langle \Psi_\theta'(w), z\right\rangle \quad \text{for any } w,z \in \mathfrak{E}.$$

Before describing the proofs of our results, we present several preliminary assertions.

Lemma 4 ([47])**.** *Assume that* (B1), (B2), *and* (G1) *hold. Then,* Ψ_θ *and* Ψ_θ' *are sequentially weakly strongly continuous.*

Definition 2. *Suppose that* $\mathfrak{X}$ *is a real Banach space. We say that the functional* $\mathcal{F}$ *satisfies the Cerami condition at level* c *(*$(C)_c$*-condition for short) in* $\mathfrak{X}$ *if any* $(C)_c$*-sequence* $\{w_n\} \subset \mathfrak{X}$, *i.e.,* $\mathcal{F}(w_n) \to c$ *and* $\|\mathcal{F}'(w_n)\|_{\mathfrak{X}^*}(1+\|w_n\|_{\mathfrak{X}}) \to 0$ *as* $n \to \infty$ *has a convergent subsequence in* $\mathfrak{X}$*.*

The following Lemmas 5 and 6 are the compactness condition for the Palais–Smale type that play a crucial role in obtaining our main results. The basic concepts behind the proofs of these logical consequences follows the analogous arguments in [30]. However, more complicated analyses have to be carried out because of the presence of the non-local Kirchhoff coefficient M.

Remark 2. *The basic concepts of the proofs for the following logical consequences use similar arguments to those in [30,31]. From this point of view, it is important that the potential function* $V \in C(\mathbb{R}^N, (0,\infty))$ *is coercive. As mentioned in the introduction, we show this condition without assuming the coercivity of the function* V*.*

Lemma 5. *Suppose that* (B1), (B2), (G1), *and* (G2) *hold. Then, the functional* $\mathcal{E}_\theta$ *ensures the* $(C)_c$*-condition for any* $\theta > 0$*.*

Proof. For $c \in \mathbb{R}$, let $\{w_n\}$ be a $(C)_c$-sequence in $\mathfrak{E}$, i.e.,

$$\mathcal{E}_\theta(w_n) \to c \text{ and } \|\mathcal{E}'_\theta(w_n)\|_{\mathfrak{X}^*}(1+\|w_n\|_{\mathfrak{X}}) \to 0 \text{ as } n \to \infty, \tag{5}$$

which show that

$$c = \mathcal{E}_\theta(w_n) + o(1) \text{ and } \langle \mathcal{E}_\theta(w_n), w_n \rangle = o(1), \tag{6}$$

where $o(1) \to 0$ as $n \to \infty$. Firstly, we verify that the sequence $\{w_n\}$ is bounded in $\mathfrak{E}$. To do this, we claim that

$$\begin{aligned}\left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy - C_1 \int_{\{|w_n| \le \mathfrak{M}_0\}} |w_n|^p + \rho_1(y)|w_n| + \rho_2 |w_n|^\ell \, dy \\ \ge \frac{1}{2}\left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy - \mathcal{K}_0\end{aligned} \tag{7}$$

for any positive constant C_1 and for some positive constant $\mathcal{K}_0$, where $\mathcal{H}_{\mathfrak{V}}$, as given in (2). Indeed, without the loss of generality, we suppose that $\mathfrak{M}_0 > 1$. By Young's inequality, we know that

$$\begin{aligned}&\left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy \\ &\quad - C_1 \int_{\{|w_n| \le \mathfrak{M}_0\}} (|w_n|^p + \rho_1(y)|w_n| + \rho_2 |w_n|^\ell)\, dy \\ &\ge \left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy \\ &\quad - C_1 \int_{\{|w_n| \le \mathfrak{M}_0\}} \left(|w_n|^p + \rho_1^{s'}(y) + |w_n|^s + \rho_2 |w_n|^\ell\right) dy \\ &\ge \frac{1}{2}\left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right)\left[\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy + \int_{\{|w_n| \le \mathfrak{M}_0\}} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy\right] \\ &\quad - C_1 \int_{\{|w_n| \le 1\}} \left(|w_n|^p + |w_n|^s + \rho_2 |w_n|^\ell\right) dy \\ &\quad - C_1 \int_{\{1 < |w_n| \le \mathfrak{M}_0\}} \left(|w_n|^p + |w_n|^s + \rho_2 |w_n|^\ell\right) dy - C_1 \|\rho_1\|^{s'}_{L^{s'}(\mathbb{R}^N)} \\ &\ge \frac{1}{2}\left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right)\left[\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy + \int_{\{|w_n| \le \mathfrak{M}_0\}} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy\right] \\ &\quad - C_1(2+\rho_2) \int_{\{|w_n| \le 1\}} |w_n|^p \, dy - C_1 \|\rho_1\|^{s'}_{L^{s'}(\mathbb{R}^N)}\end{aligned} \tag{8}$$

$$
\begin{aligned}
&- C_1\Big(1 + \mathfrak{M}_0{}^{s-p} + \mathfrak{M}_0{}^{\ell-p}\rho_2\Big) \int_{\{1<|w_n|\le\mathfrak{M}_0\}} |w_n|^p\,dy \\
&\ge \frac{1}{2}\Big(\frac{1}{\vartheta q} - \frac{1}{\mu}\Big)\Big[\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy + \int_{\{|w_n|\le\mathfrak{M}_0\}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\Big] \\
&\quad - C_1(2+\rho_2)\int_{\{|w_n|\le 1\}} \mathcal{H}(y,|w_n|)\,dy - C_1\|\rho_1\|^{s'}_{L^{s'}(\mathbb{R}^N)} \\
&\quad - C_1\Big(1 + \mathfrak{M}_0{}^{s-p} + \mathfrak{M}_0{}^{\ell-p}\rho_2\Big)\int_{\{1<|w_n|\le\mathfrak{M}_0\}} \mathcal{H}(y,|w_n|)\,dy \\
&\ge \frac{1}{2}\Big(\frac{1}{\vartheta q} - \frac{1}{\mu}\Big)\Big[\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy + \int_{\{|w_n|\le\mathfrak{M}_0\}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\Big] \\
&\quad - \widetilde{C}_0\int_{\{|w_n|\le\mathfrak{M}_0\}} \mathcal{H}(y,|w_n|)\,dy - \widetilde{C}_1,
\end{aligned}
$$

where $\mathcal{H}$, as given in (2), $\widetilde{C}_0 := C_1\Big(1 + \mathfrak{M}_0{}^{s-p} + \mathfrak{M}_0{}^{\ell-p}\rho_2\Big)$. and $\widetilde{C}_1 := C_1\|\rho_1\|^{s'}_{L^{s'}(\mathbb{R}^N)}$. We set

$$
\mathbb{B}_{r_0} = \{y \in \mathbb{R}^N : |y| < r_0\}, \quad \mathcal{A} = \{y \in \mathbb{R}^N \setminus \mathbb{B}_{r_0} : \mathfrak{V}(y) \ge \mathcal{V}_0\}
$$

and

$$
\mathcal{B} = \{y \in \mathbb{R}^N \setminus \mathbb{B}_{r_0} : \mathfrak{V}(y) < \mathcal{V}_0\}
$$

for any $\mathcal{V}_0 > 0$. Then, it is clear that $\mathcal{A} \cup \mathcal{B} = \mathbb{B}^c_{r_0}$, where $\mathcal{A}$ and $\mathcal{B}$ are disjoint. If $y \in \mathcal{A}$, then for any $\mathcal{V}_0 \ge \frac{2\vartheta q\mu\widetilde{C}_0}{\mu-\vartheta q}$, we know that

$$
\mathcal{H}_{\mathfrak{V}}(y,|w_n|) \ge \frac{2\vartheta q\mu\widetilde{C}_0}{\mu-\vartheta q}\mathcal{H}(y,|w_n|) \tag{9}
$$

for $|y| \ge r_0$. Furthermore, since $\mathfrak{V} \in L^1(\mathbb{B}_{r_0})$, we infer

$$
\int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathbb{B}_{r_0}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy < +\infty \quad\text{and}\quad \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathbb{B}_{r_0}} \mathcal{H}(y,|w_n|)\,dy < +\infty \tag{10}
$$

for some positive constants $\widetilde{C}_2, \widetilde{C}_3$. Using (V), we know $\operatorname{meas}\big(\{y \in \mathbb{R}^N : |w_n(y)| \le \mathfrak{M}_0\} \cap \mathcal{B}\big)$ is finite, and thus,

$$
\int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathcal{B}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy < +\infty \quad\text{and}\quad \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathcal{B}} \mathcal{H}(y,|w_n|)\,dy < +\infty. \tag{11}
$$

This, together with (8)–(11), yields the following:

$$
\begin{aligned}
&\Big(\frac{1}{\vartheta q} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy - C_1\int_{\{|w_n|\le\mathfrak{M}_0\}} (|w_n|^p + \rho_1(y)|w_n| + \rho_2|w_n|^\ell)\,dy \\
&\ge \frac{1}{2}\Big(\frac{1}{\vartheta q} - \frac{1}{\mu}\Big)\Big[\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy + \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathbb{B}^c_{r_0}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy \\
&\qquad\qquad + \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathbb{B}_{r_0}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\Big] \\
&\quad - \widetilde{C}_0\Big[\int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathbb{B}^c_{r_0}} \mathcal{H}(y,|w_n|)\,dy + \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathbb{B}_{r_0}} \mathcal{H}(y,|w_n|)\,dy\Big] - \widetilde{C}_1 \\
&\ge \frac{1}{2}\Big(\frac{1}{\vartheta q} - \frac{1}{\mu}\Big)\Big[\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy + \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathcal{A}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy \\
&\quad + \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathcal{B}} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\Big] - \widetilde{C}_0\Big[\int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathcal{A}} \mathcal{H}(y,|w_n|)\,dy \\
&\quad + \int_{\{|w_n|\le\mathfrak{M}_0\}\cap\mathcal{B}} \mathcal{H}(y,|w_n|)\,dy\Big] - \widetilde{\mathcal{K}}_0
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{2}\Big(\frac{1}{\vartheta q}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy+\frac{\mu-\vartheta q}{2\vartheta q\mu}\int_{\{|w_n|\leq\mathfrak{M}_0\}\cap\mathcal{A}}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\\
&\quad-\widetilde{C}_0\int_{\{|w_n|\leq\mathfrak{M}_0\}\cap\mathcal{A}}\mathcal{H}(y,|w_n|)\,dy-\mathcal{K}_0\\
&\geq \frac{1}{2}\Big(\frac{1}{\vartheta q}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy-\mathcal{K}_0
\end{aligned}
$$

where $\widetilde{\mathcal{K}}_0$ and $\mathcal{K}_0$ are suitable constants. From this, the relation (7) is proved. Combining (7) with (B1), (B2), (G1), and (G2), we find the following:

$$
\begin{aligned}
c+1 &\geq \mathcal{E}_\theta(w_n)-\frac{1}{\mu}\langle\mathcal{E}'_\theta(w_n),w_n\rangle\\
&=\mathcal{M}\bigg(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\bigg)+\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V},p,q}(y,|w_n|)\,dy\\
&\quad-\frac{1}{r}\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy-\theta\int_{\mathbb{R}^N}G(y,w_n)\,dy\\
&\quad-\frac{1}{\mu}M\bigg(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\bigg)\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy\\
&\quad-\frac{1}{\mu}\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy+\frac{1}{\mu}\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy+\frac{\theta}{\mu}\int_{\mathbb{R}^N}g(y,w_n)w_n\,dy\\
&\geq\frac{1}{\vartheta}M\bigg(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\bigg)\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\\
&\quad+\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V},p,q}(y,|w_n|)\,dy-\frac{1}{r}\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy-\theta\int_{\mathbb{R}^N}G(y,w_n)\,dy\\
&\quad-\frac{1}{\mu}M\bigg(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\bigg)\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy\\
&\quad-\frac{1}{\mu}\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy+\frac{1}{\mu}\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy+\frac{\theta}{\mu}\int_{\mathbb{R}^N}g(y,w_n)w_n\,dy\\
&\geq\bigg(\frac{1}{\vartheta q}-\frac{1}{\mu}\bigg)M\bigg(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\bigg)\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy\\
&\quad+\bigg(\frac{1}{q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy-\bigg(\frac{1}{r}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy\\
&\quad+\frac{\theta}{\mu}\int_{\mathbb{R}^N}g(y,w_n)w_n-\mu G(y,w_n)\,dy\\
&\geq\kappa_0\bigg(\frac{1}{\vartheta q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy+\bigg(\frac{1}{q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\\
&\quad-\bigg(\frac{1}{r}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy+\frac{\theta}{\mu}\int_{\{|w_n|\leq\mathfrak{M}_0\}}g(y,w_n)w_n-\mu G(y,w_n)\,dy\\
&\quad+\frac{\theta}{\mu}\int_{\{|w_n|\geq\mathfrak{M}_0\}}g(y,w_n)w_n-\mu G(y,w_n)\,dy\\
&\geq\kappa_0\bigg(\frac{1}{\vartheta q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy+\bigg(\frac{1}{q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\\
&\quad-\bigg(\frac{1}{r}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy\\
&\quad-C_1\int_{\{|w_n|\leq\mathfrak{M}_0\}}|w_n|^p+\rho_1(y)|w_n|+\rho_2|w_n|^\ell\,dy\\
&\geq\kappa_0\bigg(\frac{1}{\vartheta q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy+\frac{1}{2}\bigg(\frac{1}{q}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\\
&\quad-\bigg(\frac{1}{r}-\frac{1}{\mu}\bigg)\int_{\mathbb{R}^N}\sigma(y)|w_n|^r\,dy-\mathcal{K}_0
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{\min\{\kappa_0,1\}(\mu-\vartheta q)}{2\vartheta q\mu}\left[\int_{\mathbb{R}^N}\mathcal{H}(y,|\nabla w_n|)\,dy+\int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\right]\\
&\quad-\left(\frac{1}{r}-\frac{1}{\mu}\right)\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|w_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)}-\mathcal{K}_0\\
&\geq \frac{\min\{\kappa_0,1\}(\mu-\vartheta q)}{2\vartheta q\mu}\min\left\{\frac{\|w_n\|^p}{2^p},\frac{\|w_n\|^q}{2^{q-1}}\right\}\\
&\quad-\left(\frac{1}{r}-\frac{1}{\mu}\right)\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|w_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)}-\mathcal{K}_0\\
&\geq \frac{\min\{\kappa_0,1\}(\mu-\vartheta q)}{2\vartheta q\mu}\min\left\{\frac{\|w_n\|^p}{2^p},\frac{\|w_n\|^q}{2^{q-1}}\right\}\\
&\quad-\left(\frac{1}{r}-\frac{1}{\mu}\right)\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}C_{\gamma_0,imb}\|w_n\|^r-\mathcal{K}_0,
\end{aligned}$$

where $C_{\gamma_0,imb}$ is an embedding constant of $\mathfrak{E}\hookrightarrow L^{\gamma_0}(\mathbb{R}^N)$. Since $p>r>1$, we assert that the sequence $\{w_n\}$ is bounded in $\mathfrak{E}$, and thus, $\{w_n\}$ has a weakly convergent subsequence in $\mathfrak{E}$. Passing to the limit, if necessary, to a subsequence according to Lemma 2, we have the following:

$$w_n\rightharpoonup w_0 \text{ in } \mathfrak{E},\quad w_n(y)\to w_0(y) \text{ a.e. in } \mathbb{R}^N \quad\text{and}\quad w_n\to w_0 \text{ in } L^{\tau}(\mathbb{R}^N) \tag{12}$$

as $n\to\infty$ for any $\tau\in[p,p^*)$. To prove that $\{w_n\}$ converges strongly to w_0 in $\mathfrak{E}$ as $n\to\infty$, we let $\psi\in\mathfrak{E}$ be fixed and let $\tilde{\Phi}_\psi$ denote the linear function on $\mathfrak{E}$ as defined by

$$\tilde{\Phi}_\psi(v)=\int_{\mathbb{R}^N}|\nabla\psi|^{p-2}\nabla\psi\cdot\nabla v\,dy+\int_{\mathbb{R}^N}\nu(y)|\nabla\psi|^{q-2}\nabla\psi\cdot\nabla v\,dy \tag{13}$$

for all $v\in\mathfrak{E}$. Obviously, by the Hölder inequality, $\tilde{\Phi}_\psi$ is also continuous, as

$$\begin{aligned}
|\tilde{\Phi}_\psi(v)|&\leq C_2\Big(\||\nabla\psi|^{p-1}\|_{L^{p'}(\mathbb{R}^N)}+\||\nabla\psi|^{q-1}\|_{L^{q'}(\nu,\mathbb{R}^N)}\Big)\|v\|\\
&\leq C_2\Big(\|\nabla\psi\|^{p-1}_{L^p(\mathbb{R}^N)}+\|\nabla\psi\|^{q-1}_{L^q(\nu,\mathbb{R}^N)}\Big)\|v\|
\end{aligned}$$

for any $v\in\mathfrak{E}$ and a positive constant C_2. Hence, (12) yields

$$\lim_{n\to\infty}\left[M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)-M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_0|)\,dy\Big)\right]\tilde{\Phi}_{w_0}(w_n-w_0)=0, \tag{14}$$

as the sequence $\left\{M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)-M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_0|)\,dy\Big)\right\}$ is bounded in $\mathbb{R}$. Using (G1) and the Hölder inequality, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N}|(g(y,w_n)-g(y,w_0))(w_n-w_0)|\,dy\\
&\leq\int_{\mathbb{R}^N}\Big[2\rho_1(y)+\rho_2\Big(|w_n|^{\ell-1}+|w_0|^{\ell-1}\Big)\Big]|w_n-w_0|\,dy\\
&\leq 2\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\|w_n-w_0\|_{L^s(\mathbb{R}^N)}\\
&\quad+\rho_2\Big(\|w_n\|^{\ell-1}_{L^{\ell'}(\mathbb{R}^N)}+\|w_0\|^{\ell-1}_{L^{\ell'}(\mathbb{R}^N)}\Big)\|w_n-w_0\|_{L^{\ell}(\mathbb{R}^N)}.
\end{aligned}$$

Then, (12) implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(g(y,w_n)-g(y,w_0))(w_n-w_0)\,dy=0. \tag{15}$$

Let us denote $\gamma := \frac{\gamma_0}{\gamma_0 - r}$. Then, by Young's inequality, we obtain the following:

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left|\sigma(y)\left(|w_n|^{r-2}w_n - |w_0|^{r-2}w_0\right)\right|^{\gamma_0'} dy \\
&= \int_{\mathbb{R}^N} |\sigma(y)|^{\gamma_0} \left|\left(|w_n|^{r-2}w_n - |w_0|^{r-2}w_0\right)\right|^{\gamma_0'} dy \\
&\le \int_{\mathbb{R}^N} |\sigma(y)|^{\gamma_0} \left(|w_n|^{r-1} + |w_0|^{r-1}\right)^{\gamma_0'} dy \\
&\le \int_{\mathbb{R}^N} \left[\frac{(|\sigma(y)|^{\gamma_0})^{\frac{\gamma}{\gamma_0'}}}{\frac{\gamma}{\gamma_0'}} + \frac{\left[(|w_n|^{r-1} + |w_0|^{r-1})^{\gamma_0'}\right]^{\left(\frac{\gamma}{\gamma_0}\right)'}}{\left(\frac{\gamma}{\gamma_0}\right)'} \right] dy \qquad (16) \\
&= \int_{\mathbb{R}^N} \left[\frac{\gamma_0'}{\gamma} |\sigma(y)|^{\gamma} + \frac{r-1}{\gamma_0 - 1} \left(|w_n|^{r-1} + |w_0|^{r-1}\right)^{\frac{\gamma_0}{r-1}} \right] dy \\
&\le C_3 \int_{\mathbb{R}^N} \frac{\gamma_0'}{\gamma} |\sigma(y)|^{\gamma} + \frac{r-1}{\gamma_0 - 1} (|w_n|^{\gamma_0} + |w_0|^{\gamma_0})\, dy
\end{aligned}
$$

for a positive constant C_3. Invoking (12), (16), and the convergence principle, we have

$$
\left|\sigma(y)|w_n|^{r-2}w_n - \sigma(y)|w_0|^{r-2}w_0\right|^{\gamma_0'} \le f_1(y)
$$

for almost all $y \in \mathbb{R}^N$ and for some $f_1 \in L^1(\mathbb{R}^N)$, and thus, $\sigma(y)|w_n|^{r-2}w_n \to \sigma(y)|w_0|^{r-2}w_0$ as $n \to \infty$ for almost all $y \in \mathbb{R}^N$. This, together with Lebesgue's dominated convergence theorem, yields the following:

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} \sigma(y)\left(|w_n|^{r-2}w_n - |w_0|^{r-2}w_0\right)(w_n - w_0)\, dy = 0. \qquad (17)
$$

Because $w_n \rightharpoonup w_0$ in $\mathfrak{E}$ and $\mathcal{E}_\theta'(w_n) \to 0$ in $\mathfrak{E}^*$, as $n \to \infty$, we obtain the following:

$$
\langle \mathcal{E}_\theta'(w_n) - \mathcal{E}_\theta'(w_0), w_n - w_0 \rangle \to 0 \text{ as } n \to \infty. \qquad (18)
$$

Let us denote $\tilde{\Psi}_\psi$ in $\mathfrak{E}$ with

$$
\tilde{\Psi}_\psi(v) := \int_{\mathbb{R}^N} \mathfrak{V}(y)\left(|\psi|^{p-2}\psi + \nu(y)|\psi|^{q-2}\psi\right) v\, dy.
$$

Then, we infer

$$
\begin{aligned}
&\langle \mathcal{E}_\theta'(w_n) - \mathcal{E}_\theta'(w_0), w_n - w_0 \rangle \\
&= M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy\right) \tilde{\Phi}_{w_n}(w_n - w_0) \\
&\quad - M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_0|)\, dy\right) \tilde{\Phi}_{w_0}(w_n - w_0) \\
&\quad + \int_{\mathbb{R}^N} \mathfrak{V}(y)\left(|w_n|^{p-2}w_n + \nu(y)|w_n|^{q-2}w_n\right)(w_n - w_0)\, dy \\
&\quad - \int_{\mathbb{R}^N} \mathfrak{V}(y)\left(|w_0|^{p-2}w_0 + \nu(y)|w_0|^{q-2}w_0\right)(w_n - w_0)\, dy \\
&\quad - \int_{\mathbb{R}^N} \sigma(y)\left(|w_n|^{r-2}w_n - |w_0|^{r-2}w_0\right)(w_n - w_0)\, dy \\
&\quad - \theta \int_{\mathbb{R}^N} \left(g(y, w_n) - g(y, w_0)\right)(w_n - w_0)\, dy \\
&= M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy\right) \left[\tilde{\Phi}_{w_n}(w_n - w_0) - \tilde{\Phi}_{w_0}(w_n - w_0)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)-M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_0|)\,dy\Big)\Big]\tilde{\Phi}_{w_0}(w_n-w_0)\\
&+\int_{\mathbb{R}^N}\mathfrak{V}(y)\Big(|w_n|^{p-2}w_n-|w_0|^{p-2}w_0+\nu(y)(|w_n|^{q-2}w_n-|w_0|^{q-2}w_0)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(w_n-w_0)\,dy\\
&-\int_{\mathbb{R}^N}\sigma(y)\Big(|w_n|^{r-2}w_n-|w_0|^{r-2}w_0\Big)(w_n-w_0)\,dy\\
&-\theta\int_{\mathbb{R}^N}\Big(g(y,w_n)-g(y,w_0)\Big)(w_n-w_0)\,dy\\
=&\Big[M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)[\tilde{\Phi}_{w_n}(w_n-w_0)-\tilde{\Phi}_{w_0}(w_n-w_0)]\\
&\qquad\qquad\qquad\qquad+\tilde{\Psi}_{w_n}(w_n-w_0)-\tilde{\Psi}_{w_0}(w_n-w_0)\Big]\\
&+\Big[M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)-M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_0|)\,dy\Big)\Big]\tilde{\Phi}_{w_0}(w_n-w_0)\\
&-\int_{\mathbb{R}^N}\sigma(y)\Big(|w_n|^{r-2}w_n-|w_0|^{r-2}w_0\Big)(w_n-w_0)\,dy\\
&-\theta\int_{\mathbb{R}^N}\Big(g(y,w_n)-g(y,w_0)\Big)(w_n-w_0)\,dy.
\end{aligned}
$$

This together with Equations (14), (15), (17), and (18) yields

$$
\begin{aligned}
\lim_{n\to\infty}\Big[M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)[\tilde{\Phi}_{w_n}(w_n-w_0)-\tilde{\Phi}_{w_0}(w_n-w_0)]&\\
+\tilde{\Psi}_{w_n}(w_n-w_0)-\tilde{\Psi}_{w_0}(w_n-w_0)\Big]&=0.
\end{aligned}
$$

By convexity, (M1), and (V), we have the following:

$$
M\Big(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w_n|)\,dy\Big)[\tilde{\Phi}_{w_n}(w_n-w_0)-\tilde{\Phi}_{w_0}(w_n-w_0)]\geq 0 \tag{19}
$$

and

$$
\mathfrak{V}(y)\Big(|w_n|^{p-2}w_n-|w_0|^{p-2}w_0+\nu(y)\Big(|w_n|^{q-2}w_n-|w_0|^{q-2}w_0\Big)\Big)(w_n-w_0)\geq 0. \tag{20}
$$

It follows that

$$
\lim_{n\to\infty}[\tilde{\Phi}_{w_n}(w_n-w_0)-\tilde{\Phi}_{w_0}(w_n-w_0)]=0 \tag{21}
$$

and

$$
\lim_{n\to\infty}[\tilde{\Psi}_{w_n}(w_n-w_0)-\tilde{\Psi}_{w_0}(w_n-w_0)]=0. \tag{22}
$$

It should be noted that there are the well-known vector inequalities:

$$
|\xi-\eta|^m\leq\begin{cases}C(m)(|\xi|^{m-2}\xi-|\eta|^{m-2}\eta)\cdot(\xi-\eta) & \text{for } m\geq 2,\\ C(m)\Big[(|\xi|^{m-2}\xi-|\eta|^{m-2}\eta)\cdot(\xi-\eta)\Big]^{\frac{m}{2}} & \\ \qquad\qquad\qquad\qquad\times(|\xi|^m+|\eta|^m)^{\frac{2-m}{2}} & \text{for } 1<m<2\end{cases} \tag{23}
$$

for all $\xi,\eta\in\mathbb{R}^N$, where $C(m)$ is a positive constant depending only on m [49]. It is now assumed that $2\leq p<q$. Then, according to (23), we know the following:

$$
\int_{\mathbb{R}^N}|\nabla w_n-\nabla w_0|^p\,dy
$$

$$\leq C(p)\int_{\mathbb{R}^N}(|\nabla w_n|^{p-2}\nabla w_n - |\nabla w_0|^{p-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\,dy \tag{24}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\nu(y)|\nabla w_n - \nabla w_0|^q\,dy\\ &\quad\leq C(q)\int_{\mathbb{R}^N}\nu(y)(|\nabla w_n|^{q-2}\nabla w_n - |\nabla w_0|^{q-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\,dy.\end{aligned} \tag{25}$$

Then, based on (24), (25), and the definition of $\tilde{\Phi}_\psi$ in (13), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla w_n - \nabla w_0|^p + \nu(y)|\nabla w_n - \nabla w_0|^q\,dy\\ &\quad\leq \max\{C(p), C(q)\}\big(\tilde{\Phi}_{w_n}(w_n - w_0) - \tilde{\Phi}_{w_0}(w_n - w_0)\big).\end{aligned} \tag{26}$$

Similarly, utilizing (V) and (23),

$$\begin{aligned}&\int_{\mathbb{R}^N}\mathfrak{V}(y)|w_n - w_0|^p\,dy\\ &\quad\leq \tilde{C}(p)\int_{\mathbb{R}^N}\mathfrak{V}(y)(|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n - w_0)\,dy\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\mathfrak{V}(y)\nu(y)|w_n - w_0|^q\,dy\\ &\quad\leq \tilde{C}(q)\int_{\mathbb{R}^N}\mathfrak{V}(y)\Big(\nu(y)|w_n|^{q-2}w_n - \nu(y)|w_0|^{q-2}w_0\Big)(w_n - w_0)\,dy.\end{aligned} \tag{28}$$

Then, according to (27) and (28), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N}\mathfrak{V}(y)\big(|w_n - w_0|^p + \nu(y)|w_n - w_0|^q\big)\,dy\\ &\quad\leq \max\{\tilde{C}(p), \tilde{C}(q)\}\big[\tilde{\Psi}_{w_n}(w_n - w_0) - \tilde{\Psi}_{w_0}(w_n - w_0)\big].\end{aligned} \tag{29}$$

However, we consider the case where $1 < p < q < 2$. As $\{w_n\}$ is bounded in $\mathfrak{E}$, there exist positive constants of C_4 and C_5 such that $\int_{\mathbb{R}^N}|\nabla w_n|^p\,dy \leq C_4$ and $\int_{\mathbb{R}^N}\nu(y)|\nabla w_n|^q\,dy \leq C_5$ for all $n \in \mathbb{N}$. By (23) and the Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla w_n - \nabla w_0|^p\,dy\\ &\leq C(p)\int_{\mathbb{R}^N}\Big[(|\nabla w_n|^{p-2}\nabla w_n - |\nabla w_0|^{p-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\Big]^{\frac{p}{2}}\\ &\qquad\qquad\times(|\nabla w_n|^p + |\nabla w_0|^p)^{\frac{2-p}{2}}\,dy\\ &\leq C(p)\bigg(\int_{\mathbb{R}^N}(|\nabla w_n|^{p-2}\nabla w_n - |\nabla w_0|^{p-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\,dy\bigg)^{\frac{p}{2}}\\ &\qquad\qquad\times\bigg(\int_{\mathbb{R}^N}(|\nabla w_n|^p + |\nabla w_0|^p)\,dy\bigg)^{\frac{2-p}{2}}\\ &\leq C(p)(2C_4)^{\frac{2-p}{2}}\bigg(\int_{\mathbb{R}^N}(|\nabla w_n|^{p-2}\nabla w_n - |\nabla w_0|^{p-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\,dy\bigg)^{\frac{p}{2}}\end{aligned} \tag{30}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\nu(y)|\nabla w_n - \nabla w_0|^q\,dy\\ &\leq C(q)\int_{\mathbb{R}^N}\Big[\nu(y)(|\nabla w_n|^{q-2}\nabla w_n - |\nabla w_0|^{q-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\Big]^{\frac{q}{2}}\end{aligned}$$

$$\times \left[\nu(y)(|\nabla w_n|^q + |\nabla w_0|^q)\right]^{\frac{2-q}{2}} dy$$

$$\leq C(q)\left(\int_{\mathbb{R}^N} \nu(y)(|\nabla w_n|^{q-2}\nabla w_n - |\nabla w_0|^{q-2}\nabla w_0)\cdot(\nabla w_n - \nabla w_0)\,dy\right)^{\frac{q}{2}} \quad (31)$$

$$\times \left(\int_{\mathbb{R}^N} \nu(y)|\nabla w_n|^q + \nu(y)|\nabla w_0|^q\,dy\right)^{\frac{2-q}{2}}$$

$$\leq C(q)(2C_5)^{\frac{2-q}{2}}\left(\int_{\mathbb{R}^N} \nu(y)\left(|\nabla w_n|^{q-2}\nabla w_n - |\nabla w_0|^{q-2}\nabla w_0\right)\cdot(\nabla w_n - \nabla w_0)\,dy\right)^{\frac{q}{2}}.$$

Then, according to (30), (31), and the definition of $\tilde{\Phi}_\psi$ in (13), it follows that

$$\int_{\mathbb{R}^N} |\nabla w_n - \nabla w_0|^p + \nu(y)|\nabla w_n - \nabla w_0|^q\,dy$$
$$\leq C\big(\tilde{\Phi}_{w_n}(w_n - w_0) - \tilde{\Phi}_{w_0}(w_n - w_0)\big)^{\alpha}, \quad (32)$$

where $C := \max\left\{C(p)(2C_4)^{\frac{2-p}{2}}, C(q)(2C_5)^{\frac{2-q}{2}}\right\}$ and α is either $\frac{p}{2}$ or $\frac{q}{2}$. Similarly, from (V) and the boundedness of $\{w_n\}$ in $\mathfrak{E}$, there exist positive constants C_6 and C_7 such that $\int_{\mathbb{R}^N} \mathfrak{V}(y)|w_n|^p\,dy \leq C_6$ and $\int_{\mathbb{R}^N} \mathfrak{V}(y)\nu(y)|w_n|^q\,dy \leq C_7$ for all $n \in \mathbb{N}$. According to (23) and the Hölder inequality, we have the following:

$$\int_{\mathbb{R}^N} \mathfrak{V}(y)|w_n - w_0|^p\,dy$$
$$\leq \tilde{C}(p)\int_{\mathbb{R}^N}\left[\mathfrak{V}(y)\left(|w_n|^{p-2}w_n - |w_0|^{p-2}w_0\right)(w_n - w_0)\right]^{\frac{p}{2}}$$
$$\times\left[\mathfrak{V}(y)(|w_n|^p + |w_0|^p)\right]^{\frac{2-p}{2}} dy$$
$$\leq \tilde{C}(p)\left(\int_{\mathbb{R}^N}\mathfrak{V}(y)\left[(|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n - w_0)\right]dy\right)^{\frac{p}{2}} \quad (33)$$
$$\times\left(\int_{\mathbb{R}^N}\mathfrak{V}(y)|w_n|^p + \mathfrak{V}(y)|w_0|^p\,dy\right)^{\frac{2-p}{2}}$$
$$\leq \tilde{C}(p)(2C_6)^{\frac{2-p}{2}}\left(\int_{\mathbb{R}^N}\mathfrak{V}(y)\left[(|w_n|^{p-2}w_n - |w_0|^{p-2}w_0)(w_n - w_0)\right]dy\right)^{\frac{p}{2}}$$

and

$$\int_{\mathbb{R}^N} \mathfrak{V}(y)\nu(y)|w_n - w_0|^q\,dy$$
$$\leq \tilde{C}(q)\int_{\mathbb{R}^N}\left[\mathfrak{V}(y)\nu(y)\left(|w_n|^{q-2}w_n - |w_0|^{q-2}w_0\right)(w_n - w_0)\right]^{\frac{q}{2}}$$
$$\times\left[\mathfrak{V}(y)\nu(y)(|w_n|^q + |w_0|^q)\right]^{\frac{2-q}{2}} dy$$
$$\leq \tilde{C}(q)\left(\int_{\mathbb{R}^N}\mathfrak{V}(y)\nu(y)\left[(|w_n|^{q-2}w_n - |w_0|^{q-2}w_0)(w_n - w_0)\right]dy\right)^{\frac{q}{2}} \quad (34)$$
$$\times\left(\int_{\mathbb{R}^N}\mathfrak{V}(y)\nu(y)|w_n|^q + \mathfrak{V}(y)\nu(y)|w_0|^q\,dy\right)^{\frac{2-q}{2}}$$
$$\leq \tilde{C}(q)(2C_7)^{\frac{2-q}{2}}\left(\int_{\mathbb{R}^N}\mathfrak{V}(y)\nu(y)\left[(|w_n|^{q-2}w_n - |w_0|^{q-2}w_0)(w_n - w_0)\right]dy\right)^{\frac{q}{2}}.$$

Then, based on (33) and (34), we get that

$$\int_{\mathbb{R}^N} \mathfrak{V}(y)(|w_n - w_0|^p + \nu(y)|w_n - w_0|^q)\, dy \\ \leq \tilde{C}\left(\tilde{\Psi}_{w_n}(w_n - w_0) - \tilde{\Psi}_{w_0}(w_n - w_0)\right)^{\beta}, \tag{35}$$

where $\tilde{C} := \max\left\{\tilde{C}(p)(2C_6)^{\frac{2-p}{2}}, \tilde{C}(q)(2C_7)^{\frac{2-q}{2}}\right\}$ and β is either $\frac{p}{2}$ or $\frac{q}{2}$. Then, with the foundation of (21) and (22) and according to (26), (29), (32), and (35), we obtain $\|w_n - w_0\| \to 0$ as $n \to \infty$. Hence, $\mathcal{E}_\theta$ satisfies the $(C)_c$-condition. This completes the proof. □

Remark 3. *As mentioned in Remark 1, condition* (G3) *is weaker than* (G2). *However, to obtain the following compactness condition, we need an additional assumption on the nonlinear term g at infinity.*

Lemma 6. *Suppose that* (B1), (B2), (G1), *and* (G3) *hold. In addition,*

(G5) $\lim_{|\zeta|\to\infty} \frac{G(y,\zeta)}{|\zeta|^{\vartheta q}} = \infty$ *uniformly for almost all* $y \in \mathbb{R}^N$

holds. Then, the functional $\mathcal{E}_\theta$ fulfills the $(C)_c$-condition for any $\theta > 0$.

Proof. For $c \in \mathbb{R}$, let $\{w_n\}$ be a $(C)_c$-sequence in $\mathfrak{E}$ satisfying (5). Based on Lemma 5, it is sufficient to prove that $\{w_n\}$ is bounded in $\mathfrak{E}$. To this end, suppose, to the contrary, that $\|w_n\| > 1$ and $\|w_n\| \to \infty$ as $n \to \infty$, and a sequence $\{\varpi_n\}$ is defined by $\varpi_n = w_n/\|w_n\|$. Then, up to the subsequence denoted by $\{\varpi_n\}$, we obtain $\varpi_n \rightharpoonup \varpi_0$ in $\mathfrak{E}$ as $n \to \infty$, and due to Lemma 2,

$$\varpi_n \to \varpi_0 \text{ a.e. in } \mathbb{R}^N \text{ and } \varpi_n \to \varpi_0 \text{ in } L^t(\mathbb{R}^N) \tag{36}$$

as $n \to \infty$ for any t with $p \leq t < p^*$. By Lemma 3 and assumption (B2), we have

$$\begin{aligned}
\mathcal{E}_\theta(w_n) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w_n|)\, dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w_n|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\geq \frac{1}{\vartheta} M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy\right) \int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy \\
&\quad + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w_n|)\, dy - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w_n|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\geq \frac{\kappa_0}{\vartheta q}\int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla w_n|)\, dy + \frac{1}{q}\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w_n|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q}\left(\int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla w_n|)\, dy + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy\right) \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w_n|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w_n\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\|w_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)} - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w_n\|^p - \frac{C_8}{r}\|w_n\|^r - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy
\end{aligned} \tag{37}$$

for a positive constant C_8. Since $\mathcal{E}_\theta(w_n) \to c$ as $n \to \infty$, $\|w_n\| \to \infty$ as $n \to \infty$, and $r < p$, we assert that

$$\int_{\mathbb{R}^N} G(y, w_n)\, dy \geq \frac{1}{\theta}\left(\frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p} \|w_n\|^p - \frac{C_8}{r}\|w_n\|^r - \mathcal{E}_\theta(w_n) \right) \to \infty \quad \text{as} \quad n \to \infty. \tag{38}$$

According to Lemma 3, we have

$$\begin{aligned}
\mathcal{E}_\theta(w_n) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy \right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w_n|)\, dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w_n|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\leq \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy \right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w_n|)\, dy \\
&\quad - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\leq \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy \right) + \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy \\
&\quad - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\leq \mathcal{M}(1)\left(1 + \left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy \right)^{\vartheta} \right) \\
&\quad + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy - \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\leq C_9 \max\{\mathcal{M}(1), 1\}\left(1 + \int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla w_n|)\, dy + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w_n|)\, dy \right)^{\vartheta} \\
&\quad - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\leq C_9 \max\{\mathcal{M}(1), 1\}(1 + 2\|w_n\|^q)^{\vartheta} - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy \\
&\leq 4^{\vartheta} C_9 \max\{\mathcal{M}(1), 1\}\|w_n\|^{\vartheta q} - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy
\end{aligned} \tag{39}$$

for a positive constant C_9, where $\mathcal{M}(\tau) \leq \mathcal{M}(1)(1 + \tau^{\vartheta})$ for all $\tau \in \mathbb{R}^+$ because if $0 \leq \tau < 1$, then $\mathcal{M}(\tau) = \int_0^\tau M(s)\, ds \leq \mathcal{M}(1)$, and if $\tau > 1$, then $\mathcal{M}(\tau) \leq \mathcal{M}(1)\tau^{\vartheta}$. Furthermore,

$$4^{\vartheta} C_9 \max\{\mathcal{M}(1), 1\}\|w_n\|^{\vartheta q} \geq \mathcal{E}_\theta(w_n) + \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy. \tag{40}$$

Due to assumption (G5), there exists a $\delta > 1$ such that $G(y, \zeta) > |\zeta|^{\vartheta q}$ for all $x \in \mathbb{R}^N$ and $|\zeta| > \delta$. Taking into account (G1), we obtain $|G(y, \zeta)| \leq \hat{\mathcal{C}}$ for all $(y, \zeta) \in \mathbb{R}^N \times [-\zeta_0, \zeta_0]$ for a constant $\hat{\mathcal{C}} > 0$. Therefore, there is $\mathcal{C}_1 \in \mathbb{R}$ such that $G(y, \zeta) \geq \mathcal{C}_1$ for all $(y, \zeta) \in \mathbb{R}^N \times \mathbb{R}$, and thus,

$$\frac{G(y, w_n) - \mathcal{C}_1}{4^{\vartheta} C_9 \max\{\mathcal{M}(1), 1\}\|w_n\|^{\vartheta q}} \geq 0 \tag{41}$$

for all $y \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Combining (7) with (B1), (B2), (G1), and (G3), we have the following:

$$\begin{aligned}
c + 1 &\geq \mathcal{E}_\theta(w_n) - \frac{1}{\mu}\langle \mathcal{E}_\theta'(w_n), w_n \rangle \\
&= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_n|)\, dy \right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w_n|)\, dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w_n|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w_n)\, dy
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\mu} M\big(\mathcal{H}_{p,q}(y,|\nabla w_n|)\big) \int_{\mathbb{R}^N} \mathcal{H}(y,|\nabla w_n|)\,dy \\
&- \frac{1}{\mu} \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy + \frac{1}{\mu} \int_{\mathbb{R}^N} \sigma(y)|w_n|^r\,dy \\
&+ \frac{\theta}{\mu} \int_{\mathbb{R}^N} g(y,w_n)w_n\,dy \\
\geq{}& \kappa_0 \left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}(y,|\nabla w_n|)\,dy \\
&+ \left(\frac{1}{q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy - \left(\frac{1}{r} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \sigma(y)|w_n|^r\,dy \\
&+ \frac{\theta}{\mu} \int_{\{|w_n| \leq \mathfrak{M}_1\}} g(y,w_n)w_n - \mu G(y,w_n)\,dy \\
&+ \frac{\theta}{\mu} \int_{\{|w_n| \geq \mathfrak{M}_1\}} g(y,w_n)w_n - \mu G(y,w_n)\,dy \\
\geq{}& \kappa_0 \left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}(y,|\nabla w_n|)\,dy \\
&+ \left(\frac{1}{q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy - \left(\frac{1}{r} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \sigma(y)|w_n|^r\,dy \\
&- C_1 \int_{\{|w_n| \leq \mathfrak{M}_1\}} |w_n|^p + \rho_1(y)|w_n| + \rho_2|w_n|^\ell\,dy \\
&- \frac{\theta}{\mu} \int_{\{|w_n| \geq \mathfrak{M}_1\}} \varsigma |w_n|^p\,dy \\
\geq{}& \kappa_0 \left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}(y,|\nabla w_n|)\,dy \\
&+ \frac{1}{2}\left(\frac{1}{q} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy - \left(\frac{1}{r} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} \sigma(y)|w_n|^r\,dy \\
&- \frac{\theta}{\mu} \int_{\mathbb{R}^N} \varsigma |w_n|^p\,dy - \mathcal{K}_0 \\
\geq{}& \frac{\min\{\kappa_0,1\}}{2} \left(\frac{1}{\vartheta q} - \frac{1}{\mu}\right) \\
&\times \left[\int_{\mathbb{R}^N} \mathcal{H}(y,|\nabla w_n|)\,dy + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y,|w_n|)\,dy\right] \\
&- \left(\frac{1}{r} - \frac{1}{\mu}\right) \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \|w_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&- \frac{\theta\varsigma}{\mu} \|w_n\|^p_{L^p(\mathbb{R}^N)} - \mathcal{K}_0 \\
\geq{}& \frac{\min\{\kappa_0,1\}(\mu - \vartheta q)}{2^{p+1}\vartheta q \mu} \|w_n\|^p \\
&- \left(\frac{1}{r} - \frac{1}{\mu}\right) \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} C_{\gamma_0,imb} \|w_n\|^r \\
&- \frac{\theta\varsigma}{\mu} \|w_n\|^p_{L^p(\mathbb{R}^N)} - \mathcal{K}_0.
\end{aligned}
$$

Hence, we know that

$$
\begin{aligned}
&c + \left(\frac{1}{r} - \frac{1}{\mu}\right) \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} C_{\gamma_0,imb} \|w_n\|^r + \frac{\theta\varsigma}{\mu} \|w_n\|^p_{L^p(\mathbb{R}^N)} + \mathcal{K}_0 + 1 \\
\geq{}& \frac{\min\{\kappa_0,1\}(\mu - \vartheta q)}{2^{p+1}\vartheta q \mu} \|w_n\|^p.
\end{aligned}
$$

Dividing this by $\frac{\min\{\kappa_0,1\}(\mu-\vartheta q)}{2^{p+1}\vartheta q\mu}\|w_n\|^p$ and then taking the limit supremum of this inequality as $n \to \infty$, we find the following:

$$1 \le \frac{2^{p+1}\vartheta q\theta\varsigma}{\min\{\kappa_0,1\}(\mu-\vartheta q)} \limsup_{n\to\infty} \|\varpi_n\|^p_{L^p(\mathbb{R}^N)} = \frac{2^{p+1}\vartheta q\theta\varsigma}{\min\{\kappa_0,1\}(\mu-\vartheta q)} \|\varpi_0\|^p_{L^p(\mathbb{R}^N)}. \tag{42}$$

Hence, based on (42), it follows that $\varpi_0 \ne 0$. Set $A_1 = \{y \in \mathbb{R}^N : \varpi_0(y) \ne 0\}$. By Equation (36), we infer that $|w_n(y)| = |\varpi_n(y)|\|w_n\| \to \infty$ as $n \to \infty$ for all $y \in A_1$. Thus, by using (G5),

$$\lim_{n\to\infty} \frac{G(y,w_n)}{\|w_n\|^{\vartheta q}} = \lim_{n\to\infty} \frac{G(y,w_n)}{|w_n|^{\vartheta q}} |\varpi_n|^{\vartheta q} = +\infty, \text{ for } y \in A_1. \tag{43}$$

Hence, we obtain that $\operatorname{meas}(A_1) = 0$. Indeed, if $\operatorname{meas}(A_1) \ne 0$, according to Equations (38)–(43) and the Fatou lemma, we have the following:

$$\begin{aligned}
\frac{1}{\theta} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(y,w_n)\,dy}{\theta\int_{\mathbb{R}^N} G(y,w_n)\,dy + \mathcal{E}_\theta(w_n)} \\
&\ge \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(y,w_n)}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(y,w_n)}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy \\
&\quad - \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{\mathcal{C}_1}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy \\
&= \liminf_{n\to\infty} \int_{A_1} \frac{G(y,w_n)-\mathcal{C}_1}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy \\
&\ge \int_{A_1} \liminf_{n\to\infty} \frac{G(y,w_n)-\mathcal{C}_1}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy \\
&= \int_{A_1} \liminf_{n\to\infty} \frac{G(y,w_n)}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy \\
&\quad - \int_{A_1} \limsup_{n\to\infty} \frac{\mathcal{C}_1}{4^\vartheta C_9 \max\{\mathcal{M}(1),1\}\|w_n\|^{\vartheta q}}\,dy = \infty,
\end{aligned}$$

which is impossible. Thus, $\varpi_0(y) = 0$ for almost all $y \in \mathbb{R}^N$. Consequently, we yielded a contradiction, and thus, the sequence $\{w_n\}$ is bounded in $\mathfrak{E}$. The proof is completed. □

3. Main Results

In this section, we illustrate two existence results for a sequence of infinitely many solutions to Problem (1). The primary tools for these consequences are the fountain theorem and the dual-fountain theorem in [37]. Let $\mathfrak{X}$ be a real reflexive and separable Banach space; then, it can be known (see [50,51]) that $\{e_k\} \subseteq \mathfrak{X}$ and $\{f_k^*\} \subseteq \mathfrak{X}^*$ exist such that

$$\mathfrak{X} = \overline{\operatorname{span}\{e_k : k = 1,2,\cdots\}}, \quad \mathfrak{X}^* = \overline{\operatorname{span}\{f_k^* : k = 1,2,\cdots\}}$$

and

$$\langle f_i^*, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

Let us denote $\mathfrak{X}_k = \operatorname{span}\{e_k\}$, $\mathfrak{F}_n = \bigoplus_{k=1}^n \mathfrak{X}_k$, and $\mathfrak{G}_n = \overline{\bigoplus_{k=n}^\infty \mathfrak{X}_k}$.

Lemma 7 (Fountain Theorem [34,37]). *Assume that $(\mathfrak{X}, \|\cdot\|)$ is a Banach space, the functional $\mathcal{F} \in C^1(\mathfrak{X}, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$, and $\mathcal{F}$ is even. Therefore, if, for each sufficiently large $n \in \mathbb{N}$, there are $\beta_n > \alpha_n > 0$ such that*

(1) $\delta_n := \inf\{\mathcal{F}(\varpi) : \varpi \in \mathfrak{G}_n, \|\varpi\| = \alpha_n\} \to \infty$ *as* $n \to \infty$*;*

(2) $\rho_n := \max\{\mathcal{F}(\varpi) : \varpi \in \mathfrak{F}_n, \|\varpi\| = \beta_n\} \le 0.$

Then $\mathcal{F}$ has an unbounded sequence of critical values, i.e., there is a sequence $\{\varpi_k\} \subset \mathfrak{X}$ such that $\mathcal{F}'(\varpi_k) = 0$ and $\mathcal{F}(\varpi_k) \to +\infty$ as $k \to +\infty$.

Lemma 8. *Let us denote*

$$\chi_{\iota,n} = \sup_{\|u\|=1, u \in \mathfrak{G}_n} \|u\|_{L^{\iota}(\mathbb{R}^N)}$$

and

$$\chi_n = \max\{\chi_{\ell,n}, \chi_{s,n}, \chi_{\gamma_0,n}\}. \tag{44}$$

Then $\chi_n \to 0$ as $n \to \infty$ (see [34]).

Lemma 9. *Assume that* (B1), (B2), (G1), *and* (G5) *hold. Then, there are $\beta_n > \alpha_n > 0$ such that*

(1) $\delta_n := \inf\{\mathcal{E}_\lambda(w) : w \in \mathfrak{G}_n, \|w\| = \alpha_n\} \to \infty$ *as* $n \to \infty$*;*

(2) $t_n := \max\{\mathcal{E}_\lambda(w) : w \in \mathfrak{F}_n, \|w\| = \beta_n\} \le 0$

for a sufficiently large n .

Proof. The basic concept of the proof is carried out similarly to [52] (see also [32]). For the reader's convenience, we provide the proof. For any $w \in \mathfrak{G}_n$, suppose that $\|w\| > 1$. From assumptions (B1), (B2), (G1), and Lemma 3, as well as the similar argument in (37), it follows that

$$\begin{aligned}
\mathcal{E}_\theta(w) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w|)\, dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w)\, dy \\
&\ge \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q}\left(\int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla w|)\, dy + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |w|)\, dy\right) \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\, dy - \theta \int_{\mathbb{R}^N} G(y, w)\, dy \\
&\ge \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\|w\|_{L^s(\mathbb{R}^N)} - \frac{\theta\rho_2}{\ell}\|w\|^\ell_{L^\ell(\mathbb{R}^N)} \\
&\ge \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\chi_n^r\|w\|^r \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\| - \frac{\theta\rho_2}{\ell}\chi_n^\ell\|w\|^\ell \\
&\ge \left(\frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p} - \frac{\chi_n^\ell \theta\rho_2}{\ell}\|w\|^{\ell - p}\right)\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\chi_n^r\|w\|^r \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\|.
\end{aligned}$$

Since $p < \ell$, we obtain

$$\alpha_n = \left(\frac{\vartheta q 2^{p+1}\chi_n^\ell \theta\rho_2}{\min\{\kappa_0, \vartheta\}\ell}\right)^{\frac{1}{p-\ell}} \to \infty$$

as $n \to \infty$. Hence, if $w \in \mathfrak{G}_n$ and $\|w\| = \alpha_n$, then we find that

$$\mathcal{E}_\theta(w) \geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^{p+1}} \alpha_n^p - \frac{1}{r} \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \chi_n^r \alpha_n^r - \theta \|\rho_1\|_{L^{s'}(\mathbb{R}^N)} \chi_n \alpha_n \to \infty \quad \text{as} \quad n \to \infty,$$

which implies (1) because $\alpha_n \to \infty$, $\chi_n \to 0$ as $n \to \infty$ and $p > r > 1$.

Next, we show condition (2). To the contrary, suppose there is $n \in \mathbb{N}$ such that condition (2) is not fulfilled. Then, sequence $\{w_k\}$ exists in $\mathfrak{F}_n$ such that

$$\|w_k\| \to \infty \text{ as } k \to \infty \quad \text{and} \quad \mathcal{E}_\lambda(w_k) \geq 0. \tag{45}$$

Let $z_k = w_k / \|w_k\|$. Since $\dim \mathfrak{F}_n < \infty$, there is a $z \in \mathfrak{F}_n \setminus \{0\}$ such that, up to a subsequence still denoted by $\{z_k\}$,

$$\|z_k - z\| \to 0 \quad \text{and} \quad z_k(y) \to z(y)$$

for almost all $y \in \mathbb{R}^N$ as $k \to \infty$. We assert that $z(y) = 0$ for almost all $y \in \mathbb{R}^N$. If $z(y) \neq 0$, then $|w_k(y)| \to \infty$ for all $y \in \mathbb{R}^N$ as $k \to \infty$. Hence, in accordance with (G5), it follows that

$$\lim_{k \to \infty} \frac{G(y, w_k)}{\|w_k\|^{\vartheta q}} = \lim_{k \to \infty} \frac{G(y, w_k)}{|w_k(y)|^{\vartheta q}} |z_k(y)|^{\vartheta q} = \infty \tag{46}$$

for all $y \in \mathcal{B}_1 := \{y \in \mathbb{R}^N : z(y) \neq 0\}$. In the same fashion as in the proof of Lemma 6, we can choose a $\mathcal{C}_2 \in \mathbb{R}$ such that $G(y, \zeta) \geq \mathcal{C}_2$ for all $(y, \zeta) \in \mathbb{R}^N \times \mathbb{R}$, and so

$$\frac{G(y, w_k) - \mathcal{C}_2}{\|w_k\|^{\vartheta q}} \geq 0$$

for all $y \in \mathbb{R}^N$ and $k \in \mathbb{N}$. Using (46) and the Fatou lemma, we have the following:

$$\begin{aligned}
\liminf_{k \to \infty} \int_{\mathbb{R}^N} \frac{G(y, w_k)}{\|w_k\|^{\vartheta q}} dy &\geq \liminf_{k \to \infty} \int_{\mathcal{B}_1} \frac{G(y, w_k)}{\|w_k\|^{\vartheta q}} dy - \limsup_{k \to \infty} \int_{\mathcal{B}_1} \frac{\mathcal{C}_2}{\|w_k\|^{\vartheta q}} dy \\
&= \liminf_{k \to \infty} \int_{\mathcal{B}_1} \frac{G(y, w_k) - \mathcal{C}_2}{\|w_k\|^{\vartheta q}} dy \\
&\geq \int_{\mathcal{B}_1} \liminf_{k \to \infty} \frac{G(y, w_k) - \mathcal{C}_2}{\|w_k\|^{\vartheta q}} dy \\
&= \int_{\mathcal{B}_1} \liminf_{k \to \infty} \frac{G(y, w_k)}{\|w_k\|^{\vartheta q}} dy - \int_{\mathcal{B}_1} \limsup_{k \to \infty} \frac{\mathcal{C}_2}{\|w_k\|^{\vartheta q}} dy.
\end{aligned}$$

Thus, we infer

$$\int_{\mathbb{R}^N} \frac{G(y, w_k)}{\|w_k\|^{\vartheta q}} dy \to \infty \quad \text{as } k \to \infty.$$

We may assume that $\|w_k\| > 1$. Therefore, by (39), we have

$$\begin{aligned}
\mathcal{E}_\theta(w_k) &\leq 4^\vartheta C_9 \max\{\mathcal{M}(1), 1\} \|w_k\|^{\vartheta q} - \theta \int_{\mathbb{R}^N} G(y, w_k) \, dy \\
&\leq \|w_k\|^{\vartheta q} \left(4^\vartheta C_9 \max\{\mathcal{M}(1), 1\} - \theta \int_{\mathbb{R}^N} \frac{G(y, w_k)}{\|w_k\|^{\vartheta q}} dy \right) \to -\infty \quad \text{as } k \to \infty,
\end{aligned}$$

which contradicts (45). This completes the proof. □

With the help of Lemma 7, we are ready to establish the existence of infinitely many large-energy solutions.

Theorem 1. *Assume that* (B1), (B2), (G1), (G2), *and* (G5) *hold. If* $g(y, -\zeta) = -g(y, \zeta)$ *holds for all* $(y, \zeta) \in \mathbb{R}^N \times \mathbb{R}$, *then for any* $\theta > 0$, *Problem* (1) *yields a sequence of non-trivial weak solutions* $\{w_k\}$ *in* $\mathfrak{E}$ *such that* $\mathcal{E}_\theta(w_k) \to \infty$ *as* $k \to \infty$.

Proof. Clearly, $\mathcal{E}_\theta$ is an even functional and the $(C)_c$-condition by Lemma 5 is ensured. From Lemma 9, this assertion can be immediately derived from the fountain theorem. This completes the proof. □

Theorem 2. *Assume that* (B1), (B2), (G1), (G3), *and* (G5) *hold. If g is odd in* $\mathfrak{E}$, *then for any* $\theta > 0$, *Problem* (1) *yields a sequence of non-trivial weak solutions* $\{w_k\}$ *in* $\mathfrak{E}$ *such that* $\mathcal{E}_\theta(w_k) \to \infty$ *as* $k \to \infty$.

Proof. If we replace Lemma 5 with Lemma 6, the proof is the same as in Theorem 1. □

Definition 3. *Suppose that* $(\mathfrak{X}, \|\cdot\|)$ *is a real separable and reflexive Banach space. We say that* $\mathcal{F}$ *satisfies the* $(C)_c^*$*-condition (with respect to* $\mathfrak{F}_k$*) if any sequence* $\{w_k\}_{k\in\mathbb{N}} \subset \mathfrak{X}$ *for which* $w_k \in \mathfrak{F}_k$ *for any* $k \in \mathbb{N}$

$$\mathcal{F}(w_k) \to c \quad \text{and} \quad \|(\mathcal{F}|_{\mathfrak{F}_k})'(w_k)\|_{\mathfrak{X}^*}(1 + \|w_k\|) \to 0 \text{ as } k \to \infty,$$

possesses a subsequence converging to a critical point of $\mathcal{F}$.

Lemma 10 (Dual Fountain Theorem [34])**.** *Assume that* $(\mathfrak{X}, \|\cdot\|)$ *is a Banach space, and* $\mathcal{F} \in C^1(\mathfrak{X}, \mathbb{R})$ *is an even functional. If* $n_0 > 0$ *so that for each* $n \geq n_0$ *there exists* $\beta_n > \alpha_n > 0$ *such that the following holds:*

$(\mathcal{A}_1)$ $\inf\{\mathcal{F}(\varpi) : \varpi \in \mathfrak{G}_n, \|\varpi\| = \beta_n\} \geq 0$;
$(\mathcal{A}_2)$ $\delta_n := \max\{\mathcal{F}(\varpi) : \varpi \in \mathfrak{F}_n, \|\varpi\| = \alpha_n\} < 0$;
$(\mathcal{A}_3)$ $\phi_n := \inf\{\mathcal{F}(\varpi) : \varpi \in \mathfrak{G}_n, \|\varpi\| \leq \beta_n\} \to 0$ *as* $n \to \infty$;
$(\mathcal{A}_4)$ $\mathcal{F}$ *fulfills the* $(C)_c^*$*-condition for every* $c \in [\phi_{n_0}, 0)$,

then $\mathcal{F}$ *yields a sequence of negative critical values* $d_k < 0$ *satisfying* $d_k \to 0$ *as* $k \to \infty$.

Next, we check all the conditions of the dual fountain theorem.

Lemma 11. *Assume that* (B1), (B2), (G1), *and* (G2) *hold. Then, the functional* $\mathcal{E}_\theta$ *satisfies the* $(C)_c^*$*-condition for any* $\theta > 0$.

Proof. First, we claim that Φ' is a mapping of type (S_+). Let $\{w_k\}$ be any sequence in $\mathfrak{E}$ such that $w_k \rightharpoonup w_0$ in $\mathfrak{E}$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \langle \Phi'(w_k) - \Phi'(w_0), w_k - w_0 \rangle \leq 0.$$

Then, by using the notation in Lemma 5, we know the following:

$$\lim_{k\to\infty} \left[M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w_k|)\, dy \right) \left[\tilde{\Phi}_{w_k}(w_k - w_0) - \tilde{\Phi}_{w_0}(w_k - w_0) \right] \right.$$
$$\left. + \tilde{\Psi}_{w_k}(w_k - w_0) + \tilde{\Psi}_{w_0}(w_k - w_0) \right] \leq 0.$$

According to (19) and (20), we find the following:

$$\lim_{k\to\infty} \langle \Phi'(w_k) - \Phi'(w_0), w_k - w_0 \rangle = 0.$$

Therefore, using (12), (26), (29), (32), and (35), $w_k \to w_0$ in $\mathfrak{E}$ as $k \to \infty$ as claimed.

Let $c \in \mathbb{R}$, and let the sequence $\{w_k\}$ in $\mathfrak{E}$ be such that $w_k \in \mathfrak{F}_k$ for any $k \in \mathbb{N}$

$$\mathcal{E}_\theta(w_k) \to c \quad \text{and} \quad \|(\mathcal{E}_\theta|_{\mathfrak{F}_k})'(w_k)\|_{\mathfrak{E}^*}(1 + \|w_k\|) \to 0 \text{ as } k \to \infty.$$

Therefore, we obtain $c = \mathcal{E}_\theta(w_k) + o_k(1)$ and $\langle \mathcal{E}_\theta'(w_k), w_k \rangle = o_k(1)$, where $o_k(1) \to 0$ as $k \to \infty$. Repeating the argument from Lemma 6 proof, we derive the boundedness of $\{w_k\}$ in $\mathfrak{E}$. Therefore, there is a subsequence, still denoted by $\{w_k\}$, and a function w_0 in $\mathfrak{E}$ such that $w_k \rightharpoonup w_0$ in $\mathfrak{E}$ as $k \to \infty$.

To complete this proof, we will show that $w_k \to w_0$ in $\mathfrak{E}$ as $k \to \infty$, and also, w_0 is a critical point of $\mathcal{E}_\theta$. Though the concept of this proof follows that in [34] (Lemma 3.12), we provide it here for convenience. As $\mathfrak{E} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{F}_k}$, we can choose $v_k \in \mathfrak{F}_k, k \in \mathbb{N}$ such that $v_k \to w_0$ as $k \to \infty$. Since $\|(\mathcal{E}_\theta|_{\mathfrak{F}_k})'(w_k)\|_{\mathfrak{E}^*} \to 0$, $\{w_k - v_k\}$ is bounded, and $w_k - v_k \in \mathfrak{F}_k$, we have

$$\langle \mathcal{E}_\theta'(w_k), w_k - v_k \rangle = \langle (\mathcal{E}_\theta|_{\mathfrak{F}_k})'(w_k), w_k - v_k \rangle \to 0 \text{ as } k \to \infty. \tag{47}$$

The analogous argument in Lemma 9 [47] implies that Φ' is continuous, bounded, and strictly monotone. This, together with Lemma 4, indicates that $\{\mathcal{E}_\theta'(w_k)\}$ is bounded because $\{w_k\}$ is bounded. Thus,

$$\langle \mathcal{E}_\theta'(w_k), v_k - w_0 \rangle \to 0 \text{ as } k \to \infty. \tag{48}$$

Using (47) and (48), we find that

$$\langle \mathcal{E}_\theta'(w_k), w_k - w_0 \rangle \to 0 \text{ as } k \to \infty.$$

Therefore,

$$\langle \mathcal{E}_\theta'(w_k) - \mathcal{E}_\theta'(w_0), w_k - w_0 \rangle \to 0 \text{ as } k \to \infty. \tag{49}$$

According to Lemma 4, we know the following:

$$\langle \Psi_\theta'(w_k) - \Psi_\theta'(w_0), w_k - w_0 \rangle \to 0 \text{ as } k \to \infty. \tag{50}$$

Based on (49) and (50), we derive that

$$\langle \Phi'(w_k) - \Phi'(w_0), w_k - w_0 \rangle \to 0 \text{ as } k \to \infty.$$

Since Φ' is a mapping of type (S_+), we conclude that $w_k \to w_0$ as $k \to \infty$. Furthermore, we have $\mathcal{E}_\theta'(w_k) \to \mathcal{E}_\theta'(w_0)$ as $k \to \infty$. Then, we can prove that w_0 is a critical point of $\mathcal{E}_\theta$. Indeed, fix $k_0 \in \mathbb{N}$ and take any $u \in \mathfrak{F}_{k_0}$. For $k \geq k_0$, we find that

$$\begin{aligned}\langle \mathcal{E}_\theta'(w_0), u \rangle &= \langle \mathcal{E}_\theta'(w_0) - \mathcal{E}_\theta'(w_k), u \rangle + \langle \mathcal{E}_\theta'(w_k), u \rangle \\ &= \langle \mathcal{E}_\theta'(w_0) - \mathcal{E}_\theta'(w_k), u \rangle + \langle (\mathcal{E}_\theta|_{\mathfrak{F}_k})'(w_k), u \rangle;\end{aligned}$$

thus, passing the limit on the right side of the previous equation, as $k \to \infty$, we obtain

$$\langle \mathcal{E}_\theta'(w_0), u \rangle = 0 \text{ for all } u \in \mathfrak{F}_{k_0}.$$

As k_0 is taken arbitrarily and $\bigcup_{k\in\mathbb{N}} \mathfrak{F}_k$ is dense in $\mathfrak{E}$, we have $\mathcal{E}_\theta'(w_0) = 0$ as required. Then, we conclude that $\mathcal{E}_\theta$ satisfies the $(\mathrm{C})_c^*$-condition for any $c \in \mathbb{R}$ and for any $\theta > 0$. □

Lemma 12. *Assume that* (B1), (B2), (G3), *and* (G5) *hold. Then, the functional $\mathcal{E}_\theta$ satisfies the $(\mathrm{C})_c^*$-condition for any $\theta > 0$.*

Proof. Based on Lemma 6, we obtain that $\{w_n\}$ is a bounded sequence in $\mathfrak{E}$. The proof is the same as for Lemma 11. □

Lemma 13. *Assume that* (B1), (B2), *and* (G1) *hold. Then, there is* $n_0 > 0$ *so that for each* $n \geq n_0$*, there exists* $\beta_n > 0$ *such that*

$$\inf\{\mathcal{E}_\theta(w) : w \in \mathfrak{G}_n, \|w\| = \beta_n\} \geq 0.$$

Proof. Let $\chi_n < 1$ for a sufficiently large n. Based on (G1), Lemma 3, and the definition of χ_n, we find

$$\begin{aligned}
\mathcal{E}_\theta(w) \geq & \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q}\Big(\int_{\mathbb{R}^N}\mathcal{H}(y, |\nabla w|)\,dy + \int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V}}(y, |w|)\,dy\Big) \\
& - \frac{1}{r}\int_{\mathbb{R}^N}\sigma(y)|w|^r\,dy - \theta\int_{\mathbb{R}^N}G(y, w)\,dy \\
\geq & \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\chi_n^r\|w\|^r \\
& - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\| - \frac{\theta\rho_2}{\ell}\chi_n^\ell\|w\|^\ell \\
\geq & \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \Big(\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta\rho_2}{\ell}\Big)\chi_n^r\|w\|^\ell \\
& - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\|
\end{aligned}$$

for a sufficiently large n and $\|w\| \geq 1$. Let us choose

$$\beta_n = \left[\left(\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta\rho_2}{\ell}\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\chi_n^r\right]^{\frac{1}{p-2\ell}}. \tag{51}$$

Let $w \in \mathfrak{G}_n$ with $\|w\| = \beta_n > 1$ for a sufficiently large k . Then, there is $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathcal{E}_\theta(w) \geq & \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p \\
& - \left(\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta\rho_2}{\ell}\right)\chi_n^r\|w\|^\ell - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\| \\
\geq & \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^{p+1}}\beta_n^p \\
& - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\left[\left(\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta\rho_2}{\ell}\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\right]^{\frac{1}{p-2\ell}}\chi_n^{\frac{r+p-2\ell}{p-2\ell}} \\
\geq & 0
\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$, which implies that the conclusion holds since $\lim_{n\to\infty}\beta_n^p = \infty$ and $\chi_n \to 0$ as $n \to \infty$. $\square$

Lemma 14. *Assume that* (B1), (B2), (G1), *and* (G4) *hold. Then for each sufficiently large* $n \in \mathbb{N}$*, there exists* $\alpha_n > 0$ *with* $0 < \alpha_n < \beta_n$ *such that*

(1) $\delta_n := \max\{\mathcal{E}_\theta(w) : w \in \mathfrak{F}_n, \|w\| = \alpha_n\} < 0$*;*
(2) $\phi_n := \inf\{\mathcal{E}_\theta(w) : w \in \mathfrak{G}_n, \|w\| \leq \beta_n\} \to 0$ *as* $n \to \infty$*,*

where β_n *is given in Lemma 13.*

Proof. (1): Since $\mathfrak{F}_n$ is a finite dimensional, $\|\cdot\|_{L^d(\xi,\mathbb{R}^N)}$, $\|\cdot\|_{L^\ell(\mathbb{R}^N)}$, and $\|\cdot\|$ are equivalent on $\mathfrak{F}_n$. Then, $\varrho_{1,n} > 0$ and $\varrho_{2,n} > 0$ exist such that

$$\varrho_{1,n}\|w\| \leq \|w\|_{L^d(\xi,\mathbb{R}^N)} \text{ and } \|w\|_{L^\ell(\mathbb{R}^N)} \leq \varrho_{2,n}\|w\|$$

for any $w \in \mathfrak{F}_n$. Let $w \in \mathfrak{F}_n$ with $\|w\| \leq 1$. Based on (G1) and (G4), there are $C_{10}, C_{11} > 0$ such that

$$G(y,\zeta) \geq C_{10}\xi(y)|\zeta|^d - C_{11}|\zeta|^\ell$$

for almost all $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$. According to Lemma 3, we obtain

$$\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla w|)\,dy \leq \mathcal{K}$$

for some positive constant $\mathcal{K}$. Then, we have

$$\begin{aligned}
\mathcal{E}_\theta(w) &\leq \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla w|)\,dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y,|w|)\,dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\,dy - \theta\int_{\mathbb{R}^N} G(y,w)\,dy \\
&\leq \left(\sup_{0\leq\tilde{\zeta}\leq\mathcal{K}} M(\tilde{\zeta})\right)\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla w|)\,dy + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y,|w|)\,dy \\
&\quad - \theta C_{10}\int_{\mathbb{R}^N} \xi(y)|w|^d\,dy + \theta C_{11}\int_{\mathbb{R}^N} |w|^\ell\,dy \\
&\leq C_{12}\|w\|^p - \theta C_{10}\|w\|^d_{L^d(\xi,\mathbb{R}^N)} + \theta C_{11}\|w\|^\ell_{L^\ell(\mathbb{R}^N)} \\
&\leq C_{12}\|w\|^p - \theta C_{10}\varrho^d_{1,n}\|w\|^d + \theta C_{11}\varrho^\ell_{2,n}\|w\|^\ell
\end{aligned} \tag{52}$$

for some positive constant C_{12}. Let $f(x) = C_{12}x^p - \theta C_{10}\varrho^d_{1,n}x^d + \theta C_{11}\varrho^\ell_{2,n}x^\ell$. Since $d < p < \ell$, we infer $f(x) < 0$ for all $x \in (0,x_0)$ for sufficiently small $x_0 \in (0,1)$. Hence, we can find $\alpha_n > 0$ such that $\mathcal{E}_\theta(w) < 0$ for all $w \in \mathfrak{F}_n$ with $\|w\| = \alpha_n < x_0$ for a sufficiently large k. If necessary, we can change n_0 to a large value so that $\beta_n > \alpha_n > 0$ and

$$\delta_n := \max\{\mathcal{E}_\theta(w) : w \in \mathfrak{F}_n, \|w\| = \alpha_n\} < 0$$

for all $n \geq n_0$.

(2): Because $\mathfrak{F}_n \cap \mathfrak{G}_n \neq \phi$ and $0 < \alpha_n < \beta_n$, we have $\phi_n \leq \delta_n < 0$ for all $n \geq n_0$. For any $w \in \mathfrak{G}_n$ with $\|w\| = 1$ and $0 < t < \beta_n$, we have

$$\begin{aligned}
\mathcal{E}_\theta(tw) &\geq \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla tw|)\,dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y,|tw|)\,dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|tw|^r\,dy - \theta\int_{\mathbb{R}^N} G(y,tw)\,dy \\
&\geq -\frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|tw|^r\,dy - \theta\int_{\mathbb{R}^N} G(y,tw)\,dy \\
&\geq -\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|tw\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta\int_{\mathbb{R}^N} \rho_1(y)|tw|dy - \frac{\theta\rho_2}{\ell}\int_{\mathbb{R}^N} |tw|^\ell dy \\
&\geq -\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\beta_n^r\|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \beta_n\theta\int_{\mathbb{R}^N} \rho_1(y)|w|dy - \frac{\theta\rho_2}{\ell}\beta_n^\ell\int_{\mathbb{R}^N} |w|^\ell dy \\
&\geq -\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\beta_n^r\chi_n^r - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\beta_n\chi_n - \frac{\theta\rho_2}{\ell}\beta_n^\ell\chi_n^\ell
\end{aligned} \tag{53}$$

for a sufficiently large n, where χ_n and β_n are given in (44) and (51), respectively. Hence, based on the definition of β_n, it follows that

$$
\begin{aligned}
0 > \phi_n \geq & -\frac{\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}}{r} \beta_n^r \chi_n^r - \theta \|\rho_1\|_{L^{s'}(\mathbb{R}^N)} \beta_n \chi_n - \frac{\theta \rho_2}{\ell} \beta_n^\ell \chi_n^\ell \\
= & -\frac{\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}}{r} \left[\left(\frac{1}{r} \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta \rho_2}{\ell} \right) q 2^{p+1} \right]^{\frac{r}{p-2\ell}} \chi_n^{\frac{(r+p-2\ell)r}{p-2\ell}} \\
& - \|\rho_1\|_{L^{s'}(\mathbb{R}^N)} \left[\left(\frac{1}{r} \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta \rho_2}{\ell} \right) q 2^{p+1} \right]^{\frac{1}{p-2\ell}} \chi_n^{\frac{r+p-2\ell}{p-2\ell}} \\
& - \frac{\rho_2}{\ell} \left[\left(\frac{1}{r} \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta \rho_2}{\ell} \right) q 2^{p+1} \right]^{\frac{\ell}{p-2\ell}} \chi_n^{\frac{(r+p-2\ell)\ell}{p-2\ell}}.
\end{aligned}
$$

Because $p < p + r < 2\ell$ and $\chi_n \to 0$ as $n \to \infty$, we derive that $\lim_{n\to\infty} \phi_n = 0$. □

With the aid of Lemmas 10 and 11, we are in a position to establish our final consequences.

Theorem 3. *Under the assumptions in Theorem 1, if* (G4) *holds, then Problem* (1) *yields a sequence of non-trivial weak solutions* $\{w_k\}$ *in* $\mathfrak{E}$ *such that* $\mathcal{E}_\theta(w_k) \to 0$ *as* $k \to \infty$ *for any* $\theta > 0$.

Proof. Due to Lemma 11, we note that the functional $\mathcal{E}_\theta$ is even and fulfills the $(C)_c^*$-condition for every $c \in [\phi_{n_0}, 0)$. Based on Lemmas 13 and 14, we ensure that properties $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and $(\mathcal{A}_3)$ in the dual fountain theorem hold. Therefore, problem (1) possesses a sequence of weak solutions $\{w_k\}$ with a sufficiently large k. The proof is complete. □

Theorem 4. *Under the assumptions in Theorem 2, if* (G4) *holds, then Problem* (1) *yields a sequence of non-trivial weak solutions* $\{w_k\}$ *in* $\mathfrak{E}$ *such that* $\mathcal{E}_\theta(w_k) \to 0$ *as* $k \to \infty$ *for any* $\theta > 0$.

Proof. Similar to Theorem 3, instead of Lemma 11, we apply Lemma 12 to obtain this result. □

Finally, we demonstrate the existence of a sequence of infinitely many weak solutions to (1) that converges to 0 in L^∞-space. To accomplish this, we needed the following additional assumptions regarding g:

(G6) There exists a constant $\zeta_1 > 0$ such that $g(y,\zeta)$ is odd in $\mathbb{R}^N \times (-\zeta_1, \zeta_1)$ and $pG(y,\zeta) - g(y,\zeta)\zeta > 0$ for all $y \in \mathbb{R}^N$ and for $0 < |\zeta| < \zeta_1$;

(G7) $\lim_{|\zeta|\to 0} \frac{g(y,\zeta)}{|\zeta|^{p-2}\zeta} = +\infty$ uniformly for all $y \in \mathbb{R}^N$.

The following assertion follows upon the analogous arguments of Proposition 1 in [40] and Proposition 3.1 in [39].

Proposition 1. *Assume that* (G1) *holds. If w is a weak solution of Problem* (1)*, then* $w \in L^\infty(\mathbb{R}^N)$*, and there exist positive constants* C, η *independent of* w *such that*

$$\|w\|_{L^\infty(\mathbb{R}^N)} \leq C \|w\|_{L^\ell(\mathbb{R}^N)}^\eta.$$

With the help of Lemma 10 and Proposition 1, we are in a position to derive our final major result.

Theorem 5. *Suppose that* (B1), (B2), (G1), (G6), *and* (G7) *hold. In addition, suppose that*

(M5) $\mathcal{M}(t) \leq M(t)t$ *for any* $t \geq 0$.

Then, there exists an interval Γ *such that problem* (1) *has a sequence of non-trivial solutions* $\{w_n\}$ *in* $\mathfrak{E}$ *whose* $\mathcal{E}_\theta(w_n) \to 0$ *and* $\|w_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ *as* $n \to \infty$ *for every* $\theta \in \Gamma$.

Proof. To obtain the desired properties of the energy functional, as in Lemma 10, we modify the nonlinear term g as follows. According to (G6) and (G7), for any $\mathfrak{M}_3 > 0$, there exists $\zeta_2 \in (0, \min\{\zeta_1, 1\})$ such that

$$G(y,\zeta) \geq \mathfrak{M}_3|\zeta|^p \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } |\zeta| < \zeta_2. \tag{54}$$

Fix $\zeta_3 \in (0, \zeta_2/2)$, and let $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ be such that φ is even, $\varphi(\zeta) = 1$ for $|\zeta| \leq \zeta_3$, $\varphi(\zeta) = 0$ for $|\zeta| \geq 2\zeta_3$, $|\varphi'(\zeta)| \leq 2/\zeta_3$, and $\varphi'(\zeta)\zeta \leq 0$. We then define the modified function $\widetilde{g} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{g}(y,\zeta) := \frac{\partial}{\partial \zeta}\widetilde{G}(y,\zeta),$$

where

$$\widetilde{G}(y,\zeta) := \varphi(\zeta)G(y,\zeta) + (1-\varphi(\zeta))\xi|\zeta|^p$$

for some fixed $\xi \in \left(0, \min\left\{\frac{1}{p}, \frac{1}{qC_{p,imb}^p}\right\}\right)$ with $C_{p,imb}$ being the embedding constant for the embedding $\mathfrak{E} \hookrightarrow L^p(\mathbb{R}^N)$ by means of Lemma 2. Clearly, $\widetilde{G}$ is even in ζ,

$$\widetilde{g}(y,\zeta) = \varphi'(\zeta)G(y,\zeta) + \varphi(\zeta)g(y,\zeta) - \varphi'(\zeta)\xi|\zeta|^p + (1-\varphi(\zeta))\xi p|\zeta|^{p-2}\zeta, \tag{55}$$

and

$$p\widetilde{G}(y,\zeta) - \widetilde{g}(y,\zeta)\zeta = \varphi(\zeta)\left[pG(y,\zeta) - g(y,\zeta)\zeta\right] - \varphi'(\zeta)\zeta\left[G(y,\zeta) - \xi|\zeta|^p\right].$$

Thus, the definition of φ and (54) yield the following:

$$p\widetilde{G}(y,\zeta) - \widetilde{g}(y,\zeta)\zeta \geq 0 \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } \zeta \in \mathbb{R}, \tag{56}$$

and

$$p\widetilde{G}(y,\zeta) - \widetilde{g}(y,\zeta)\zeta = 0 \quad \text{if and only if } \zeta = 0 \text{ or } |\zeta| \geq 2\zeta_3. \tag{57}$$

By the definition of $\widetilde{G}$ and (G1), we infer

$$\widetilde{G}(y,\zeta) \leq \rho_1(y)|\zeta| + \frac{\rho_2}{\ell}|\zeta|^\ell + \xi|\zeta|^p \tag{58}$$

for a.e. $y \in \mathbb{R}^N$ and all $\zeta \in \mathbb{R}$. Consider the modified energy functional $\widetilde{\mathcal{E}}_\theta : \mathfrak{E} \to \mathbb{R}$ given by

$$\widetilde{\mathcal{E}}_\theta(w) := \Phi(w) - \widetilde{\Psi}_\theta(w),$$

where

$$\widetilde{\Psi}_\theta(w) = \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\,dy + \theta\int_{\mathbb{R}^N} \widetilde{G}(y,w)\,dy.$$

Subsequently, by a standard argument invoking the embedding $\mathfrak{E} \hookrightarrow L^p(\mathbb{R}^N)$ and the differentiability of Φ, we can show that $\widetilde{\mathcal{E}}_\theta \in C^1(\mathfrak{E}, \mathbb{R})$ is an even functional. Furthermore, we have

$$\widetilde{\mathcal{E}}_\theta(u) = 0 = \langle \widetilde{\mathcal{E}}_\theta'(u), u\rangle \quad \text{if and only if} \quad u = 0. \tag{59}$$

Indeed, let $\widetilde{\mathcal{E}}_\theta(u) = \langle \widetilde{\mathcal{E}}_\theta'(u), u\rangle = 0$. Then, according to (M5), we find that

$$\begin{aligned}
0 &= -p\widetilde{\mathcal{E}}_\theta(u) \\
&= -p\mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla u|)\,dy\right) - p\int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y,|u|)\,dy \\
&\quad + \frac{p}{r}\int_{\mathbb{R}^N} \sigma(y)|u|^r\,dy + \theta p\int_{\mathbb{R}^N} \widetilde{G}(y,u)\,dy \\
&\geq -pM\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla u|)\,dy\right)\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y,|\nabla u|)\,dy
\end{aligned} \tag{60}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |\nabla u|)\, dy + \int_{\mathbb{R}^N} \sigma(y)|u|^r\, dy + \theta \int_{\mathbb{R}^N} p\widetilde{G}(y,u)\, dy \\
&\geq -M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla u|)\, dy\right) \int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla u|)\, dy \\
&\quad - \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |\nabla u|)\, dy + \int_{\mathbb{R}^N} \sigma(y)|u|^r\, dy + \theta \int_{\mathbb{R}^N} p\widetilde{G}(y,u)\, dy
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \widetilde{\mathcal{E}}_\theta'(u), u\rangle &= M\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla u|)\, dy\right) \int_{\mathbb{R}^N} \mathcal{H}(y, |\nabla u|)\, dy \\
&\quad + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V}}(y, |\nabla u|)\, dy - \int_{\mathbb{R}^N} \sigma(y)|u|^r\, dy - \theta \int_{\mathbb{R}^N} \widetilde{g}(y,u)u\, dy = 0.
\end{aligned} \tag{61}
$$

Based on Equations (60) and (61), it follows that

$$
\int_{\mathbb{R}^N} \left(p\widetilde{G}(y,u) - \widetilde{g}(y,u)u\right) dy \leq 0.
$$

Consequently, the relations (56) and (57) imply $u = 0$.

($\mathcal{A}_1$): Let $\chi_n < 1$ for a sufficiently large n. Based on Lemmas 1 and 3 as well as the similar argument in (37), it follows that

$$
\begin{aligned}
\widetilde{\mathcal{E}}_\theta(w) &= \Phi(w) - \widetilde{\Psi}_\theta(w) \\
&= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w|)\, dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\, dy - \theta \int_{\mathbb{R}^N} \widetilde{G}(y,w)\, dy \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q}\left[\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy + \mathcal{H}_{\mathfrak{V},p,q}(y, |w|)\, dy\right] \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|w|^r\, dy - \theta \int_{\mathbb{R}^N} (G(y,w) + \xi|w|^p)\, dy \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta \int_{\mathbb{R}^N} G(y,w)\, dy - \theta\xi \int_{\mathbb{R}^N} |w|^p\, dy \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta \int_{\mathbb{R}^N} \left(\rho_1(y)|w| + \frac{\rho_2}{\ell}|w|^\ell\right) dy - \theta\xi\chi_n^p\|w\|^p \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\|w\|_{L^s(\mathbb{R}^N)} - \frac{\theta\rho_2}{\ell}\|w\|^\ell_{L^\ell(\mathbb{R}^N)} - \theta\xi\chi_n^p\|w\|^p \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\chi_n^r\|w\|^r \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\| - \frac{\theta\rho_2}{\ell}\chi_n^\ell\|w\|^\ell - \theta\xi\chi_n^p\|w\|^p \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}\chi_n^r\|w\|^r \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\chi_n\|w\| - \theta\left(\frac{\rho_2}{\ell} + \xi\right)\chi_n^p\|w\|^\ell.
\end{aligned}
$$

for a sufficiently large n and $\|w\| \geq 1$. Let us choose

$$\widetilde{\beta}_n = \left[\theta\left(\frac{\rho_2}{\ell} + \xi\right) \frac{\vartheta q 2^{p+1} \chi_n^p}{\min\{\kappa_0, \vartheta\}} \right]^{\frac{1}{p-2\ell}}$$

and let $w \in \mathfrak{G}_n$ with $\|w\| = \widetilde{\beta}_n > 1$ for a sufficiently large n. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\widetilde{\mathcal{E}}_\theta(w) &\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^p} \|w\|^p - \frac{1}{r} \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \chi_n^r \|w\|^r \\
&\quad - \theta \|\rho_1\|_{L^{s'}(\mathbb{R}^N)} \chi_n \|w\| - \theta\left(\frac{\rho_2}{\ell} + \xi\right) \chi_n^p \|w\|^\ell \\
&\geq \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^{p+1}} \widetilde{\beta}_n^p - \frac{1}{r} \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \left[\theta\left(\frac{\rho_2}{\ell} + \xi\right) \frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}} \right]^{\frac{r}{p-2\ell}} \chi_n^{\frac{2r(p-\ell)}{p-2\ell}} \\
&\quad - \theta \|\rho_1\|_{L^{s'}(\mathbb{R}^N)} \left[\theta\left(\frac{\rho_2}{\ell} + \xi\right) \frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}} \right]^{\frac{1}{p-2\ell}} \chi_n^{\frac{2(p-\ell)}{p-2\ell}} \\
&\geq 0
\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ by being

$$\lim_{n \to \infty} \frac{\min\{\kappa_0, \vartheta\}}{\vartheta q 2^{p+1}} \widetilde{\beta}_n^p = \infty.$$

Then, we find the following:

$$\inf\{\widetilde{\mathcal{E}}_\theta(w) : w \in \mathfrak{G}_n, \|w\| = \widetilde{\beta}_n\} \geq 0.$$

$(\mathcal{A}_2)$: Observe that $\|\cdot\|_{L^\infty(\mathbb{R}^N)}$, $\|\cdot\|_{L^p(\mathbb{R}^N)}$, and $\|\cdot\|$ are equivalent on $\mathfrak{F}_n$. Then, there are positive constants $\widetilde{\varrho}_{1,n}$ and $\widetilde{\varrho}_{2,n}$ such that

$$\widetilde{\varrho}_{1,n} \|w\|_{L^\infty(\mathbb{R}^N)} \leq \|w\| \leq \widetilde{\varrho}_{2,n} \|w\|_{L^p(\mathbb{R}^N)} \tag{62}$$

for any $w \in \mathfrak{F}_n$. From (G6) and (G7), for any $\mathfrak{M}_3 > 0$, there exists $\zeta_3 \in (0, \zeta_2/2)$ such that

$$G(y, \zeta) \geq \frac{\mathfrak{M}_3 \widetilde{\varrho}_{2,n}^p}{p} |\zeta|^p$$

for almost all $y \in \mathbb{R}^N$ and all $|\zeta| \leq \zeta_3$. Choose $\widetilde{\alpha}_n := \min\{\frac{1}{2}, \zeta_3 \widetilde{\varrho}_{1,n}\}$ for all $n \in \mathbb{N}$. Then, we know that $\|w\|_{L^\infty(\mathbb{R}^N)} \leq \zeta_3$ for $w \in \mathfrak{F}_n$ with $\|w\| = \widetilde{\alpha}_n$, and so $\widetilde{G}(y, w) = G(y, w)$. From the analogous argument in (52) and based on (62), we derive the following:

$$\begin{aligned}
\widetilde{\mathcal{E}}_\theta(w) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy \right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w|)\, dy \\
&\quad - \frac{1}{r} \int_{\mathbb{R}^N} \sigma(y) |w|^r\, dy - \theta \int_{\mathbb{R}^N} \widetilde{G}(y, w)\, dy \\
&\leq \left(\sup_{0 \leq \tilde{\zeta} \leq \mathcal{K}} M(\tilde{\zeta}) \right) \int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla w|)\, dy \\
&\quad + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |w|)\, dy - \theta \int_{\mathbb{R}^N} \frac{\mathfrak{M}_3 \widetilde{\varrho}_{2,n}^p}{p} |w|^p\, dy \\
&\leq C_{12} \|w\|^p - \frac{\theta \mathfrak{M}_3 \widetilde{\varrho}_{2,n}^p}{p} \|w\|_{L^p(\mathbb{R}^N)}^p
\end{aligned}$$

$$\leq C_{12}\|w\|^p - \frac{\theta\mathfrak{M}_3}{p}\|w\|^p$$
$$\leq \frac{pC_{12} - \theta\mathfrak{M}_3}{p}\widetilde{\alpha}_n^p$$

for any $w \in \mathfrak{F}_n$ with $\|w\| = \widetilde{\alpha}_n$. If we choose a sufficiently large $\mathfrak{M}_3$ such that $1 < \theta\mathfrak{M}_3$, we obtain the following:

$$\widetilde{\delta}_n = \max\{\widetilde{\mathcal{E}}_\theta(w) : w \in \mathfrak{F}_n, \|w\| = \widetilde{\alpha}_n\} < 0.$$

If necessary, we can change n_0 to a larger value so that $\widetilde{\beta}_n > \widetilde{\alpha}_n > 0$ for all $n \geq n_0$.

($\mathcal{A}_3$): Because $\mathfrak{Y}_n \cap \mathfrak{G}_n \neq \phi$ and $0 < \widetilde{\alpha}_n < \widetilde{\beta}_n$, we have $\widetilde{\phi}_n \leq \widetilde{\delta}_n < 0$ for all $n \geq n_0$. For any $w \in \mathfrak{G}_n$ with $\|w\| = 1$ and $0 < t < \widetilde{\beta}_n$, we have

$$\begin{aligned}
\widetilde{\mathcal{E}}_\theta(tw) &= \mathcal{M}\left(\int_{\mathbb{R}^N} \mathcal{H}_{p,q}(y, |\nabla tw|)\,dy\right) + \int_{\mathbb{R}^N} \mathcal{H}_{\mathfrak{V},p,q}(y, |tw|)\,dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} \sigma(y)|tw|^r\,dy - \theta\int_{\mathbb{R}^N} \widetilde{G}(y, tw)\,dy \\
&\geq -\frac{1}{r}\widetilde{\beta}_n^r \int_{\mathbb{R}^N} \sigma(y)|w|^r\,dy - \theta\int_{\mathbb{R}^N} (G(y, tw) + \xi|tw|^p)\,dy \\
&\geq -\frac{1}{r}\widetilde{\beta}_n^r \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta\int_{\mathbb{R}^N} G(y, tw)\,dy - \theta\xi\int_{\mathbb{R}^N} |tw|^p\,dy \\
&\geq -\frac{1}{r}\widetilde{\beta}_n^r \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta\int_{\mathbb{R}^N} \rho_1(y)|tw|\,dy - \frac{\theta\rho_2}{\ell}\int_{\mathbb{R}^N} |tw|^\ell\,dy - \theta\xi\int_{\mathbb{R}^N} |tw|^p dy \\
&\geq -\frac{1}{r}\widetilde{\beta}_n^r \|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&\quad - \theta\widetilde{\beta}_n\int_{\mathbb{R}^N} \rho_1(y)|w|\,dy - \frac{\theta\rho_2}{\ell}\widetilde{\beta}_n^\ell\int_{\mathbb{R}^N} |w|^\ell\,dy - \theta\xi\widetilde{\beta}_n^p\int_{\mathbb{R}^N} |w|^p\,dy \\
&\geq -\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \widetilde{\beta}_n^r\chi_n^r - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\widetilde{\beta}_n\chi_n - \frac{\theta\rho_2}{\ell}\widetilde{\beta}_n^\ell\chi_n^\ell - \theta\xi\widetilde{\beta}_n^p\chi_n^p,
\end{aligned}$$

where χ_n is given in (44). Hence, we achieve

$$\begin{aligned}
0 > \widetilde{\phi}_n &\geq -\frac{\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}}{r}\widetilde{\beta}_n^r\chi_n^r - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\widetilde{\beta}_n\chi_n - \frac{\theta\rho_2}{\ell}\widetilde{\beta}_n^\ell\chi_n^\ell - \theta\xi\widetilde{\beta}_n^p\chi_n^p \\
&\geq -\frac{\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}}{r}\left[\theta\left(\frac{\rho_2}{\ell} + \xi\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\right]^{\frac{r}{p-2\ell}} \chi_n^{\frac{2r(p-\ell)}{p-2\ell}} \\
&\quad - \theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\left[\theta\left(\frac{\rho_2}{\ell} + \xi\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\right]^{\frac{1}{p-2\ell}} \chi_n^{\frac{2(p-\ell)}{p-2\ell}} \\
&\quad - \frac{\theta\rho_2}{\ell}\left[\theta\left(\frac{\rho_2}{\ell} + \xi\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\right]^{\frac{\ell}{p-2\ell}} \chi_n^{\frac{2\ell(p-\ell)}{p-2\ell}} \\
&\quad - \theta\xi\left[\theta\left(\frac{\rho_2}{\ell} + \xi\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\right]^{\frac{p}{p-2\ell}} \chi_n^{\frac{2p(p-\ell)}{p-2\ell}}.
\end{aligned}$$

Because $p < \ell$ and $\chi_n \to 0$ as $n \to \infty$, we conclude that $\lim_{n\to\infty} \widetilde{\phi}_n = 0$.

($\mathcal{A}_4$): Before proving that $\widetilde{\mathcal{E}}_\theta$ ensures the $(C)_c^*$-condition, we have to show that $\widetilde{\Psi}'_\theta$ is sequentially weakly strongly continuous on $\mathfrak{E}$ for any $\theta > 0$ and that $\widetilde{\mathcal{E}}_\theta$ is coercive.

Therefore, we first derive that $\widetilde{\mathcal{E}}_\theta$ ensures the $(C)_c$-condition for any $c \in \mathbb{R}$ and for every $\theta \in \Gamma$. Let $\{w_k\}$ be a sequence in $\mathfrak{E}$ such that $w_k \rightharpoonup w$ in $\mathfrak{E}$ as $k \to \infty$. Since $\{w_k\}$ is bounded in $\mathfrak{E}$, Lemma 3 guarantees that there exists a subsequence $\left\{w_{k_j}\right\}$ such that

$$w_{k_j}(y) \to w(y) \text{ a.e. in } \mathbb{R}^N \text{ and } w_{k_j} \to w \text{ in } L^m(\mathbb{R}^N) \text{ as } j \to \infty, \tag{63}$$

where $p \le m < p^*$. By the convergence principle, there exists a subsequence $\{w_{k_j}\}$ and a non-negative function $v \in L^p(\mathbb{R}^N) \cap L^\ell(\mathbb{R}^N) \cap L^{\gamma_0}(\mathbb{R}^N)$ such that $w_{k_j}(y) \to v(y)$ as $j \to \infty$ for almost all $y \in \mathbb{R}^N$, and $|w_{k_j}(y)| \le v(y)$ for all $j \in \mathbb{N}$ and for almost all $y \in \mathbb{R}^N$. For any $u \in \mathfrak{E}$, we have

$$\begin{aligned}
&\left|\langle \widetilde{\Psi}_\theta'\left(w_{k_j}\right) - \widetilde{\Psi}_\theta'(w), u\rangle\right| \\
&= \Big| \int_{\mathbb{R}^N} \left(\sigma(y)\left|w_{k_j}\right|^{r-2} w_{k_j} - \sigma(y)|w|^{r-2}w\right) u \, dy \\
&\qquad + \theta \int_{\mathbb{R}^N} \left(\widetilde{g}\left(y, w_{k_j}\right) - \widetilde{g}(y,w)\right) u \, dy \Big| \\
&\le \left(\int_{\mathbb{R}^N} \left|\sigma(y)\left|w_{k_j}\right|^{r-2} w_{k_j} - \sigma(y)|w|^{r-2}w\right|^{r'} dy\right) \|u\|_{L^r(\mathbb{R}^N)} \\
&\qquad + \theta \left|\int_{\mathbb{R}^N} \left(\widetilde{g}\left(y, w_{k_j}\right) - \widetilde{g}(y,w)\right) u \, dy\right|.
\end{aligned}$$

By Young's inequality, we infer that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|\sigma(y)\left|w_{k_j}\right|^{r-2} w_{k_j} - \sigma(y)|w|^{r-2}w\right|^{r'} dy \\
&\le C_{13} \int_{\mathbb{R}^N} |\sigma(y)|^{\frac{1}{r-1}} |\sigma(y)| \left(\left|w_{k_j}\right|^r + |w|^r\right) dy \\
&\le C_{14} \int_{\mathbb{R}^N} |\sigma(y)| \left(\left|w_{k_j}\right|^r + |w|^r\right) dy \\
&\le C_{15} \int_{\mathbb{R}^N} \left(\frac{2(\gamma_0 - r)}{\gamma_0} |\sigma(y)|^{\frac{\gamma_0}{\gamma_0 - r}} + \frac{r}{\gamma_0}|v|^{\gamma_0} + \frac{r}{\gamma_0}|w|^{\gamma_0}\right) dy
\end{aligned} \tag{64}$$

for some positive constants C_{13}, C_{14}, and C_{15}. By the definition of φ and (G1) and based on (55), we deduce that

$$|\widetilde{g}(y,\zeta)| \le C_{16}\left(\rho_1(y) + \rho_2|\zeta|^{\ell-1} + \xi p|\zeta|^{p-1}\right). \tag{65}$$

Due to (65), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \left(\widetilde{g}\left(y, w_{k_j}\right) - \widetilde{g}(y,w)\right) u \, dy\right| \\
&\le \int_{\mathbb{R}^N} \left(\left|\widetilde{g}\left(y, w_{k_j}\right)\right| + |\widetilde{g}(y,w)|\right)|u| \, dy \\
&\le C_{17} \int_{\mathbb{R}^N} (2\rho_1(y) + \rho_2\left|w_{k_j}\right|^{\ell-1} + \xi p\left|w_{k_j}\right|^{p-1} + \rho_2|w|^{\ell-1} + \xi p|w|^{p-1})|u| \, dy \\
&\le C_{17} \int_{\mathbb{R}^N} \left(2\rho_1(y) + \rho_2\left(|v|^{\ell-1} + |w|^{\ell-1}\right) + \xi p\left(|v|^{p-1} + |w|^{p-1}\right)\right)|u| \, dy
\end{aligned} \tag{66}$$

for some positive constants C_{16} and C_{17}. Invoking (63)–(66) and the convergence principle, we find the following:

$$\left|\sigma(y)\left|w_{k_j}\right|^{r-2} w_{k_j} - \sigma(y)|w|^{r-2}w\right|^{r'} \le f_1(y) \text{ and } \left|\left(\widetilde{g}\left(y, w_{k_j}\right) - \widetilde{g}(y,w)\right)u\right| \le f_2(y)$$

for almost all $y \in \mathbb{R}^N$ and for some $f_1, f_2 \in L^1(\mathbb{R}^N)$, and also, $\sigma(y)\left|w_{k_j}\right|^{r-2} w_{k_j} \to \sigma(y)|w|^{r-2}w$ and $\left|\left(\widetilde{g}\left(y, w_{k_j}\right) - \widetilde{g}(y,w)\right)u\right| \to 0$ as $j \to \infty$ for almost all $y \in \mathbb{R}^N$. This, together with Lebesgue's dominated convergence theorem, yields that

$$\begin{aligned}
&\|\widetilde{\Psi}'_\theta\left(w_{k_j}\right) - \widetilde{\Psi}'_\theta(w)\|_{\mathfrak{E}^*} \\
&= \sup_{\|u\|\le 1}\left|\left\langle \widetilde{\Psi}'_\theta\left(w_{k_j}\right) - \widetilde{\Psi}'_\theta(w), u\right\rangle\right| \\
&= \sup_{\|u\|\le 1}\left|\int_{\mathbb{R}^N}\left(\sigma(y)\left|w_{k_j}\right|^{r-2} w_{k_j} - \sigma(y)|w|^{\gamma-2}w\right)u\,dy\right. \\
&\qquad\qquad \left. + \theta\int_{\mathbb{R}^N}\left(\widetilde{g}\left(y, w_{k_j}\right) - \widetilde{g}(y,w)\right)u\,dy\right| \to 0
\end{aligned}$$

as $j \to \infty$. Therefore, we derive that $\widetilde{\Psi}'_\theta\left(w_{k_j}\right) \to \widetilde{\Psi}'_\theta(w)$ in $\mathfrak{E}^*$ as $j \to \infty$. Let $w \in \mathfrak{E}$ with $\|w\| \ge 1$. We set $\Lambda_1 := \{y \in \mathbb{R}^N : |w(y)| \le \zeta_3\}$, $\Lambda_2 := \{y \in \mathbb{R}^N : \zeta_3 \le |w(y)| \le 2\zeta_3\}$, and $\Lambda_3 := \{y \in \mathbb{R}^N : 2\zeta_3 \le |w(y)|\}$, where ζ_3 is given in (57). From the condition of φ, we have

$$\begin{aligned}
\widetilde{\mathcal{E}}_\theta(w) &= \mathcal{M}\left(\int_{\mathbb{R}^N}\mathcal{H}_{p,q}(y,|\nabla w|)\,dy\right) + \int_{\mathbb{R}^N}\mathcal{H}_{\mathfrak{V},p,q}(y,|w|)\,dy \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N}\sigma(y)|w|^r\,dy - \theta\int_{\mathbb{R}^N}\widetilde{G}(y,w)\,dy \\
&\ge \frac{\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|w\|^r_{L^{\gamma_0}(\mathbb{R}^N)} - \theta\int_{\Lambda_1}|G(y,w)|\,dy \\
&\quad - \theta\int_{\Lambda_2}\varphi(w)|G(y,w)| + (1-\varphi(w))\xi|w|^p\,dy - \theta\int_{\Lambda_3}\xi|w|^p\,dy \\
&\ge \frac{\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}C^r_{\gamma_0,imb}\|w\|^r \\
&\quad - \theta\int_{\Lambda_1\cup\Lambda_2}|G(y,w)|\,dy - \theta\int_{\Lambda_2\cup\Lambda_3}\xi|w|^p\,dy \\
&\ge \frac{\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}C^r_{\gamma_0,imb}\|w\|^r \\
&\quad - \theta\int_{\Lambda_1\cup\Lambda_2}\rho_1(y)|w|\,dy - \theta\int_{\Lambda_1\cup\Lambda_2}\frac{\rho_2}{\ell}|w|^\ell\,dy - \theta\int_{\Lambda_2\cup\Lambda_3}\xi|w|^p\,dy \\
&\ge \frac{\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}C^r_{\gamma_0,imb}\|w\|^r \\
&\quad - 2\theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\|w\|_{L^s(\mathbb{R}^N)} - \theta\left(\frac{\rho_2}{\ell}+\xi\right)\int_{\mathbb{R}^N}|w|^p\,dy \\
&\ge \frac{\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p}\|w\|^p - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}C^r_{\gamma_0,imb}\|w\|^r \\
&\quad - 2C_{s,imb}\theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\|w\| - \theta\left(\frac{\rho_2}{\ell}+\xi\right)\|w\|^p_{L^p(\mathbb{R}^N)} \\
&\ge \left[\frac{\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p} - \theta\left(\frac{\rho_2}{\ell}+\xi\right)C_{p,imb}\right]\|w\|^p \\
&\quad - \frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}C^r_{\gamma_0,imb}\|w\|^r - 2C_{s,imb}\theta\|\rho_1\|_{L^{s'}(\mathbb{R}^N)}\|w\|
\end{aligned}$$

where $C_{m,imb}$ is an embedding constant of $\mathfrak{E} \hookrightarrow L^m(\mathbb{R}^N)$ for any m with $p \le m < p^*$. Therefore, we deduce that for any

$$\theta \in \Gamma := \left(0, \frac{\ell\min\{\kappa_0,\vartheta\}}{\vartheta q 2^p(\rho_2+\ell\xi)C_{p,imb}}\right),$$

the functional $\widetilde{\mathcal{E}}_\theta$ is coercive in $\mathfrak{E}$; that is, $\widetilde{\mathcal{E}}_\theta(w) \to \infty$ as $\|w\| \to \infty$. Based on the analogous argument in Lemma 9 in [47], it follows that Φ' is strictly monotone and coercive. Similar to the proof of Lemma 11, Φ' is a mapping of type (S_+). According to the Browder–Minty theorem, the inverse operator of Φ' exists (see Theorem 26.A in [53]). Since Φ' is of type (S_+), it is clear that it has a continuous inverse. From the compactness of the operator $\widetilde{\Psi}'_\theta$ and the coercivity of $\widetilde{\mathcal{E}}_\theta$, it follows that the functional $\widetilde{\mathcal{E}}_\theta$ satisfies the $(C)_c$-condition for any $c \in \mathbb{R}$ and for every $\theta \in \Gamma$ as required.

Finally, we show that $(\mathcal{A}_4)$ is verified. Let $c \in \mathbb{R}$ and let the sequence $\{w_k\}$ in $\mathfrak{E}$ be such that $w_k \in \mathfrak{F}_k$ for any $k \in \mathbb{N}$,

$$\widetilde{\mathcal{E}}_\theta(w_k) \to c \quad \text{and} \quad \|(\widetilde{\mathcal{E}}_\theta|_{\mathfrak{F}_k})'(w_k)\|_{\mathfrak{E}^*}(1 + \|w_k\|) \to 0 \text{ as } k \to \infty.$$

Then, based on the coercivity of $\widetilde{\mathcal{E}}_\theta$, it follows that $\{w_k\}$ is bounded in $\mathfrak{E}$ for every $\theta \in \Gamma$. Following the concept of the proof of Lemma 11, we deduce that $w_k \to w_0$ in $\mathfrak{E}$ as $k \to \infty$ and also that w_0 is a critical point of $\widetilde{\mathcal{E}}_\theta$. Therefore, we conclude that the functional $\widetilde{\mathcal{E}}_\theta$ satisfies the $(C)^*_c$-condition for any $c \in \mathbb{R}$ and for any $\theta > 0$. This shows the condition $(\mathcal{A}_4)$.

Consequently, all conditions of Proposition 10 hold, and thus, for $\theta \in \Gamma$, we find a sequence of negative critical values d_k for $\widetilde{\mathcal{E}}_\theta$ satisfying $d_k \to 0$ when k goes to ∞. Then, for any $\{w_k\} \in \mathfrak{E}$ with $\widetilde{\mathcal{E}}_\theta(w_k) = d_k$ and $\|\widetilde{\mathcal{E}}'_\theta(w_k)\|_{\mathfrak{E}^*} = 0$, the sequence $\{w_k\}$ is a $(C)_0$-sequence of $\widetilde{\mathcal{E}}_\theta(w)$, and $\{w_k\}$ yields a convergent subsequence. Thus, up to the subsequence denoted by $\{w_k\}$, we have $w_k \to w$ in $\mathfrak{E}$ as $k \to \infty$. Equations (56), (57), and (59) imply that 0 is the only critical point with 0 energy and the subsequence $\{w_k\}$ has to converge to 0 in $\mathfrak{E}$; thus, $\|w_k\|_{L^t(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for any t with $p \le t \le p^*$. By virtue of Proposition 1, any weak solution w of (1) belongs to the space $L^\infty(\mathbb{R}^N)$, and there are positive constants of C, η independent of w such that

$$\|w\|_{L^\infty(\mathbb{R}^N)} \le C\|w\|^\eta_{L^\ell(\mathbb{R}^N)}.$$

Therefore, we know $\|w_k\|_{L^\infty(\mathbb{R}^N)} \to 0$. Hence, by applying (56) and (57) once again, we achieve $\|w_k\|_{L^\infty(\mathbb{R}^N)} \le \zeta_3$ for a sufficiently large k. Thus, $\{w_k\}$ with a sufficiently large k is a sequence of weak solutions to (1). The proof is complete. □

4. Conclusions

In order to use the dual fountain theorem, the authors of [23,36,37,40,47] considered the existence of two sequences $0 < \alpha_n < \beta_n \to 0$ as $n \to \infty$. However, our approach differs from the above papers. In view of the papers [32–35], we adopted the conditions (G5) and

(g) $G(y,\zeta) = o(|\zeta|^q)$ as $\zeta \to 0$ uniformly for all $y \in \mathbb{R}^N$.

These conditions play an important role in proving the assumptions of the dual fountain theorem, and the authors of [30,32–35] established the existence of two sequences $0 < \alpha_n < \beta_n$, which are both sufficiently large. However, when utilizing the analogous argument from [33,34], we cannot ensure property (2) in Lemma 14. More precisely, if we replace β_n in (51) with

$$\hat{\beta}_n = \left[\left(\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta\rho_2}{\ell}\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\chi_n^r\right]^{\frac{1}{p-\ell}},$$

and $r + p > \ell$, then in Equation (53),

$$\hat{\beta}_n\chi_n = \left[\left(\frac{1}{r}\|\sigma\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{\theta\rho_2}{\ell}\right)\frac{\vartheta q 2^{p+1}}{\min\{\kappa_0, \vartheta\}}\right]^{\frac{1}{p-\ell}}\chi_n^{\frac{r+p-\ell}{p-\ell}} \to \infty \text{ as } n \to \infty.$$

However, the authors of [32,35] overcame this difficulty with a new setting for β_n, as in (51). Although the basic idea for proving Lemmas 13 and 14 is analogous to [32,35], in this paper, we derive these conditions without assuming (G5) and (g). For this reason, our approach is slightly different from those of previous related studies [23,32–37,40,47].

Additionally, a new research direction is the study of Kirchhoff-Schrödinger-type problems with Hardy potentials:

$$\begin{aligned} &-M\left(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla w|^p + \frac{\nu(y)}{q}|\nabla w|^q\,dy\right)\operatorname{div}(|\nabla w|^{p-2}\nabla w + \nu(y)|\nabla w|^{q-2}\nabla w) \\ &+ \mathfrak{V}(y)(|w|^{p-2}w + \nu(y)|w|^{q-2}w) = \lambda\left(\frac{|w|^{p-2}w}{|y|^p} + \nu(y)\frac{|w|^{q-2}w}{|y|^q}\right) + \theta g(y,w) \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where $N \geq 2$, $1 < p < q < N$, $\lambda \in (-\infty, \lambda^*)$ for some $\lambda^* > 0$, θ is a positive real parameter, $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function,

$$\frac{q}{p} \leq 1 + \frac{1}{N}, \quad \nu : \mathbb{R}^N \to [0,\infty) \text{ is Lipschitz continuous,}$$

and $\mathfrak{V} : \mathbb{R}^N \to (0,\infty)$ is a potential function satisfying (V), and a Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfies the conditions (M1) and (M2).

Because of the term $\lambda(|w|^{p-2}w|y|^{-p} + \nu(y)|w|^{q-2}w|y|^{-q})$, when $\lambda \neq 0$, the classical variational approach is not applicable to our focus in the present paper. The reason is that the Hardy inequality only guarantees the embeddings of the Musielak–Orlicz–Sobolev space $W_0^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, |y|^{-p})$ and $W_0^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, \nu(y), |y|^{-q})$. However, these embeddings are not compact. Hence, problems with $\lambda \neq 0$ must be handled more carefully due to the lack of compactness.

Also, we indicate some further research for degenerated Kirchhoff coefficients as follows.

$$\begin{cases} -M(\varphi_{\mathcal{H}}(|\nabla u|))\operatorname{div}\big((|\nabla u|^{p-2} + b(y)|\nabla u|^{q-2})\nabla u\big) = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the modular function $\varphi_{\mathcal{H}}$ is defined by $\varphi_{\mathcal{H}}(|\nabla u|) := \int_\Omega |\nabla u|^p + b(y)|\nabla u|^q\,dy$ for all $u \in W_0^{1,\mathcal{H}}(\Omega)$, g is a continuous function with suitable conditions, and the exponents p, q and the weight function $b : \Omega \to [0,+\infty)$ satisfy the following condition:

(K1) $1 < p < N$, $p < q < p^* := \frac{Np}{N-p}$, and $b \in L^\infty(\Omega; [0,+\infty))$.

Also, $M : [0,+\infty) \to [0,+\infty)$ is the Kirchhoff function satisfying the condition:

(K2) M is continuous and there are constants $0 = s_0 < s_1 < s_2 < \cdots < s_R$ such that $M(s_\ell) = 0$ for each $\ell \in \{0,1,\ldots,R\}$ and $M(s) > 0$ for all $s \in [0,s_R] \setminus \{s_0,s_1,\ldots,s_R\}$.

Regarding this problem, the authors of [54] considered a nonlinear elliptic equation involving a nonlocal term that vanishes at finitely many points, a double phase differential operator that satisfies unbalanced growth, and a nonlinear reaction term. The model is referred as the double phase degenerate Kirchhoff problem, as it involves a nonlocal Kirchhoff term, too. The major contribution of this paper is to establish a multiplicity theorem in which the main method is based on a truncation technique and variational method.

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