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Article

Spectral Decomposition of Gramians of Continuous Linear Systems in the Form of Hadamard Products

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Abstract: New possibilities of Gramian computation, by means of canonical transformations into diagonal, controllable, and observable canonical forms, are shown. Using such a technique, the Gramian matrices can be represented as products of the Hadamard matrices of multipliers and the matrices of the transformed right-hand sides of Lyapunov equations. It is shown that these multiplier matrices are invariant under various canonical transformations of linear continuous systems. The modal Lyapunov equations for continuous SISO LTI systems in diagonal form are obtained, and their new solutions based on Hadamard decomposition are proposed. New algorithms for the element-by-element computation of Gramian matrices for stable, continuous MIMO LTI systems are developed. New algorithms for the computation of controllability Gramians in the form of Xiao matrices are developed for continuous SISO LTI systems, given by the equations of state in the controllable and observable canonical forms. The application of transformations to the canonical forms of controllability and observability allowed us to simplify the formulas of the spectral decompositions of the Gramians. In this paper, new spectral expansions in the form of Hadamard products for solutions to the algebraic and differential Sylvester equations of MIMO LTI systems are obtained, including spectral expansions of the finite and infinite cross-Gramians of continuous MIMO LTI systems. Recommendations on the use of the obtained results are given.

Keywords: spectral decompositions; linear continuous systems; Gramians; Sylvester and Lyapunov equations; Xiao matrices; Hadamard product

MSC: 11C08; 11C20; 11E39; 11F22; 44A10; 45D05

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1. Introduction

The first Gramian spectral decompositions for linear continuous and discrete systems with simple spectra were obtained in [1] by the spectral decomposition of the integral representation of the solution to the Lyapunov or Sylvester equations. It is well-known that Gramians are solutions to the Sylvester and Lyapunov equations, to which a great number of scientific papers have been devoted, among which we note [2–13]. These equations also play a fundamental role in control theory. Research in the field of linear control systems is closely related to the problem of lowering the model order by constructing an approximating model of lower dimensionality. Even in the case of linear systems of high dimensionality, the use of projection methods allows us to significantly reduce the dimensionality of the approximating model [6,10]. Among such methods, we note balanced truncation, singular value decomposition, the Krylov subspace method, methods for constructing a simplified model based on the Gramian H2-norm, optimal methods, and hybrid methods. Iterative algorithms for their realization have been developed for most of the methods. The Sylvester and Lyapunov matrix equations in applied problems of control theory were studied in [12,13]. In recent years, there has been an interest in developing methods for computing various energy metrics to analyze the stability and degree of controllability, reachability, and observability of these systems. A number of papers proposed such metrics for linear and unstable linear systems [14–21]. In another paper [16], simplified models of large networks based on controllability Gramians were proposed, allowing the computation of energy metrics. Some further papers considered the important problem of the optimal placement of sensors and actuators based on different energy functionals [15,17,18,20]. In [17], a general approach to solving the problem of the optimal placement of sensors and actuators for multivariable control systems was formulated, which is based on the decomposition of the system into stable and unstable subsystems. It is shown that the degree of controllability of the system is determined on the basis of energy metrics based on the use of finite and infinite controllability Gramians. A general method for computing the inverse Gramian of controllability for the equations of state given in the canonical forms of controllability is proposed. In [18], a method for the optimal placement of virtual inertia devices on the graph of an energy system is proposed. This method is based on the use of the energy metrics of the coherence of generators and the square of the H2-norm of the system transfer function, which is given by a standard dynamic model in the state space. The problem is formalized as a nonconvex optimization problem with constraints in the form of the values of the observability Gramians. It is known that energy-efficient control problems are also solved using Gramians. In recent years, these approaches have been developed for complex energy systems, as well as social, transportation, and biological networks in [17–19]. In [16,17], it was shown that the closer the eigenvalues of the dynamics matrix are located to the imaginary axis, the less energy is required to make the system or network fully controllable. In [19–21], these ideas were developed for digital ecosystems, vibroacoustic control systems, and thermal plant control systems. Thus, the degree of controllability (reachability) of a network is related to the minimum amount of energy, which allows us to introduce new metrics in the form of the minimum eigenvalue of the controllability Gramian and the maximum eigenvalue of its inverse Gramian, as well as the traces of these Gramians. Note that most of the mentioned works use the spectrum of the system dynamics matrix, which makes it quite natural to apply spectral analysis methods to solve the above problems. The use of canonical forms of controllability previously gave rise to a new approach to Gramian computation based on the use of Rouse–Gurwitz tables and Xiao matrices [22–26]. Almost 30 years ago, a paper [23] appeared in which a method and algorithm for computing solutions to Lyapunov equations are proposed for the case when the equations of state of a linear system are given in the canonical form of controllability or observability. It was shown that the solution matrices of the Lyapunov equations can be computed in a new way based on the use of Rouse tables that depends only on the coefficients of the characteristic equation of the dynamic matrix of the system. In recent years, due to the rapid growth of renewable generation in electric power systems, serious problems of controlling the modes of these systems have arisen due to the integration of renewable generation with conventional generation based on the use of conventional synchronous generators [27–30]. One of the directions for solving the problems of ensuring the stability of the modes of electric power systems is the use of virtual inertia devices in distributed power systems [27–29]. The main idea of virtual inertia is to use the synchronous generator model to stabilize the frequency control modes of distributed power systems. In [27], a method and algorithm for the optimal control of the grid frequency based on the use of a simplified synchronous generator model for a reduced-order model based on the model optimization criterion using the H2-norm of the transfer functions of the full and simplified models were proposed. It is well-known that the Gramian method [1,6] is used to solve this problem. In [30], a new method for the simultaneous estimation of unbalanced power and generation based on the inertia index of the system was developed. A mathematical model of a virtual inertia device was proposed, which guarantees accurate power imbalance estimation. The effectiveness of the proposed solutions was confirmed by the fact that the approach takes into account uncertainties in measurement errors, signal delays when using GPS networks, and telecommunication system failures. In large dynamical systems subject to noise or forced oscillations, the stability under small perturbations is determined by their energy that is stored in the power system [31]. To analyze this perturbation energy, this paper proposes a new physically motivated modal Lyapunov analysis (LMA), which combines selective modal analysis with spectral decompositions of specially selected Lyapunov stability indicators. New modal indicators are proposed that characterize individual modes and modal interactions and their relationships with specific state variables. These indicators estimate the integral energy associated with states and signals over an infinite or finite time interval. The new indicators characterize the pairwise interaction between modals in terms of their mutual actions produced in the states of the system over time. It is shown that the proposed modal indicators characterize the stability of individual modes and resonant modal interactions in linear systems with a variable parameter. In this paper, the application of these indices to analyze the stability of a two-zone model of a power system is investigated. This approach is based on the application of the Gramian method in power engineering and is innovative.

Main Contribution

In Section 2, the formulation of the problems of computing Gramian controllability and observability is considered within the framework of a unified concept. An important feature of the concept of this paper is the consideration of Hadamard products for spectral Gramian decompositions, which allows us to reduce the computation of sub-Gramian and Gramian matrices to the computation of numerical sequences of their elements. In this article we propose to improve this approach using Gramian spectral decompositions by extending its application to multidimensional linear control systems given by a standard (A,B,C) representation of the state space. In Section 3, we introduce modal Lyapunov equations of the second type for the state equations of MIMO LTI systems in diagonal canonical form. These equations allow us to compute various sub-Gramians in closed form. They obtain their spectral decompositions in the form of Hadamard products and derive formulas for the multiplier matrices. These equations also obtain spectral decompositions for the SISO LTI system in the canonical forms of controllability and observability. The multiplier matrices of these spectral decompositions, which also are Xiao matrices, play an important role in the following presentation. These equations allow us to compute various sub-Gramians in closed form. Their spectral decompositions in the form of Hadamard products are obtained, and formulas for the multiplier matrices are derived. It is proved that for stable systems the Xiao matrices are positively defined and invariant under similarity transformations. In the rest of this section, we consider the general case of linear continuous MIMO LTI systems represented by (A,B,C)-equations of state. New spectral decompositions of controllability and observability Gramians in the form of Hadamard products are obtained. It is shown that the multiplier matrices are the same in both the MIMO LTI and SISO LTI cases, provided that the system is stable, fully controllable, and observable for both the simple and the pairwise spectra of the dynamics matrix. A new analysis of the properties of multiplier matrices is given. An important property of multiplier matrices is their positive definiteness, which manifests itself in the positivity of the energy metric associated with this property [15,16]. In Section 4, the obtained results are developed to construct spectral decompositions of solutions to a wide class of matrix Sylvester differential equations. In particular, we obtain closed formulas for the Hadamard products of the spectral expansions matrices of cross-Gramian MIMO LTI systems, as well as their traces and diagonal elements.

2. Discussion of the Results and Problem Statement

We consider the Lyapunov equations for a continuous stationary MIMO LTI in diagonal canonical form

$$AP + PA^T = -BB^T,$$

$$A^TP + PA = -C^TC.$$

$$\begin{aligned} &x_d = Tx,\ \dot{x}_d = A_d x_d + B_d u,\ y_d = C_d x_d, \\ &A_d = TAT^{-1},\ B_d = TB,\ C_d = CT^{-1}, \end{aligned} \tag{1}$$

or

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix} \begin{bmatrix} \nu_1^* \\ \nu_2^* \\ \vdots \\ \nu_n^* \end{bmatrix},$$

where the matrix T is composed of the right eigenvector su_i, and the matrix T^{-1} is composed of the left eigenvectors $\nu_{\rm i}^*$, corresponding to the eigenvalue s_i. Let us introduce the notations

$$\beta_{ij} = e_i TBB^T T^* e_j^T,\ \gamma_{ij} = e_i (CT^{-1})^* CT^{-1} e_j^T.$$

Let us further consider the SISO LTI systems in controllability canonical form [9]

$$\begin{gathered} x_c(t) = R_c^F x(t), \\ \dot{x}_c(t) = A_c^F x_c(t) + b^F u(t),\ x_c(0) = 0, \\ y_c(t) = c_c^F x_c(t), \end{gathered} \tag{2}$$

$$A_c^F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, b^F = \begin{bmatrix} 0 & 0 & .. & 0 & 1 \end{bmatrix}^{\rm T},$$

$$c^F = \begin{bmatrix} \xi_0 & \xi_1 & .. & \xi_{n-2} & \xi_{n-1} \end{bmatrix}.$$

The following relations are valid [15]

$$R_c^F A \left(R_c^F\right)^{-1} = A_c^F,\ R_c^F b = b_\gamma^F,\ c\left(R_c^F\right)^{-1} = c^F,$$

$$P_c = \left(R_c^F\right)^{-1} \overset{F}{\underset{c}{P}} (\left(R_c^F\right)^{-1})^T,$$

where the matrix P_c is a solution of the corresponding Lyapunov equation. With respect to systems (1) and (2), we assume that various structural conditions for the stability, controllability, observability, and spectrum properties of the dynamic matrix are fulfilled. In [26], the following spectral decomposition of the controllability Gramian was obtained:

$$P_c^F = \sum_{k=1}^n \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} \mathbf{1}_{j+1\eta+1}.$$

Let us consider the further SISO LTI of a linear system in observability canonical form [9]. In this case, the following formulas are valid:

$$x_o(t) = R_o^F x(t) \dot{x}_o(t) = A_o^F x_o(t) + b_o^F u(t),\ x_o(0) = 0, y_o^F(t) = c_o^F x_o(t),$$

According to the principle of duality, we obtain the expressions [26]

$$P_o^F = \sum_{k=1}^n \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} \mathbf{1}_{j+1\eta+1},$$

In addition,

$$P_o = \left(R_o^F\right)^T P_o^F R_o^F.$$

Let us call the Xiao matrix (zero plaid structure) a matrix of the form [23]

$$Y = \begin{bmatrix} y_1 & 0 & -y_2 & 0 & y_3 \\ 0 & y_2 & 0 & -y_3 & 0 \\ -y_2 & 0 & y_3 & 0 & \dots \\ 0 & -y_3 & 0 & \dots & 0 \\ y_3 & 0 & \dots & 0 & y_n \end{bmatrix}. \tag{3}$$

The corresponding matrix elements are calculated by the formulas

$$y_{j\eta} = \begin{cases} 0, & if \;\; j+\eta = 2k+1,\; k = 1,2\dots n; \\ (-1)^{\frac{j-\eta}{2}}, & if \;\; j+\eta = 2k,\; k = 1,2\dots n. \end{cases} \tag{4}$$

The aim of this article is to develop a general approach and study the properties of spectral decompositions of solutions to differential and algebraic Sylvester and Lyapunov equations in the form of Hadamard products, including modal equations.

3. Main Results

Spectral Gramian decompositions allow us to represent the Gramian matrix as a sum of summands containing multiples of different indices. The role of indices can be different. Some indices play the role of leading indices, others play the role of slave indices. The distribution of the indices' roles is determined by the specifics of the applied tasks of monitoring and controlling the system state. In addition, computations in the real or complex domain require a different approach to the choice of method and algorithm for computing or analyzing Gramian properties. The main idea of the derivation of modal Lyapunov equations is to decompose the matrix of the right-hand side of the Lyapunov equation into the sum of matrices corresponding to the individual eigenvalues of the dynamics matrix or their combinations and to transform the matrices of the left-hand side accordingly. The main types of spectral decompositions are decompositions by simple, multiple, or Raman (pairwise) spectra. The Gramian matrix is, in general, a complex Hermite matrix, which can be represented as the sum of a symmetric matrix and a oblique symmetric matrix. Many applications of the Lyapunov equations are based on the use of dynamics matrices, input–output matrices, and Faddeev matrices, and, in this case, the matrices of Gramian spectral expansions are valid [30].

$$A^T P_i + P_i A = -\frac{1}{2}(R_i^* Q + Q R_i), \tag{5}$$

$$A P_i + P_i A^T = -\frac{1}{2}(R_i^* Q + Q R_i), \tag{6}$$

or

$$A^T P_{ij} + P_{ij} A = -\frac{1}{2}(R_i^* Q R_j + R_j^* Q R_i), \tag{7}$$

$$A P_{ij} + P_{ij} A^T = -\frac{1}{2}(R_i^* Q R_j + R_j^* Q R_i), \tag{8}$$

where Q is the matrix of the right-hand side of the Lyapunov equations, R_i, R_j are residues of the dynamics matrix, their resolvent in its corresponding eigenvalue. Let us call Equations (5)–(8) modal Lyapunov equations of the first type. In contrast, there are applications in which it is possible to use complex matrices of Lyapunov equation solutions

$$A^T P_i + P_i A = -R_i^* Q, \tag{9}$$

$$A P_i + P_i A^T = -R_i^* Q, \tag{10}$$

or

$$A^T P_{ij} + P_{ij} A = -R_i^* Q R_j, \tag{11}$$

$$AP_{ij} + P_{ij}A^T = -R_i^* Q R_j, \tag{12}$$

Let us call Equations (9)–(12) modal Lyapunov equations of the second type.

Theorem 1 ([32]). *Consider the modal Lyapunov equations of the second type for a continuous stationary MIMO LTI system in diagonal canonical form.*

$$\begin{aligned} A_d P_{cij} + P_{cij} A_d^* &= -\beta_{ij} e_i e_j^T, \\ A_d P_{ci} + P_{ci} A_d^* &= -\textstyle\sum_{j=1}^n \beta_{ij} e_i e_j^T \end{aligned} \tag{13}$$

$$\begin{aligned} A_d P_{ci} + P_{ci} A_d^* &= -\textstyle\sum_{j=1}^n \beta_{ij} e_i e_j^T, \\ A_d P_{oi} + P_{oi} A_d^* &= -\textstyle\sum_{j=1}^n \gamma_{ij} e_i e_j^T \end{aligned} \tag{14}$$

Above, the corresponding unit vectors are denoted by e_i, e_j^T. Suppose that the system is stable and has a simple spectrum. Then, the controllability and observability Gramians exist, are singular, and can be represented in the form of Hadamard products as

$$P_c = \Omega_c \circ \Psi_c, P_o = \Omega_o \circ \Psi_o, \tag{15}$$

$$\begin{gathered} \Psi_c = \left[\beta_{ij}\right]_{n\times n},\ \Omega_c = \left[-\tfrac{1}{\lambda_i+\lambda_j}\right]_{n\times n}, \\ \Psi_o = \left[\gamma_{ij}\right]_{n\times n}, \Omega_o = \left[-\tfrac{1}{\lambda_i+\lambda_j}\right]_{n\times n}, \\ P_{cij} = \Omega_c \circ \Psi_{cij},\ \Psi_{cij} = e_i \left[\beta_{ij}\right]_{n\times n} e_j^T, \\ P_{ci} = \textstyle\sum_{j=1}^n \Omega_c \circ \Psi_{cij}, \end{gathered} \tag{16}$$

If, in addition, the pair (A,B) is controllable and the pair (A,C) is observable, then the matrices of multipliers Ω_c and Ω_o are definitely positive, and their diagonal elements and traces are positive numbers. The Hermite components of the Gramians have the form [2]

$$P_c^H = \frac{1}{2}(P_c + P_c^*),\ P_o^H = \frac{1}{2}(P_o + P_o^*).$$

For Gramians and sub-Gramians of controllability and observability in the form of Hadamard's products, the next formulas are valid

$$P_{cj\eta}^H = \Omega_{cj\eta}^H \circ \Psi_{cj\eta}^H,\ P_{oj\eta}^H = \Omega_{oj\eta}^H \circ \Psi_{oj\eta}^H, \tag{17}$$

$$\Omega_{cj\eta}^H = \Omega_{oj\eta}^H = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} Re\left[-\frac{1}{\lambda_i + \lambda_\eta}\right] e_{j+1} e_{\eta+1}^T,$$

$$\Psi_{cj\eta}^H = \tfrac{1}{2}\left(\beta_{j\eta} + \beta_{j\eta}{}^*\right),\ \Psi_{oj\eta}^H = \tfrac{1}{2}\left(\gamma_{j\eta} + \gamma_{j\eta}{}^*\right), \tag{18}$$

$$P_c^H = \sum_{j=1}^n \sum_{\eta=1}^n P_{cj\eta}^H,\ P_o^H = \sum_{j=1}^n \sum_{\eta=1}^n P_{oj\eta}^H. \tag{19}$$

Proof. The proof of the general formulas is based on the results of [32], taking into account the separability properties of spectral expansion ns of Gramians. The validity of the formulas of the modal Lyapunov Equations (16)–(19) is established by substituting the formulas into the original Lyapunov equations and taking into account the equalities

$$\begin{gathered} P_c = \textstyle\sum_{i=1}^n \sum_{j=1}^n P_{cij},\ P_o = \sum_{i=1}^n \sum_{j=1}^n P_{oij}, \\ P_c = \textstyle\sum_{i=1}^n P_{ci}, P_o = \sum_{i=1}^n P_{oi} \end{gathered}$$

In [26], the general formulas for computing the spectral expansion of Gramians are derived, which are also applicable to the modal equations of MIMO LTI systems

$$P^c = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho^\eta}{\dot{N}(s_k)\dot{N}(s_\rho)} A_j BB^T (A_\eta)^T. \tag{20}$$

$$P^c = \sum_{j=0}^{n-1}\sum_{\rho=0}^{n-1}\sum_{k=1}^{\mathrm{n}} \frac{\lambda_k^j(-\lambda_k)^\rho}{\dot{N}(\lambda_k)N(-\lambda_k)} A_j BB^T A_\rho^T, \tag{21}$$

$$P^o = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho^\eta}{\dot{N}(s_k)\dot{N}(s_\rho)} A_j C^T C (A_\eta)^T. \tag{22}$$

$$P^o = \sum_{j=0}^{n-1}\sum_{\rho=0}^{n-1}\sum_{k=1}^{\mathrm{n}} \frac{\lambda_k^j(-\lambda_k)^\rho}{\dot{N}(\lambda_k)N(-\lambda_k)} A_j C^T C A_\rho^T, \tag{23}$$

Above, A_j denotes the Faddeev matrices, which are expressed through linear combinations of the products of the coefficients of the characteristic equations and the degree of the dynamics matrix of the system [33,34]. When performing the transformations, it should be taken into account that the residues of the resolvent of the dynamics matrix in its eigenvalues for the diagonal canonical form are strongly simplified: $\mathrm{Res}\left[(Is-A_d)^{-1},\lambda_k\right] = e_k e_k^T$. Therefore, Formulas (20)–(23) pass to the following formulas:

$$P^c = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho^\eta}{\dot{N}(s_k)\dot{N}(s_\rho)} e_j BB^T e_\eta^T.$$

$$P^c = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n} \frac{\lambda_k^j(-\lambda_k)^\rho}{\dot{N}(\lambda_k)N(-\lambda_k)} e_j BB^T e_\eta^T,$$

$$P^o = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho^\eta}{\dot{N}(s_k)\dot{N}(s_\rho)} e_j C^T C e_\eta^T,$$

□

Theorem 2. *Consider the modal Lyapunov equations for a continuous stationary SISO LTI system in the canonical forms of controllability and observability,*

$$\begin{aligned} A^F P_{cij} + P_{cij}\left(A^F\right)^T &= -e_i e_j^T, \\ \left(A^F\right)^T P_{oij} + P_{oij} A^F &= -e_i e_j^T, \end{aligned} \tag{24}$$

$$\begin{aligned} A^F P_{ci} + P_{ci}\left(A^F\right)^T &= -\sum_{j=1}^{n} e_i e_j^T, \\ \left(A^F\right)^T P_{oi} + P_{oi} A^F &= -\sum_{j=1}^{n} e_i e_j^T, \end{aligned} \tag{25}$$

Suppose that the system is stable and has a simple spectrum; then, pair (A,B) is controllable, and pair (A,C) is observable.

Then, the modal Gramians of controllability and observability exist and are singular. The modal Gramians of controllability for equations of state in the canonical form of controllability coincide with the Gramians of observability for equations of state in the canonical form of observability. The following decompositions of the Gramian matrices in the form of Hadamard's products are valid:

$$P_c = \widetilde{\Omega}_o \circ \widetilde{\Psi}_c, \qquad P_o = \widetilde{\Omega}_o \circ \widetilde{\Psi}_o, \tag{26}$$

The Hadamard decomposition on the pair and simple spectrum has the form

$$\widetilde{\Psi}c = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j^T,$$

$$\widetilde{\Omega}_c = \left[\sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho^\eta}{\dot{N}(\lambda_k)\dot{N}(s_\rho)}\right]_{n\times n} = \left[\sum_{k=1}^{n}\frac{\lambda_k^j(-\lambda_k)^\eta}{\dot{N}(\lambda_k)N(-\lambda_k)}\right]_{n\times n} \tag{27}$$

$$\widetilde{\Psi}_o = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j^T, \widetilde{\Omega}_o = \widetilde{\Omega}_c, \tag{28}$$

The Hadamard decomposition of the controllability and observability sub-Gramians over a simple spectrum has the form

$$P_{ci} = \sum_{j=1}^{n}\widetilde{\Omega}_c \circ \widetilde{\Psi}_{cij},\ P_{oi} = \sum_{j=1}^{n}\widetilde{\Omega}_o \circ \widetilde{\Psi}_{oij}. \tag{29}$$

Hadamard products are invariant under the similarity transformations.

Remark 1. *The upper sign of the wave is used in the matrices of the co-multipliers in the Hadamard decomposition for the modal Gramians of controllability and observability for a continuous stationary SISO LTI system.*

Proof. The singularity of solutions to modal equations follows from the stability of these equations. The coincidence of the solution matrices of the modal equations follows from the coincidence of the solution matrices of the original equations P_c and P_o [26]. In this article, analytical expressions of the solution matrices in the form of spectral expansions for a simple spectrum were derived as

$$P_c = P_o = \sum_{k=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\frac{\lambda_k{}_k^j(-\lambda_k)^\eta}{\dot{N}(\lambda_k)N(-\lambda_k)}1_{j+1\eta+1},$$

And, for a pair spectrum, in the form

$$P_c = P_o = \sum_{k=1}^{n}\sum_{\rho=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho{}^\eta}{\dot{N}(\lambda_k)\dot{N}(\lambda_\rho)}\ 1_{j+1\eta+1},$$

Let us represent the matrix factors of the Hadamard decomposition in the form

$$\widetilde{\Psi}c = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j^T\ \widetilde{\Psi}_o = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j^T.$$

We have scalar matrices of multipliers in the form

$$\widetilde{\Omega}_o = \left[\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho^\eta}{\dot{N}(\lambda_k)\dot{N}(s_\rho)}\right]_{n\times n} = \left[\sum_{k=1}^{n}\frac{\lambda_k^j(-\lambda_k)^\eta}{\dot{N}(\lambda_k)N(-\lambda_k)}\right]_{n\times n}.$$

Hence, Formulas (24)–(29) follow. Since multiplier matrices are the known functions of eigenvalues that serve as invariants under similarity transformations, multiplier matrices and Hadamard products are invariants under these transformations. □

Corollary 1. *The controllability and observability Gramians for the equations of state in the canonical forms of controllability and observability are Xiao matrices that are invariants under similarity transformations. The Xiao matrix is positively defined.*

Proof. The following formulas are valid

$$P_c = \widetilde{\Omega}_c \circ \widetilde{\Psi}_c,\ \ P_o = \widetilde{\Omega}_o \circ \widetilde{\Psi}_o, \tag{30}$$

$$P_c = \widetilde{\Omega}_c \circ \begin{bmatrix} 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \end{bmatrix},\ P_o = \widetilde{\Omega}_o \circ \begin{bmatrix} 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \\ 1 & 1 & 1 & 1 & ..1 \end{bmatrix},$$

Let us prove the fulfillment of the first property of Xiao matrices (3). The fulfillment for zero elements of matrices is proved in [26]. The alternation of signs of the side diagonal elements passing through diagonal element p_{jj} follows from the sequence of these elements in the form

$$\sum_{k=1}^{n} \frac{s_k^{j+2}(-s_k)^{j-2}}{\dot{N}(s_k)N_1(-s_k)},$$
$$\sum_{k=1}^{n} \frac{s_k^{j+1}(-s_k)^{j-1}}{\dot{N}(s_k)N_1(-s_k)},$$
$$\sum_{k=1}^{n} \frac{s_k^{j}(-s_k)^{j}}{\dot{N}(s_k)N_1(-s_k)},$$
$$\sum_{k=1}^{n} \frac{s_k^{j-1}(-s_k)^{j+1}}{\dot{N}(s_k)N_1(-s_k)}$$

The fulfillment of property (4) is similarly checked. Since the multiplier matrices are known functions of the eigenvalues, the Xiao matrices are invariant under the similarity transformation. We show the validity of this statement for its controllability Gramians. The transformation matrix R_c^F can be represented as the product of the Kalman controllability matrix and the Hankel matrix [9,24]

$$R_c^F = \begin{bmatrix} e_n & A_c^F e_n & \dots & \dots & \left(A_c^F\right)^{n-1} e_n \end{bmatrix} H_c,$$
$$H_c = \begin{bmatrix} a_{n-1} & \dots & a_1 & a_o & ..1 \\ \vdots & a_1 & a_o & 1 & ..0 \\ a_1 & a_o & 1 & 0 & ..0 \\ a_o & 1 & 0 & 0 & ..0 \\ 1 & 0 & 0 & 0 & ..0 \end{bmatrix},$$

A substitution when calculating the controllability matrix leads to the equality

$$\begin{bmatrix} e_n & A_c^F e_n & \dots & \dots & \left(A_c^F\right)^{n-1} e_n \end{bmatrix} = H_c^{-1}.$$

It follows that the controllability matrix is nondegenerate when transforming state equations into the canonical form of controllability or observability. So the multiplier matrix, which is the Xiao matrix, is positively defined. □

Theorem 3. *Let us consider the spectral decompositions of solutions to the equations of linear continuous stationary MIMO LTI systems. Suppose that the system is stable, matrices A, B, C are real, matrix A has a simple spectrum, the pair (A,B) is controllable, and the pair (A,C) is observable. Then, the following statements are true.*

1. *Spectral decompositions of its controllability and observability Gramians and controllability or observability sub-Gramians in the form of Hadamard products for the case of pair spectrum of the dynamics matrix have the following form*

$$P_{cj\eta} = \widetilde{\Omega}_{cj\eta} \circ \Psi_{cj\eta},\ \Psi_{cj\eta} = A_j BB^T (A_\eta)^T, P_{\boldsymbol{c}} = \widetilde{\Omega}_c \circ \Psi_c, \tag{31}$$

$$\widetilde{\Omega}_c = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \widetilde{\Omega}_{cj\eta} = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \omega(n, \lambda_k, \lambda_\rho, j, \eta) e_{j+1} e_{\eta+1}^T \tag{32}$$

$$\omega(n,\lambda_k,\lambda_\rho,j,\eta)=\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{-1}{\lambda_\rho+\lambda_k}\frac{\lambda_k^j\lambda_\rho{}^\eta}{\dot{N}(\lambda_k)\dot{N}(\lambda_\rho)}, \tag{33}$$

$$\begin{gathered}\Psi_{cj\eta}=\sum_{\nu=1}^{n}\sum_{\mu=1}^{n}\beta_{\nu\mu}^{(j\eta)}e_\nu e_\mu^T,\\ e_\nu A_j BB^T(A_\eta)^T e_\mu^T=\left[\beta_{\nu\mu}^{(j\eta)}\right]_{n\times n},\\ e_\nu A_j^T C^T CA_\eta e_\mu^T=\left[\gamma_{\nu\mu}^{(j\eta)}\right]_{n\times n},\end{gathered} \tag{34}$$

$$\Psi_c=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{\nu=1}^{n}\sum_{\mu=1}^{n}\beta_{\nu\mu}^{(j\eta)}e_\nu e_\mu^T \tag{35}$$

2. *For the case of the decomposition of the controllability Gramian by a simple spectrum of the dynamics matrix in the form of Hadamard products, we obtain the same Formulas (32)–(35), except for the formulas of the multiplier matrix $\widetilde{\Omega}_c$, which takes the form*

$$\widetilde{\Omega}_c=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\widetilde{\Omega}_{cj\eta}=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\omega(\lambda_k,-\lambda_k,j,\eta)e_{j+1}e_{\eta+1}^T \tag{36}$$

$$\omega(\lambda_k,-\lambda_k,j,\eta)=\sum_{k=1}^{n}\frac{\lambda_k^j(-\lambda_k)^\eta}{\dot{N}(\lambda_k)N(-\lambda_k)} \tag{37}$$

3. *Exactly the same formulas as (31)–(35) are valid for the observability Gramians in the form of Hadamard products. Only the formulas for the matrices* Ψo *are changing:*

$$P_{\boldsymbol{o}}=\widetilde{\Omega}_c\circ\Psi_o,\ \Psi_o=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{\nu=1}^{n}\sum_{\mu=1}^{n}\gamma_{\nu\mu}^{(j\eta)}e_\nu e_\mu^T. \tag{38}$$

4. *The Hermite component of the controllability and observability Gramians has the form [2]*

$$P_c^H=\frac{1}{2}(P_{\boldsymbol{c}}+P_c^*),\ P_o^H=\frac{1}{2}(P_{\boldsymbol{o}}+P_o^*), \tag{39}$$

$$P_{cj\eta}^H=\frac{1}{2}\left(P_{cj\eta}+P_{cj\eta}^*\right),\ P_{oj\eta}^H=\frac{1}{2}\left(P_{oj\eta}+P_{oj\eta}^*\right), \tag{40}$$

Spectral decompositions of the Hermite components of the controllability and observability Gramians have the form of Hadamard matrices

$$P_{cj\eta}^H=\Omega_{cj\eta}^H\circ\Psi_{cj\eta}^H,\ P_{oj\eta}^H=\Omega_{oj\eta}^H\circ\Psi_{oj\eta}^H, \tag{41}$$

$$\begin{aligned}\Omega_{cj\eta}^H&=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}Re\left[\omega(n,\lambda_k,\lambda_\rho,j,\eta)\right]e_{j+1}e_{\eta+1}^T\\ &=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}Re\left[\omega(n,\lambda_k,-\lambda_k,j,\eta)\right]e_{j+1}e_{\eta+1}^T,\end{aligned} \tag{42}$$

$$\begin{gathered}\Psi_{cj\eta}^H=\tfrac{1}{2}\left(A_jBB^TA_\eta^T+A_\eta BB^TA_j^T\right),\\ \Psi_{oj\eta}^H=\tfrac{1}{2}\left(A_j^TC^TCA_\eta+A_\eta C^TCA_j^T\right),\end{gathered} \tag{43}$$

$$\begin{gathered}\Psi_{oj\eta}^H=\tfrac{1}{2}\left(A_j^TC^TCA_\eta+A_\eta C^TCA_j^T\right),\\ \Omega_{oj\eta}^H=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}Re\left[\omega(n,\lambda_k,\lambda_\rho,j,\eta)\right]e_{j+1}e_{\eta+1}^T,\end{gathered} \tag{44}$$

$$\begin{gathered}\Psi_{oj\eta}^H=\tfrac{1}{2}\left(A_j^TC^TCA_\eta+A_\eta C^TCA_j^T\right),\\ P_c^H=\sum_{j=1}^{n}\sum_{\eta=1}^{n}P_{cj\eta}^H,\ P_o^H=\sum_{j=1}^{n}\sum_{\eta=1}^{n}P_{oj\eta}^H,\end{gathered} \tag{45}$$

$$\begin{gathered}P_c^H=\sum_{j=1}^{n}\sum_{\eta=1}^{n}P_{cj\eta}^H,\\ P_o^H=\sum_{j=1}^{n}\sum_{\eta=1}^{n}P_{oj\eta}^H,\end{gathered} \tag{46}$$

The multiplier matrices in all Gramian decompositions are Xiao matrices.

Proof. Let us return to the general formulas for Gramian spectral expansions (20)–(23). Let us first consider pairwise Gramian spectral expansions. We divide the summation indices into two groups: the first group covers the summation over the "j,η" indices of the resolvent decompositions into Faddeev series, and the second group covers the summation of the "j,η" indices over the pair spectrum. Let us distinguish the controllability sub-Gramian $P_{cj\eta}$ and represent its spectral decomposition as

$$P_{cj\eta} = \widetilde{\Omega}_{cj\eta} \circ \Psi_{cj\eta},\ \Psi_{cj\eta} = A_j BB^T (A_\eta)^T \tag{47}$$

$$\widetilde{\Omega}_c = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \widetilde{\Omega}_{cj\eta} = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \omega(n,\lambda_k, \lambda_\rho, j, \eta) e_{j+1} e_{\eta+1}^T \tag{48}$$

$$\omega(n, \lambda_k, \lambda_\rho, j, \eta) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} \frac{-1}{\lambda_\rho + \lambda_k} \frac{\lambda_k^j \lambda_\rho{}^\eta}{\dot{N}(\lambda_k)\dot{N}(\lambda_\rho)}. \tag{49}$$

Taking into account the designation

$$e_\nu A_j BB^T (A_\eta)^T e_\mu^T = \left[\beta_{\nu\mu}^{(j\eta)}\right]_{n\times n},\ e_\nu A_j^T C^T C A_\eta e_\mu^T = \left[\gamma_{\nu\mu}^{(j\eta)}\right]_{n\times n},$$

we have

$$\Psi_{cj\eta} = \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \beta_{\nu\mu}^{(j\eta)} e_\nu e_\mu^T \tag{50}$$

Taking into account the previous calculations, we obtain the spectral decomposition of the controllability Gramian of the system in the form

$$P_{\boldsymbol{c}} = \widetilde{\Omega}_c \circ \Psi_c,\ \Psi_c = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \beta_{\nu\mu}^{(j\eta)} e_\nu e_\mu^T \tag{51}$$

Repeating similar reasoning for the case of the decomposition of the controllability Gramian over the simple spectrum of the dynamics matrix, we obtain the same Formulas (47)–(51) as in the previous case, except for the formulas for the matrix $\widetilde{\Omega}_c$

$$\widetilde{\Omega}_c = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \widetilde{\Omega}_{cj\eta} = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \omega(\lambda_k, -\lambda_k, j, \eta) e_{j+1} e_{\eta+1}^T \tag{52}$$

$$\omega(n, \lambda_k, -\lambda_k, j, \eta) = \sum_{k=1}^{n} \frac{\lambda_k^j (-\lambda_k)^\eta}{\dot{N}(\lambda_k) N(-\lambda_k)}.$$

In [27], it is proved that the multiplier matrices $\widetilde{\Omega}_c$ are Xiao matrices. They coincide with Formulas (27) and (28) of Theorem 2. It is easy to find that exactly the same formulas are true for the observability Gramian's multipliers if the conditions of the theorem are preserved. Only the formulas for the matrices Ψ_o are changing

$$P_{\boldsymbol{o}} = \widetilde{\Omega}_c \circ \Psi_o,\ \Psi_o = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \gamma_{\nu\mu}^{(j\eta)} e_\nu e_\mu^T.$$

Note that the developed method and algorithms for computing Gramians in the form of Hadamard products provide a convenient way to compute and subsequently analyze elements of Gramian matrices, which is an advantage when computing diagonal elements and traces of Gramians and sub-Gramians as well as spectral decompositions of energy functionals [31].

In all the cases discussed above, we are talking about the complex Gramians and sub-Gramians of controllability and observability. Under the conditions of the theorem, the controllability and observability Gramians are always real matrices, but the sub-Gramians can be complex. As can be seen from the last expressions, when calculating the Hermite components of the Hadamard products of sub-Gramians, we obtain the formulas

$$P_{cj\eta}^{H} = \frac{1}{2}\left(P_{cj\eta} + P_{cj\eta}^{*}\right),\ P_{oj\eta}^{H} = \frac{1}{2}\left(P_{oj\eta} + P_{oj\eta}^{*}\right),$$

Therefore, the matrix part of the sub-Gramians in the form of the Hadamard product becomes a symmetric matrix, and its multiplier matrix becomes a real matrix. As a result of these transformations, we obtain Formulas (41)–(46). □

4. Spectral Expansions of Solutions to Sylvester Differential Equations on a Finite Interval

Let us consider two linear stationary continuous MIMO LTI dynamic systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = 0, \tag{53}$$

$$y(t) = Cx(t),$$

where $\mathrm{x}(t) \in R^{n}, u(t) \in R^{d}, \mathrm{y}(t) \in R^{d}$. We consider real matrices of corresponding sizes A, B, C. Let us assume that system (53) is stable, unless otherwise stated, completely controllable, and observable, and all eigenvalues of matrix A are different.

$$x_m(t) = A_m x_m(t) + B_m u(t), x_m(0) = 0, \tag{54}$$

$$y_m(t) = C_m x_m(t),$$

where $x_m(t) \in R^{n_1}, u(t) \in R^{d}, y_m(t) \in R^{d}$. We consider real matrices of corresponding sizes A, B, C, A_m, B_m, C_m. Let us assume that system (54) is stable, unless otherwise stated, completely controllable, and observable, and all eigenvalues of matrix A_m are different and do not coincide with the eigenvalues of matrix A. Following [27], consider the following continuous differential equations associated with these systems of the form

$$\frac{dP(t)}{dt} = AP(t) + P(t)A^{T} + R,\ P(0) = 0_{n\times n}, \tag{55}$$

$$\frac{dP(t)}{dt} = A_m P(t) + P(t)B_m + R_1,\ P(0) = 0_{n\times n}, \tag{56}$$

where R_1 is a real matrix of size $(n \times n_1)$. This section focuses on the Sylvester differential equation. The main method for constructing a solution and its spectral expansions is operational calculus and the expansion of the resolvents of dynamics matrices A_m and B_m into the Faddeev–Leverrier series. The latter have the form [33,34]

$$(Is - A_m)^{-1} = \textstyle\sum_{j=0}^{n-1} A_{mj} s^{j} [N_m(s)]^{-1},\ A_{mj} = \sum_{i=j+1}^{n} a_{mi} A_m^{i-j+1},$$

$$(Is - B_m)^{-1} = \textstyle\sum_{j=0}^{n_1} B_{mj} s^{j} [N_m(s)]^{-1},\ B_{mj} = \sum_{i=j+1}^{n_1} b_{mi} B_m^{i-j+1},$$

where A_{mj}, B_{mj} are Faddeev matrices constructed for resolvent matrices A_m, B_m using the Faddeev–Leverrier algorithm; $N_m(s), N_{m1}(s)$—are the characteristic polynomials of matrices A_m, B_m; a_{mi}, b_{mi} are the coefficients of these polynomials.

The first method for spectral expansions of solutions to Sylvester differential equations is based on the following lemma:

Lemma 1 ([32]). *Let us consider solving equations on a finite interval [0,t) ∈ [0,T]. Let us assume that systems (53) and (54) are stable, matrices A_m, B_m, R, R_1 are real, matrices A_m, B_m have a* *simple spectrum, their eigenvalues s_k, s_ρ are different, they do not belong to the imaginary axis of the eigenvalue plane, and the conditions are valid.*

$$s_k + s_\rho \neq 0,\ k = \overline{1,n};\ \rho = \overline{1,n};\ s_k \in spec\ A_m,\ s_\rho \in spec\ B_m.$$

Let us transform the dynamics matrices to diagonal form

$$A_{md} = diag\{\ldots s_k \ldots\} = Q_1 A_m Q_1^{-1},\ B_{md} = diag\{\ldots s_\rho \ldots\} = Q_2 B_m Q_2^{-1},$$

where $Q_1,\ Q_2$ are matrices of dimensions $[n \times n]$ u $[n_1 \times n_1]$.

Then, the Sylvester differential equation solution on finite interval $[0,t) \in [0,\ T]$ has the form

$$P_d(t) = \left[p_{dj\eta}(t)\right],$$
$$p_{dj\eta}(t) = \tfrac{r_{dj\eta} e^{(s_j + s_\eta)t}}{s_j + s_\eta} + p_{dj\eta},\ p_{dj\eta} = -\tfrac{r_{dj\eta}}{s_j + s_\eta},$$
$$P(t) = Q_1^{-1} P_d(t) \left(Q_2^T\right)^{-1}.$$

The second method of spectral decompositions of solutions to the Sylvester differential equations is based on using the Laplace transform to compute the Lyapunov integral and decomposing the resolvents of dynamics matrices A_m and B_m into a Faddeev–Leverier series.

Theorem 4. *Let us consider spectral expansions of solutions to Sylvester differential equations for MIMO LTI systems (53) and (54). Let us assume that these systems are stable, matrices A_m, B_m and R_1 are real, have a simple spectrum, their eigenvalues $s_{mk}, s_{m\rho}$ are different, they do not belong to the imaginary axis of the eigenvalue plane, and the conditions are met.*

$$s_{mk} + s_{m\rho} \neq 0,\ k = \overline{1,n};\ \rho = \overline{1,n_1};\ s_{mk} \in spec\ A_m,\ s_{m\rho} \in spec\ B_m.$$

Then, the following statements are true.

1. *Spectral expansions of solutions to Sylvester differential Equation (56) in the form of Hadamard products for the case of the combination spectrum of dynamics matrices have the form*

$$P_{j\eta}(t) = \Omega_{j\eta}(t) \circ \Psi_{j\eta},\ \Psi_{j\eta} = A_{mj} R_1 B_{m\eta}, \tag{57}$$

$$P_{j\eta}(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \frac{s_{mk}^{j} s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk}) \dot{N}(s_{m\rho})} \left[\frac{1 - \exp(s_{mk} + s_{m\rho})t}{s_{mk} + s_{m\rho}}\right] A_{mj} R_1 B_{m\eta}, \tag{58}$$

$$\Omega_{j\eta}(t) = \textstyle\sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \frac{s_{mk}^{j} s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk}) \dot{N}(s_{m\rho})} \left[\frac{1 - \exp(s_{mk} + s_{m\rho})t}{s_{mk} + s_{m\rho}}\right],$$
$$P(t) = \Omega(t) \circ \Psi,$$
$$\Psi = \textstyle\sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} A_{mj} R_1 B_{m\eta},$$
$$\Omega(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \tfrac{s_{mk}^{j} s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk}) \dot{N}(s_{m\rho})} \left[\tfrac{1 - \exp(s_{mk} + s_{m\rho})t}{s_{mk} + s_{m\rho}}\right] e_j e_\eta^T.$$

2. *For the case of the expansion of solutions to Sylvester's differential equations over the simple spectrum of the dynamics matrix, the same Formulas (57) and (58) are valid, but with new multiplier matrices:*

$$P_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_{mk}^{j} (-s_{mk})^{\eta}}{\dot{N}(s_{mk}) N_1(-s_{mk})} (\exp s_{mk} t - 1) A_{mj} R_1 B_{m\eta} = \Omega_{j\eta}(t) \circ \Psi_{j\eta} \tag{59}$$

$$\Omega_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_{mk}^{j}(-s_{mk})^{\eta}}{\dot{N}(s_{mk})N_1(-s_{mk})}(\exp s_{mk}t - 1),\ \Psi_{j\eta} = A_{mj}R_1B_{m\eta}, \tag{60}$$

$$P(t) = \Omega(t) \circ \Psi,\ \Psi = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} A_{mj}R_1B_{m\eta} \tag{61}$$

$$\Omega(t) = \sum_{k=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \frac{s_{mk}^{j}(-s_{mk})^{\eta}}{\dot{N}(s_{mk})N_1(-s_{mk})}(\exp s_{mk}t - 1) \tag{62}$$

The Hermitian component of spectral expansions of solutions to the Sylvester equations has the form

$$P^H(t) = \frac{1}{2}(P(t) + P^*(t)),\ P_{j\eta}^H(t) = \frac{1}{2}\left(P_{j\eta}(t) + P_{j\eta}{}^*(t)\right),$$

where the spectral decompositions of matrices $P, P^*,\ P_{j\eta},\ P_{j\eta}{}^*$ *are determined by Formulas (59)–(62).*

Proof. The solution to the differential equation (56) is an integral of the form [1,3]:

$$P(t) = \int_o^t e^{A_m\tau}Re^{B_m\tau}d\tau.$$

Let us apply the Laplace transform to both sides of the equation, considering the initial conditions to be zero and using the theorem on the Laplace transform of the product of real functions of time, the image of which is a fractional–rational algebraic fraction. In our case, this fraction contains one zero pole, and all other poles are simple. In this case, the direct transformation has the form

$$\frac{f(s)}{sF(s)} = \frac{f(0)}{sF(0)} + \sum_{i=1}^{q} \frac{f(s_i)}{s_iF(s_i)}, \tag{63}$$

where functions $\frac{f(0)}{sF(0)}$and $F(s)$ have the form

$$\frac{f(0)}{sF(0)} = \frac{1}{s}\left[\sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \frac{-1}{s_{mk}+s_{m\rho}} \frac{s_{mk}^{j}s_{m\rho}{}^{\eta}}{\dot{N}(s_k)\dot{N}\left(s_{m\rho}\right)} A_{mj}R_1B_{m\eta}\right],$$

$$F(s) = \sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \frac{-1}{s_{mk}+s_{m\rho}} \frac{s_{mk}^{j}s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}\left(s_{m\rho}\right)} \frac{1}{s-s_{mk}-s_{m\rho}}.$$

Substituting these expressions into (63), we obtain an image of the expansion of the solution to Sylvester's differential Equation (56), in terms of the combination spectrum of the dynamics matrices, in the form

$$P(s) =$$

$$\sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \frac{-1}{s_{mk}+s_{m\rho}} \frac{s_{mk}^{j}s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}\left(s_{m\rho}\right)} A_{mj}R_1B_{m\eta} +$$

$$\sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \frac{-1}{s_{mk}+s_{m\rho}} \frac{s_{mk}^{j}s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}\left(s_{m\rho}\right)} A_{mj}R_1B_{m\eta} \frac{1}{s-s_{mk}-s_{m\rho}}.$$

Having performed the inverse transformation, we obtain the spectral expansion of the solution to the Sylvester differential Equation (56), in the combination spectrum of the dynamics matrices, in the time domain:

$$P_{j\eta}(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \frac{s_{mk}^{j}s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}\left(s_{m\rho}\right)} \left[\frac{-1}{s_{mk}+s_{m\rho}}\right] A_{mj}R_1B_{m\eta} = \Omega_{j\eta}(t) \circ \Psi_{j\eta},$$

$$\Omega_{j\eta}(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \frac{s_{mk}^{j}s_{m\rho}{}^{\eta}}{\dot{N}(s_k)\dot{N}\left(s_{m\rho}\right)} \left[\frac{\exp\left(s_{mk}+s_{m\rho}\right)t - 1}{s_{mk}+s_{m\rho}}\right], \Psi_{j\eta} = A_{mj}R_1B_{m\eta},$$

$$\Omega(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n_1-1} \frac{s_{mk}^{j} s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}(s_{m\rho})} \frac{s_{mk}^{j} s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}(s_{m\rho})} \left[\frac{\exp(s_{mk}+s_{m\rho})t-1}{s_{mk}+s_{m\rho}}\right] e_j e_\eta^T,$$
$$P(t) = \Omega(t) \circ \Psi, \tag{64}$$

Equality (64) expresses the spectral expansion of the Sylvester differential equations' solutions in the combination spectrum of matrices A_m and B_m. This proves the first statement of the theorem.

Using the identity

$$\sum_{k=1}^{n} \sum_{\rho=1}^{n_1} \frac{-1}{s_{mk}+s_{m\rho}} \frac{s_{mk}^{j} s_{m\rho}{}^{\eta}}{\dot{N}(s_{mk})\dot{N}(s_{m\rho})} \equiv \sum_{k=1}^{n} \frac{s_{mk}^{j}(-s_{mk})^{\eta}}{\dot{N}(s_{mk})N(-s_{mk})} \tag{65}$$

one can obtain similar expansions for the simple spectrum of matrix A_m:

$$P_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_{mk}^{j}(-s_{mk})^{\eta}}{\dot{N}(s_{mk})N_1(-s_{mk})}(\exp s_{mk}t-1)A_{mj}R_1B_{m\eta} = \Omega_{j\eta}(t) \circ \Psi_{j\eta},$$
$$\Omega_{j\eta}(t) = \sum_{k=1}^{n} \frac{s_{mk}^{j}(-s_{mk})^{\eta}}{\dot{N}(s_{mk})N_1(-s_{mk})}(\exp s_{mk}t-1) = \Omega_{j\eta}(t) \circ \Psi_{j\eta}, \tag{66}$$
$$\Psi_{j\eta} = A_{mj}R_1B_{m\eta},$$
$$P(t) = \Omega(t) \circ \Psi,$$

$$\Psi = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} A_{mj}R_1B_{m\eta}, \tag{67}$$

The resulting expansions prove the second statement of the theorem. The third statement follows from statements 1 and 2. Equality (66) expresses the spectral expansion of solutions to the Sylvester equations in the simple spectrum of matrix A_m. □

Let us apply the results of the theorem to the calculation of spectral decompositions of the finite cross-Gramian of a continuous stable MIMO LTI system,

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = 0, \tag{68}$$
$$y(t) = Cx(t),$$

which is a solution to the simple Sylvester differential equation

$$\frac{dP(t)}{dt} = AP(t) + P(t)A + BC,\ P(0) = 0. \tag{69}$$

Corollary 2. *Let us consider the spectral expansions of solutions to Sylvester differential equations and renumber for the MIMO LTI system (69). Let us assume that the system is stable, matrix A, B and C, are real, their dimensions have been harmonized, matrix A has a simple spectrum, and the conditions are met.*

$$s_k + s_\rho \neq 0,\ k = \overline{1,n};\ s_k \in spec\ A.$$

Then the following statements are true.

1. *The spectral decomposition of the cross-Gramian image has the form*

$$P(s) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{-1}{s_k+s_\rho} \frac{s_k^j s_\rho{}^{\eta}}{\dot{N}(s_k)\dot{N}(s_\rho)} A_j BCA_\eta \frac{1}{s-s_k-s_\rho}$$

2. *The spectral decomposition of the cross-Gramian over the pair spectrum of matrix A in the time domain has the form*

$$P(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{-1}{s_k+s_\rho} \frac{s_k^j s_\rho{}^\eta}{\dot{N}(s_k)\dot{N}(s_\rho)} \left[\frac{\exp{(s_k+s_\rho)t}-1}{s_k+s_\rho} \right]$$

The Hadamard decomposition for a finite cross-Gramian has the form

$$P(t) = \Omega_{cr}(t) \circ \Psi_{cr},\ \Omega_{cr}(t)$$

$$\Omega_{cr}(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{s_k^j s_\rho{}^\eta}{\dot{N}(s_k)\dot{N}(s_\rho)}\ A_j BCB_j \left[\frac{\exp{(s_k+s_\rho)t}-1}{s_k+s_\rho} \right]$$

$$\Psi_{cr} = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} A_j BCB_\eta,$$

3. *The diagonal terms and trace of the cross-Gramian have the for*

$$p_{jj}(t) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} \frac{s_k^{j-1} s_\rho{}^{j-1}}{\dot{N}(s_k)\dot{N}(s_\rho)} \left[\frac{e^{(s_k+s_\rho)t}-1}{s_k+s_\rho} \right] A_{j-1} BCA_{j-1},\ \ j = \overline{1,n}.$$

$$trP(t) = tr\sum_{k=1}^{n} \sum_{\rho=1}^{n} \sum_{j=0}^{n-1} \frac{s_k^j s_\rho{}^j}{\dot{N}(s_k)\dot{N}(s_\rho)} \left[\frac{\exp{(s_k+s_\rho)t}-1}{s_k+s_\rho} \right] A_j BCA_\eta.$$

Example 1. *Let us consider the linear stationary continuous SISO LTI dynamic system of the form*

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = 0, \tag{70}$$

$$y(t) = Cx(t),$$

where

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix},\ b = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Let us, then, obtain spectral expansions of the cross-Gramian in Hadamard form for the system (70). In this case, it is possible to calculate the expressions:

$$N(s) = a_2 s^2 + a_1 s + a_0 = (s - s_1)(s - s_2),$$

$$s_1 = -0.5,\ s_2 = -1,$$

$$a_2 = 1,\ a_1 = 1.5,\ a_0 = 0.5,$$

$$\dot{N}(s) = 2s + 1.5,$$

$$(Is - A)^{-1} = (A_1 s + A_0),\ N^{-1}(s),$$

$$\begin{bmatrix} s+5 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} s+1 & 0 \\ 0 & s+0.5 \end{bmatrix} N^{-1}(s),$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$bc = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix},$$

$$\dot{N}(s_1) = 0.5,\ \dot{N}(s_2) = -0.5,\ N(-s_1) = 1.5,$$

$$N(-s_2) = 3,$$

$$AP + PA + bc = 0,\ \frac{dP}{dt} = AP + PA + bc,\ P(0) = 0,$$

$$P(\infty) = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \sum_{k=1}^{\mathrm{n}} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} A_j bcA_\eta,$$

$$\Psi_{00} = A_0 bcA_0 = \begin{bmatrix} 0 & 0.25 \\ 0 & 0.25 \end{bmatrix}, \ \Psi_{01} = A_0 bcA_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.5 \end{bmatrix},$$

$$\Psi_{10} = A_1 bcA_0 = \begin{bmatrix} 0 & 0.25 \\ 0 & 0.5 \end{bmatrix}, \ \Psi_{11} = A_1 bcA_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix}.$$

Let us calculate multiplier matrices $\Omega_{j\eta} = \sum_{k=1}^{2} \omega(1, s_k, -s_k, j, \eta) e_{j+1} e_{\eta+1}^T$, $k = 1, 2$.

$$\omega(1, s_1, -s_1, j, \eta):$$

$$\omega(1, s_1, -s_1, 0, 0) = \tfrac{4}{3}, \ \omega(1, s_1, -s_1, 1, 0) = -\tfrac{2}{3},$$

$$\omega(1, s_1, -s_1, 0, 1) = \tfrac{2}{3}, \ \omega(1, s_1, -s_1, 1, 1) = -\tfrac{1}{3},$$

$$\omega(1, s_2, -s_2, j, \eta):$$

$$\omega(1, s_2, -s_2, 0, 0) = -\tfrac{2}{3}, \ \omega(1, s_2, -s_2, 1, 0) = \tfrac{2}{3},$$

$$\omega(1, s_2, -s_2, 0, 1) = -\tfrac{2}{3}, \ \omega(1, s_2, -s_2, 1, 1) = \tfrac{2}{3}.$$

Finally, we obtain the spectral decomposition of the cross-Gramian in Hadamard form:

$$P_{j\eta}(\infty) = \Omega_{j\eta} \circ \Psi_{j\eta},$$

$$\Omega_{00} = \frac{2}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \ \Omega_{01} = 0, \ \Omega_{10} = 0, \ \Omega_{11} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{71}$$

$$P(\infty) = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{72}$$

It is easy to see that this matrix is a solution to the original algebraic Sylvester equation. Note that the multiplier matrix is the Xiao matrix. Let us proceed to consider solutions to the Sylvester differential equations that are finite cross-Gramians. Consider spectral decompositions of cross-Gramians over the simple spectrum of the dynamics matrix A. For this purpose, we use Formulas (59)–(62).

$$P_{j\eta}(t) = \Omega_{j\eta}(t) \circ \Psi_{j\eta},$$

$$\Omega_{j\eta}(t) = \sum_{k=1}^{2} \{\omega(1, s_k, -s_k, j, \eta)[1 - \exp(s_k t)]\Big\} e_{j+1} e_{\eta+1}^T,$$

$$\Omega_{00}(t) = \frac{2}{3} \sum_{k=1}^{2} [1 - \exp(s_k t)] e_1 e_1^T = \left[\frac{2}{3} - \frac{4}{3}\exp(-0.5t) - \frac{2}{3}\exp(-t)\right], \tag{73}$$

$$\Omega_{01}(t) = 0, \ \Omega_{10}(t) = 0, \tag{74}$$

$$\Omega_{11}(t) = \frac{1}{3} \sum_{k=1}^{2} [1 - \exp(s_k t)] e_2 e_2^T = \left[\frac{1}{3} + \frac{1}{3}\exp(-0.5t) - \frac{2}{3}\exp(-t)\right], \tag{75}$$

Finally, we obtain the formulas of the finite cross-Gramian in Hadamard form:

$$P(t) = \sum_{j=0}^{1} \sum_{\eta=0}^{1} \{\omega(1, s_1, -s_1, j, \eta[1 - \exp(s_1 t)]\} + \{\omega(1, s_2, -s_2, j, \eta[1 - \exp(s_2 t)]\} A_j bcA_\eta.$$

The matrices of multipliers of finite cross-Gramians preserve the structure of Xiao matrices of infinite controllability and observability Gramians (zero plaid structure [23–25]), but their elements depend not only on the eigenvalues of the dynamics matrix but also on time. Therefore, such matrices can be called generalized Xiao matrices, in contrast to the controllability and observability Gramians defined in [23–25].

Limit formulas are valid for the elements of matrices multipliers of finite and infinite cross-Gramians defined by Formula (72):

$$\lim_{t \to \infty} \Omega_{j\eta}(t) = \Omega_{j\eta}(\infty), \ \forall i, j = \overline{1, 2}, \ \forall t \in [0, \infty).$$

Note that, unlike the controllability and observability Gramians of the original system, the cross-Gramians matrix is not symmetric, although the matrices of their multipliers are symmetric.

5. Conclusions

This paper shows that the Gramians' Hadamard decomposition and their multiplier matrices play an important role in the problems of analyzing structural properties for a wide class of continuous linear dynamical systems given by their different equations of state. In the Introduction, we noted the important role of the scientific direction of works [22–27] based on the use of Rouse tables and Xiao matrices to compute finite and infinite Gramians by transforming the equations of state into canonical forms. Let us discuss the question, what is the difference between the methods and algorithms developed in this paper and those of the first direction? In [23], iterative procedures for computing solutions to algebraic Lyapunov equations using elements of Rouse tables and coefficients of the system's characteristic equation, leading to a Gramian representation in the form of Xiao matrices, are proposed. The methods and algorithms developed in this paper are based on direct methods for computing Gramian elements in closed form, for which Hadamard products and expansions of the resolvent dynamics matrix into the Faddeev–Leverrier series are used. Spectral expansions of infinite Gramians for the cases of simple and pair spectra were first obtained in the monograph [1]. In the monograph [25], there appeared formulas for computing the elements of the controllability Gramian for a system in which the equations of state are represented in diagonal canonical form. A similar result was previously obtained, by a different method in [32], for computing the controllability and observability Gramians in a broader setting, taking into account the multiplicity of the eigenvalues of the dynamics matrix. It is shown that Gramian matrices are pseudo-Hankel matrices—Xiao matrices. The advantages of this approach are obvious:

- Rouse tables are easier to compute compared to computing the eigenvalues of the matrix;
- the computation of Gramians using spectral decompositions leads to cumbersome expressions for the multiple spectra of the dynamics matrix, which makes it problematic to apply this method for high-dimensional systems, while in the first direction such problems do not arise;
- the computation of inverse controllability Gramians is reduced to solving systems of linear algebraic equations [24];
- the method can be used not only to compute Gramians but also to analyze the stability of the system according to the Rouse–Gurwitz criterion.

The disadvantage of the method is its limited application: it is recommended for use in systems of small and medium dimensionalities [24]. The works of the first direction did not use the Hadamard decomposition of Gramian matrices, the decomposition of the resolvent of the dynamics matrix into the Faddeev–Leverrier series, or the Laplace transform for computing Gramians. The results obtained in this paper also present a new method for computing Xiao matrices in closed form based on the information of the spectrum of the dynamics matrix and its characteristic equation. The decomposition of the Hadamard for Gramians and cross-Gramians of continuous systems given by equations of state in basic canonical forms changes the very paradigm of computation from the computation of matrices to the computation of their elements. This is a useful technique for computing the Gramians of weakly filled dynamics matrices and computing mixed Gramians for unstable systems [35]. Another advantage of the Hadamard decomposition: the proposed method gets rid of Faddeev matrices in spectral decompositions by transforming the equations of state into canonical forms of controllability and observability. The disadvantage of the second direction method compared to the first one is the cumbersome formulas of Gramian spectral decompositions for the case of the multiple spectra of the dynamics matrix. New possibilities of Gramian computation by using canonical transformations into diagonal, controllable, and observable canonical forms are shown in this article. In this case, Gramian matrices can be represented as the Hadamard product matrices of the multiplier matrices and matrices of the transformed right-hand side of the Lyapunov equations. It is shown that the multiplier matrices are invariant under various canonical transformations of linear continuous systems. We obtain modal Lyapunov equations for continuous SISO LTI systems in diagonal form and new algorithms for the elementwise computation of Gramian matrices for stable continuous MIMO LTI systems. We develop new algorithms for the computation of controllability Gramians and their traces in the form of Hadamard products of Xiao matrices for continuous SISO LTI systems in controllable and observable canonical forms. The use of transformations into canonical forms of controllability and observability made it possible to simplify the formulas of spectral decompositions in the form of Xiao matrices and simplify the calculations of Gramians. This article obtains new spectral expansions in Hadamard form for solutions to algebraic and differential Sylvester equations and spectral expansions of finite and infinite cross-Gramians of continuous MIMO LTI systems. The obtained results can be used for the optimal selection of locations for sensors and actuators in multivariable control systems and dynamic networks, for calculations and analysis of empirical Gramians, for assessing the risk of loss of stability in electric power systems, and in problems of analysis and synthesis of modal control systems [23].

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Nomenclature

SISO LTI system	a linear time invariant system with one input and one output
MIMO LTI system	a linear time invariant system with many inputs and many outputs
Xiao matrix	a pseudo-Hankel matrix depending on the coefficients of the characteristic polynomial of the dynamics matrix [23,24]
Faddeev matrix	a matrix arising from the decomposition of the resolvent of the dynamics matrix of a linear dynamical system [33,34]
Faddeev's series	a recursive method for computing the coefficients of the characteristic polynomial of a matrix [33,34]
Kalman controllability matrix	a matrix used in Kalman decomposition to transform the system state equations into the canonical form of controllability [8,9]
Gramian	a matrix that is a solution of a special kind of Lyapunov equation [1]
Sub-Gramian	a matrix that is a summand of the sum of matrices in the spectral decomposition of the Gramian matrix [1,4,6]
Hadamard's product	a matrix whose every element is the product of the corresponding elements of the input matrices (wiki) [2,4,6]
Hermite component	a matrix that is a complex square matrix equal to its conjugate transpose matrix [2]

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