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Review

From Quantum Automorphism of (Directed) Graphs to the Associated Multiplier Hopf Algebras

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Abstract: This is a noticeably short biography and introductory paper on multiplier Hopf algebras. It delves into questions regarding the significance of this abstract construction and the motivation behind its creation. It also concerns quantum linear groups, especially the coordinate ring of $M_q(n)$ and the observation that $\mathbb{K}[M_q(n)]$ is a quadratic algebra, and can be equipped with a multiplier Hopf $*$-algebra structure in the sense of quantum permutation groups developed by Wang and an observation by Rollier–Vaes. In our next paper, we will propose the study of multiplier Hopf graph algebras. The current paper can be viewed as a precursor to this upcoming work, serving as a crucial intermediary bridging the gap between the abstract concept of multiplier Hopf algebras and the well-developed field of graph theory, thereby establishing connections between them! This survey review paper is dedicated to the 78th birthday anniversary of Professor Alfons Van Daele.

Keywords: multiplier Hopf algebras; discrete quantum groups; compact quantum groups; quantum permutation group; quantum isometry group; quadratic algebra

MSC: 01A99; 16T05; 46L65; 46L67; 46L54

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1. Introduction

Alfons Van Daele was born in Sint-Niklaas, a city and municipality located in the Flemish province of East Flanders, on Thursday, 29 June 1945. From 1963 to 1967, he attended the mathematics department of the University of Leuven and received a specialist degree in mathematics. Later, he continued his Ph.D. studies under the direction of Frans Armand Cerulus at the Theoretical Physics Institute of the University of Leuven, and he defended his dissertation in 1970 on the applications of Lie algebras in nuclear physics.

Initially, he was very interested in working on operator algebras, and after completing his Ph.D. degree, he started working in this direction. One of his early significant accomplishments included simplifying Tomita's theory of generalized Hilbert algebras [1]. In 1989, he began researching quantum groups, and, together with his coauthors, attempted to develop the theory of locally compact quantum groups. One of the early outcomes of this collaboration was a result in discrete quantum groups [2]. This led him and his research group to introduce the concept of multiplier Hopf algebras, initially motivated by the C^*-algebraic approach to Quantum Groups. In the setting used by him, there were always some approximated identities, which allowed for the simplification of arguments rooted in Hopf algebra theory.

As a brief introduction, let us recall what it means to say that a certain object is a multiplier algebra, which mainly has been borrowed from ([3], Appendix A). Before delving into the main subject, we need to establish some preliminary groundwork to provide our foundation.

It is known that an algebra A over a field $\mathbb{K}$ is endowed with a non-degenerate product when it has the property that $a = 0$ if $ab = 0$ for all $b \in A$ and $b = 0$ if $ab = 0$ for all $a \in A$, and if our algebra has a non-degenerate product, then we can define the multiplier algebra of A and denote it by $M(A)$, where it is the usual $\mathbb{K}$-vector space of all ordered pairs (U, V) of linear maps on A, in a way that $V(a)b = aU(b)$, for all $a, b \in A$. In $M(A)$, the product will be given by the following rule:

$$(U, V)(U', V') = (U \circ U', V' \circ V).$$

Naturally, this algebra is associative and unital, with the identity element represented by the pair $1 = (\iota, \iota)$ where ι is the identity map taking values in A. Moreover, there exists a canonical algebra monomorphism $\jmath : A \to M(A)$ given by $a \mapsto (U_a, V_a)$, where U_a (resp., V_a) denotes the left (resp., right) multiplication by a, for all $a \in A$. Furthermore, if A is unital, then $\jmath$ is an isomorphism.

By the construction above and as a result of working within the theory of operator algebras, and using techniques derived from this theory, it was quite natural for him to decide to work on algebras without identity. Instead of looking at their multiplier algebras, he realized that for defining the dual of any Hopf algebra, which you obtain by taking the trivial co-product on the group algebra of an infinite group G, you will not be able to stay inside the theory of Hopf algebras anymore. An extended version of the notion of Hopf algebras will be needed!

After their well-developed study on integrals on the multiplier Hopf algebras, particularly the positive integrals on multiplier Hopf $*$-algebras, together with J. Kustermans, he lifted the theory to the operator algebra context. This meant that they employed all the existing ideas and improved them to obtain the notion of a locally compact quantum group based on multiplier Hopf algebras.

The structure of locally compact quantum groups, studied and introduced by Kustermans–Vaes–Van Daele, is somehow complicated, despite being quite well-developed. Working on them requires extensive skills in operator algebras, such as the Tomita–Takasaki theory, and also C^*-algebraic methods developed to study the unbounded operators on Hilbert spaces. Hence, this makes it challenging to work with them to develop other directions or to use them to study old theoretical results.

Fortunately, there are some objects that are called "algebraic quantum groups" in the category of multiplier Hopf ($*$-) algebras coming with (positive) integrals (this will be discussed later). The importance of discussing the (positive) integrals can be realized by noting that some of the locally compact quantum groups can be viewed as a multiplier Hopf $*$-algebra with a positive integral. For instance, this includes the compact and discrete quantum groups, and the Drinfel'd double of a compact quantum group.

Going back to algebraic quantum groups, it is worth mentioning that since they possess purely algebraic structures, they are a good option to work with. This is because they avoid deep analysis and contain almost all features of the general locally compact quantum groups, without imposing any extra restrictions on these algebraic quantum groups [4,5].

2. Conventions

Throughout the paper, we will use the Sweedler notation [6]

$$\Delta(a) = \sum_i a_{1_i} \otimes a_{2_i} \quad \text{for any } a \text{ in the coalgebra } C \tag{1}$$

or simply, we will write $\Delta(a) = \sum_{(a)} a_1 \otimes a_2$.

Throughout this paper, id will be referred to as the identity map, and $\mathbb{K}$ will be considered an arbitrary field unless otherwise stated.

By $\mathcal{G}\big(\mathbb{K}(M_q(n))\big)$, we mean the oriented connected graph associated with the relations of the coordinate ring of $M_q(n)$, the quantum $n \times n$ matrices.

3. From Non-Unital to Unital Algebras

In Hopf algebra theory, when we equip the underlying algebra A with a coalgebra structure, we need to define a map, Δ, from A to $A \otimes A$, such that the relation, $\Delta(A) \subset A \otimes A$, satisfies. Now, as a well-known fact, by considering the set of $\mathbb{K}$-valued functions $\mathbb{K}(S) = \mathbb{K}^S$, and equipping it with the specific unital, associative multiplication $f \cdot g = s \mapsto f(s)g(s)$, we will obtain a commutative algebra over $\mathbb{K}$. Furthermore, by letting S be a finite group, say G, and due to the natural algebraic structure of the domain of these functions, it is straightforward to define and demonstrate the existence of the coalgebra structure. The comultiplication will be the map $\Delta\colon \mathbb{K}(G) \to \mathbb{K}(G) \otimes \mathbb{K}(G)$, and since G is finite, we will have the following isomorphism

$$\mathbb{K}(G \times G) \cong \mathbb{K}(G) \otimes \mathbb{K}(G),$$

meaning that what we need is just to produce a function in two variables somehow taking values in G. Hence, for $f\colon G \to \mathbb{K}$, $\Delta(f)$ needs to take two group elements, and the most natural way to do this is to have $\Delta(f)(g,h) := f(gh)$, by simply receiving help from the group multiplication.

Comultiplication must possess a counit and satisfy the co-associativity axiom. It will be co-commutative if G is Abelian, and in addition to the above structures, Hopf algebras are equipped with an antipode, which relates the algebra and coalgebra structures. For $\mathbb{K}(G)$, we may take $S(f)(g) := f(g^{-1})$ to play the role of antipode of f at g in $\mathbb{K}(G)$.

But what we described above does not work under all conditions and everywhere! The above construction of the comultiplication breaks down when working with an infinite-dimensional group G, where everything changes suddenly, and we encounter a significant problem in defining such a comultiplication structure. This is because equality no longer holds, and we find that $\mathbb{K}(G) \otimes \mathbb{K}(G)$ is a proper subset of $\mathbb{K}(G \times G)$ instead of being equal, and when we try to define Δ as a co-multiplication map, then the image falls in $\mathbb{K}(G \times G)$. In this case, it is uncertain whether or not the image of our map belongs to $\mathbb{K}(G) \otimes \mathbb{K}(G)$.

To come up with a solution for this problem, instead of looking for functions f, such that $\Delta(f) \subset \mathbb{K}(G) \otimes \mathbb{K}(G)$, Van Daele [3] offered the following solution:

Take A as a unital or non-unital algebra with a non-degenerate product (meaning that if $a \cdot b = 0$ for all $a \in A$ we should obtain $b = 0$ and vice versa), then define $M(A)$ to be the set

$$\{\rho : A \to A \mid \rho \text{ is a right and left multiplier satisfying in (2)}\}$$
$$:= \{x = (\rho_1, \rho_2) \mid \text{for } \rho_1 a \text{ left multiplier and } \rho_2 a \text{ right multiplier}\},$$

where the statement "ρ is a left multiplier" means that ρ is a linear functional on A, such that for all $a, b \in A$, we have $\rho(a \cdot b) := a\rho(b)$ (in this case, we write $\rho \in L(A)$).

Remark 1 ([3])**.** *We will call ρ a multiplier of A if it satisfies the following compatibility relation:*

$$c(\rho(a)b) = (c\rho(a))b \tag{2}$$

for all $a, b, c \in A$.

Notation 1. For $x \in L(A)$ and all $a \in A$, we write $x \cdot a := x(a)$, and for $x \in R(A)$ and all $a \in A$, we write $a \cdot x := x(a)$, and if $x \in M(A)$, then for all $a \in A$, we write $x \cdot a := L(a)$ and $a \cdot x := R(a)$.

At this point, let us recall some facts regarding the multiplier algebras from Appendix A in [3].

$M(A)$ is a unital algebra with the unit denoted by $1_{M(A)} := (\mathrm{id}_A, \mathrm{id}_A)$ and the product

$$x \cdot y := (\rho_{1_x} \circ \rho_{1_y}, \rho_{2_y} \circ \rho_{2_x}),$$

for all $x = (\rho_{1_x}, \rho_{2_x})$ and $y = (\rho_{1_y}, \rho_{2_y})$ in $M(A)$ (with $l_x \in L(A)$ and $l_x(a) := x \cdot a$ and $r_x(a) := a \cdot x$ and vice versa for l_y and r_y).

We embed A in a natural way in $L(A)$, $R(A)$, and $M(A)$, in the following way:

Define maps $l_a : A \to A : b \mapsto l_a(b) := a \cdot b$ and $r_a : A \to A : b \mapsto r_a(b) := b \cdot a$ for all $a, b \in A$, and the map $\rho : A \to A : a \mapsto \rho(a) := (l_a, r_a)$. Then we prove that these maps are injective. To do so, suppose that for all $c \in A$, we have $a \cdot c = l_a(c) = l_b(c) = b \cdot c$. Then we have that $l_{(a-b)}(c) = (a-b)c = 0$ and because of the non-degeneracy of A, we obtain $c = 0$. The injectivity of r_a can be satisfied in almost a similar way. So, we obtain the injectivity of $\rho \in M(A)$.

If the algebra A contains an identity, then the product in A will automatically be non-degenerate in the following way:

Let $a \in A$, such that for all $b \in A$, we have $a \cdot b = 0$, then we have $a = a \cdot 1 = 0$, because A is unital. This shows that the product in A is non-degenerate. In other words, if the algebra A has an identity, we will have $A = L(A) = R(A) = M(A)$, and we will have the surjectivity of ρ, because if we suppose that $(L, R) \in M(A)$ and if we set $a = L(1) = R(1)$, then we will have $l_a = L$ and $r_a = R$; this is because if $b \in A$, we have:

$$l_a(b) = L(1)b = L(1b) = L(b) \text{ and } r_a(b) = bR(1) = R(1b) = R(b). \tag{3}$$

This gives the surjectivity of ρ, L, and R.

Remark 2. *If A is an ideal in the algebra B, meaning that for $b \in B$ and $a \in A$, we have $a \cdot b$ and $b \cdot a$ in A, then we say that A is an essential or dense ideal in B if for all $b \in B$ and $b \cdot A = \{b \cdot a \mid a \in A\} = 0$ we have $b = 0$.*

Lemma 1 ([3]). *$M(A)$ is the largest unital algebra that contains A as an essential ideal.*

Proof. To prove this, let us take B, a unital algebra containing A as an essential ideal. Then, we prove that $B \subset M(A)$. We define the map $\varphi : B \hookrightarrow M(A) : b \mapsto \varphi(b) : a \mapsto b \cdot a$, meaning that $\varphi(b)a := b \cdot a$ implies that φ is a left multiplier. Similarly, we define $\varphi : B \hookrightarrow M(A) : b \mapsto \varphi(b) : a \mapsto a \cdot b$ i.e., $a\varphi(b) := a \cdot b$; this means that φ is a right multiplier and we have the compatibility of these definitions, which means that φ is a multiplier and our definition is true and $M(A)$ is the largest unital algebra containing A.

Now, it remains to prove that A is contained in $M(A)$ as an essential ideal. Suppose there exists $\rho \in M(A)$ such that $\rho \cdot A = \{\rho \cdot a \mid \forall a \in A\} = 0$. Because of the definition of the multiplier algebra, we have that ρ is in $L(A)$, such that the above-mentioned product is satisfied in A, and because of the non-degeneracy of the product in A, we have that $\rho = 0$ and this means that A is an essential ideal in $M(A)$. □

Proposition 1. *For $A = \mathbb{K}_f(G) = \{f : G \to \mathbb{K} \mid G$ is any group and f has finite support$\}$, we have $M(A) = \mathbb{K}(G) = \{f : G \to \mathbb{K} \mid G$ is any group$\}$.*

Proof. To see this, we prove that $\mathbb{K}(G)$ is a unital algebra containing A as an essential ideal. We have $A \subset \mathbb{K}(G)$. For $f \in \mathbb{K}(G)$, let $S := \mathrm{supp}(f) = \{x \in G \mid f(x) \neq 0\}$ and if S is finite, then f will be in A. On the other hand, we note that there is another equivalent definition for finite support, which can be stated as follows:

$$\mathrm{supp}(f) \text{ is finite} \Leftrightarrow \exists S \subset G \text{ such that } S \text{ is finite and } f\mid_{S^c} = 0.$$

Let $h\colon G \to \mathbb{K}$, such that $hA = 0$. We prove that $h = 0$.

For all $x \in G$, set $S = \{x\}$ and define $f\colon G \to \mathbb{K}$ as follows for all $p \in G$:

$$f(p) = \begin{cases} 1 & \text{if } p = x \\ 0 & \text{if } p \neq x. \end{cases}$$

We see that f has finite support and, hence, is in A. From the assumption $hA = \{hf \mid$ for all $f \in A\} = 0$, we have that $hf = 0$, but as we have $f \neq 0$, we should have $h = 0$, and this proves that $\mathbb{K}(G)$ contains A as an essential ideal.

But we know that the largest unital algebra that contains A as an essential ideal is $M(A)$; hence, there are two possible cases $\mathbb{K}(G), \subset M(A)$ or $\mathbb{K}(G) = M(A)$.

On the contrary, let us suppose that $\mathbb{K}(G) \neq M(A)$ satisfies. Hence, we should have $\mathbb{K}(G) \subset M(A)$, and from there, we can conclude that there is a $\rho \in M(A)$ that is not contained in $\mathbb{K}(G)$. But since the elements of $M(A)$ are functions with values in $\mathbb{K}$, and $\mathbb{K}(G)$ is the set of all functions with values in $\mathbb{K}$, the latest statement cannot happen. So, we have $\mathbb{K}(G) = M(A)$. $\square$

4. The Birth of Multiplier Hopf Algebra

For A, as in Proposition 1, let us identify $A \otimes A$ with $\mathbb{K}_f(G \times G)$ in the following way:

$$A \otimes A \to \mathbb{K}_f(G \times G) : f_1 \otimes f_2 \mapsto (f_1 \otimes f_2)(g_1, g_2) := f_1(g_1) f_2(g_2). \tag{4}$$

Then, by looking at $\mathbb{K}_f(G \times G)$ as a subspace of $\mathbb{K}(G \times G)$, we can equip them with an algebra structure by using pointwise multiplication. For $f \in A$, we will also consider $\Delta(f)$ in $\mathbb{K}(G \times G)$, defining relation $\Delta(f)(p, q) := f(pq)$.

Let $f \in \mathbb{K}(G)$ and $h \in A$, such that $h\mid_{S^c} = 0$. Then for all $x \in S^c$, we have $K(x) := f(x)h(x) = 0$. So, K has finite support, and we have $K \in \mathbb{K}_f(G)$; from this, for all $g, h \in A$, we observe that $\Delta(h)(1 \otimes g)$ and $(h \otimes 1)\Delta(g)$ are in $A \otimes A$, and for $p, q \in G$, we have the bijections $G \times G \to G \times G : (p, q) \mapsto (pq, q)$ and $G \times G \to G \times G : (p, q) \mapsto (p, pq)$. We have $\Delta(f)(1 \otimes g)(p, q) = f(pq)g(q)$, which is dual to the map $(p, q) \mapsto (pq, q)$ and $(f \otimes 1)\Delta(g)(p, q) = f(p)g(pq)$ is dual to the map $(p, q) \mapsto (p, pq)$, so they are bijective, and at this point, let us call them T_1 and T_2, respectively, defined over the non-unital algebra $A \otimes A$, as has been defined in [3].

Hence, in this process, we obtain from a non-unital algebra, a unital algebra, and we define the co-multiplication for this algebra in a way that T_1 and T_2 are bijections and are satisfied in the following co-associativity relation in $M(A \otimes A \otimes A)$:

$$(f_1 \otimes 1_{M(A)} \otimes 1_{M(A)})(\Delta \otimes i)(\Delta(f_2)(1_{M(A)} \otimes f_3))$$
$$= (i \otimes \Delta)((f_1 \otimes 1_{M(A)})\Delta(f_2))(1_{M(A)} \otimes 1_{M(A)} \otimes f_3),$$

for all f_1, f_2, and f_3 in A [3].

Remark 3. *From the above explanation, we find that this relation coincides with the co-associativity relation of Hopf algebras, and it is evident that we are working within multiplier spaces. Therefore, we will prove this by utilizing the facts that are satisfied in multiplier spaces.*

Lemma 2 ([3])**.** *$T_2 := (i \otimes \Delta) \circ \Delta$ and $T_1 := (\Delta \otimes i) \circ \Delta$ are maps from A to $M(A \otimes A \otimes A)$.*

Proof. For all f_1, f_2, and f_3 in A, let $C := (i \otimes \Delta) \circ \Delta = (\Delta \otimes i) \circ \Delta$ and define $\rho_1(1 \otimes 1 \otimes f_1) := C(1 \otimes 1 \otimes f_1)$ as a left multiplier and $\rho_1(f_2 \otimes 1 \otimes 1) := (f_2 \otimes 1 \otimes 1)C$ as a right multiplier. We prove that $((f_2 \otimes 1 \otimes 1)C)(1 \otimes 1 \otimes f_1) = (f_2 \otimes 1 \otimes 1)(C(1 \otimes 1 \otimes f_1))$ is a true relation and because of the right relation $(i \otimes \Delta) \circ \Delta = (\Delta \otimes i) \circ \Delta$, we obtain the co-associativity relation for multiplier Hopf algebras.

Now, for all $c \in A$ and $x \in A \otimes A$, we need to prove that $m(x(1 \otimes c)) = m(x)c$.

To prove this assertion, let us assume $x = a_1 \otimes a_2$, and from the above, we have $(1 \otimes c) = (l_{1 \otimes c}, r_{1 \otimes c}) = (\mathrm{id} \otimes l_c, \mathrm{id} \otimes r_c)$, so $x(1 \otimes c) = (a_1 \otimes a_2)(1 \otimes c) = r_{1 \otimes c}(a_1 \otimes a_2) = (\mathrm{id} \otimes r_c)(a_1 \otimes a_2) = a_1 \otimes a_2 c$. So, we obtain $m(x(1 \otimes c)) = a_1(a_2 c) = (a_1 a_2)c = m(a_1 \otimes a_2)c = m(x)c$.

In the same way, we find that for all $c \in A$ and $x \in A \otimes A$, we have $T_1(x(1 \otimes c)) = T_1(x)(1 \otimes c)$, because if we assume $x = a_1 \otimes a_2$, then we have $T_1(x(1 \otimes c)) = T_1((a_1 \otimes a_2)(1 \otimes c)) = T_1(r_{1 \otimes c}(a_1 \otimes a_2)) = T_1((\mathrm{id} \otimes r_c)(a_1 \otimes a_2)) = T_1(a_1 \otimes a_2 c) =$

$\Delta(a_1)(1 \otimes a_2 c) = \Delta(a_1)((1 \otimes a_2)(1 \otimes c)) = (\Delta(a_1)(1 \otimes a_2))(1 \otimes c) = r_{1 \otimes c}(\Delta(a_1)(1 \otimes a_2)) = r_{1 \otimes c}(T_1(a_1 \otimes a_2)) = r_{1 \otimes c}(T_1(x)) = T_1(x)(1 \otimes c)$. □

Proposition 2 ([3], Appendix A)**.** *We have natural embeddings,* $L(A) \otimes L(B) \hookrightarrow L(A \otimes B)$ *and* $R(A) \otimes R(B) \hookrightarrow R(A \otimes B)$ *and* $M(A) \otimes M(B) \hookrightarrow M(A \otimes B)$*.*

Proof. For all $c \in L(A)$, $d \in L(B)$, $a \in A$, and $b \in B$, we define $\varphi : L(A) \otimes L(A) \to A \otimes B : c \otimes d \mapsto \varphi(c \otimes d) : a \otimes b \mapsto \varphi(c \otimes d)(a \otimes b) := ca \otimes db$. Now, the claim is that $\varphi(c \otimes d)$ is a left multiplier. The proof of this claim will proceed as follows:

$(\varphi(c \otimes d))((a \otimes b)(a' \otimes b')) = (\varphi(c \otimes d))(aa' \otimes bb') = c(aa') \otimes d(bb') = (ca)a' \otimes (db)b' = (ca \otimes db)(a' \otimes b') = (\varphi(c \otimes d)(a \otimes b))(a' \otimes b')$.

Now, we prove that $\varphi(c \otimes d)$ is an injective map. We assume that $\varphi(c \otimes d)(a \otimes b) := ca \otimes db = 0$. For an arbitrary linear map ω_1 in $B*$, and for all $c \in L(A)$, we have $(i \otimes \omega_1)(ca \otimes db) = ca \otimes \omega_1(db) = (ca)\omega_1(db) = 0$. Then, due to the non-degeneracy, we deduce $a\omega_1(db) = 0$, which leads to $a \otimes db = 0$; then, if we apply this to an arbitrary linear functional, $\omega_2 \in A*$, for all $d \in L(B)$, we have that $0 = (\omega_2 \otimes i)(a \otimes db) = \omega_2(a) \otimes db = \omega_2(a)db$, then we have that $d\omega_2(a)b = 0$, then $\omega_2(a)b = 0$, and then $a \otimes b = 0$. So, $\varphi(c \otimes d)$ provides an embedding into the space of left multipliers on $A \otimes B$ and we can extend it to $L(A \otimes B)$. For the second assertion, we can define for all $c \in R(A)$, $d \in R(B)$, $a \in A$, and $b \in B$:

$\psi : R(A) \otimes R(B) \to A \otimes B : c \otimes d \mapsto \psi(c \otimes d) : a \otimes b \mapsto \psi(c \otimes d)(a \otimes b) := ca \otimes db$. The proof will proceed exactly in a similar way as to the case of left multipliers. □

5. Multiplier Hopf Algebras

The main references for this section are [2–5] and the Definitions have been adapted accordingly.

Working with a $\mathbb{K}$-algebra (it might be unital or non-unital), dualizing it in a very natural way, by changing the direction of the arrows, and obtaining a $\mathbb{K}$-bialgebra structure in a way that satisfies the compatibility condition between the algebra and the coalgebra structures, and equipping it with an antipode as the inverse of the identity map in the convolution algebra of $\mathbb{K}$-endomorphisms of the $\mathbb{K}$-bialgebra, we obtain a Hopf algebra over $\mathbb{K}$.

Many people over the last decades have tried to provide a general or even a partial generalization of the Hopf algebra category. By considering it from the module theoretic view and as a $\mathbb{K}$-bialgebra, it could be understood as a $\mathbb{K}$-algebra by turning its category of left (or, equivalently right) modules into a monoidal category In such a way, the forgetful functor to the category of $\mathbb{K}$-modules would be a strict monoidal functor. In pursuing this, several generalizations have emerged, such as quasi-Hopf algebras [7], weak Hopf algebras [8], Hopf algebroids [9], and Hopf group (co)algebras [10].

As has been highlighted, motivated by the theory of (discrete) quantum groups, multiplier Hopf algebras were introduced by A. Van Daele in [3]. To define a multiplier Hopf algebra, we start with a non-unital algebra A, and a map $\Delta : A \to M(A \otimes A)$, the so-called comultiplication map, and two certain bijective endomorphisms T_1, T_2 on $A \otimes A$. Here, $M(A \otimes A)$ is the multiplier algebra of $A \otimes A$, considered the largest unital algebra containing $A \otimes A$ as a two-sided dense ideal. By dense, we mean that if for any $x \in M(A \otimes A)$, we have $x \cdot a = 0$ for all $a \in A$, then we have $x = 0$. The interested reader is referred to ([11], appendix), for a topological interpretation of these properties.

Definition 1. *Let A be an algebra over $\mathbb{C}$ with a non-degenerate unital or non-unital product. And let $\Delta : A \to M(A \otimes A)$ be a homomorphism. Assume that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ belong to $A \otimes A$ for all a and b in A. We say that Δ is co-associative if:*

$$(a \otimes 1 \otimes 1)(\Delta \otimes i)(\Delta(b)(1 \otimes c)) = (i \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all $a, b,$ and c in A and $\iota : A \to A$ and 1 the unit element of $M(A)$. Then, Δ will be called a comultiplication on A.

Now, we can define a multiplier Hopf algebra according to A. Van Daele [3].

Definition 2. *Let A be as in Definition 1 and Δ be a comultiplication on A. We call A a multiplier Hopf algebra if the linear maps $T_1, T_2 : A \otimes A \to A \otimes A$, defined by*

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

are bijective. A will be called regular if $\sigma\Delta$, where σ is the flip map, is again a comultiplication, such that $(A, \sigma\Delta)$ is also a multiplier Hopf algebra.

These conditions imply that Δ is a nondegenerate homomorphism. As discovered above, the homomorphisms $\iota \otimes \Delta$ and $\Delta \otimes \iota$ will have unique extensions to $M(A \otimes A)$. Then, the co-associativity condition in Definition 1 means nothing else but $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. But, we will always use co-associativity, as formulated in Definition 1.

In [3], it has been proven that for A, a multiplier Hopf algebra with an identity, the structures of a Hopf algebra also satisfy and, hence, A will automatically be a Hopf algebra, and $\sigma\Delta$ will automatically be a comultiplication. Hence, the multiplier Hopf algebra category is a natural generalization of the Hopf algebra category.

Remark 4. *The multiplier Hopf algebra A will be regular if and only if S has an inverse, and in general, if A is Abelian, then automatically, A will be regular.*

6. The Concept of an Integral and the Algebraic and Locally Compact Quantum Groups

No new results are presented in this section and the definitions, propositions, and theorems were adapted from Prof. Van Daele's new papers [4,5], written on the subject.

In order to motivate the definition of integrals, it is important to reformulate the concept of a Haar measure in terms of Hopf algebras.

For G, a locally compact group with a left Haar measure λ, if G is discrete (with a discrete topology), then the counting measure on G is a (left) Haar measure. Now, consider $A \subseteq \mathbb{C}(G)$ a Hopf algebra of functions on G, with the following structures

$$\Delta : \mathbb{C}(G) \to \mathbb{C}(G \times G) : f \mapsto \Delta(f)((x, y)) := f(xy);$$
$$\epsilon : \mathbb{C}(G) \to \mathbb{C} : f \mapsto \epsilon(f) := f(e);$$
$$S : \mathbb{C}(G) \to \mathbb{C}(G) : f \mapsto S(f)(x) := f(x^{-1}),$$

for $x, y \in G$ and e the identity element. If $A \subseteq L^1(G, \lambda)$, then the left Haar integral defines a linear functional

$$\Psi : A \to \mathbb{C} : f \mapsto \int_G f(y) d\lambda(y). \tag{5}$$

The left invariance of λ amounts to the fact that for each $f \in A$, the function F on G, such that

$$F(\cdot) = \int_G f(\cdot y) d\lambda(y) \tag{6}$$

satisfies $F(x) = \Psi(f)$ for all $x \in G$. We replace the multiplication of G by the comultiplication of A, using the relation

$$f(xy) = \Delta(f)((x, y)) = \Sigma f_{(1)}(x) f_{(2)}(y),$$

for x and y, as before, and accordingly, we obtain

$$F = \Sigma f_{(1)} \int_G f_{(2)}(y) d\lambda(y) = (\mathrm{id} \otimes \Psi)\Delta(f),$$

which means that the invariance condition "$F(x) = \Psi(f)$" for all $x \in G$ takes the form

$$(\mathrm{id} \otimes \Psi)\Delta(f) = \Psi(f)1_A.$$

Now, let (A, Δ) be a regular multiplier Hopf algebra. Given $\Psi \in A'$ (the space of complex A-valued functions) and $a, b \in A$, we define

$$((\mathrm{id} \otimes \Psi)\Delta(a))b = (\mathrm{id} \otimes \Psi)(\Delta(a)(b \otimes 1)) = \Sigma a_{(1)} b \Psi(a_{(2)});$$

$$b((\mathrm{id} \otimes \Psi)\Delta(a)) = (\mathrm{id} \otimes \Psi)((b \otimes 1)\Delta(a)) = \Sigma b a_{(1)} \Psi(a_{(2)}).$$

It is easy to see that $(\mathrm{id} \otimes \Psi)\Delta(a) \in M(A)$.

Similarly, we define $(\Psi \otimes \mathrm{id})\Delta(a) \in M(A)$ by

$$((\Psi \otimes \mathrm{id})\Delta(a))b = (\Psi \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = \Sigma \Psi(a_{(1)}) a_{(2)} b,$$

and

$$b((\Psi \otimes \mathrm{id})\Delta(a)) = (\Psi \otimes \mathrm{id})((1 \otimes b)\Delta(a)) = \Sigma \Psi(a_{(1)}) b a_{(2)}.$$

Hence, the above notations will give rise to the following definition:

Definition 3 ([12]). *For (A, Δ), a regular multiplier Hopf algebra, a linear map $\Psi : A \to \mathbb{C}$ will be called a left-invariant (resp. right-invariant) if $(\mathrm{id} \otimes \Psi)\Delta(a) = \Psi(a)1_{M(A)}$ (if $(\Psi \otimes \mathrm{id})\Delta(a) = \Psi(a)1_{M(A)}$) satisfies for all $a \in A$.*

A non-zero left/right-invariant will be called a left/right integral. And if a left integral, Ψ, is simultaneously a right integral, then Ψ will be just called an integral.

Example 1.

(i) *For G a finite group, let $(\mathbb{C}(G), \Delta)$ be as above. Then, from the equality $S(\delta_x) = \delta_{x^{-1}}$, we can deduce that $S : \mathbb{C}(G) \to \mathbb{C}(G)$ is invertible, and, hence, $(\mathbb{C}(G), \Delta)$ is a regular multiplier Hopf algebra with $\mathbb{C}(G) = M(\mathbb{C}(G))$ and $1_{M(\mathbb{C}(G))} = 1_G$ (where 1 stands for the identity function on G). Now, consider*

$$\Psi \to \mathbb{C} : f \mapsto \Sigma_{x \in G} f(x),$$

which is linear. Moreover, for all $x \in G$, we have

$$\begin{aligned}
(\mathrm{id} \otimes \Psi)\Delta(\delta_x) &= \Sigma_{y,z \in G, yz=x}(\mathrm{id} \otimes \Psi)(\delta_y \otimes \delta_z) \\
&= \Sigma_{y,z \in G, yz=x} \delta_y \Psi(\delta_z) \\
&= \Sigma_{y \in G} \delta_y \\
&= 1_G \\
&= \Psi(\delta_x)\, 1_{M(\mathbb{C}(G))}
\end{aligned}$$

and in the same way, we have $(\Psi \otimes \mathrm{id})\Delta(\delta_x) = 1_G = \Psi(\delta_x) = 1_{M(\mathbb{C}(G))}$, and hence, $\Psi : \mathbb{C}(G) \to \mathbb{C}$ will be an integral on $(\mathbb{C}(G), \Delta)$.

(ii) *More generally, let $A = \mathbb{C}_f(G)$, be the algebra of complex functionals with finite support on a (discrete) group G. Then it is not too difficult to see that $M(A) \cong \mathbb{C}(G)$ and $1_{M(A)} = 1_G$, and $\Psi : A \to \mathbb{C}$, defined by*

$$\Psi(f) = \Sigma_{x \in G} f(x),$$

will satisfy the conditions of being an integral. Now, consider $f, g \in A$ and $x \in G$. Then,

$$\begin{aligned}(f((\mathrm{id}\otimes\Psi)\Delta(g)))(x) &= ((\mathrm{id}\otimes\Psi)((f\otimes 1)\Delta(g)))(x)\\ &= \Sigma((\mathrm{id}\otimes\Psi)(f\otimes 1)\Delta(g)))(x)\\ &= \Sigma((\mathrm{id}\otimes\Psi)(fg_{(1)}\otimes g_{(2)}))(x)\\ &= \Sigma_{y\in G}f(x)g_{(1)}(x)g_{(2)}(y)\\ &= \Sigma_{y\in G}f(x)g(xy)\\ &= \Sigma_{y\in G}f(x)g(y)\\ &= f(x)\Psi(g),\end{aligned}$$

and as a result, we will obtain $f((\mathrm{id}\otimes\Psi)\Delta(g)) = f\Psi(g)$. *In a similar way, we can obtain* $((\mathrm{id}\otimes\Psi)\Delta(g))f = \Psi(g)f$, *meaning that for all* $g \in A$,

$$(\mathrm{id}\otimes\Psi)\Delta(g) = \Psi(g)\,1_G,$$

satisfies and, therefore, Ψ *will be a left-invariant. In this case, since for all* $f \in A$ *and* $y \in G$, $\Sigma_x f(xy) = \Sigma_x f(x)$; *hence,* Ψ *will also be a right-invariant.*

Below, you can find some characterizations of the left and right invariants.

Remark 5.

(i) *For* (A,Δ), *a regular multiplier Hopf algebra,* $\Psi \in A'$ *will be a left-invariant if and only if for all* $f \in A'$ *and* $a, b \in A$, *we have*

$$\begin{aligned}(f\otimes\Psi)(\Delta(a)(b\otimes 1)) &= f((\mathrm{id}\otimes\Psi)(\Delta(a)(b\otimes 1)))\\ &= f(b((\mathrm{id}\otimes\Psi)\Delta(a)))\\ &= f(b\Psi(a))\\ &= \Psi(a)f(b),\end{aligned}$$

and

$$(f\otimes\Psi)(\Delta(a)(1\otimes b)) = \Psi(a)f(b) \quad \textit{all} \quad (\Psi\otimes f)((1\otimes b)\Delta(a)) = \Psi(a)f(b).$$

(ii) *Let* (A,Δ) *be a regular Hopf algebra. Then, it is possible to characterize the invariance of the linear maps in terms of the convolution of two (linear complex-valued) functions, denoted by* $*$. *Let* $\Psi \in A'$. *Then* Ψ *is left-invariant if and only if* $f * \Psi = f(1_A)\Psi$ *for all* $f \in A'$; *and, respectively,* Ψ *is right-invariant if and only if* $\Psi * f = f(1_A)\Psi$ *for all* $\in A'$. *These follow from the relations*

$$(f\otimes\Psi)(\Delta(a)) = (f*\Psi)(a) \quad \textit{all} \quad (\Psi\otimes f)(\Delta(a)) = (\Psi * f)(a).$$

Let us recall that a linear map $\Psi : A \to \mathbb{C}$ *on an algebra* A *will be called*

- *Faithful if* $\Psi(aA) \neq 0$ *and* $\Psi(Aa) \neq 0$ *for every non-zero* $a \in A$;
- *Positive if* A *is a* $*$*-algebra and* $\Psi(a^*a) \geq 0$ *for all* $a \in A$;
- *Normalized if* A *is unital and* $\Psi(1_A) = 1$.

Lemma 3 (Cauchy–Schwarz Inequality)**.** *For* $*$*-algebra* A *and a positive linear functional* $\Psi : A \to \mathbb{C}$ *on* A, *we have*

$$\mid \Psi(a^*b) \mid^2 \leq \Psi(a^*a)\Psi(b^*b) \qquad \textit{for all} \quad a, b \in A. \tag{7}$$

Remark 6. *In completion of the above discussion, we recall that there are three cases in which the multiplier Hopf algebra* (A,Δ) *will automatically be regular*

(a) *When A is co-commutative, then we have $\Delta = \sigma \circ \Delta$ and, hence, $(A, \Delta)^{cop} = (A, \Delta)$ and, hence, (A, Δ) will be regular. In this case, we have $S = S^{cop} = S^{-1}$ and, therefore, $S^2 = \mathrm{id}$ satisfies.*

(b) *When A is commutative, then we have $(A, \Delta)^{op} = (A, \Delta)$ and, therefore, (A, Δ) will also be regular.*

(c) *For (A, Δ), a multiplier Hopf $*$-algebra, we have that $S(A) \subseteq A$ and $S \circ * \circ S \circ * = \mathrm{id}_A$. Hence, the antipode $S : A \to A$ will be bijective and, therefore, (A, Δ) will be regular. Furthermore, in this case, the relation $S^{cop} = S^{-1} = * \circ S \circ *$ also satisfies.*

Remark 7. *For a multiplier Hopf $*$-algebra (A, Δ), (which is also automatically regular by Remark 6), Let Ψ be a non-zero positive linear functional. The following results are well-known:*

(a) *Ψ is $*$-linear.*

(b) *If A is unital, then there exists a real number $\lambda > 0$, such that $\lambda\Psi$ will be normalized.*

(c) *Ψ is faithful if and only if $\Psi(r^* r) > 0$ for all non-zero $r \in A$.*

Proposition 3 ([12])**.** *Every left (right) integral on a regular multiplier Hopf algebra will be faithful.*

Proposition 4 ([12])**.**

(i) *Let (A, Δ) be a regular multiplier Hopf algebra with a left/right integral Ψ. Then $\Psi \circ S/\Psi \circ S^{-1}$ will be a right/left integral on (A, Δ).*
Hence, we can say that a regular multiplier Hopf algebra (A, Δ) will have a left integral if and only if it has a right integral.

(ii) *Every regular Hopf algebra will obey, at most, one normalized left/right integral Ψ, such that it will be simultaneously a right/left integral satisfying $\Psi = \Psi \circ S = \Psi \circ S^{-1}$.*

It is natural to consider the positivity of integrals for multiplier Hopf $*$-algebras. In this context, regarding the results obtained in Proposition 4 and those concerning the preservation of positivity, it is unfortunate that the correspondence between the left and right integrals in Proposition 4 does not necessarily preserve positivity! But, we still have the following proposition, which can somehow be considered as a motivation behind the concept of an algebraic quantum group!

Proposition 5 ([4])**.** *A multiplier Hopf $*$-algebra has a positive left integral if and only if it has a positive right integral.*

And we have the following Definition.

Definition 4 ([4])**.** *A multiplier Hopf $*$-algebra with a positive left integral and a positive right integral, will be called an algebraic quantum group.*

In [12], multiplier Hopf algebras that allow a non-zero left-invariant functional, as in Definition 4, have been considered. Such multiplier Hopf algebras are eventually called algebraic quantum groups, featuring very rich structures such as duality theory. They could be regarded as an algebraic model for locally compact quantum groups, despite not having a proper definition initially at that time. In [6], it has been shown that a $*$-algebraic quantum group will naturally give rise to a C^*-algebraic quantum group as defined by Woronowicz, Masuda, and Nakagami. This is considered a definition of a locally compact quantum group, as coined by Kustermans and Vaes [13]. The definition proposed by Kustermans and Vaes was based on the same set of axiomatic relations as those proposed by Woronowicz, Masuda, and Nakagami. However, it was much simpler and smaller, inspired by the axioms of the theory of $*$-algebraic quantum groups. Although it became a bit more analytical later, this made them somehow not sufficiently powerful to achieve a theory that could satisfy all desired properties!

Following this, by looking at the C^*-algebras, as quantized locally compact spaces, a framework based on C^*-algebras has been proposed by Woronowicz, in order to define locally compact quantum groups [14]. A long list of axiomatic relations was proposed, leading to the most general C^*-algebra version of the locally compact quantum group, which was formulated in the von Neumann algebra framework by Masuda and Nakagami [15], and later by Masuda, Nakagami, and Woronowicz in some lectures, based on the paper mentioned above.

Revisiting Proposition 4, for A as an $*$-algebraic quantum group, it is still possible to choose Φ (right integral) to be positive. However, note that in order to arrive at this function, using the *GNS* construction for Ψ seemed a little bit inevitable. The problem is that the right-invariant functional $\Phi = \Psi \circ S$ will not necessarily be positive! In order to obtain Φ, one might use the square root of the modular element, or a polar decomposition of the antipode (see [6]).

The theory of multiplier Hopf algebras, particularly the duality for regular multiplier Hopf algebras with integrals, is not only a framework that allows for results not possible within the usual Hopf algebras. It also serves as a model for an analytical theory of locally compact quantum groups (see [13,16]).

Now, we are ready to formulate the Definition of a reduced C^*-algebraic quantum group for $M(A)$, the multiplier algebra of a C^*-algebra A, as was done in [17]. When $\Delta : A \to M(A \otimes A)$ is a non-degenerate $*$-homomorphism, we will call a proper weight ϕ on (A, Δ) left-invariant (resp. right-invariant) when

$$\phi((\omega \otimes \iota)\Delta(a)) = \omega(1)\phi(a) \qquad \text{resp.} \qquad \phi((\iota \otimes \omega)\Delta(a)) = \omega(1)\phi(a)$$

for all $a \in M_\phi^+$ and $\omega \in M_+^*$.

Definition 5 ([4])**.** *Consider a C^*-algebra A and a non-degenerate $*$-homomorphism Δ as above, such that*

- *Δ is coassociative, meaning that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.*
- *The following density conditions are satisfied: the closed linear spans of*

$$\{(\omega \otimes \iota)\Delta(a) \mid \omega \in A^*, a \in A\} \qquad \text{and} \qquad \{(\iota \otimes \omega)\Delta(a) \mid \omega \in A^*, a \in A\}$$

are equal to A.

Moreover, by assuming the existence of

- *a faithful left-invariant approximate KMS weight ϕ on (A, Δ).*
- *a right-invariant approximate KMS weight ψ on (A, Δ),*

the pair (A, Δ) will be called a reduced C^*-algebraic quantum group.

The above Definition 5 can be regarded as a much simpler version of the definition of a locally compact quantum group in its reduced form, on which, the reduced means having a faithful Haar weight [17].

In summary, we have the following:

Remark 8 ([4,5])**.**

1. *For a (unital or non-unital) $*$-algebra A with a nondegenerate product, and a comultiplication map Δ, (A, Δ) will be called a multiplier Hopf $*$-algebra, if the linear maps T_1 and T_2 defined on $A \otimes A$ by*

$$T_1(a \otimes a') = \Delta(a)(1 \otimes a'), \qquad \text{and} \qquad T_2(a \otimes a') = (a \otimes 1)\Delta(a)$$

are one-to-one and have a range equal to $A \otimes A$.

2. *For any unital $*$-algebra A, the multiplication is automatically non-degenerate.*

3. *For any algebra A (with a non-degenerate product), one can associate (as explained before) the so-called multiplier algebra (unital), containing A as an essential ideal, and the largest algebra with these properties.*
4. *For a $*$-algebra A, $M(A)$ will also be a $*$-algebra, and as the tensor product $A \otimes A$ will also be a $*$-algebra with a non-degenerate product, $M(A \otimes A)$ can be constructed, in a way that $1 \otimes a$ and $a \otimes 1$ will be in $M(A \otimes A)$ for all $a \in A$.*
5. *The $*$-non-degenerate and coassociative homomorphism $\Delta : A \to M(A \otimes A)$ will be called a comultiplication. Here, by non-degenerate, we mean that $\Delta(A)(A \otimes A) = A \otimes A$.*
6. *And the non-degeneracy of Δ will ensure the possible extension of the maps $\Delta \otimes \iota$ and $\iota \otimes \Delta$ (for ι the identity map) on $A \otimes A$ to maps from $M(A \otimes A)$ to $M(A \otimes A \otimes A)$.*
7. *And the last point is that T_1 and T_2, defined in (1), with the requirement of being injective and having range in $A \otimes A$, will be maps from $A \otimes A$ to $M(A \otimes A)$.*

There are relations between the multiplier Hopf $*$-algebra and the notion of a Hopf $*$-algebra, which can be summarized as the following proposition (see [3]):

Proposition 6. *Any Hopf $*$-algebra (A, Δ) is a multiplier Hopf $*$-algebra. And conversely, for the unital algebra A, if (A, Δ) is a multiplier Hopf $*$-algebra, then it is a Hopf $*$-algebra.*

As we already have defined in Definition 3, an integral simply means a non-zero left-/right-invariant. We have the same definition for multiplier Hopf $*$-algebras.

After this point, by a multiplier Hopf algebra, we mean a multiplier Hopf $*$-algebra equipped with a positive integral. Also, we will use the notion $*$-algebraic quantum group to refer to the purely algebraic framework of the method used for studying this kind of locally compact quantum group, distinguishing it from algebraic groups!

By ϕ, we will denote the positive left integral, and by ψ, the positive right integral of a multiplier Hopf $*$-algebra (A, Δ).

Remark 9. *Note that any multiplier Hopf $*$-algebra with positive integrals will easily and straightforwardly provide a locally compact quantum group (in the sense of Kustermans and Vaes). However, not all locally compact quantum groups fall into this category. For instance, the compact quantum groups of Woronowicz, the discrete quantum groups of Effros and Ruan, and some combinations of them (like the Drinfel'd double of a compact quantum group) belong to this class. It might be possible to characterize the ones originating from a multiplier Hopf algebra, as indicated by a private discussion between A. Van Daele and M. Landstad, 2001)! But what we said needs to be proven, and remains an open question, awaiting a bright mind to take on the challenge!*

The class of locally compact quantum groups that arise from such multiplier Hopf $*$-algebras is 'self-dual'.

For the class of locally compact quantum groups described above, i.e., constructed within the framework of multiplier Hopf $*$-algebras, the following properties are of importance:

- They are purely algebraic.
- Quite similar to general locally compact quantum groups, they possess significant complexity!

This means that, from an algebraic point of view and as a toy model for general objects, they can play a vital role in studying various directions in the general case, without being bound to the complexity imposed by the analytic structure of the general locally compact quantum group!

Remark 10. *Continuing from the above concerns and observations, it is noteworthy that the well-developed work on algebraic quantum groups by Kustermans and Van Daele [6], has significantly motivated, and even made possible, the study and construction of the general theory of locally compact quantum groups by Kustermans and Vaes [13,17]. This connection is not far from reality. Therefore, before confirming a result in the general theory of locally compact quantum groups,* *it would be prudent to first attempt to apply it to algebraic quantum groups (multiplier Hopf $*$-algebras with positive integrals) and then extend these findings to the general case!*

7. Thinking Quadratic and Becoming Quantum

The main reference for this section is [18] and the Definitions have been adapted accordingly.

This section is a very important part of the paper in which a very important link between the coordinate ring of $M_q(n)$ and the multiplier Hopf $*$-algebras will be created, and its devotion to the algebras defined by quadratic relations will be verified.

Let us fix, once and for all, a ground field $\mathbb{K}$ with $Char\mathbb{K} \neq 2$, where all tensor products will be taken (unless stated otherwise). For a linear $\mathbb{K}$-vector space V, let $\mathbb{T}(V) := \oplus_{i=0}^{\infty}\mathbb{T}^i(V)$, where $\mathbb{T}^i(V) = V^{\otimes i}$, be denoted as the tensor algebra (the free-associative algebra) generated by V, and for a subset $R \subset T(V)$, the quotient algebra $T(V)/ < R >$ will be identified with $\mathbb{K}[x_i]$ if we consider $V = \bigoplus\limits_i \mathbb{K}[x_i]$.

According to Y. I. Manin, an associative algebra A generated by $\{x_1, \cdots, x_n\}$ over $\mathbb{K}$, with some quadratic commutation relations, is called quadratic if it is isomorphic to a quotient algebra of the form $A = T(V)/\langle R\rangle \equiv \{V, R\}$, where $\langle R\rangle$ is the ideal is generated by algebras

$$R = \{\sum_{i,j=1}^{n} A_{ij}^{k\ell} e_i \otimes e_j \mid e_i, e_j \text{ generators of } V = (\mathbb{K}^n)^*\} \subset V^{\otimes 2}$$

defined in terms of a finite number of generators satisfying some homogeneous quadratic relations. As the number of monomials $x_i x_j$ is equal to n^2, the number of independent quadratic relations is less or equal to n^2, and we have $\sum_{i,j=1}^{n} A_{ij}^{k\ell} x_i x_j = 0$ for $k, \ell = 1, \cdots, n$, where $A_{ij}^{k\ell}$ are the entries of a matrix A acting on $\mathbb{K}^n \otimes \mathbb{K}^n$ as above [19].

Definition 6 ([18])**.** *A quadratic algebra is an associative $\mathbb{Z}$-graded $\mathbb{K}$-algebra $A = \sum\limits_{i=0}^{\infty} A_i$ with the following properties:*

(i) $A_0 = \mathbb{K}$, $\dim A_1 < \infty$;

(ii) *A is generated by A_1 over $\mathbb{K}$, and the ideal generated by the relations between elements of A_1, i.e., the kernel of the homomorphism $T(A_1) \to A$ is generated by a subspace $R(A) \subset A_1 \otimes A_1$.*

A morphism of quadratic algebras $f : A \to B$ is a grading-preserving $\mathbb{K}$-homomorphism. There exists a bijection between such morphisms and $\mathbb{K}$-linear maps $f_1 : A_1 \to B_1$ for which $(f_1 \otimes f_1)(R(A)) \subset R(B)$. We denote by $\mathcal{QA}$ the category of quadratic algebras. It is often convenient to write A as

$$A \leftrightarrow \{A_1, R(A) \subset A_1 \otimes A_1\}.$$

Example 2. *Of course, the first example to mention is $T(V)$ itself.*

Example 3. *The algebra $\mathbb{K}[x_1, \cdots, x_n]$ of polynomials in n indeterminates with coefficients in $\mathbb{K}$ is quadratic with $V = \mathbb{K}^n = \mathrm{span}_{\mathbb{K}}\{x_1, \ldots, x_n\}$ and $R = \mathrm{span}_{\mathbb{K}}\{x_i \otimes x_j - x_j \otimes x_i \mid i, j = 1 \ldots, n\}$.*

By changing R to $R = \mathrm{span}_{\mathbb{K}}\{x_i \otimes x_j + x_j \otimes x_i \mid i, j = 1 \ldots, n\}$, we will obtain the exterior algebra $\Lambda(V)$.

By Drinfel'd and Manin's works on quantum groups, we can think of the ring $A = T(V)/\langle R\rangle$, as the ring of functions Spec A.

Definition 7. *Let V be a complex vector space and $R(u)$ be a function of $u \in \mathbb{C}$ taking values in* $\mathsf{End}_{\mathbb{C}}(V \otimes V)$*. Then, the following equation for $R(u)$ is called the Yang–Baxter equation(YBE):*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{8}$$

where R_{ij} signifies the matrix on $V^{\otimes 3}$, acting as $R(u)$ on the ith and jth components, and as the identity on the other components, e.g., $R_{23}(u) = I \otimes R(u)$. The variable u will be called the spectral parameter, and usually, a solution of (20) will be referred to as an R-matrix.

Quasi-Yang–Baxter Algebras as Quadratic Algebras

The main reference for this section is [18] and some of the definitions, propositions, and theorems are adapted accordingly; otherwise, they are stated.

Before starting this section, let us recall some quasi-Hopf algebras. Quasi-Hopf algebras were introduced and studied in [7,20] and many of the ideas and constructions from the theory of Hopf algebras have analogs in the quasi-Hopf algebra setting. Examples include the quantum double construction, the Tannaka–Krein theorem, the existence of integrals, the construction of link invariants and extensions to the superalgebra case, and many others. Quasi-Hopf algebras have applications in conformal field theory and the theory of integrable models (via elliptic quantum groups).

Definition 8 ([21])**.** *A quasi-Hopf algebra A is a unital associative algebra over a field $\mathbb{K}$, equipped with algebra homomorphisms $\epsilon : A \to \mathbb{K}$, $\Delta : A \to A \otimes A$ (comultiplication) and an invertible element $\xi \in A \otimes A \otimes A$, and an algebra anti-homomorphism $S : A \to A$ (antipode), satisfying*

$$(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta, \tag{9}$$

$$(1 \otimes \Delta)\Delta(a) = \xi^{-1}(\Delta \otimes 1)\Delta(a)\xi, \qquad \forall a \in A, \tag{10}$$

$$(\Delta \otimes 1 \otimes 1)\xi(1 \otimes 1 \otimes \Delta)\xi = (\xi \otimes 1)(1 \otimes \Delta \otimes 1)\xi(1 \otimes \xi), \tag{11}$$

$$(1 \otimes \epsilon \otimes 1)\xi = 1, \tag{12}$$

$$m \cdot (1 \otimes \alpha)(S \otimes 1)\Delta(a) = \epsilon(a)\alpha, \qquad \forall a \in A, \tag{13}$$

$$m \cdot (1 \otimes \beta)(1 \otimes S)\Delta(a) = \epsilon(a)\beta, \qquad \forall a \in A, \tag{14}$$

$$m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)\xi^{-1} = 1, \tag{15}$$

$$m \cdot (m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha \otimes \beta)(1 \otimes 1 \otimes S)\xi = 1. \tag{16}$$

In the above definition, m stands for the usual multiplication map on A.

Definition 9 ([21])**.** *A quasi-Hopf algebra $(A, \Delta, \epsilon, \xi)$ is called quasi-triangular if there exists an invertible homogeneous element $R \in A \otimes A$, such that*

$$\Delta^{op}(a)R = R\Delta(a), \qquad \forall a \in A, \tag{17}$$

$$\begin{aligned}(\Delta \otimes 1)R &= (1 \otimes \tau)\xi_{13}^{-1}R_{13}(1 \otimes \tau)\xi_{23}R_{23}(1 \otimes \tau)\xi_{32} \\ &= \xi_{31}^{-1}R_{13}\xi_{32}R_{23}\xi_{23}^{-1},\end{aligned} \tag{18}$$

$$\begin{aligned}(1 \otimes \Delta)R &= (\tau \otimes 1)\xi_{13}R_{13}(1 \otimes \tau)\xi_{31}^{-1}R_{12}(1 \otimes \tau)\xi_{32} \\ &= \xi_{13}R_{31}\xi_{13}^{-1}R_{12}\xi_{23},\end{aligned} \tag{19}$$

and R is referred to as the universal R-matrix.

we can easily see that equations (i)–(iii) imply the Yang–Baxter type (quasi-Yang–Baxter) equation

$$R_{12}\xi_{31}^{-1}R_{13}\xi_{32}R_{23}\xi_{23}^{-1} = \xi_{32}^{-1}R_{23}\xi_{31}R_{13}\xi_{21}^{-1}R_{12}, \tag{20}$$

in other words, we have

$$\begin{aligned}(1\otimes\tau\circ\Delta)R &= (1\otimes\tau)(1\otimes\Delta)R\\ &= (1\otimes\tau)(\xi_{13}R_{13}\xi_{13}^{-1}R_{12}\xi_{23})\\ &= \xi_{31}R_{12}\xi_{31}^{-1}R_{13}\xi_{32},\end{aligned}\tag{21}$$

and if we attempt to consider $(1\otimes\tau\circ\Delta)R$ in its usual way, then we obtain

$$\begin{aligned}(1\otimes\tau\circ\Delta)R &= R_{(1)}\otimes\tau\circ\Delta(R_{(2)})\\ &= R_{(1)}\otimes(R'^{(1)}\otimes R'^{(2)}\Delta(R_{(2)})R''^{-1})\\ &= (R_{(1)}\otimes R'^{(1)}R_{(2)_{(1)}}\otimes R'^{(2)}R_{(2)_{(2)}})(1\otimes R''^{-1})\\ &= (1\otimes R'^{(1)}\otimes R'^{(2)})(R_{(1)}\otimes R_{(2)_{(1)}}\otimes R_{(2)_{(2)}})(1\otimes R''^{-1})\\ &= (1\otimes R)(R\Delta(R))(1\otimes R^{-1})\\ &= (1\otimes R)(\tau\Delta(R))(1\otimes R^{-1})\\ &= (1\otimes R)(\tau\Delta(R))(1\otimes R^{-1})\\ &= (1\otimes R)((\tau\otimes 1)(\Delta\otimes 1)R)(1\otimes R^{-1})\\ &= (1\otimes R)(\tau\otimes 1)(\xi_{31}^{-1}R_{13}\xi_{32}R_{23}\xi_{23}^{-1})(1\otimes R^{-1})\\ &= R_{23}(\tau\otimes 1)(\xi_{31}^{-1}R_{13}\xi_{32}R_{23}\xi_{23}^{-1})(1\otimes R^{-1}),\end{aligned}$$

which means that

$$\xi_{31}R_{12}\xi_{31}^{-1}R_{13}\xi_{32} = R_{23}\xi_{32}^{-1}R_{23}\xi_{31}R_{13}\xi_{21}^{-1}(1\otimes R^{-1}),$$

satisfies, and we have

$$\xi_{31}R_{12}\xi_{31}^{-1}R_{13}\xi_{32}R_{23} = R_{23}\xi_{32}^{-1}R_{23}\xi_{31}R_{13}\xi_{21}^{-1},$$

$$\begin{aligned}R_{12}\xi_{31}^{-1}R_{13}\xi_{32}R_{23} &= \xi_{31}^{-1}R_{23}\xi_{32}^{-1}R_{23}\xi_{31}R_{13}\xi_{21}^{-1}\\ &= (1\otimes\tau)(\tau\otimes 1)\xi_{23}^{-1}R_{23}\xi_{32}^{-1}R_{23}\xi_{31}R_{13}\xi_{21}^{-1}\\ &\Rightarrow\end{aligned}$$

and this concludes the proof of the assertion:

$$R_{12}\xi_{31}^{-1}R_{13}\xi_{32}R_{23}\xi_{23}^{-1} = \xi_{32}^{-1}R_{23}\xi_{31}R_{13}\xi_{21}^{-1}R_{12},$$

An important class of quadratic algebras arises from the quasi-Yang–Baxter operators. Let n be a positive integer and consider $N\in\{1,\ldots,n\}$. Suppose we have an n-dimensional vector space V with a fixed basis $v_1,\ldots,v_n$. Recall that an element in the tensor algebra $T(V)$ is called a monomial if it is of the form $v_{i1}\otimes v_{i2}\otimes\ldots v_{im}$. The order of the monomial will be defined as an $(n+1)$-tuple $(t_n,t_{n-1},\ldots,t_1,t_0)$ of non-negative integers, where t_i for $i\geq 1$ is the number of i_js that are equal to i, and i_0 is the number of inversions in the sequence $(i_1,i_2,\ldots,i_m)$. We may order the set of all such $(n+1)$-tuples lexicographically, meaning that $(t_n,t_{n-1},\ldots,t_1,t_0)\leq(t_n^{'},t_{n-1}^{'},\ldots,t_1^{'},t_0^{'})$ if $t_n=t_n^{'},\ldots,t_k^{'}$ and $t_{k-1}<t_{k-1}^{'}$ for some $0<k\leq n$. We say that a monomial in $T(V)$ is smaller than the other one if the former has a smaller order. This defines a pre-order on the set of monomials in $T(V)$. In this case, a linear combination of monomials is smaller than another linear combination if the supremum of the monomial orders that appear with nonzero coefficients in the former is smaller than that for the latter.

Definition 10 ([18]). *Let $\rho : V \otimes V \to V \otimes V$ be a linear operator. We call ρ a quasi-Yang–Baxter operator if it satisfies the following relations:*

(i) $\rho^2 = 1,$ *(QYB1)*

(ii) $v_i \otimes v_j$ *is an eigenvector of* ρ *for* $i = 1, 2, \ldots, n,$ *(QYB2)*

(iii) *If* $i > j$, *then* $v_i \otimes v_j > \rho(v_i \otimes v_j)$ *with respect to the above pre-order,* *(QYB3)*

(iv) *If* $i > j \geq k$, *then* $(1 \otimes (1-\rho))(v_i \otimes v_j \otimes v_k) = ((1-\rho) \otimes 1)(f) + (1 \otimes (1-\rho))(g)$ *for some* $f, g \in V \otimes V \otimes V$ *with* $g < v_i \otimes v_j \otimes v_k$. *(QYB4)*

Now, we are ready to define the quasi-Yang–Baxter algebra according to [18]:

Definition 11 ([18]). *For ρ a quasi-Yang–Baxter operator on the vector space V, let $R(\rho) = Ker(1+\rho) = \operatorname{Im}(1-\rho)$ be a subspace of $V \otimes V$. Then, we can define the quadratic algebra $\mathbb{K}[A(\rho)] \equiv \{V, R(\rho)\}$. A quadratic algebra obtained in this way will be called a quasi-Yang–Baxter algebra.*

Remark 11.

1. *Because of QYB1 (i), the eigenvalues λ_i of $v_i \otimes v_j$ are equal to ± 1, and if $\lambda_i = 1$, then i is called a symmetric index. Otherwise, i will be called a skew-symmetric index. The number of symmetric indices will be called the symmetric rank of ρ or $\mathbb{K}[A(\rho)]$, and the number of skew-symmetric indices will be called the skew-symmetric rank.*
2. *Sometimes, we write $\mathbb{K}[A(\rho)^{p|q}]$ to indicate that ρ (or $\mathbb{K}[A(\rho)]$) has symmetric rank p and skew-symmetric rank q.*
3. *If ρ has symmetric rank n, then ρ (or $\mathbb{K}[A(\rho)]$) will be called symmetric, and if it has symmetric rank 0, then it will be called skew-symmetric.*

Example 4. *For example, the switch map $\mathrm{s} : V \otimes V \to V \otimes V : v \otimes w \mapsto w \otimes v$ for $v, w \in V$ is a symmetric quasi-Yang–Baxter operator, and $\mathbb{K}[A(\mathrm{s})]$ is the ordinary symmetric algebra over V. Also, $-\mathrm{s}$ is a skew-symmetric quasi-Yang–Baxter operator, and $\mathbb{K}[A(-\mathrm{s})]$ is the ordinary exterior algebra over V.*

Remark 12. *As defined above, a Yang–Baxter operator ρ on a vector space V is a linear operator on $V \otimes V$, satisfying QYB1 (i), and the following relation*

$$(1 \otimes \rho) \circ (\rho \otimes 1) \circ (1 \otimes \rho) = (\rho \otimes 1) \circ (1 \otimes \rho) \circ (\rho \otimes 1). \tag{22}$$

In most interesting cases, a Yang–Baxter operator satisfies (QYB2) and (QYB3) (for a suitable choice of the basis). It also satisfies (QYB4), such that we have

$$\begin{aligned} 1 \otimes (1-\rho) = &((1-\rho) \otimes 1) \circ (1 - 1 \otimes \rho + (1 \otimes \rho) \circ (\rho \otimes 1)) \\ &+ (1 \otimes (1-\rho)) \circ (\rho \otimes 1 - (\rho \otimes 1) \circ (1 \otimes \rho)) \end{aligned}$$

and in this case, we may let $f = (1 - 1 \otimes \rho + (1 \otimes \rho) \circ (\rho \otimes 1))(v_i \otimes v_j \otimes v_k)$ and $g = (\rho \otimes 1 - (\rho \otimes 1) \circ (1 \otimes \rho))(v_i \otimes v_j \otimes v_k)$ and then by QYB3 (iii), we have $g < v_i \otimes v_j \otimes v_k$, because we have

$$\begin{aligned} g &= (\rho \otimes 1 - (\rho \otimes 1) \circ (1 \otimes \rho))(v_i \otimes v_j \otimes v_k) \\ &= \rho(v_i \otimes v_j) \otimes v_k - (\rho \otimes 1)(v_i \otimes \rho(v_j \otimes v_k)) \\ &< v_i \otimes v_j \otimes v_k - (\rho \otimes 1)(v_i \otimes v_j \otimes v_k) \\ &< v_i \otimes v_j \otimes v_k \end{aligned}$$

and we have that g is well-defined, and for QYB4 (iv), we have

$$
\begin{aligned}
&((1-\rho)\otimes 1)\circ(1-1\otimes\rho+(1\otimes\rho)\circ(\rho\otimes 1))\\
&=(1\otimes 1-\rho\otimes 1)\circ((1\otimes 1-1\otimes\rho)+(1\otimes\rho)\circ(\rho\otimes 1))\\
&=(1\otimes 1-\rho\otimes 1)\circ(1\otimes 1-1\otimes\rho)+(1\otimes 1-\rho\otimes 1)\circ((1\otimes\rho)\circ(\rho\otimes 1))\\
&=1\otimes 1-1\otimes\rho-\rho\otimes 1+(\rho\otimes 1)\circ(1\otimes\rho)+(1\otimes\rho)\circ(\rho\otimes 1)-(\rho\otimes 1)\\
&\circ(1\otimes\rho)\circ(\rho\otimes 1)
\end{aligned}
$$

and

$$
\begin{aligned}
&(1\otimes(1-\rho))\circ(\rho\otimes 1-(\rho\otimes 1)\circ(1\otimes\rho))\\
&=(1\otimes 1-1\otimes\rho)\circ(\rho\otimes 1-(\rho\otimes 1)\circ(1\otimes\rho))\\
&=\rho\otimes 1-(\rho\otimes 1)\circ(1\otimes\rho)-(1\otimes\rho)\circ(\rho\otimes 1)+(1\otimes\rho)\circ(\rho\otimes 1)+(1\otimes\rho)\\
&\circ(\rho\otimes 1)\circ(1\otimes\rho),
\end{aligned}
$$

and it is easy to see that the above statements are equal to $1\otimes(1-\rho)$*, and we are done.*

To continue, we need to study the quantum analog $M_q(n)$ of the space of all $n\times n$-matrices, $M(n)$. This can be done by defining the coordinate ring of $M_q(n)$. Let $\mathbb{K}[M_q(n)]$ be the associative algebra over $\mathbb{K}$ generated by n^2 elements $X_{ij}, i,j=1,2,\dots,n$, with relations

$$
\begin{aligned}
X_{ri}X_{rj} &= q^{-1}X_{rj}X_{ri}, && \forall i<j;\\
X_{ri}X_{si} &= q^{-1}X_{si}X_{ri}, && \forall r<s;\\
X_{ri}X_{sj} &= X_{sj}X_{ri}, && \text{if } r<s \text{ and } i>j;\\
X_{ri}X_{sj}-X_{sj}X_{ri} &= \widehat{q}X_{si}X_{rj}, && \text{if } r<s \text{ and } i<j,
\end{aligned}
\tag{23}
$$

where we have $\widehat{q}=q^{-1}-q$. We observe that $\mathbb{K}[M_q(n)]$ is also a quadratic algebra, meaning that $\mathbb{K}[M_q(n)]=\{\widetilde{V},\widetilde{R}\}$, where $\widetilde{V}$ is an n^2-dimensional vector space with basis X_{ij} for $i,j\in\{1,2,\dots,n\}$ and $\widetilde{R}$ is spanned by elements corresponding to the relations in (23).

The relations of the last type in (23) can be explained in a much more natural way. We have linear isomorphisms $\varphi,\psi: V^*\otimes V\to\widetilde{V}$ defined by

$$
\varphi(\xi_i\otimes a_j)=X_{ij}\qquad\text{and}\qquad\psi(\xi_i\otimes a_j)=X_{ji},
$$

respectively. These isomorphisms extend to isomorphisms between the tensor algebras; this will make us able to define a linear operator $\widetilde{\rho}$ on $\widetilde{V}\otimes\widetilde{V}$ as in Theorem 1 for $r,s,i,j\in\{1,2,\dots,n\}$ with relations $r\overset{1}{\sim}s$ and $i\overset{2}{\sim}j$, where $\overset{1}{\sim},\overset{2}{\sim}\in\{<,=>\}$, and then we have the following detailed proof of a result by Brian Parshall, which has been stated in ([18], Theorem 3.5.1):

Theorem 1 ([18])**.** *The operator*

$$
\widetilde{\rho}(X_{ri}\otimes X_{sj})=\begin{cases}
q^{-1}X_{sj}\otimes X_{ri}, & \textit{if } (\overset{1}{\sim},\overset{2}{\sim})=(=,<) \textit{ or } (<,=);\\
qX_{sj}\otimes X_{ri}, & \textit{if } (\overset{1}{\sim},\overset{2}{\sim})=(=,>) \textit{ or } (>,=);\\
X_{sj}\otimes X_{ri}, & \textit{if } (\overset{1}{\sim},\overset{2}{\sim})=(<,>) \textit{ or } (>,<)\textit{or}(=,=);\\
X_{sj}\otimes X_{ri}+\widetilde{q}X_{si}\otimes X_{rj}, & \textit{if } (\overset{1}{\sim},\overset{2}{\sim})=(<,<);\\
X_{sj}\otimes X_{ri}-\widetilde{q}X_{si}\otimes X_{rj}, & \textit{if } (\overset{1}{\sim},\overset{2}{\sim})=(>,>);
\end{cases}
$$

is a symmetric quasi-Yang–Baxter operator and, therefore, $\mathbb{K}[M_q(n)]$ *is an integral domain with basis*

$$
\widetilde{B}=\{\prod_{i,j}X_{ij}^{t_{ij}}\mid t_{ij}\in\mathbb{Z}^+\},
$$

where the products have been formed concerning any fixed order of X_{ij}'s.

Proof. Let us start by exploring the definition of $\widetilde{\rho}$, which consists of five main relations, and let us call them (i) to (v), respectively, from top to bottom. Let us just use this ordering to refer to them.

The first step is to prove that $\widetilde{\rho}^2 = \mathrm{Id}$:

(i) $\widetilde{\rho}\widetilde{\rho}(X_{ri} \otimes X_{sj}) = \widetilde{\rho}(q^{-1}X_{sj} \otimes X_{ri}) = q^{-1}(\widetilde{\rho}(X_{sj} \otimes X_{ri})) = q^{-1}q(X_{ri} \otimes X_{sj}) = X_{ri} \otimes X_{sj}$.

(ii) $\widetilde{\rho}\widetilde{\rho}(X_{ri} \otimes X_{sj}) = \widetilde{\rho}(qX_{sj} \otimes X_{ri}) = q\widetilde{\rho}(X_{sj} \otimes X_{ri}) = qq^{-1}(X_{ri} \otimes X_{sj}) = X_{ri} \otimes X_{sj}$.

(iii) $\widetilde{\rho}\widetilde{\rho}(X_{ri} \otimes X_{sj}) = \widetilde{\rho}(X_{sj} \otimes X_{ri}) = X_{ri} \otimes X_{sj}$.

(iv) $\widetilde{\rho}\widetilde{\rho}(X_{ri} \otimes X_{sj}) = \widetilde{\rho}(X_{sj} \otimes X_{ri} + \widetilde{q}X_{si} \otimes X_{rj}) = \widetilde{\rho}(X_{sj} \otimes X_{ri}) + \widetilde{q}\widetilde{\rho}(X_{si} \otimes X_{rj}) = X_{ri} \otimes X_{sj} - \widetilde{q}X_{rj} \otimes X_{si} + \widetilde{q}X_{rj} \otimes X_{si} = X_{ri} \otimes X_{sj}\widetilde{\rho}(X_{sj} \otimes X_{ri}) + \widetilde{q}\widetilde{\rho}(X_{si} \otimes X_{rj}) = X_{ri} \otimes X_{sj} - \widetilde{q}X_{rj} \otimes X_{si} + \widetilde{q}X_{rj} \otimes X_{si} = X_{ri} \otimes X_{sj}$.

(v) $\widetilde{\rho}\widetilde{\rho}(X_{ri} \otimes X_{sj}) = \widetilde{\rho}(X_{sj} \otimes X_{ri} - \widetilde{q}X_{si} \otimes X_{rj}) = \widetilde{\rho}(X_{sj} \otimes X_{ri}) + \widetilde{q}\widetilde{\rho}(X_{si} \otimes X_{rj}) = X_{ri} \otimes X_{sj} + \widetilde{q}X_{rj} \otimes X_{si} - \widetilde{q}X_{rj} \otimes X_{si} = X_{ri} \otimes X_{sj}$.

So, we have $(QYB1)$. Relation (iii) will give us $\widetilde{\rho}(X_{ri} \otimes X_{ri}) = X_{ri} \otimes X_{ri}$, and, hence, we have $(QYB2)$, and in order to verify $(QYB3)$, we need to have some ordering on the basis elements X_{ij} of $\widetilde{V}$, and this can be done by employing the lexicographic ordering, meaning that $X_{ri} < X_{sj}$ if and only if $r < s$ or $r = s$ and $i < j$. And by doing so, $(QYB3)$ will become almost clear since the defined lexicographical order will provide us with a strict total order on $\widetilde{V}$, and by using this order, we can define a lexicographical order on $\widetilde{V} \otimes \widetilde{V}$, such that:

$$X_{ij} \otimes X_{kl} < X_{lm} \otimes X_{gh} if X_{ij} < X_{lm} \text{ or } X_{ij} = X_{lm} \text{ and } X_{kl} < X_{gh},$$

and this will yield $(QYB3)$ as follows:

- For relation (26), we have $ri < sj$, and $(QYB3)$ concerns the situation where $ri > sj$.
- For relation (27), we have $ri > sj$ and $X_{sj} < X_{ri}$; thus, we have $\widetilde{\rho}(X_{ri} \otimes X_{sj}) = qX_{sj} \otimes X_{ri} < X_{ri} \otimes X_{sj}$.
- For relation (iii), we have $ri = sj$.
- For relation (iv), we have $ri < sj$.
- For relation (v), we have $ri > sj$ and $X_{sj} < X_{ri}$. So, we have $X_{sj} \otimes X_{ri} < X_{ri} \otimes X_{sj}$ and $X_{si} < X_{ri}$, and so, $X_{si} \otimes X_{rj} < X_{ri} \otimes X_{sj}$, and we have $\widetilde{\rho}(X_{ri} \otimes X_{sj}) = X_{sj} \otimes X_{ri} - \widetilde{q}X_{si} \otimes X_{rj} < (1 - \widetilde{q})X_{ri} \otimes X_{sj}$.

So, we obtain $(QYB3)$.

In order to verify $(QYB4)$, let us assume $X_{ri} > X_{sj} > X_{tk}$, because if $X_{ri} > X_{sj} = X_{tk}$, then we have

$$\begin{aligned}(1 \otimes (1 - \widetilde{\rho}))(X_{ri} \otimes X_{sj} \otimes X_{tk}) &= X_{ri} \otimes (1 - \widetilde{\rho})(X_{sj} \otimes X_{tk}) \\ &= X_{ri} \otimes (X_{sj} \otimes X_{tk} - X_{sj} \otimes X_{tk}) \\ &= 0,\end{aligned}$$

and in this case, $(QYB4)$ is evident.

According to our restriction, six cases need to be checked:

(1) $s = t < r, k < j$ and $k < i$,
(2) $t < s \leq r, j < i$ and $j \leq k$,
(3) $t < r = s$ and $k < j < i$,
(4) $t < s < r$ and $i \leq k < j$,
(5) $t < s < r$ and $k < i \leq j$,
(6) $t < s < r$ and $k < j < i$.

Let us see how it works, and let us start with the last one:

6. We have

$$
\begin{aligned}
(1\otimes(1-\widetilde{\rho}))(X_{ri}\otimes X_{sj}\otimes X_{tk}) &= X_{ri}\otimes(1-\widetilde{\rho})(X_{sj}\otimes X_{tk}) \\
&= X_{ri}\otimes(X_{sj}\otimes X_{tk}-\widetilde{\rho}(X_{sj}\otimes X_{tk})) \\
&= X_{ri}\otimes(X_{sj}\otimes X_{tk}-X_{tk}\otimes X_{sj}+\widetilde{q}X_{tj} \\
&\otimes X_{sk}),
\end{aligned}
$$

now for the rest of the proof, we need to use a trick. By trick, we mean that, instead of finding f and g in ($QYB4$), the plan is to use the modulo operation in $\mod (1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})+(1\otimes(1-\widetilde{\rho}))(W)$, for W, a subspace of $\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V}$ spanned by monomials that are smaller than $X_{ri}\otimes X_{tk}\otimes X_{sj}$; in this case, if we obtain 0, then we are done.

$$
\begin{aligned}
&X_{ri}\otimes X_{sj}\otimes X_{tk}-X_{ri}\otimes X_{tk}\otimes X_{sj}+\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk} \\
&\equiv ((X_{ri}\otimes X_{sj}\otimes X_{tk}) \mod (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) \\
&+(X_{ri}\otimes X_{sj}\otimes X_{tk}) \mod ((1\otimes(1-\widetilde{\rho}))(W)) \\
&-((X_{ri}\otimes X_{tk}\otimes X_{sj}) \mod (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) \\
&-(X_{ri}\otimes X_{tk}\otimes X_{sj}) \mod (((1\otimes(1-\widetilde{\rho}))(W)) \\
&+(\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk}) \mod (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) \\
&+(\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk}) \mod ((1\otimes(1-\widetilde{\rho}))(W)) \\
&=(X_{ri}\otimes X_{sj}\otimes X_{tk}) \mod (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) \\
&+0 \\
&-((X_{ri}\otimes X_{tk}\otimes X_{sj}) \mod (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) \\
&-0 \\
&+(\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk}) \mod (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) \\
&+0
\end{aligned}
$$

since the monomials are of a higher order than $X_{ri}\otimes X_{tk}\otimes X_{sj}$, the above statement will be equal to

$$
\begin{aligned}
&=(((1-\widetilde{\rho})\otimes 1)(X_{ri}\otimes X_{sj}\otimes X_{tk})-X_{ri}\otimes X_{sj}\otimes X_{tk}) \\
&-(((1-\widetilde{\rho})\otimes 1)(X_{ri}\otimes X_{tk}\otimes X_{sj})-X_{ri}\otimes X_{tk}\otimes X_{sj}) \\
&+((1-\widetilde{\rho})\otimes 1)(\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk})-\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk} \\
&=\widetilde{\rho}(X_{ri}\otimes X_{sj})\otimes X_{tk}-\widetilde{\rho}(X_{ri}\otimes X_{tk})\otimes X_{sj}+\widetilde{\rho}(\widetilde{q}X_{ri}\otimes X_{tj})\otimes \\
&X_{sk}=X_{sj}\otimes X_{ri}\otimes X_{tk}-\widetilde{q}X_{si}\otimes X_{rj}\otimes X_{tk}-X_{tk}\otimes X_{ri}\otimes X_{sj} \\
&+\widetilde{q}X_{ti}\otimes X_{rk}\otimes X_{sj}+\widetilde{q}X_{tj}\otimes X_{ri}\otimes X_{sk}-\widetilde{q}^2X_{ti}\otimes X_{rj}\otimes X_{sk}
\end{aligned}
$$

and we see that, now, all the monomials are smaller than $X_{ri}\otimes X_{tk}\otimes X_{sj}$; considering the order, the module operation will just take place on $(1\otimes(1-\widetilde{\rho}))(W)$, and the above statement will be equivalent to the following:

$$
\begin{aligned}
&\equiv X_{sj}\otimes X_{tk}\otimes X_{ri}-\widetilde{q}X_{sj}\otimes X_{ti}\otimes X_{rk}-\widetilde{q}X_{si}\otimes X_{tk}\otimes X_{rj}+\widetilde{q}^2X_{si}\otimes X_{tj} \\
&\otimes X_{rk}-X_{tk}\otimes X_{sj}\otimes X_{ri}+\widetilde{q}X_{tk}\otimes X_{si}\otimes X_{rj}+\widetilde{q}X_{ti}\otimes X_{sj}\otimes X_{rk}+\widetilde{q}X_{tj} \\
&\otimes X_{sk}\otimes X_{ri}-\widetilde{q}^2X_{tj}\otimes X_{si}\otimes X_{rk}-\widetilde{q}^2X_{ti}\otimes X_{sk}\otimes X_{rj}+\widetilde{q}^3X_{ti}\otimes X_{sj}\otimes X_{rk} \\
&=(X_{sj}\otimes X_{tk}-X_{tk}\otimes X_{sj}+\widetilde{q}X_{tj}\otimes X_{sk})\otimes X_{ri}-\widetilde{q}(X_{sj}\otimes X_{ti}-X_{ti}\otimes X_{sj}) \\
&\otimes X_{rk}-\widetilde{q}(X_{si}\otimes X_{tk}-X_{tk}\otimes X_{si}+\widetilde{q}X_{ti}\otimes X_{sk})X_{rj}+\widetilde{q}^2(X_{si}\otimes X_{tj}-X_{tj}\otimes \\
&X_{si}+\widetilde{q}X_{ti}\otimes X_{sj})\otimes X_{rk} \\
&\equiv 0 \mod ((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})+(1\otimes(1-\widetilde{\rho}))(W),
\end{aligned}
$$

and we are complete with (6).

Note that, in all of the above computations, we just use relations (i) to (v) and the lexicographical ordering.

5.

$$\begin{aligned}(1\otimes(1-\widetilde{\rho}))(X_{ri}\otimes X_{sj}\otimes X_{tk}) &= X_{ri}\otimes(1-\widetilde{\rho})(X_{sj}\otimes X_{tk})\\ &= X_{ri}\otimes(X_{sj}\otimes X_{tk}-\widetilde{\rho}(X_{sj}\otimes X_{tk}))\\ &= *\end{aligned}$$

Here, we still see that $s > t$ and $j > k$, so we use the relation (v), and the above computation will follow

$$\begin{aligned}* &= X_{ri}\otimes X_{sj}\otimes X_{tk}-X_{ri}\otimes X_{tk}\otimes X_{sj}+\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk}\\ &\equiv ((X_{ri}\otimes X_{sj}\otimes X_{tk}-X_{ri}\otimes X_{tk}\otimes X_{sj}+\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk})\\ &\quad \text{mod } (((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})) = **\end{aligned}$$

and since all monomials are bigger than $(X_{ri}\otimes X_{tk}\otimes X_{sj})$, by order, we have

$$\begin{aligned}** &= (X_{ri}\otimes X_{sj}\otimes X_{tk}-((1-\widetilde{\rho})\otimes 1)(X_{ri}\otimes X_{sj}\otimes X_{tk}))\\ &\quad -(X_{ri}\otimes X_{tk}\otimes X_{sj}-((1-\widetilde{\rho})\otimes 1)(X_{ri}\otimes X_{tk}\otimes X_{sj}))\\ &\quad +(\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk}-((1-\widetilde{\rho})\otimes 1)(\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk})\\ &= \widetilde{\rho}(X_{ri}\otimes X_{sj})\otimes X_{tk}-\widetilde{\rho}(X_{ri}\otimes X_{tk})\otimes X_{sj}+\widetilde{q}\widetilde{\rho}(X_{ri}\otimes X_{tj})\otimes X_{sk}\\ &= X_{sj}\otimes X_{ri}\otimes X_{tk}-X_{tk}\otimes X_{ri}\otimes X_{sj}+\widetilde{q}X_{ti}\otimes X_{rk}\otimes X_{sj}+\widetilde{q}X_{tj}\otimes\\ &\quad X_{ri}\otimes X_{sk} = X_{sj}\widetilde{\rho}(X_{ri}\otimes X_{tk})-X_{tk}\widetilde{\rho}(X_{ri}\otimes X_{sj})+\widetilde{q}X_{ti}\widetilde{\rho}(X_{rk}\otimes X_{sj})\\ &\quad +\widetilde{q}X_{tj}\widetilde{\rho}(X_{ri}\otimes X_{sk}) = X_{sj}\otimes X_{tk}\otimes X_{ri}-\widetilde{q}X_{sj}\otimes X_{ti}\otimes X_{rk}-X_{tk}\otimes\\ &\quad X_{sj}\otimes X_{ri}+\widetilde{q}X_{ti}\otimes X_{sj}\otimes X_{rk}+\widetilde{q}X_{tj}\otimes X_{sk}\otimes X_{ri}-\widetilde{q}^2X_{tj}\otimes X_{si}\otimes X_{rk}\\ &= (X_{sj}\otimes X_{tk}-X_{tk}\otimes X_{sj}+\widetilde{q}X_{tj}\otimes X_{sk})\otimes X_{ri}-\widetilde{q}(X_{sj}\otimes X_{ti}-X_{ti}\otimes\\ &\quad X_{sj}+\widetilde{q}X_{tj}\otimes X_{si})\otimes X_{rk}\equiv 0 \quad \text{mod } ((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})+(1\otimes\\ &\quad (1-\widetilde{\rho})(W)),\end{aligned}$$

and the proof of (5) is completed.

4.

$$\begin{aligned}&(1\otimes(1-\widetilde{\rho}))(X_{ri}\otimes X_{sj}\otimes X_{tk}) = X_{ri}\otimes(1-\widetilde{\rho})(X_{sj}\otimes X_{tk}) = X_{ri}\otimes(X_{sj}\\ &\otimes X_{tk}-\widetilde{\rho}(X_{sj}\otimes X_{tk})) = X_{ri}\otimes X_{sj}\otimes X_{tk}-X_{ri}\otimes X_{tk}\otimes X_{sj}+\widetilde{q}X_{ri}\otimes X_{tj}\\ &\otimes X_{sk}\equiv\widetilde{\rho}(X_{ri}\otimes X_{sj})\otimes X_{tk}-\widetilde{\rho}(X_{ri}\otimes X_{tk})\otimes X_{sj}+\widetilde{q}\widetilde{\rho}(X_{ri}\otimes X_{tj})\otimes X_{sk}\\ &= X_{sj}\otimes X_{ri}\otimes X_{tk}-X_{tk}\otimes X_{ri}\otimes X_{sj}+\widetilde{q}X_{tj}\otimes X_{ri}\otimes X_{sk} = X_{sj}\widetilde{\rho}(X_{ri}\otimes\\ &X_{tk})-X_{tk}\widetilde{\rho}(X_{ri}\otimes X_{sj})+\widetilde{q}X_{tj}\widetilde{\rho}(X_{ri}\otimes X_{sk}) = X_{sj}\otimes X_{tk}\otimes X_{ri}-X_{tk}\otimes\\ &X_{sj}\otimes X_{ri}+\widetilde{q}X_{tj}\otimes X_{sk}\otimes X_{ri}\equiv 0 \quad \text{mod } ((1-\widetilde{\rho})\otimes 1)(\widetilde{V}\otimes\widetilde{V}\otimes\widetilde{V})+(1\\ &\otimes(1-\widetilde{\rho})(W)),\end{aligned}$$

and (4) is evident.

3. The proof is almost in the same line as the proof of (2):

2.

$$
\begin{gathered}
(1\otimes(1-\widetilde{\rho}))(X_{ri}\otimes X_{sj}\otimes X_{tk}) = X_{ri}\otimes X_{sj}\otimes X_{tk} - X_{ri}\otimes X_{tk}\otimes \\
X_{sj}+\widetilde{q}X_{ri}\otimes X_{tj}\otimes X_{sk} \equiv \widetilde{\rho}(X_{ri}\otimes X_{sj})\otimes X_{tk}-\widetilde{\rho}(X_{ri}\otimes X_{tk})\otimes \\
X_{sj}+\widetilde{q}X_{ri}\otimes\widetilde{\rho}(X_{tj}\otimes X_{sk}) = X_{sj}\otimes X_{ri}\otimes X_{tk} - \widetilde{q}X_{si}\otimes X_{rj}\otimes X_{tk} \\
- X_{tk}\otimes X_{ri}\otimes X_{sj} + \widetilde{q}X_{ti}\otimes X_{rk}\otimes X_{sj}+\widetilde{q}X_{ri}\otimes X_{sk}\otimes X_{tj}+\widetilde{q}^2 X_{ri} \\
\otimes X_{sj}\otimes X_{tk} \equiv X_{sj}\otimes\widetilde{\rho}(X_{ri}\otimes X_{tk}) - \widetilde{q}X_{si}\otimes\widetilde{\rho}(X_{rj}\otimes X_{tk}) - X_{tk}\otimes \\
\widetilde{\rho}(X_{ri}\otimes X_{sj}) + \widetilde{q}X_{ti}\otimes\widetilde{\rho}(X_{rk}\otimes X_{sj}) + \widetilde{q}\widetilde{\rho}(X_{ri}\otimes X_{sk})\otimes X_{tj}+\widetilde{q}^2\widetilde{\rho}(\\
X_{ri}\otimes X_{sj})\otimes X_{tk} = X_{sj}\otimes X_{tk}\otimes X_{ri} - \widetilde{q}X_{sj}\otimes X_{ti})\otimes X_{rk} - \widetilde{q}X_{si}\otimes \\
X_{tk})\otimes X_{rj} - X_{tk}\otimes X_{sj})\otimes X_{ri}+\widetilde{q}X_{tk}\otimes X_{si})\otimes X_{rj}+\widetilde{q}X_{ti}\otimes X_{sj}) \\
\otimes X_{rk} - \widetilde{q}X_{ti}\otimes X_{sk})\otimes X_{rj}+\widetilde{q}X_{sk}\otimes X_{ri})\otimes X_{tj} - \widetilde{q}^2X_{si}\otimes X_{rk})\otimes \\
X_{tj}+\widetilde{q}^2X_{sj}\otimes X_{ri})\otimes X_{tk} - \widetilde{q}^3X_{si}\otimes X_{rj})\otimes X_{tk}\equiv X_{sj}\otimes\widetilde{\rho}(X_{tk}\otimes \\
X_{ri}) - \widetilde{q}X_{sj}\otimes\widetilde{\rho}(X_{ti}\otimes X_{rk}) - \widetilde{q}X_{si}\otimes\widetilde{\rho}(X_{tk}\otimes X_{rj}) - X_{tk}\otimes\widetilde{\rho}(X_{sj} \\
\otimes X_{ri}) + \widetilde{q}X_{tk}\otimes\widetilde{\rho}(X_{si}\otimes X_{rj}) + \widetilde{q}X_{ti}\otimes\widetilde{\rho}(X_{sj}\otimes X_{rk}) - \widetilde{q}X_{ti}\otimes\widetilde{\rho}(\\
X_{sk}\otimes X_{rj}) + \widetilde{q}X_{sk}\otimes\widetilde{\rho}(X_{ri}\otimes X_{tj}) - \widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{rk}\otimes X_{tj}) + \widetilde{q}^2X_{sj} \\
\otimes\widetilde{\rho}(X_{ri}\otimes X_{tk}) - \widetilde{q}^3X_{si}\otimes\widetilde{\rho}(X_{rj}\otimes X_{tk}) = X_{sj}\otimes X_{ri}\otimes X_{tk} + \widetilde{q}X_{sj} \\
\otimes X_{rk}\otimes X_{ti} - \widetilde{q}X_{sj}\otimes X_{rk}\otimes X_{ti} - \widetilde{q}X_{si}\otimes X_{rj}\otimes X_{tk} - X_{tk}\otimes X_{ri}\otimes \\
X_{sj}+\widetilde{q}X_{tk}\otimes X_{rj}\otimes X_{si}+\widetilde{q}X_{ti}\otimes X_{rk}\otimes X_{sj}+\widetilde{q}^2X_{ti}\otimes X_{rj}\otimes X_{sk}- \\
\widetilde{q}X_{ti}\otimes X_{rj}\otimes X_{sk}+\widetilde{q}X_{sk}\otimes X_{tj}\otimes X_{ri} - \widetilde{q}^2X_{sk}\otimes X_{ti}\otimes X_{rj} - \widetilde{q}^2X_{si} \\
\otimes X_{tj}\otimes X_{rk}+\widetilde{q}^3X_{si}\otimes X_{tk}\otimes X_{rj}+\widetilde{q}^2X_{sj}\otimes X_{tk}\otimes X_{ri} - \widetilde{q}^3X_{sj}\otimes \\
X_{ti}\otimes X_{rk} - \widetilde{q}^3X_{si}\otimes X_{tk}\otimes X_{rj} = \widetilde{q}X_{sk}\otimes X_{tj}\otimes X_{ri} - \widetilde{q}^2X_{sk}\otimes X_{ti} \\
\otimes X_{rj} - \widetilde{q}^2X_{si}\otimes X_{tj}\otimes X_{rk}+\widetilde{q}^2X_{si}\otimes X_{tk}\otimes X_{ri} - \widetilde{q}^3X_{sj}\otimes X_{ti}\otimes \\
X_{rk}\equiv X_{sk}\otimes\widetilde{\rho}(X_{tj}\otimes X_{ri}) - \widetilde{q}^2X_{sk}\otimes\widetilde{\rho}(X_{ti}\otimes X_{rj}) - \widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{tj} \\
\otimes X_{rk}) + \widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{tk}\otimes X_{ri}) - \widetilde{q}^3X_{sj}\otimes\widetilde{\rho}(X_{ti}\otimes X_{rk}) = X_{sk}\otimes \\
X_{ri}\otimes X_{tj} - \widetilde{q}X_{sk}\otimes X_{rj})\otimes X_{ti} - \widetilde{q}^2X_{sk}\otimes X_{rj}\otimes X_{ti} - \widetilde{q}^2X_{si}\otimes X_{rk} \\
\otimes X_{tj} - \widetilde{q}^3X_{si}\otimes X_{rj}\otimes X_{tk}+\widetilde{q}^2X_{si}\otimes X_{ri}\otimes X_{tk}+\widetilde{q}^3X_{si}\otimes X_{rk}\otimes \\
X_{ti} - \widetilde{q}^3X_{sj}\otimes X_{rk}\otimes X_{ti}\equiv -\widetilde{q}^2X_{si}\otimes X_{rk}\otimes X_{tj} - \widetilde{q}^3X_{si}\otimes X_{rj}\otimes X_{tk} \\
+\widetilde{q}^2X_{si}\otimes X_{ri}\otimes X_{tk}+\widetilde{q}^3X_{si}\otimes X_{rk}\otimes X_{ti} - \widetilde{q}^3X_{sj}\otimes X_{rk}\otimes X_{ti}\equiv \\
-\widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{rk}\otimes X_{tj}) - \widetilde{q}^3X_{si}\otimes\widetilde{\rho}(X_{rj}\otimes X_{tk})+\widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{ri}\otimes \\
X_{tk})+\widetilde{q}^3X_{si}\otimes\widetilde{\rho}(X_{rk}\otimes X_{ti}) - \widetilde{q}^3X_{sj}\otimes\widetilde{\rho}(X_{rk}\otimes X_{ti}) = -\widetilde{q}^2X_{si}\otimes X_{tj} \\
\otimes X_{rk}+\widetilde{q}^3X_{si}\otimes X_{tk}\otimes X_{rj} - \widetilde{q}^3X_{si}\otimes X_{tk}\otimes X_{rj}+\widetilde{q}^3X_{si}\otimes X_{tk}\otimes X_{ri} \\
-\widetilde{q}^3X_{si}\otimes X_{ti}\otimes X_{rk}+\widetilde{q}^3X_{si}\otimes X_{ti}\otimes X_{rk} - \widetilde{q}X_{si}\otimes X_{ti}\otimes X_{rj}\equiv -\widetilde{q}^2 \\
X_{si}\otimes\widetilde{\rho}(X_{tj}\otimes X_{rk})+\widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{tk}\otimes X_{ri}) - \widetilde{q}^3X_{sj}\otimes\widetilde{\rho}(X_{ti}\otimes X_{rk}) = \\
-\widetilde{q}^2X_{si}\otimes X_{rk}\otimes X_{tj} - \widetilde{q}^3X_{si}\otimes X_{rj}\otimes X_{tk}+\widetilde{q}^2X_{si}\otimes X_{ri}\otimes X_{tk}+\widetilde{q}^3 \\
X_{si}\otimes X_{rk}\otimes X_{ti} - \widetilde{q}^3X_{sj}\otimes X_{rk}\otimes X_{ti}\equiv -\widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{rk}\otimes X_{tj}) - \widetilde{q}^3 \\
X_{si}\otimes\widetilde{\rho}(X_{rj}\otimes X_{tk})+\widetilde{q}^2X_{si}\otimes\widetilde{\rho}(X_{ri}\otimes X_{tk})+\widetilde{q}^3X_{si}\otimes\widetilde{\rho}(X_{rk}\otimes X_{ti}) \\
-\widetilde{q}X_{si}\otimes\widetilde{\rho}(X_{rj}\otimes X_{ti}) = -\widetilde{q}^2X_{si}\otimes X_{rk}\otimes X_{tj} - \widetilde{q}^3X_{si}\otimes X_{rj}\otimes X_{tk} \\
+\widetilde{q}^2X_{si}\otimes X_{ri}\otimes X_{tk}+\widetilde{q}^3X_{si}\otimes X_{rk}\otimes X_{ti} - \widetilde{q}X_{si}\otimes X_{rj}\otimes X_{ti} = -\widetilde{q}^2
\end{gathered}
$$

$$X_{si} \otimes X_{tj} \otimes X_{rk} + \widetilde{q}^3 X_{si} \otimes X_{tk} \otimes X_{rj} - \widetilde{q}^3 X_{si} \otimes X_{tk} \otimes X_{rj} + \widetilde{q}^2 X_{si} \otimes X_{tk} \otimes X_{ri} - \widetilde{q}^3 X_{si} \otimes X_{ti} \otimes X_{rk} + \widetilde{q}^3 X_{si} \otimes X_{ti} \otimes X_{rk} - \widetilde{q}^4 X_{si} \otimes X_{tk} \otimes X_{ri} - \widetilde{q}^3 X_{sj} \otimes X_{ti} \otimes X_{rk} + \widetilde{q}^2 X_{tj} \otimes X_{si} \otimes X_{rk} \equiv \widetilde{q}^2 (X_{tj} \otimes X_{si} \otimes X_{rk} - X_{si} \otimes X_{tj} \otimes X_{rk} - \widetilde{q} X_{sj} \otimes X_{ti} \otimes X_{rk}) \equiv 0 mod((1-\widetilde{\rho}) \otimes 1)(\widetilde{V} \otimes \widetilde{V} \otimes \widetilde{V}) + (1 \otimes (1-\widetilde{\rho})(W)),$$

and we have (4).

1.

$$(1 \otimes (1-\widetilde{\rho}))(X_{ri} \otimes X_{sj} \otimes X_{tk}) = X_{ri} \otimes X_{sj} \otimes X_{tk} - X_{ri} \otimes X_{tk} \otimes X_{sj} + \widetilde{q} X_{ri} \otimes X_{tj} \otimes X_{sk} \equiv \widetilde{\rho}(X_{ri} \otimes X_{sj}) \otimes X_{tk} - \widetilde{\rho}(X_{ri} \otimes X_{tk}) \otimes X_{sj} + \widetilde{q}\widetilde{\rho} (X_{ri} \otimes X_{tj}) \otimes X_{sk} = X_{sj} \otimes X_{ri} \otimes X_{tk} - \widetilde{q} X_{si} \otimes X_{rj} \otimes X_{tk} - X_{tk} \otimes X_{ri} \otimes X_{sj} + \widetilde{q} X_{ti} \otimes X_{rk} \otimes X_{sj} + \widetilde{q} X_{tj} \otimes X_{ri} \otimes X_{sk} - \widetilde{q}^2 X_{ti} \otimes X_{rj} \otimes X_{sk} \equiv X_{sj} \otimes \widetilde{\rho}(X_{ri} \otimes X_{tk}) - \widetilde{q} X_{si} \otimes \widetilde{\rho}(X_{rj} \otimes X_{tk}) - X_{tk} \otimes \widetilde{\rho}(X_{ri} \otimes X_{sj}) + \widetilde{q} X_{ti} \otimes \widetilde{\rho}(X_{rk} \otimes X_{sj}) + \widetilde{q} X_{tj} \otimes \widetilde{\rho}(X_{ri} \otimes X_{sk}) - \widetilde{q}^2 X_{ti} \otimes \widetilde{\rho}(X_{rj} \otimes X_{sk}) = X_{sj} \otimes X_{tk} \otimes X_{ri} - \widetilde{q} X_{sj} \otimes X_{ti} \otimes X_{rk} - \widetilde{q} X_{si} \otimes X_{tk} \otimes X_{rj} + \widetilde{q}^2 X_{si} \otimes X_{tj} \otimes X_{rk} - X_{tk} \otimes X_{sj} \otimes X_{ri} + \widetilde{q} X_{tk} \otimes X_{si} \otimes X_{rj} + \widetilde{q} X_{ti} \otimes X_{sj} \otimes X_{rk} + \widetilde{q} X_{tj} \otimes X_{sk} \otimes X_{ri} - \widetilde{q}^2 X_{tj} \otimes X_{si} \otimes X_{rk} - \widetilde{q}^2 X_{ti} \otimes X_{sk} \otimes X_{rj} + \widetilde{q}^3 X_{ti} \otimes X_{sj} \otimes X_{rk},$$

It is evident that the rest of the process is almost identical to (6) and we can conclude with reference to (1).

The remainder just follows from ([18], Theorem 3.3.1), and we have the desired result. □

8. Quantum Automorphism of Locally Finite Graphs

Some philosophical (types) parts of this section are quoted from [22].

If a compact space G is equipped with a continuous associative map $\cdot : G \times G \to G$, then $(G, \cdot)$ will be a compact semigroup by definition. Then, on the level of the function algebras, the map $\cdot$ will induce a unital $*$-homomorphism

$$\Delta : C(G) \to C(G \times G) \equiv C(G) \otimes C(G)$$
$$\Delta(f)(g_1 \cdot g_2) = f(g_1 \cdot g_2).$$

Moreover, we have the following definition.

Definition 12 ([23])**.** *The algebra of continuous functions on a compact quantum semigroup is a unital C^*-algebra $\mathcal{A}$ equipped with a unital $*$-algebra homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that the following co-associativity condition holds:*

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta,$$

and as before, Δ will be called a comultiplication or a coproduct.

But, here, the main question is:

Question 1 ([23])**.** *What are compact quantum groups in general, and how do we define them? [14]*

One way to look at the above question could be by looking at the multiplication of the inverse operation and the neutral element, which has led people to the theory of compact Kac algebras [24].

Now, assume that G is a cancellative semigroup, such that the translations $x \to xg$ and $x \to gx$ are continuous, and assume that it contains an idempotent e. Then, since for every $g \in G$, one has $gee = ge$ and $eg = eeg$, we obtain $eg = g = ge$, meaning that e is an identity for G; G is a monoid, its identity 1 is a unique idempotent, and g has a right inverse and a left inverse via a dual argument, meaning that G is a compact group. This will lead us to an alternative way, which is the subject of this paper.

Definition 13 ([14,23])**.** *A unital C^*-algebra $\mathcal{A}$ is the algebra of continuous functions on a compact quantum group if it admits a unital $*$-algebra homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, such that*

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad (\textit{co-associativity})$$

and

$$\overline{\Delta(\mathcal{A})(\mathcal{A} \otimes 1_{\mathcal{A}})} = \mathcal{A} \otimes \mathcal{A} = \overline{\Delta(\mathcal{A})(1_{\mathcal{A}} \otimes \mathcal{A})} \qquad (\textit{quantum cancellation rules}),$$

and then, we write $\mathcal{A} = C(\mathbb{G})$, and we will call $\mathbb{G}$ a compact quantum group.

However, the main question has been asked by A. Connes, marking the starting point:

Question 2 (Connes)**.** *What is a quantum permutation group? [22]*

To solve this question, one needs to acquire good knowledge about quantum groups, and there is a need to go back in time and think about some questions, such as the one that has been coined by Heisenberg:

Question 3 (Heisenberg)**.** *What is a quantum space? [22]*

"Regarding this latter question, there are as many answers as quantum physicists, starting with Heisenberg himself in the early 1920s, then Schrödinger and Dirac shortly after, with each coming with his answer to the question. Not to forget Einstein, who labeled all these solutions as "nice, but probably fundamentally wrong"." [22]

So, due to the lack of a good answer, let us take as a starting point something that is nice and mathematical, widely agreed upon in the 1930s, coming from Dirac's work, as follows:

Answer 1 (von Neumann)**.** *A quantum space is the dual of an operator algebra. [22]*

"With a fast forward to the 90s and to Connes' question, this remains something nontrivial, even when knowing what a quantum space is, and this is for a myriad of technical reasons. You have to work a bit on that question and try all sorts of things that do not work until you hit a good answer. This good answer is as follows" [22]:

Answer 2 (Wang)**.** *The quantum permutation group S_n^+ is the largest compact quantum group acting on the set $\{1, \ldots, N\}$. [22]*

Here, the key word is "compact". What happens is that $\{1, \ldots, N\}$ has all sorts of "quantum permutations", and there is an infinity of such quantum permutations, and the quantum group formed by this infinity of quantum permutations is compact. So, by doing some reverse engineering, we are led to the above answer.

The notion of the quantum group was coined at the International Congress of Mathematics in 1986 by V. Drinfel'd, emerging from an extensive search for potential solutions to the quantum-Yang–Baxter equation. Since then, quantum groups have been the subject of many studies in various areas of mathematics and physics.

Despite their introduction and applications, these fascinating objects still lack a rigorous and universally accepted axiomatic definition that could be considered general for the category of quantum groups. However, on the other side, there are the C^*-algebraic compact quantum groups (CQG) introduced and developed by Woronowicz. These possess a rigorous and well-defined definition, along with a powerful representation theory. Actions of quantum groups on C^*-algebras dualize the usual group actions, qualifying them as descriptors of symmetries in noncommutative cases and motivating the notion of "quantum symmetries"!

In [25], Wang proposed studying quantum symmetries, demonstrating that even classical objects can exhibit quantum symmetry not apparent when restricted to classical groups. For instance, by considering a set of n points, and regarding them as vertices of graphs, Wang showed that the corresponding graph might have a quantum automorphism group, thereby exhibiting non-trivial quantum symmetries, and termed it S_n^+. It has already been proved that for $n \geq 4$, S_n^+ is not a group and is infinite-dimensional. Since we work within the C^*-algebraic framework, with the induced topology (and the adjacency matrices of planar graphs and the associated commuting matrices, which should be a magic unitary with certain C^*-algebraic properties), it is not very important to us how the projected non-crossing planer graph looks like, and we look for isomorphic graphs, for example, a rectangle and a square! Later on, we will delve into the definition of a magic unitary matrix and its properties.

8.1. Thinking Noncommutative and Becoming Quantum!

By a C^*- algebra, we mean a complex algebra with a norm and an involution, such that the Cauchy sequences converge, and we have $||aa^*|| = ||a||^2$. One of the basic examples to mention can be the algebra of bounded operators on a Hilbert space H, and by a universal C^*-algebra, we mean a C^*-algebra that is presented by a set of generators and relations, constructed as follows:

(i) Consider $X = \{x_i \mid i \in I\}$ as a set of generators, for I, the index set;
(ii) Consider $P(X)$ the set of non-commutative polynomials in x_i and x_i^*;
(iii) Consider $R \subseteq P(X)$ a set of relations;
(iv) Consider $I(R) \subseteq p(X)$ the ideal generated by the set of relations R;
(v) Consider $A(X,R) := P(X)/I(R)$ the quotient of $P(X)$ by $I(R)$ (the universal $*$-algebra generated by X and R;
(vi) Consider $||x|| := sup\{p(x) \mid p \text{ a } C^* - \text{seminorm on } A(X,R)\}$;
(vii) And now, if for all $x \in A(X,R)$, $||x|| < \infty$ satisfies, then the universal C^*-algebra $C^*(X,R)$, generated by X and R, could be defined as the completion of $A(X,R)/\{x \mid ||x|| = 0\}$ in norm $|| \cdot ||$.

Some of the simplest examples and non-examples that can be mentioned here are the universal C^*-algebra $C^*(u, 1 \mid u^*u = uu^* = 1)$ isomorphic to the algebra of continuous functions $C(S^1)$. To have an intuition of a non-example, as there are no bounded operators x and y satisfying the CCR relation $xy - yx = 1$, $C^*(x, y \mid xy - yx = 1)$ cannot be considered as a universal C^*-algebra.

Now, let $X_n := \{x_1, x_2, \cdots, x_n\}$ be a finite set of points. Then as we know, its automorphism group $\mathrm{Aut}(X_n)$ is exactly the permutation group S_n, and the question is what will happen if we view X_n as a quantum space, and what will be its quantum symmetry group?

To come up with a solution to the above question, the first step is to dualize the set X_n and obtain

$$C(X_n) \sim C^*\left(p_1, p_2, \cdots, p_n \mid \sum_{i=1}^{n} p_i = 1, \text{ for } p_i \text{ projections}\right),$$

as a universal C^*-algebra, and since p_i forms a basis, any action of a CQG (A, Δ) on $C(X_n)$ is of the form

$$\begin{aligned}\alpha :&C(X_n) \to C(X_n) \otimes A \\ &p_j \mapsto \sum_{i=1}^{n} p_i \otimes a_{ij},\end{aligned}$$

and in order for α to be an action, the elements a_{ij} need to satisfy several relations:

$$\begin{aligned}
\alpha(p_j) = \alpha(p_j)^* &\Rightarrow a_{ij} = a_{ij}^* \\
\alpha(p_j) = \alpha(p_j)^2 &\Rightarrow \sum_i p_i \otimes a_{ij} = \sum_{i,k} p_i p_k \otimes a_{ij} a_{kj} = \sum_i p_i \otimes a_{ij}^2 \\
&\Rightarrow a_{ij} = a_{ij}^2 \\
\alpha \text{ is unital } &\Rightarrow 1 \otimes 1 = \alpha(1) = \alpha(\sum_j p_j) = \sum_{i,j} p_i \otimes a_{ij} = \sum_i p_i \otimes (\sum_j a_{ij}) \\
&\Rightarrow \sum_j a_{ij} = 1,
\end{aligned}$$

this has led Wang [25] in 1998 to the definition of the quantum symmetric (permutation) group S_n^+, as follows:

Definition 14. *The quantum symmetric (permutation) group $S_n^+ = (C(S_n^+), u)$ is the compact matrix quantum group, where*

$$C(S_n^+) := C^*\left(u_{ij}, i,j = 1, \cdots, n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^{n} u_{kj} = \sum_{k=1}^{n} u_{ik} = 1\right).$$

Remark 13.

1. *Matrix $u = (u_{ij})_{i,j}$ with entries u_{ij}s from a non-trivial unital C^*-algebra satisfying relations $u_{ij} = u_{ij}^* = u_{ij}^2$ and $\sum_{k=1}^{n} u_{kj} = \sum_{k=1}^{n} u_{ik} = 1$, as in Definition 14, will be called a magic unitary, such that all its entries are projections, all distinct elements of the same row or same column are orthogonal, and sums of rows and columns are equal to 1.*
2. *A magic unitary matrix u is orthogonal, meaning that we have $u = \overline{u}$ and $uu^t = I_n = u^t u$.*
3. *If an element p satisfies $p^2 = p^* = p$, then it will be called a projection, and two projections will be orthogonal if we have $pq = 0$, and a partition of the unity is a finite set of mutually orthogonal projections, and sum up to 1.*
4. *The generators of the algebra of continuous functions on the quantum automorphism group of a (finite) graph Γ can be arranged in a matrix, known as a magic unitary. This matrix has the distinctive property that it commutes with the adjacency matrix of the graph Γ, and we will use this property later on when we attempt to construct the multiplier Hopf graph algebra.*

We may equip the C^*-algebra $C(S_n^+)$ with a comultiplication Δ by naturally defining $\Delta(u_{ij}) = u_{ij}^{'} := \sum_k u_{ik} \otimes u_{kj}$, and by using the orthogonality of the projections u_{ik} and $u_{i\ell}$ for $k \neq \ell$ (this can be deduced from the fact that $\sum_k u_{ik} = 1$), we have

$$u_{ij}^{'\,2} = \sum_{k,\ell} u_{ik} u_{i\ell} \otimes u_{kj} u_{\ell j} = \sum_k u_{ik} \otimes u_{kj} = u_{ij}^{'}$$

and,

$$\sum_k u_{ik}^{'} = \sum_k u_{kj}^{'} = 1 \otimes 1 = 1,$$

and, hence, by the universal property of $C(S_n^+)$, the map Δ is a $*$-homomorphism from $C(S_n^+)$ to $C(S_n^+) \otimes C(S_n^+)$, indeed, and it is easy to see that it is co-associative

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta(u_{ij}) &= \sum_k \Delta(u_{ik}) \otimes u_{kj} \\ &= \sum_{k,\ell} u_{i\ell} \otimes u_{\ell k} \otimes u_{kj} \\ &= \sum_\ell u_{i\ell} \otimes \Delta(u_{\ell j}) \\ &= (\mathrm{id} \otimes \Delta) \circ \Delta(u_{ij}).\end{aligned}$$

The density condition holds as well, since

$$\Delta(u_{ij})(1 \otimes u_{mj}) = \sum_k u_{ik} \otimes u_{kj} u_{mj} = u_{im} \otimes u_{mj}$$

implies

$$u_{im} \otimes 1 = \sum_j u_{im} \otimes u_{mj} \in \Delta(A)(1 \otimes A),$$

which means that $A \otimes 1 \subset \Delta(A)(1 \otimes A)$, and in the same way, we can obtain $1 \otimes A \subset \Delta(A)(1 \otimes A)$, which implies $A \otimes A = \Delta(A)(1 \otimes A)$ and proves that $(C(S_n^+), \Delta)$ is a *CQG* and the quantum automorphism group of X_n. Thus, in the category of *CQG*s, space X_n has more automorphisms than the classical one, and its automorphism group in the category of *CQG*s will be S_n^+ versus the classical space, which is S_n.

The following key construction is due to Wang [26]:

Proposition 7. *We have a compact quantum group O_n^+, defined as*

$$\begin{aligned} C(O_n^+) &= C^*\Big((u_{ij})_{i,j=1,\cdots,N} \mid u = \overline{u}, u^t = u^{-1} \Big) \\ &= C^*\left(u_{ij}, i,j = 1, \cdots, n \mid u_{ij} = u_{ij}^*, \sum_{k=1}^n u_{ik} u_{jk} = \sum_{k=1}^n u_{ki} u_{kj} = \delta_{i,j} 1 \right). \end{aligned} \tag{24}$$

This quantum group contains O_N, and the inclusion $O_N \subset O_N^+$ is not an isomorphism; it is non-commutative for all $n \geq 2$.

The simplest symmetry groups in the quantum framework and the classic one are (quantum) permutation groups, viewed as universal (quantum) groups, acting on a given finite set proven by Wang [25], and as a result, we have that the category $\mathfrak{C}(\mathbb{C}^n)$ of quantum groups acting on the n-point set X_n admits a universal object denoted by S_n^+, as in Definition 14, and is the quantum isometry group of the simplex with n points. But this statement is not true when dealing with the space of $n \times n$ matrices $M_n(\mathbb{R})$ and it has been proven that the category $\mathfrak{C}(M_n(\mathbb{R}))$ does not admit a universal object if $n > 1$ [25]; this is because of the existence of a universal object in the category of compact quantum semigroups acting on $M_n(\mathbb{R})$, while not being a compact quantum group [23].

However, there is also a positive answer to the problem related to $M_n(\mathbb{R})$ [23].

Theorem 2 (S. Wang [27])**.** *For $\mathcal{A}$, a finite-dimensional C^*-algebra with a faithful state ω, the category $\mathfrak{C}(\mathcal{A}, \omega)$ of quantum groups acting on $\mathcal{A}$ and preserving the state ω admits a universal object.*

For example, by considering $\mathcal{A} = M_2$, P. Sołtan [28] showed that the universal compact quantum group in $\mathfrak{C}(M_2, \omega) := A_{\mathrm{aut}}(M_2)$ is isomorphic to $SO_q(3)$, with q dependent on the choice of ω.

Hence, in general, to verify the existence of a quantum symmetry group of a C^*-algebra, we might need to have some more structures on it; this observation has led to the development of the theory of quantum isometry groups of non-commutative manifolds by Goswami, Banica, Bhowmick, Skalski, and others.

8.2. Looking for a Connection

Recall that, for the complex-valued functions a, b satisfying relation $a^*a + b^*b = 1$, $C(SU(2))$ is the commutative C^*-algebra generated by a and b, equipped with a comultiplication Δ, such that $\Delta(a) = a \otimes a - b^* \otimes b$ and $\Delta(b) = b \otimes a + a^* \otimes b$, induced by the group multiplication in $SU(2)$. Now, let $q \in [-1, 1)/\{0\}$, and define $C(SU_q(2))$ as the unital C^*-algebra generated by operators a, b, such that:

$$\begin{gathered} a^*a + b^*b = 1, \qquad aa^* + q^2b^*b = 1, \\ b^*b = bb^*, \quad qba = ab, \quad qb^*a = ab^*, \end{gathered}$$

and by defining Δ on $SU_q(2)$, such that

$$\Delta(a) = a \otimes a - qb^* \otimes b, \qquad \Delta(b) = b \otimes a + a^* \otimes b,$$

we will have a CQG structure on $SU_q(2)$.

$C(SU(2))$ is the commutative C^*-algebra generated by the complex-valued functions, a, b, satisfying the relation $a^*a + b^*b = 1$, and the group multiplication in $SU(2)$ induces a comultiplication Δ on $C(SU(2))$, such that we have $\Delta(a) = a \otimes a - b^* \otimes b$ and $\Delta(b) = b \otimes a + a^* \otimes b$

We have seen that the quantum permutation group S_n^+ can be viewed, on the one hand, as the quantum symmetry group of the n-point set, and on the other, as the liberation of the classical permutation group S_n. Now, the question is, are there any more examples of this type?

By following the literature on graph automorphisms, let $\Gamma = (V, E)$ be a locally finite (directed) graph with no multiple edges, and let $\pi \in M_n(\{0, 1\})$ be its adjacency matrix, and consider by $\mathrm{Aut}(\Gamma) = \{\sigma : V \to V \in S_n \mid \sigma\pi = \pi\sigma\} \subset S_n$ its automorphism group with the property that $(i, j) \in E$ if and only if $(\sigma(i), \sigma(j)) \in E$ and $C(\mathrm{Aut}(\Gamma)) = C(S_n)/\langle u\pi = \pi u\rangle$ [29], quantized by Banica [30] with the quantized version considered by $\mathsf{QAut}(\Gamma) := C(S_n^+)/\langle u\pi = \pi u\rangle$, such that the diagram

$$\begin{array}{ccc} \mathrm{Aut}(\Gamma) & \subseteq & \mathsf{QAut}(\Gamma) \\ \cap\!\!| & & \cap\!\!| \\ S_n & \subseteq & S_n^+ \end{array}$$

satisfies. In any case, if the inclusion on the lower line is strict, one has that Γ has quantum symmetries [29,30].

Definition 15 ([30])**.** *For a locally finite graph Γ with a vertex set $[n] = \{1, \cdots, n\}$, its quantum automorphism group, denoted by $\mathsf{QAut}(\Gamma)$, is the compact matrix quantum group given by $(C(\mathsf{QAut}(\Gamma)), \mathcal{U})$. Here, $C(\mathsf{QAut}(\Gamma))$ is the universal C^*-algebra with generators u_{ij}, where $1 \le i, j \le n$, satisfying the relations of Definition 14, in addition to the following relation*

$$\sum_{k\sim j} u_{ik} = \sum_{\ell\sim i} u_{\ell j}. \tag{25}$$

Let us have a closer look at Definition 15 by considering an example:

Example 5 ([31,32])**.** *Let Γ be the directed graph on $V = [4] = \{1, 2, 3, 4\}$, such that (Figure 1)*

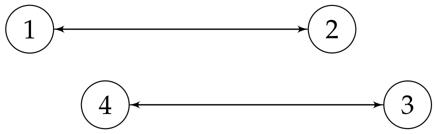

Figure 1. Directed locally connected graph related to $V = [4]$.

And then consider the following block matrix

$$B = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}.$$

If we set $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$*, then we will have the following matrix:*

$$\mathcal{U} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}$$

, and it is easy to see that u_{ij} *satisfies in relation to Definition 14 and Equation (25), and, hence,* $\mathcal{U}$ *is a quantum automorphism matrix for* Γ*. For example, to observe how Equation (25) works, let us perform some computations. We have*

$$u_{11}u_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{26}$$

and on the other hand, we have

$$u_{33}u_{43} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{27}$$

and, hence, we see that Equations (26) = (27) satisfy; this can be extended to all complying conditions of u_{ij} *of Equation (25).*

In general, we have the following definition.

Definition 16. *Let* $N \in \mathbb{N}$*, and* H *be a Hilbert space, and for* $V = [N] = \{1, \cdots, N\}$ *let* Γ_1 *and* Γ_2 *be finite (simple) graphs. Then a quantum isomorphism matrix of* Γ_1 *and* Γ_2 *is a matrix* $\mathcal{U} \in M_N(B(H))$ *consisting of entries* $u_{ij} \in B(H)$*,* $i, j = 1, \cdots, N$*, such that*

(a) $u_{ij} = u_{ij}^* = u_{ij}^2, \quad \forall i, j = 1, \cdots, N,$
(b) $\sum_{k=1}^N u_{ik} = \sum_{k=1}^N u_{kj} = 1 \quad \forall i, j = 1, \cdots, N,$ *and* $i \neq j$*,*
(c) $u_{ij}u_{k\ell} = u_{k\ell}u_{ij} = 0$ *if* $i \sim_2 k$ *and* $j \not\sim_1 \ell$*,*
(d) $u_{ij}u_{k\ell} = u_{k\ell}u_{ij} = 0$ *if* $i \not\sim_2 k$ *and* $j \sim_1 \ell$*,*

and if we have $\Gamma = \Gamma_1 = \Gamma_2$*, then we say that* $\mathcal{U}$ *is a quantum automorphism matrix of* Γ*.*

We have the following Proposition

Proposition 8 ([29,30])**.** *The necessity for a graph* Γ *to have quantum symmetries is that the quotient* $S_n^+ / \langle u\pi = \pi u \rangle$ *has to be a non-commutative algebra.*

Remark 14. *So, according to Proposition 8, a locally finite (directed) graph* Γ *has no quantum symmetry if* $\mathrm{Aut}(\Gamma) = \mathrm{QAut}(\Gamma)$*, meaning that the unital canonical* $*$*-homomorphism from*

$C(\mathsf{QAut}(\Gamma))$ onto $C(\mathrm{Aut}(\Gamma))$ is injective [33], which is equivalent to saying that $C(\mathsf{QAut}(\Gamma))$ is a commutative C^*-algebra [33]. For example, for $n \in \{1,2,3\}$, the graph with no edges and n vertices has no quantum symmetry [25], and we have $S_n^+ := C(\mathsf{QAut}(\Gamma)) = C(\mathrm{Aut}(\Gamma)) := S_n$. For $n = 4$, consider the block matrix B from Example 5, and then by using the universal property of the universal C^*-algebra $C(S_4^+)$, and considering the $*$-homomorphism from $C(S_4^+)$ to B sending u_{ij} to its respective entry from B, and noting the non-commutativity of B, one obtains the non-commutativity of $C(S_4^+)$, meaning that $S_4 \neq S_4^+$.

8.3. Some Open Directions

1. Geometric aspects. Groups S_n, O_n, and their quantum (free) versions, S_n^+, O_n^+, were involved in many other “classical vs. free” considerations. Notable examples include the Poisson boundary results in [34] and the quantum isometry groups in [35]. We note that the easy quantum groups can lead to some new results here.
2. Eigenvalue computations. The key results of Diaconis and Shahshahani in [36] concerning S_n, O_n have been obtained as well by using Weingarten functions and cumulants; an extension to all easy quantum groups has been constructed, and the original philosophy suggested in [37], namely the fact that “any result which holds for S_n, O_n should have an extension to easy quantum groups”, has been illustrated. Now the question is, “What are the eigenvalues of a random quantum group matrix?”.
3. The problem of computing the 3-orbitals of O_N looks purely combinatorial, and for H_N, involves some analysis coming from triangle inequalities for the edges of the triangles; hence, the combinatorics are not the same, and so the results of the computations should not be the same, so the claim is that the quantum groups $H_N \subset \overline{O}_N$ are distinguished by their 3-orbitals [37].

9. Into Multiplier Hopf Algebras

Following the constructions from the previous section 8, and considering compact spaces, the question arose as to how one could extend the construction of permutation quantum groups to obtain infinite versions. This question was addressed by Goswami and Skalski [38], who addressed this question by introducing two quantum semigroups of infinite quantum permutations on an infinite set. Addressing this question, many frameworks have been developed, from which, the two quantum semigroups of infinite quantum permutations have been addressed by Goswami and Skalski [38] by just moving finitely many points of S_n^+ by working under the framework of von Neumann algebras, and the second one was a universal von Neumann algebra generated by the entries of an infinite magic unitary matrix. However, it was unclear if any of those constructed objects could fit into the theory of locally compact quantum groups in the sense of Kustermans and Vaes [13]. This could be an interesting open direction to explore, to address this, in 2023, infinite quantum permutation groups have been introduced and studied by Ch. Voigt in [39] with a slightly different approach that allows for the creation of genuine quantum groups. The definition of these groups can be summarized as follows:

Definition 17 (Voigt [39])**.** *For a set X, a quantum permutation of X will be defined as a pair $\alpha = (H_\alpha, p^\alpha)$ consisting of a Hilbert space H_α and a family $p^\alpha = (p^\alpha_{xy})_{x,y \in X}$ of projections $p^\alpha_{xy} \in B(H_\alpha)$, such that*

1. *For every $x \in X$, the projections p^α_{xz} for $z \in X$ are pairwise orthogonal.*
2. *For every $y \in X$, the projections p^α_{zy} for $z \in X$ are pairwise orthogonal.*
3. *We have $\sum_{z \in X} p^\alpha_{xz} = 1 = \sum_{z \in X} p^\alpha_{zy}$ for all $x, y \in X$, with convergence understood in the strong operator topology.*

It is almost clear from Definition 17 that the key ingredient is the $*$-algebra generated by the entries of an infinite magic unitary matrix with different classes of representations inspired by the theory of non-local games and their associated game algebras [40]. Studying the structures of such representations will essentially amount to understanding matrix models for S_n^+ [22], which is very interesting!

Definition 18 (Voigt [39])**.** *For graph $\Gamma = (V_\Gamma, E_\Gamma)$, the quantum automorphism group $\mathrm{Qut}_\delta(\Gamma)$ is the quantum subgroup of $S_n^+(V_\Gamma)$ corresponding to the rigid C^*-tensor category of finite dimensional quantum automorphisms of Γ.*

Remark 15. *In the above definition, $\mathrm{Qut}_\delta(\Gamma)$ is shorthand for the discrete quantum automorphism group of Γ, which is not the same as the quantum automorphism group as defined by Banica–Bichon for a finite graph.*

Hence, it is said that a graph Γ has no quantum symmetry in the sense of Banica and Bichon, if every irreducible quantum automorphism of Γ is one-dimensional; this can happen if and only if the entries p_{xy} of every quantum automorphism $\alpha = (H, p)$ with respect to Γ pairwise commute; otherwise, we say that Γ possesses quantum symmetry.

For example, it has been proven that the infinite Johnson and Kneser graphs, respectively denoted as $J(\infty, 2)$ and $K(\infty, 2)$, have no quantum symmetries [39]. There are unit distance graphs U_d associated with Euclidean space $\mathbb{R}^d$, and for U_1, we can write $U_1 \simeq U_{x\in\mathbb{R}/\mathbb{Z}} L$ as the disjoint union of, uncountably, many copies of the "infinite line" graph L, i.e., the unit distance graph of $\mathbb{Z} \subset \mathbb{R}$; it has been shown [39] that it possesses quantum symmetry. But for $d > 1$, the situation seems much less clear, and we have the following question:

Question 4 (Voigt [39])**.** *Does U_d for $d > 1$ have quantum symmetry?*

Regarding this question, it might be helpful to relate quantum symmetries with the study of quantum automorphism groups of metric spaces [41]. Also, there is the Rado graph, known as the Erdös-Rényi graph or random graph, which can be defined as the countable graph R with vertex set V_R consisting of prime numbers congruent to $1 \mod 4$, and with $(p, q) \in E_R$ if and only if p is a quadratic residue mod q; the same question also waits to be explored and developed for this graph:

Question 5 (Voigt [39])**.** *Do Rado graphs R have quantum symmetry?*

Since for connected, locally finite graphs, the compatibility between the infinite magic unitary and the adjacency matrix can be expressed entirely algebraically, Rollier and Vaes devised an impressive constructive solution [42]. They constructed the associated multiplier $*$-algebra, equipped with Haar weights, by using the compatibility of the infinite magic unitary matrix with the adjacency matrix despite being interpreted purely algebraically. To construct those weights, Rollier–Vaes studied a certain unitary tensor category associated with the graph, extending the work by Mančinska-Roberson [43].

To such a graph, one can naturally associate a multiplier Hopf $*$-algebra in the sense of Van Daele [3], especially in the case of connected, locally finite graphs, where the relations for an infinite magic unitary compatible with the adjacency relations can be interpreted purely algebraically. The key result of [42] indicates that this multiplier Hopf $*$-algebra admits Haar weights, and to construct these weights, Rollier–Vaes studied a certain unitary tensor category associated with the graph, extending the work by Mančinska-Roberson [43].

Quantized Matrix Algebra $M_q(n)$

Let us start by exploring the $n = 2$ case. We write $\mathbb{K}\langle x_1, \cdots, x_n\rangle$ for the $\mathbb{K}$-algebra of polynomials in non-commuting indeterminates $x_1, \cdots, x_n$. Then the coordinate algebra of the algebra of quantum 2×2 matrices is defined by

$$\mathbb{K}[M_q(2)] = \mathbb{K}\langle x_{11}, x_{12}, x_{21}, x_{22}\rangle / R,$$

where R is the system of equations

$$x_{11}x_{12} = q^{-1}x_{12}x_{11}, \qquad x_{21}x_{22} = q^{-1}x_{22}x_{21}, \qquad x_{11}x_{21} = q^{-1}x_{21}x_{11},$$
$$x_{12}x_{22} = q^{-1}x_{22}x_{12}x_{12}x_{21} = x_{21}x_{12}, \qquad x_{11}x_{22} - x_{22}x_{11} = (q^{-1} - q)x_{21}x_{12}$$

Remark 16. *For relations for the coordinate algebra $M_q(n)$, we can easily relate a directed locally connected finite graph. For example, in the case of $\mathbb{K}(M_q(2))$, we have the following related graph (Figure 2):*

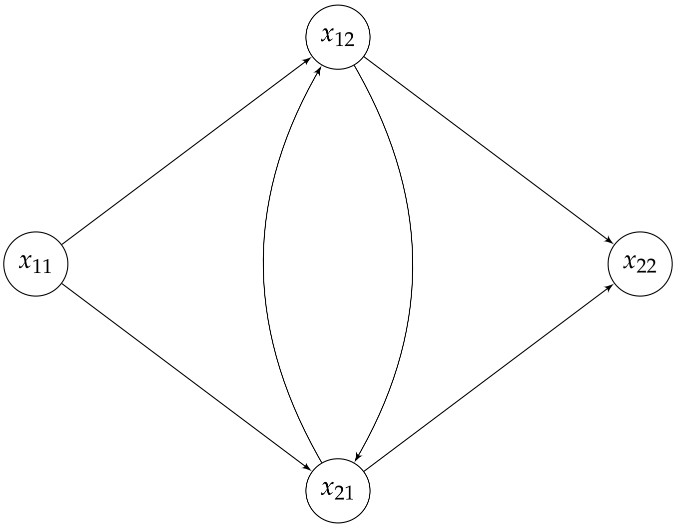

Figure 2. Directed locally connected graph related to $\mathbb{K}(M_q(2))$.

In the case of $\mathbb{K}(M_q(3))$, we have the following directed locally connected graph (Figure 3):

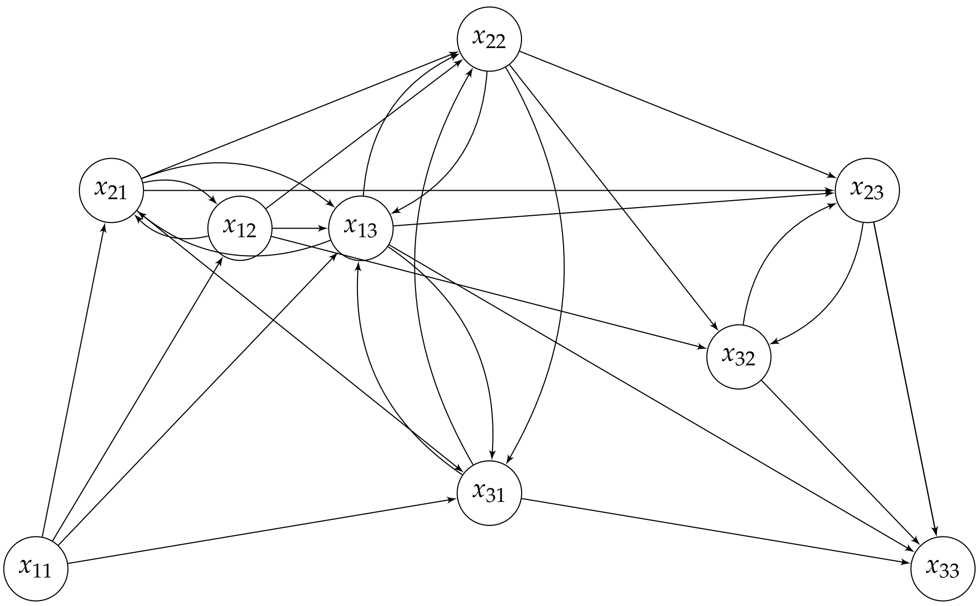

Figure 3. Directed locally connected graph related to $\mathbb{K}(M_q(3))$.

Now, let $h := ij, h' := i'j'$, and consider $A = (u_{hh'})_{h,h' \in I^2}$ for $I = \{1, \cdots, n\}$ our index set. Then, by ([42], Theorem A), there is a unique multiplier Hopf $*$-algebra $(\mathcal{A}, \Delta)$, consisting of self-adjoint idempotent elements $(u_{ij})_{i,j \in I}$ (forming "magic unitary"), commuting with the adjacency matrix $\Pi \in \mathcal{G}(\mathbb{K}(M_q(n)))$ introduced above, as in the following proposition:

Proposition 9. *For Π, a locally finite connected graph associated with coordinate algebra $\mathbb{K}(M_q(n))$ with vertex set $\{x_{11}, x_{12}, \cdots, x_{ij}\}$ for $i, j \in \{1, 2, \cdots, n\}$ and the index set $I := \{11, 12, \cdots, ij\}$, there exists a unique universal nondegenerate $*$-algebra $\mathcal{A}$ generated by elements $(u_{hh'})_{h,h' \in I}$, satisfying the relations of quantum permutation in Definition 17, and a unique nondegenerate $*$-homomorphism $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ satisfying $\Delta(u_{hh'}) = \sum_{k \in I}(u_{hk} \otimes u_{kh'})$ for all $h, h' \in I$, such that the pair $(\mathcal{A}, \Delta)$ is a multiplier Hopf $*$-algebra in the sense of ([3], Definition 2.4), and since it admits a positive faithful left-invariant (resp. right-invariant) functional, it is an algebraic quantum group in the sense of [2].*

Proof. It is a direct result of ([42], Theorem A). □

Note that matrix $\Pi_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ can be associated with the graph in Figure 1, as its adjacency matrix and the only commuting matrix with Π_2 satisfying relations of Definition 17 will be $\pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and this algorithm will hold for all matrices associated with $\mathbb{K}(M_q(n))$, for any n, meaning that in the commuting matrices, the entries associated with the row related to x_{ij}, will be 1 in the $(ij)(ji)$ position and 0 elsewhere. But the matrix space $(u_{hh'})_{h,h' \in I}$ associated with $\mathbb{K}(M_q(n))$ for any n will not produce an algebra because of the multiplication rule. Hence, we need to look at the associated graph algebra by the new overlay and connect binary operations as follows:

Definition 19. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 \cap V_2$ can be nonempty (as it is in our case). The overlay of G_1 and G_2, denoted by $G_1 + G_2$ is defined to be the union $G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2)$, and the connect $\to$ operation will be defined similarly, unless otherwise stated (usually the connect operation also consists of the new edges between new vertices). Formally, this means*

$$G_1 + G_2 := (V_1 \cup V_2, E_1 \cup E_2) \tag{28}$$
$$G_1 \to G_2 := (V_1 \cup V_2, E_1 \cup E_2). \tag{29}$$

This will give us an algebra that is called the algebra of parametrized graphs, with the empty graph $\zeta = (\emptyset, \emptyset)$ considered as the identity element for both operations.

Remark 17. *The structure $(G, +, \to, \zeta)$ introduced above satisfies many usual laws:*

(i) $(G, +, \zeta)$ *is an idempotent commutative monoid.*
(ii) $(G, \to, \zeta)$ *is a monoid.*
(iii) $\to$ *distributes over* $+$, *e.g.,* $1 \to (2 + 3) = 1 \to 2 + 1 \to 3$ *(but, as in our case, we no longer use the operation* $\to$*, since in our graphs, we have a sequence of increasing subgraphs with* $\cap_i V_i \neq \emptyset$*).*

The following decomposition axiom is the only law that makes the algebra of graphs different from a semiring:

$$x \to y \to z = x \to y + x \to z + y \to z.$$

Indeed, in a semiring the two operators have different identity elements, let us denote them as ζ_+ and $\zeta_\to$, respectively. By using the decomposition axiom, we can prove that they coincide:

$$\begin{aligned}
\zeta_+ &= \zeta_+ \to \zeta_\to \to \zeta_\to && (identity\ of\ \to) \\
&= \zeta_+ \to \zeta_\to + \zeta_+ \to \zeta_\to + \zeta_\to \to \zeta_\to && (decomposition) \\
&= \zeta_+ + \zeta_+ + \zeta_\to && (identity\ of\ \to) \\
&= \zeta_\to && (identity\ of\ +)
\end{aligned}$$

The idempotence of $+$ also follows from the decomposition axiom. We also have the following minimal set of axioms that describes the graph algebra:

(a) $+$ is commutative and associative.
(b) $(G, \to, \zeta)$ is a monoid, i.e., $\to$ is associative and ζ is the identity element.
(c) $\to$ distributes over $+$.
(d) $\to$ can be decomposed: $x \to y \to z = x \to y + x \to z + y \to z$.

Remark 18. *To switch from directed to undirected graphs, it is sufficient to add the axiom of commutativity of* $\to$.

Remark 19. *In [42], Definition 19 is considered for graphs without orientations (but can have loops). Here, Definition 19 works for any directed graphs, with or without loops.*

Consider by $\mathcal{G}_i = \{\mathcal{G}(\pi_i) \mid i \in \{1, \cdots n\}\}$ the set of $(n^2 - 2)$-connected oriented graphs associated with π_is. For instance, $\mathcal{G}(\pi_2)$ and $\mathcal{G}(\pi_3)$ are as follows (Figures 4 and 5):

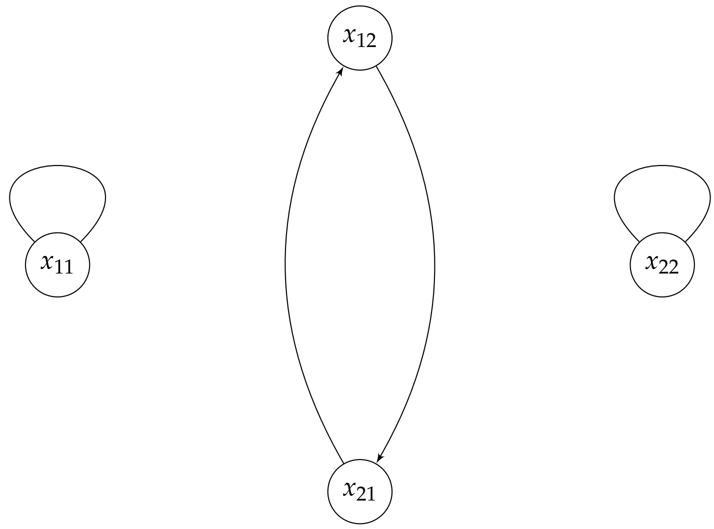

Figure 4. Directed 2-connected graph related to π_2.

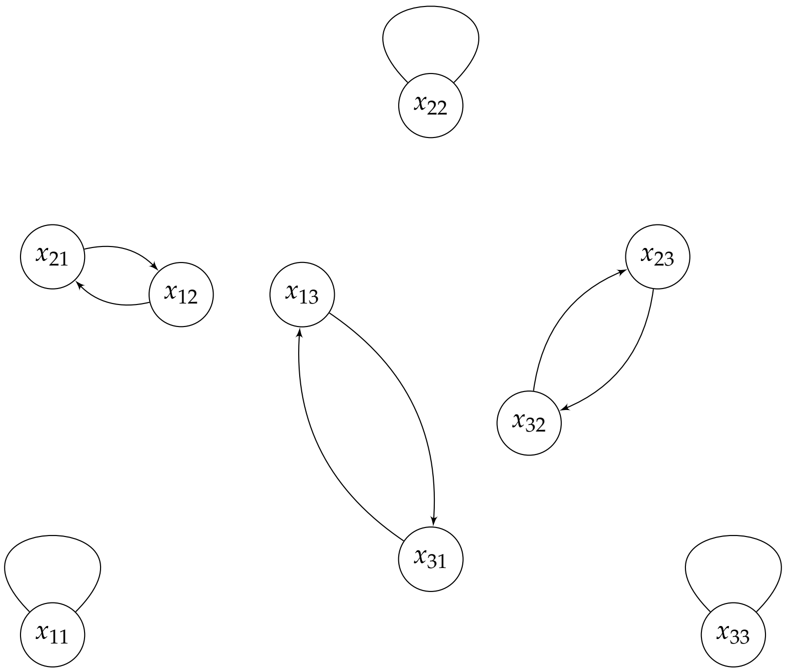

Figure 5. Directed 7-connected graph related to π_3.

There is an algebra structure equipped on this set by overlay and connect operations defined in Definition 19.

Claim 1. *The claim is that this algebra possesses a multiplier Hopf $*$-algebra structure.*

To start a logical proof of the above claim, one can start by looking at the directed graphs as operators on the Hilbert space $\mathcal{H}$ by following the construction developed in [44], looking at the vertices as the orthogonal projections and the edges as partial isometries, and taking into account the definition of the quantum symmetry (permutation) group by Wang [26]. We will discuss this in our next work [45].

Now, consider $G = Aut\Pi$, for Π as before, the adjacency matrix of the graph associated with $\mathbb{K}\big(M_q(n)\big)$. Then, the $*$-algebra $\mathcal{O}(G)$ of $\mathbb{K}$-valued locally constant functions on G with comultiplication $\Delta : \mathcal{O}(G) \to \mathcal{O}(G \times G) : f \mapsto \Delta(f)(x, y) := f(x \cdot y)$, is a multiplier Hopf $*$-algebra, and with $(\mathcal{A}, \Delta)$ defined by Proposition 9, there is the surjective multiplier Hopf $*$-algebra homomorphism

$$\mathfrak{P} : \mathcal{A} \to \mathcal{O}(G) : \mathfrak{P}(u_{ij}) = 1_{\{\sigma \in G | \sigma(j) = i\}}. \tag{30}$$

G will become a closed quantum subgroup of the locally compact quantum group defined by $(\mathcal{A}, \Delta)$.

10. Looking for Quantum Symmetries

For a locally finite (directed) graph Γ, let π be its adjacency matrix, and let $u = (u_{ij})_{i,j}$ be such that the condition $u\pi = \pi u$ satisfies. Then by Proposition 8, if the condition of being noncommutative algebra for $S_n^+ / \{u\pi = \pi u\}$ still works, then the necessity for the graph Γ to have quantum symmetries is provided, and if for non-zero elements u_{ij} we have $u_{ij}u_{k\ell} \neq u_{k\ell}u_{ij}$ for $i \neq k$ and $j \neq \ell$, then the sufficient part will also be provided, and we have $\mathrm{Aut}(\Gamma) \neq \mathrm{QAut}(\Gamma)$, and Γ will have quantum symmetries.

In this regard, let us take a look at the adjacency matrix of $\mathbb{K}[M_q(n)]$. For example, for the graph presented in Figure 2, the adjacency matrix is associated with $\pi_2 =$

$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and by performing some simple computations, it is easy to see that for $u = \begin{pmatrix} q & 0 & 0 & p \\ 0 & 0 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$, we have $u\pi = \pi u$, for p, q some projections, and for u to be a magic unitary, meaning that its row and column sums have to be equal to 1, we require $q = 1$ and $p = 0$, which means that the matrix only has commuting entries, we do not have quantum symmetry, and the quantum automorphism group is trivial.

Theorem 3 ([45])**.** *For $i, j \in \{1, 2, \cdots, n\}$, there are matrices $u = \{u_{(ij)(ji)} = u_{(ii)(ii)} = q$, and $u_{(11)(nn)} = p \mid p, q$ are projections$\}$, such that it commutes with π_i, the adjacency matrix of $\mathbb{K}[M_q(n)]$, and for u to be a magic unitary, we require $q = 1$ and $p = 0$, meaning that its entries commute, and, hence, for any i, the graphs $\mathcal{G}(\pi_i)$ possess no quantum symmetries.*

Example 6. *Consider the following directed locally connected graph (Figure 6),*

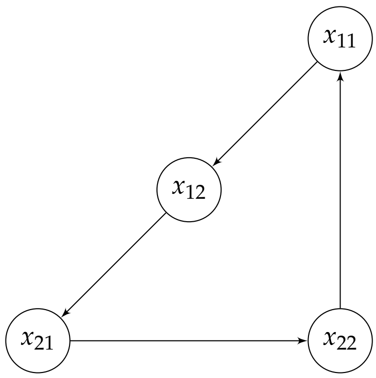

Figure 6. Triangular cyclical directed locally connected graph.

Where $\overrightarrow{\rightleftarrows}$ denotes the direction from vertex u_{ij} to vertex $u_{k\ell}$, and the rule for specifying the direction can be specified as follows:

We have $u_{ij} \overrightarrow{\rightleftarrows} u_{k\ell}$ if and only if the following conditions are satisfied

1. *$i = k$ and $j < \ell$,*
2. *$i < k$ and $j > \ell$,*
3. *$i > k$ and $j > \ell$.*

Now, let us consider the matrix $\pi_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, which can be associated with the graph shown in Figure 6, and it is not too difficult to see that its space of commuting matrices will only consist of the following matrices, where p and q are some projections (for example, they can be considered as 2×2 matrices, such that the summation of each row and column entries becomes twice the identity matrix):

$$u = \frac{1}{2}\begin{pmatrix} 1-q & p & q & 1-p \\ 1-p & 1-q & p & q \\ q & 1-p & 1-q & p \\ p & q & 1-p & 1-q \end{pmatrix},$$

but by considering the coefficient $\frac{1}{2}$, the entries of u will no longer be projections, and, hence, u will not be a magic unitary, which is necessary to find the quantum symmetries.

And, hence, we obtain Theorem 4, which is almost the same as Theorem 3, as a non-regular generalization of Example 6 for matrices of the above type by using relations (1) to (3):

Theorem 4 ([45])**.** *For $i, j \in \{1, 2, \cdots, n\}$, there are no non-trivial commuting magic unitary matrices for the adjacency matrices of the graphs associated with relations (1) to (3); hence, they do not possess any quantum symmetries, despite having the non-trivial automorphism group.*

These observations embrace the following open directions:

1. Are there any (undirected) graphs with trivial symmetry groups and non-trivial quantum symmetry groups?
2. Are there are any quantum groups sitting between S_n and S_n^+?
   In this case, for example, for $n = 4$ and $n = 5$, we know that there is no such intermediate quantum group, but for the others, it is unknown!
3. For A_n, the alternating group, how can we define A_n^+?
   For example, the quantum automorphism group of a folded n-dimensional cube, when n is odd, is known to be SO_n^{-1}, but this is an open problem for the case where n is even!
   So, one formal approach to the above problem is to find a graph whose automorphism group is the alternating group A_n, and then show that this graph possesses quantum symmetries. From there, we can proceed to define A_n^+.

11. Concluding Remarks

We believe that the research conducted in this paper is very interesting, and if we want to describe it in just one sentence, it would be "from simplicity to complexity"!

We started with our toy example $\mathbb{K}[M_q(n)]$, in Section 7, by demonstrating that it is a quadratic algebra in the sense of Y. I. Manin. This was accomplished by providing a very detailed proof, following B. Parshall's work on quasi-Yang–Baxter algebras. We then attempted to impose a monoid graph algebra structure on the set of entangled $(n^2 - 2)$-connected oriented graphs $\mathcal{G}_i = \{\mathcal{G}(\pi_i) \mid i \in \{1, \cdots n\}\}$, associated with π_is, the commuting matrices with the adjacency matrices associated with $\mathbb{K}[M_q(n)]$. This structure is equipped with a nondegenerate binary operation, encouraging us to claim that the introduced graph algebra possesses the characteristics of a multiplier Hopf $*$-algebra!

Returning to the example theorem introduced and studied by Rollier–Vaes ([42], Theorem A), and once again working on our toy example in Proposition 9, it is not too difficult to see that the universal nondegenerate C^*-algebra

$$\mathcal{A} = C^*\Big(u_{hh'} \mid u = (u_{hh'})_{h,h' \in I = \{11, \cdots, nn\}} \text{ a magic unitary}\Big),$$

consisting of just 0 and 1, equipped with a unique nondegenerate $*$-homomorphism $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A}) : u_{hh'} \mapsto \sum_{k \in I}(u_{hk} \otimes u_{kh'})$, satisfies the essential requirements of being a multiplier Hopf $*$-algebra in the sense of ([3], Definition 2.4). Since it admits a positive faithful left-invariant (resp., right-invariant) functional, it qualifies as an algebraic quantum group in the sense of [2].

We have also shown that the graphs associated with $\mathbb{K}[M_q(n)]$ and the triangular cyclical directed locally connected graphs, as studied in Example 6, despite having nontrivial automorphism groups, do not possess any quantum symmetries!

For future work, as we have pointed out in Sections 8–10, there are many possibilities, but above all, we are primarily interested in pursuing the open directions mentioned at the end of Section 10 to extend our results and potentially conclude with a positive answer to the claim raised in Claim 1, and we aim to classify such graph algebras that obey a multiplier $*$-algebra structure!

Another direction not investigated in this paper concerns Woronowicz Hopf C^*-dynamical systems, studied by S. Wang in [46], and given that our studied objects are also Woronowicz algebras, they could be applied to Wang's constructions, which also looks very interesting and applicable!

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