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On Another Type of Convergence for Intuitionistic Fuzzy Observables

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Abstract: The convergence theorems play an important role in the theory of probability and statistics and in its application. In recent times, we studied three types of convergence of intuitionistic fuzzy observables, i.e., convergence in distribution, convergence in measure and almost everywhere convergence. In connection with this, some limit theorems, such as the central limit theorem, the weak law of large numbers, the Fisher–Tippet–Gnedenko theorem, the strong law of large numbers and its modification, have been proved. In 1997, B. Riečan studied an almost uniform convergence on D-posets, and he showed the connection between almost everywhere convergence in the Kolmogorov probability space and almost uniform convergence in D-posets. In 1999, M. Jurečková followed on from his research, and she proved the Egorov's theorem for observables in MV-algebra using results from D-posets. Later, in 2017, the authors R. Bartková, B. Riečan and A. Tirpáková studied an almost uniform convergence and the Egorov's theorem for fuzzy observables in the fuzzy quantum space. As the intuitionistic fuzzy sets introduced by K. T. Atanassov are an extension of the fuzzy sets introduced by L. Zadeh, it is interesting to study an almost uniform convergence on the family of the intuitionistic fuzzy sets. The aim of this contribution is to define an almost uniform convergence for intuitionistic fuzzy observables. We show the connection between the almost everywhere convergence and almost uniform convergence of a sequence of intuitionistic fuzzy observables, and we formulate a version of Egorov's theorem for the case of intuitionistic fuzzy observables. We use the embedding of the intuitionistic fuzzy space into the suitable MV-algebra introduced by B. Riečan. We formulate the connection between the almost uniform convergence of functions of several intuitionistic fuzzy observables and almost uniform convergence of random variables in the Kolmogorov probability space too.

Keywords: intuitionistic fuzzy event; intuitionistic fuzzy observable; intuitionistic fuzzy state; MV-algebra; almost everywhere convergence; almost uniform convergence; Egorov's theorem; function of several intuitionistic fuzzy observables

MSC: 03B52; 60A86; 60B10

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1. Introduction

In 1983, K.T. Atanassov first introduced the theory of intuitionistic fuzzy sets in the paper [1]. So, in this year, 2023, we celebrate the 40th anniversary of the foundation of this theory. By an intuitionistic fuzzy set on the set Ω, he means a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $0_\Omega \leq \mu_A + \nu_A \leq 1_\Omega$. The concept of the intuitionistic fuzzy sets is an extension of the concept of fuzzy sets introduced by L. Zadeh (see [2,3]). Namely, if f is a fuzzy set, then $(f, 1-f)$ is the corresponding intuitionistic fuzzy set. The inequality $\mu_A + \nu_A \leq 1_\Omega$ means that there is room for a third function $\pi_A = 1_\Omega - \mu_A - \nu_A > 0$, which stays for the degree of uncertainty. So, there are three functions: membership, nonmembership and uncertainty (hesitation).

The year 2023 is the 20th anniversary of the research of the intuitionistic fuzzy sets in Slovakia. In 2003, B. Riečan formulated the descriptive definition of probability for intuitionistic fuzzy sets in the paper [4]. He was inspired by the research of P. Grzegorzewski and E. Mrówka (see [5]). In papers [6–10], the authors formulated several types of convergence of intuitionistic fuzzy observables, i.e., an almost everywhere convergence, a convergence in distribution and a convergence in measure, and they proved some limit theorems for a sequence of intuitionistic fuzzy observables, i.e., a variation of the central limit theorem, a variation of the weak law of large numbers, a variation of the Fisher–Tippet–Gnedenko theorem, a variation of the strong law of large numbers and its modification.

B. Riečan studied an almost uniform convergence on D-posets in the paper [11]. He showed the connection between an almost everywhere convergence in the Kolmogorov probability space and an almost uniform convergence in D-posets. M. Jurečková followed on from this research, and she proved the Egorov's theorem for observables in MV-algebra using results from D-posets (see [12]). In [13], B. Riečan formulated and proved the variations of Egorov's theorem for small systems, submeasures, lattice-valued nonadditive measures, maxitive measures and MV-algebras. Note that Egorov's theorem can also be found in the literature under the name Egoroff's theorem (see [13]) or Jegorov's theorem (see [12]). In [14], the authors studied an almost uniform convergence and the Egorov's theorem for fuzzy observables in the fuzzy quantum space. Since the intuitionistic fuzzy sets are an extension of fuzzy sets, it is interesting to study an almost uniform convergence on the family of the intuitionistic fuzzy sets.

In this paper, we define an almost uniform convergence for intuitionistic fuzzy observables. We show the relation between an almost uniform convergence and an almost everywhere convergence of a sequence of intuitionistic fuzzy observables. We formulate a version of Egorov's theorem too. Finally, we prove the relation between the almost uniform convergence of functions of several intuitionistic fuzzy observables and random variables in the Kolmogorov probability space too. Since the intuitionistic fuzzy observable $x : \mathcal{B}(R) \to \mathcal{F}$ is an extension of the random variable $\xi : \Omega \to R$, I am inspired by an almost uniform convergence of random variables: *The sequence of random variables $(\xi_n)_1^\infty$ converges to 0 almost uniformly on A if for every $\alpha > 0$ there exists a measurable set A such that $P(A) > 1 - \alpha$ and such that for every $\beta > 0$ there exists k such that $A \subset \{t \in \Omega : |\xi_n(t)| < \beta\}$ for every $n \geq k$.*

Now, we recall an almost everywhere convergence of random variables: *The sequence $(\xi_n)_1^\infty$ converges P-almost everywhere to 0 if*

$$P\left(\bigcap_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\xi_n^{-1}\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) = 1,$$

i.e.,

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i}\xi_n^{-1}\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) = 1.$$

And we are inspired by the Egorov's theorem for random variables: *Let $(\Omega, \mathcal{S}, P)$ be a probability space and $(\xi_n)_1^\infty$ be a sequence of random variables. If a sequence $(\xi_n)_1^\infty$ converges P-almost everywhere to 0, then the sequence $(\xi_n)_1^\infty$ converges almost uniformly to 0.* See [13].

We note that in the whole text we use the notation IF as an abbreviation for intuitionistic fuzzy.

2. IF-State, IF-Observable and m-Almost Everywhere Convergence

In this part, we explain the basic terms from IF-probability theory, like the IF-event, IF-state, IF-observable and term of **m**-almost everywhere convergence.

Definition 1 ([15–17])**.** *Let us have a space $(\Omega, \mathcal{S})$, which is measurable. Hence, $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$, where $0_\Omega \leq \mu_A + \nu_A \leq 1_\Omega$, such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable functions.*

Denote by $\mathcal{F}$ the family of all IF-events on $(\Omega, \mathcal{S})$. In this paper, we will work with Łukasiewicz binary operations $\oplus, \odot$ given by

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))\end{aligned}$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$ and $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$. The partial ordering is defined by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In this paper, we use the max-min connectives defined by $\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$, $\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$. We work with the De Morgan rules: $\neg(a \vee b) = \neg a \wedge \neg b$ and $\neg(a \wedge b) = \neg a \vee \neg b$, where $\neg a = 1 - a$.

Definition 2 ([18])**.** *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$ *and* $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) *If* $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, *then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) *If* $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.,* $\mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Let $\mathcal{J}$ be the family of all intervals in R of the form $[a, b) = \{x \in R : a \leq x < b\}$. Then, the σ-algebra $\sigma(\mathcal{J})$ is denoted by $\mathcal{B}(R)$. It is called the σ-algebra of Borel sets, and its elements are called Borel sets.

Definition 3 ([18])**.** *By an n-dimensional IF-observable on $\mathcal{F}$, we understand each mapping $x : \mathcal{B}(R^n) \to \mathcal{F}$, satisfying the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$ *and* $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *If* $A \cap B = \emptyset$ *and* $A, B \in \mathcal{B}(R^n)$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *If* $A_i \nearrow A$, *then* $x(A_i) \nearrow x(A)$ *for all* $A, A_i \in \mathcal{B}(R^n)$.

When $n = 1$, we simply say that x is an IF-observable.

In [17], B. Riečan defined the notion of a joint IF-observable and proved its existence.

Definition 4 ([17])**.** *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be two IF-observables. The joint IF-observable of the IF-observables x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *If* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) \oplus h(B)$ *and* $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$;

(iii) *If* $A, A_n \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$.

Recall that $\cdot$ is the product operation defined by $\mathbf{A} \cdot \mathbf{B} = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B)$ for each $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ (see [19]).

Theorem 1 ([17])**.** *For each two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their joint IF-observable.*

In paper [6], we defined the notion of almost everywhere convergence for IF-observables.

Definition 5 ([6])**.** *Let $(x_n)_1^\infty$ be a sequence of IF-observables on an IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(x_n)_1^\infty$ converges* **m***-almost everywhere to* 0 *if*

$$\mathbf{m}\left(\bigwedge_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\mathbf{m}\left(\bigwedge_{n=k}^{k+i} x_n\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) = 1.$$

3. MV-Algebras and Embedding

In this section, we show that any IF-space $\mathcal{F}$ can be embedded into a particular MV-algebra. First, we recall the basic notions. Using the Mundici theorem, any MV-algebra can be defined with the help of an ℓ-group (see [20]), as defined below.

Definition 6 ([20])**.** *By an ℓ-group, I shall mean the structure $(G, +, \leq)$, such that the following properties are satisfied:*

(i) $(G, +)$ *is an Abelian group;*
(ii) $(G, \leq)$ *is a lattice;*
(iii) $a \leq b \Longrightarrow a + c \leq b + c$.

For each ℓ-group G, an element $u \in G$ is said to be a strong unit of G if for all $a \in G$ there is an integer $n \geq 1$ such that $nu \geq a$ (nu is the sum $u + \ldots + u$ with n).

Definition 7 ([20])**.** *An MV-algebra is an algebraic system $(M, \oplus, \odot, \neg, 0, u)$, where $\oplus, \odot$ are binary operations, $\neg$ is a unary operation and $0, u$ are fixed elements, which can be obtained by the following way: there exists a lattice group $(G, +, \leq)$ such that $M = \{x \in G; 0 \leq x \leq u\}$, where 0 is the neutral element of G, u is a strong unit of G and*

$$\begin{aligned} a \oplus b &= (a + b) \wedge u, \\ a \odot b &= (a + b - u) \vee 0, \\ \neg a &= u - a. \end{aligned}$$

Here, $\vee, \wedge$ are the lattice operations with respect to the order and $\neg a$ is the opposite element of the element a with respect to the operation of the group.

In this paper, we will work with the following MV-algebra connected with IF-sets.

Example 1. *Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ be a σ-algebra and $\mathcal{M}$ be the family of all pairs $\mathbf{A} = (\mu_A, \nu_A)$, where $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable functions,*

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \\ \neg \mathbf{A} &= (1_\Omega - \mu_A, 1_\Omega - \nu_A). \end{aligned}$$

Then, the system $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ is an MV-algebra. There, the particular ℓ-group is $(\mathcal{G}, +, \leq)$, where

$$\begin{aligned} \mathcal{G} &= \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to R \text{ are } \mathcal{S}\text{-measurable functions}\}, \\ \mathbf{A} + \mathbf{B} &= (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega), \\ \mathbf{A} \leq \mathbf{B} &\iff \mu_A \leq \mu_B, \nu_A \geq \nu_B, \end{aligned}$$

and $\mathbf{0} = (0_\Omega, 1_\Omega)$ is the neutral element of ℓ-group $(\mathcal{G}, +, \leq)$,

$$\mathbf{A} - \mathbf{B} = (\mu_A - \mu_B, \nu_A - \nu_B + 1_\Omega).$$

The lattice operations of ℓ-group $(\mathcal{G}, +, \leq)$ are given by $\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$ and $\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

Definition 8 ([20])**.** *An MV-algebra M is said to be σ-complete if its underlying lattice is σ-complete, i.e., every nonempty countable subset of M has a supremum in M.*

Recall that every finite MV-algebra M is σ-complete. Indeed, M is complete in the sense that every nonempty subset of M has a supremum in M. Now, we explain a notion of state in MV-algebra and its properties.

Definition 9 ([21])**.** *Let $(M, \oplus, \odot, \neg, 0, u)$ be an MV-algebra. By a finitely additive state on an MV-algebra M is considered each monotone mapping $m : M \to [0,1]$ (i.e., $a \leq b \Rightarrow m(a) \leq m(b)$) satisfying the following conditions:*

(i) $m(u) = 1, m(0) = 0$;
(ii) $a \odot b = 0 \Rightarrow m(a \oplus b) = m(a) + m(b)$.

A finitely additive state is a state if, moreover,

(iii) $a_n \nearrow a \Rightarrow m(a_n) \nearrow m(a)$.

m is faithful (also called, strictly positive) if $m(x) \neq 0$ whenever $x \neq 0$, $x \in M$.

Proposition 1 ([20])**.** *Let m be a finitely additive state on an MV-algebra M. Then, we have the following:*

(i) $m(\neg a) = 1 - m(a)$ *for all* $a \in M$;
(ii) *m is a valuation:* $m(a) + m(b) = m(a \oplus b) + m(a \odot b)$ *for all* $a, b \in M$;
(iii) *If m is faithful, then m is strictly monotone: if $a < b$, then $m(a) < m(b)$;*
(iv) *m is also a valuation with respect to the underlying lattice order of M; stated otherwise, for all $a, b \in M$, $m(a) + m(b) = m(a \vee b) + m(a \wedge b)$;*
(v) *m is subadditive, in the sense that $m(a \vee b) \leq m(a \oplus b) \leq m(a) + m(b)$.*

Due to Proposition 1, we prove that each state on MV-algebra is sub-σ-additive.

Lemma 1. *Let m be a state on MV-algebra M. Then,*

$$m\left(\bigvee_{n=1}^{\infty} a_n\right) \leq \sum_{n=1}^{\infty} m(a_n)$$

for each sequence $(a_n)_1^{\infty}$, $a_n \in M$.

Proof. By mathematical induction, we can show that for every $n \geq 1$ the following inequality holds:

$$m\left(\bigvee_{i=1}^{n} a_i\right) \leq \sum_{i=1}^{n} m(a_i). \tag{1}$$

According to property (v) in Lemma 1, for $n = 2$ the condition (1) holds. Suppose that condition (1) is true for n. Then, for $k = n + 1$ we calculate the following:

$$\begin{aligned} m\big((a_1 \vee a_2 \vee \ldots \vee a_n) \vee a_{n+1}\big) &\leq m(a_1 \vee a_2 \vee \ldots \vee a_n) + m(a_{n+1}) \\ &\leq m(a_1) + m(a_2) + \ldots + m(a_{n+1}). \end{aligned}$$

Then, using (iii) in the Definition 9,

$$\begin{aligned} m\left(\bigvee_{n=1}^{\infty} a_n\right) &= \lim_{n\to\infty} m\left(\bigvee_{i=1}^{n} a_i\right) \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{n} m(a_i) \\ &= \sum_{n=1}^{\infty} m(a_n). \end{aligned}$$

This completes the proof. □

Now, we recall some important results about the embedding of the IF-space to the particular MV-algebra.

Theorem 2 ([22])**.** *Let the system $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ be the MV-algebra constructed in Example 1. Then, $\mathcal{F} \subset \mathcal{M}$ and to each finitely additive IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists a finitely additive state $m : \mathcal{M} \to [0,1]$ such that it is an extension of $\mathbf{m}$ (i.e., $m|\mathcal{F} = \mathbf{m}$).*

Theorem 3 ([23])**.** *The family $\mathcal{F}$ can be embedded into an MV-algebra $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ constructed in Example 1 such that for each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists a state $m : \mathcal{M} \to [0,1]$ such that $m|\mathcal{F} = \mathbf{m}$.*

Really, if $(\mu_A, \nu_A) \in \mathcal{M}$, then $(\mu_A, 0_\Omega) \in \mathcal{F}$ and $(0_\Omega, 1_\Omega - \nu_A) \in \mathcal{F}$. It is reasonable to define

$$m\big((\mu_A, \nu_A)\big) = \mathbf{m}\big((\mu_A, 0_\Omega)\big) - \mathbf{m}\big((0_\Omega, 1_\Omega - \nu_A)\big).$$

It is not difficult to prove that $m : \mathcal{M} \to [0,1]$ is a state of MV-algebra $\mathcal{M}$ and $m|\mathcal{F} = \mathbf{m}$. For details of proof, see [23]. We will use these results later.

4. Almost Uniform Convergence for IF-Observables and a Variation of Egorov's Theorem

In this section, we formulate almost uniform convergence for a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$.

Definition 10. *Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with an IF-state $\mathbf{m}$. The sequence $(x_n)_1^\infty$ of IF-observables converges $\mathbf{m}$-almost uniformly to 0 if to every $\alpha > 0$ there exists an IF-set $\mathbf{A} \in \mathcal{F}$ such that $\mathbf{m}(\mathbf{A}) > 1 - \alpha$ and such that to every $\beta > 0$ there exists k such that $\mathbf{A} \leq x_n\big((-\beta, \beta)\big)$ for every $n \geq k$.*

The following theorem mentions a relation between the $\mathbf{m}$-almost uniform convergence and $\mathbf{m}$-almost everywhere convergence of IF-observables.

Theorem 4 (A variation of Egorov's Theorem)**.** *Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with an IF-state $\mathbf{m}$. If a sequence $(x_n)_1^\infty$ of IF-observables converges $\mathbf{m}$-almost everywhere to 0, then the sequence $(x_n)_1^\infty$ converges $\mathbf{m}$-almost uniformly to 0.*

Proof. Let us assume that a sequence of IF-observables $(x_n)_1^\infty$ converges $\mathbf{m}$-almost everywhere to 0. Using Definition 5, we have

$$\mathbf{m}\left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1.$$

Put

$$\mathbf{A}_k^p = \bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right).$$

Then, $\mathbf{A}_k^p \leq \mathbf{A}_{k+1}^p$ and

$$\mathbf{m}\left(\bigvee_{k=1}^{\infty} \mathbf{A}_k^p\right) = \mathbf{m}\left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1 \tag{2}$$

for every p, i.e., $\lim_{p\to\infty} \mathbf{m}\left(\mathbf{A}_k^p\right) = 1$.

If $\mathbf{m}$ is an IF-state, then from Theorem 3 there exists a state $m : \mathcal{M} \to [0,1]$ such that $m|\mathcal{F} = \mathbf{m}$ and $\mathcal{F} \subset \mathcal{M}$. By (2), for every $\alpha > 0$ and every p there exists $\mathbf{A}_{k(p)}^p \in \mathcal{F} \subset \mathcal{M}$ such that

$$m\left(\neg\mathbf{A}_{k(p)}^p\right) < \frac{\alpha}{2^p} \tag{3}$$

and $\neg\mathbf{A}_{k(p)}^p \in \mathcal{M}$. Put

$$\mathbf{A} = \bigwedge_{p=1}^{\infty} \mathbf{A}_{k(p)}^p \text{ and } \mathbf{A} \in \mathcal{F},$$

then, using De Morgan rules,

$$\neg\mathbf{A} = \bigvee_{p=1}^{\infty} \neg\mathbf{A}_{k(p)}^p \text{ and } \neg\mathbf{A} \in \mathcal{M}.$$

Now, we use the sub-σ-additivity of state m (see Lemma 1) and the inequality (3); Therefore, we obtain

$$\begin{aligned} m(\neg\mathbf{A}) &= m\left(\bigvee_{p=1}^{\infty} \neg\mathbf{A}_{k(p)}^p\right) \\ &\leq \sum_{p=1}^{\infty} m\left(\neg\mathbf{A}_{k(p)}^p\right) \\ &< \sum_{p=1}^{\infty} \frac{\alpha}{2^p} = \alpha. \end{aligned}$$

Since, by (i) of Proposition 1, $m(\neg\mathbf{A}) = 1 - m(\mathbf{A})$ and, moreover, $m(\mathbf{A}) = \mathbf{m}(\mathbf{A})$, because $m|\mathcal{F} = \mathbf{m}$ and $\mathbf{A} \in \mathcal{F}$,

$$\begin{aligned} m(\neg\mathbf{A}) &< \alpha, \\ 1 - m(\mathbf{A}) &< \alpha, \\ \mathbf{m}(\mathbf{A}) &> 1 - \alpha. \end{aligned}$$

For every $\beta > 0$, choose p such that $\frac{1}{p} < \beta$. Then,

$$\begin{aligned} \mathbf{A} &= \bigwedge_{p=1}^{\infty} \mathbf{A}_{k(p)}^p \\ &\leq \mathbf{A}_{k(p)}^p \\ &= \bigwedge_{n=k(p)}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right) \\ &\leq x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right) \\ &\leq x_n(-\beta, \beta), \end{aligned}$$

i.e., by Definition 10 the sequence $(x_n)_1^{\infty}$ of IF-observables converges $\mathbf{m}$-almost uniformly to 0. □

In the following text, we describe the construction of the Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$ and we show its connection to the IF-space $(\mathcal{F}, \mathbf{m})$.

Let R^N be a space of all sequences $(t_i)_1^\infty$ of real numbers. A set $C \subset R^N$ given by

$$C = \{(t_i)_1^\infty \in R^N : (t_1, \ldots, t_n) \in A\}$$

is called a cylinder, where $n \in N$ and $A \in \mathcal{B}(R^n)$. Denote by $\mathcal{C}$ the family of all cylinders in R^N and by $\sigma(\mathcal{C})$ the σ-algebra generated by $\mathcal{C}$. The cylinder C can be expressed in the form $C = \pi_n^{-1}(A)$, where a mapping $\pi_n : R^N \to R^n$ is n-th coordinate random vector given by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$. Therefore,

$$\mathcal{C} = \{\pi_n^{-1}(A) \mid n \in N, A \in \mathcal{B}(R^n)\}$$

and there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P(\pi_n^{-1}(A)) = P_n(A) = \mathbf{m}\big(h_n(A)\big)$$

for each $A \in \mathcal{B}(R^n)$, where h_n is a joint IF-observable of IF-observables $x_1, \ldots, x_n$. Hence, we can define the random variable $\xi_n : R^N \to R$ with respect to $\sigma(\mathcal{C})$ by $\xi_n((t_i)_1^\infty) = t_n$ such that $P_{\xi_n} = \mathbf{m} \circ x_n = \mathbf{m}_{x_n}$ (see [6,24]).

In limit theorems, we used the functions of several IF-observables and studied their convergence (see [6–8]). The function of several IF-observables is the IF-observable, which is a composition of several IF-observables and a Borel measurable function (see [8]).

Definition 11 ([8])**.** *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n their joint IF-observable and $g_n : R^n \to R$ a Borel measurable function. Then, we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ using the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

In the following example, we show the definitions of functions of several observables for some limit theorems.

Example 2. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables and $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be their joint IF-observable. Then, we use the following:*

1. *For the function of several IF-observables $y_n = g_n(x_1, \ldots, x_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)$ in the central limit theorem and in the weak law of large numbers, is the particular Borel function $g_n(t_1, \ldots, t_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} t_i - a\Big)$;*
2. *For the function of several IF-observables $y_n = g_n(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mathbf{E}(x_i))$ in the strong law of large numbers, is the particular Borel function $g_n(t_1, \ldots, t_n) = \frac{1}{n}\sum_{i=1}^{n}(t_i - \mathbf{E}(x_i))$;*
3. *For the function of several IF-observables $y_n = g_n(x_1, \ldots, x_n) = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big)$ in the Fisher–Tippett–Gnedenko theorem, is the particular Borel function $g_n(t_1, \ldots, t_n) = \frac{1}{a_n}\big(\max(t_1, \ldots, t_n) - b_n\big)$.*

There $t_1, \ldots, t_n$ are real numbers.

The next proposition discusses the connection between an almost everywhere convergence of the functions of several IF-observables in the IF-space and an almost everywhere convergence of random variables in the Kolmogorov probability space (see [24]).

Proposition 2 ([24])**.** *Let $(x_i)_1^\infty$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be the joint IF-observable of $x_1, \ldots, x_n$ and $g_n : R^n \to R$ be a Borel measurable function. Let IF-observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ be given by $y_n = h_n \circ g_n^{-1}$ and random variable $\eta_n = g_n(t_1, \ldots, t_n) : R^N \to R$ be defined by $\eta_n = g_n \circ \pi_n$, where $\pi_n : R^N \to R^n$ is the n-th coordinate random vector defined by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$. It follows that*

$$P_{\eta_n} = P \circ \eta_n^{-1} = \mathbf{m} \circ y_n = \mathbf{m}_{y_n} \text{ and}$$

if the sequence $(\eta_n)_1^\infty$ converges P-almost everywhere to 0, then the sequence $(y_n)_1^\infty$ converges $\mathbf{m}$-almost everywhere to 0.

In Theorem 4, we proved for IF-observables that almost everywhere convergence implies almost uniform convergence. Will it also apply to functions of several IF-observables? What is the relationship between the uniform convergence of functions of several IF-observables and uniform convergence of random variables in the Kolmogorov probability space? The following theorem discusses this.

Theorem 5. *Let $(x_i)_1^\infty$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be the joint IF-observable of $x_1, \ldots, x_n$ and $g_n : R^n \to R$ be a Borel measurable function. Let IF-observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ be given by $y_n = h_n \circ g_n^{-1}$ and random variable $\eta_n = g_n(t_1, \ldots, t_n) : R^N \to R$ be defined by $\eta_n = g_n \circ \pi_n$, where $\pi_n : R^N \to R^n$ is the n-th coordinate random vector defined by $\pi_n\big((t_i)_1^\infty\big) = (t_1, \ldots, t_n)$. Then, the following applies:*

(i) The sequence $(y_n)_1^\infty$ converges $\mathbf{m}$-almost uniformly to 0 if and only if the sequence $(\eta_n)_1^\infty$ converges P-almost uniformly to 0;

(ii) If the sequence $(\eta_n)_1^\infty$ converges P-almost everywhere to 0, then the sequence $(y_n)_1^\infty$ converges $\mathbf{m}$-almost uniformly to 0.

Proof. (i) "$\Leftarrow$" Let the sequence $(\eta_n)_1^\infty$ converge P-almost uniformly to 0 in the Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$. Then, by definition, for every $\alpha > 0$ there exists $A \in \sigma(\mathcal{C})$ such that $P(A) > 1 - \alpha$ and such that for every $\beta > 0$ there exists k such that $|\eta_n(t)| < \beta$ for every $n \geq k$ and every $t \in A$.

Since $A \in \sigma(\mathcal{C})$, then there exist $n \in N$ and $B \in \mathcal{B}(R^n)$ such that $A = \pi_n^{-1}(B)$. But $A \subset \eta_n^{-1}\big((-\beta, \beta)\big)$; therefore,

$$\pi_n^{-1}(B) \subset \{(t_i)_1^\infty \in R^N : (t_1, \ldots, t_n) \in g_n^{-1}\big((-\beta, \beta)\big)\};$$

i.e.,

$$\pi_n^{-1}(B) \subset \pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big).$$

Put $\mathbf{A} = h_n(B)$. Then,

$$\begin{aligned} \mathbf{m}(\mathbf{A}) &= \mathbf{m}\big(h_n(B)\big) \\ &= P\big(\pi_n^{-1}(B)\big) \\ &= P(A) \\ &> 1 - \alpha \end{aligned}$$

and

$$\begin{aligned} \pi_n^{-1}(B) &\subset \pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big), \\ h_n\Big(\pi_n^{-1}(B)\Big) &\leq h_n\Big(\pi_n^{-1}\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big)\Big), \\ h_n(B) &\leq h_n\Big(g_n^{-1}\big((-\beta, \beta)\big)\Big), \\ \mathbf{A} &\leq y_n\big((-\beta, \beta)\big). \end{aligned}$$

Hence, the sequence $(y_n)_1^\infty$ converges **m**-almost uniformly to 0. "$\Rightarrow$" is an analogy to proof "$\Leftarrow$".

(ii) Let $(\eta_n)_1^\infty$ converge P-almost everywhere to 0. Then, by Proposition 2 the sequence $(y_n)_1^\infty$ converges **m**-almost everywhere to 0. Using Theorem 4, the sequence $(y_n)_1^\infty$ converges **m**-almost uniformly to 0. $\square$

Remark 1. *The condition (ii) in Theorem 5 is a variation of Egorov's theorem for functions of several IF-observables.*

5. Conclusions

This paper concerns aspects of a probability theory for the intuitionistic fuzzy sets. We defined the **m**-almost uniform convergence for a sequence of IF-observables. We proved a variation of Egorov's theorem for an intuitionistic fuzzy case. We showed the connection between almost everywhere convergence and almost uniform convergence for IF-observables. We formulated the connection between the almost uniform convergence of functions of several IF-observables and almost uniform convergence of random variables in the Kolmogorov probability space too. The results are a generalization of the results in [14], because if f is a fuzzy set, then $(f, 1-f)$ is the corresponding IF-set, i.e., fuzzy sets are the special case of IF-sets. In further research, we shall study another type of convergence of IF-observables, like the convergence in a mean p, where $1 \leq p \leq \infty$.

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Abbreviation

The following abbreviation is used in this manuscript:

IF	Intuitionistic Fuzzy

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