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On the Bias of the Unbiased Expectation Theory

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Abstract: The unbiased expectation theory stipulates that long-term interest rates are determined by the market's expectations of future short-term interest rates. According to this hypothesis, if investors have unbiased expectations about future interest rate movements, the forward interest rates should be good predictors of future spot interest rates. This hypothesis of the term structure of interest rates has long been a subject of debate due to empirical and theoretical challenges. Despite extensive research, a satisfactory explanation for the observed systematic difference between future spot interest rates and forward interest rates has not yet been identified. In this study, we approach this issue from an arbitrage theory perspective, leveraging on the connection between the expectation hypothesis and changes in probability measures. We propose that the observed bias can be explained by two adjustments: a risk premia adjustment, previously considered in the literature, and a stochastic adjustment that has been overlooked until now resulting from two measure changes. We further demonstrate that for specific instances of the Vasicek and Cox, as well as the Ingersoll and Ross, stochastic interest rate models, quantifying these adjustments reveals that the stochastic adjustment plays a significant role in explaining the bias, and ignoring it may lead to an overestimation of the required risk premia/aversion adjustment. Our findings extend beyond the realm of financial economic theory to have tangible implications for interest rate modelling. The capacity to quantify and distinguish between risk and stochastic adjustments empowers modellers to make more informed decisions, leading to a more accurate understanding of interest rate dynamics over time.

Keywords: spot rates; forward rates; expectation hypothesis; Vasicek model; CIR model

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1. Introduction

The unbiased expectations theory (UET), also known as the interest rate expectation hypothesis, is commonly used in fixed-income markets, where forward rates are often used to estimate future interest rates and make investment decisions. The theory suggests that investors do not have a systematic bias in their expectations and that they do not prefer one type of investment horizon over another. In other words, long-term forward rates are expected to be, on average, the same as the compounded average of short-term interest rates over the same period. According to this hypothesis, investors form their expectations about future interest rates based on all available information, and the forward rates, which are the market's best estimate of future rates, are unbiased predictors of actual future spot rates. This theory has been developed and refined over time through contributions from various scholars in the field of finance and economics. Some early works that have been associated with its development took place in the 1930s [1,2] and later on the 1970s/1980s [3–6].

However, it is important to note that the relationship proposed between forward and spot rates is a theoretical concept and may not always hold true in practice. Various factors, such as market sentiment, investor sentiment, and changing economic conditions, can cause deviations from the hypothesis, leading to differences between expected and actual interest rate movements.

This hypothesis has been subject to empirical research and debate in the academic literature. Some studies have provided evidence in support of the theory, suggesting that forward rates do indeed provide unbiased predictions of future interest rates. Other studies, however, have found evidence of deviations from the theory, indicating that market participants may have systematic biases in their expectations, leading to discrepancies between forward rates and actual future rates.

For example, research by [7,8] found that forward rates are generally good predictors of future short-term interest rates, but they tend to overestimate long-term rates. Similarly, [9] proposed the "excess volatility puzzle," which suggests that long-term interest rates are more volatile than what the UET would predict, implying that market participants may have time-varying expectations.

More recent empirical studies have shown that there seems to be a systematic difference or *bias* between forward interest rates and expected future spot interest rates; see, for instance, [10–14] or [15]. This bias has, thus, been a topic of extensive research and has been found to persist despite various modifications and refinements to the expectation hypothesis.

The theory of the classical expectation hypothesis postulates that forward rates are unbiased predictors of future spot rates under the real-life probability measure, P. The empirical evidence, however, shows this does not hold in practice and proposes a risk premia (or risk aversion) explanation for the identified bias. In other words, the literature implicitly assumes the expectation hypothesis would work in a risk-neutral world, where the risk aversion effect does not influence the value of interest rates, or equivalently, when we take the expectation under the so-called risk neutral martingale measure Q. A "puzzle", however, arises when one realizes only abnormally high levels of risk aversion would be able to explain the observed bias. [16] analysed the expectation hypothesis using U.S. Treasury bills data. Based upon a representative agent with constant relative risk aversion (CRRA), they concluded that only CRRA coefficients greater than 8 would support risk aversion as an explanation for the bias. However, commonly observed CRRA coefficients are much lower than these; see, for instance, [17], who refer to values of between 1 and 2 as bounds for the CRRA coefficient.

Our research is motivated by observed discrepancies between forward and spot rates, indicating a gap in the literature. We aim to systematically investigate and quantify the factors contributing to this bias, challenging the traditional perspective embedded in the UET.

The existence of the expectation hypothesis bias has important implications for financial markets, as well as implications for pricing fixed-income securities, interest rate derivatives, and risk management strategies. Understanding the sources and implications of this bias is crucial for financial practitioners, policymakers, and researchers.

Here we approach the expectation hypothesis bias from a different perspective based upon arbitrage theory, probability measure changes, in the context of stochastic spot rate models. From standard arbitrage theory, we know forward rates are the expected future spot rates under the T-forward probability measures, so forward rates with different maturities T are martingales under a different measure. Moreover, we also know the risk-neutral measure Q will only coincide with the T-forward measures if we assume interest rates are deterministic. However, in reality we know they are not; they are stochastic. Thus, we take into account both risk aversion (the change of the measure from P to Q—risk aversion adjustment, $RA(t,T)$) and stochastic effects (the change of the measure from Q to the appropriate T-forward measure—stochastic adjustment, $SA(t,T)$) and show that considering the stochastic effect, something not considered in the previous literature, helps explain the puzzling results of [16].

Our research outcomes include theoretical and numerical results for two popular instantaneous spot rate models: the Vasicek [18] and Cox–Ingersoll–Ross (CIR) [19] (the CIR model henceforward) models. We present closed-form expressions for expected instantaneous spot rates under the different measures P, Q, and T, decomposing the bias into SA and RA and offering insights into the underlying mechanisms behind the expectation hypothesis bias. This approach aims to provide a deeper understanding of the nature, direction, and magnitude of the bias, as well as its implications for market participants. Above all, we seek to contribute to the development of more accurate pricing models and risk management strategies in fixed-income markets.

In summary, our research contributes novel insights by addressing gaps in the existing literature and providing a nuanced understanding of the expectation hypothesis bias. The remaining of this paper is organised as follows. Section 2 presents a literature review on the expectation hypothesis. Section 3 sets the notation and introduces preliminary concepts about interest rate theory and measure changes. Section 4 presents the theoretical and numerical results for the Vasicek and CIR models. Section 5 discusses the main results and implications, and Section 6 concludes and presents avenues for future research. All formal proofs are presented in the Appendix A.

2. Literature Review

Unbiased expectation theory (UET), which posits that long-term interest rates are determined by the market's expectations of future short-term interest rates, is a fundamental concept in the field of fixed-income securities. According to the expectation hypothesis, forward interest rates should be unbiased estimates of expected future spot interest rates. However, empirical studies have consistently documented the existence of a systematic difference or bias between forward interest rates and expected future spot interest rates, known as the expectation hypothesis bias. This literature review aims to provide a comprehensive overview of the existing literature on the expectation hypothesis bias, including its sources, implications, and various approaches that have been proposed to explain and quantify this phenomenon.

There are several potential sources of the expectation hypothesis bias in fixed-income markets that have been proposed in the literature. The commonly cited source is risk aversion, which suggests that investors may require a premium for bearing uncertainty about future interest rate movements. This risk aversion adjustment results in a bias in the forward interest rates relative to expected future spot interest rates. Several studies have explored the role of risk aversion in explaining the expectation hypothesis bias and have found that it can account for only part of the observed bias [8,16,20,21]. Another potential source of the expectation hypothesis bias is market segmentation, which suggests that different market participants may have different expectations about future interest rate movements, leading to differences between forward interest rates and expected future spot interest rates. Market segmentation can arise from differences in information, trading strategies, and market liquidity, among other factors. Some studies have found evidence of market segmentation in fixed-income markets and have suggested that it can contribute to the expectation hypothesis bias [22]. Additionally, time-varying risk premia have also been proposed as a source of the expectation hypothesis bias. Time-varying risk premia may result from changes in market conditions, macroeconomic factors, or investor sentiment and can affect the relative pricing of forward interest rates and expected future spot interest rates. Several studies have examined the role of time-varying risk premia in explaining the expectation hypothesis bias and have found evidence of their significance [15,23,24]. More recently, it has been proposed that examining investors' well-known behavioural biases when computing expectations could help explain what risk aversion seems to be unable to explain [25].

Empirical methods, such as regression analysis and time series econometrics, have also been employed to explain and quantify the expectation hypothesis bias. These methods involve analysing historical data on interest rates, macroeconomic variables, and other relevant factors to identify patterns and relationships that may explain the observed bias. Empirical methods provide valuable insights into the potential sources of the expectation hypothesis bias and their quantitative effects, but they are also subject to limitations such as data availability, model specification, and potential confounding factors.

Here we use arbitrage theory and probability measure changes to explain the phenomenon, showing that the literature has been missing the fact that interest rates themselves are stochastic, and that, in that case, we need to also consider the measure change from the risk neural measure Q to the forward measure T that depends on the maturity of each forward rate.

In the context of the UET, interest rates are, by definition, stochastic. In the above-mentioned literature, the most common choices to model interest rates are either (i) to consider dynamic versions of well-known deterministic models, such as the Nelson–Siegel [26], Svensson [27] and Bjork–Christensen [28] models, assuming their factors to be stochastic [29,30], or (ii) to adopt a pure stochastic interest rate model. Among the pure stochastic models, short-rate models are the most used; see, e.g., [31,32] and the references there in, as opposed to the alternative Heath, Jarrow, and Morton (HJM) [33] type of models or the so-called market models [34,35]. For an empirical overview on short-rate models, we refer to [36]. The common choice is, thus, to model the instantaneous spot rate as a stochastic process and derive, by no arbitrage, the interest rate term structure and any other variables of interest. It is also well-established that only short-rate models of the affine class allow for closed-form solutions for zero-coupon bond prices [37]. Well-known extensions that leave the affine class, and thus must reply on numerics, are the Black–Derman–Toy [38] , Black–Karasinski [39], Mercurio–Moraleda [40], and Cox CEV models [41], to mention just a few.

Here, we opted to rely on an affine short-rate class of models and to choose the most classical and popular—the Vasicek [18] and CIR [19] models. Although there have been extensions of both models to time-varying drifts [42] and multi-dimensional factors [30], the original versions allow for better parameter interpretations. Extensions of our results within the class of affine term structure models are, in principle, feasible.

3. Preliminary Notes

3.1. Concepts

Let $p(t,T)$ denote the price, at time t, of a non-defaultable zero-coupon bond (ZCB) that pays 1 at maturity T. For all possible maturities T, we have by definition, $p(T,T)=1$. In general, the price $p(t,T)$ is both time-, t, and maturity-, T, dependent.

For a fixed value of t, the price is a function of T, which provides prices for ZCB for all possible maturities. The graph of this function is called "the discount curve at t" or "term structure at time t". This graph is differentiable with respect to T. For a fixed maturity T, $p(t,T)$ is a stochastic process. This process gives the prices at different moments in time for the ZCB with a particular maturity T, and the trajectory is typically very irregular.

Once ZCBs are defined, we can think of several definitions of interest rates. Here, we consider only the instantaneous version of rates, which is defined as follows.

Definition 1. *The instantaneous forward rate, contracted at t with maturity T is defined as*

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} \tag{1}$$

The instantaneous spot rate or short rate at time t is defined as

$$r(t) = f(t,t)\,. \tag{2}$$

Stochastic interest rate models focus on the dynamics of the instantaneous forward rate defined in Equation (1) (forward rate models) or on the dynamics of the short rate in Equation (2) (short-rate models). Here, we focus on short-rate models, and in that context, the bank account is defined as

$$B(T) = \exp\left\{\int_0^T r(s)\,ds\right\} \qquad \text{i.e.,} \qquad \begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}, \tag{3}$$

and the short-rate dynamics are modelled directly under the risk neutral measure Q,

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t^Q \tag{4}$$

where μ and σ are adapted processes under Q.

The general no-arbitrage pricing formula of any T claim $\mathcal{X}$ is given as follows:

$$\Pi(t; \mathcal{X}) = E_{t,r}^Q\left[\exp\left\{-\int_t^T r(s)\, ds\right\}\mathcal{X}\right] = E_{t,r}^Q\left[\frac{\mathcal{X}}{B(t,T)}\right] \tag{5}$$

where $E_{t,r}^Q[\cdot]$ denotes the conditional expected value under the measure Q, and $B(t,T) = B(T) - B(t)$ denotes the discount factor between t and T.

So, in terms of pricing, we obtain that ZCB prices are given by

$$p(t,T) = E_t^Q\left[\exp\left(-\int_t^T r(s)\, ds\right)\right] \tag{6}$$

where $E_t^Q(\cdot)$ is the expectation under the risk-neutral measure Q, which can be solved by solving the so-called term structure equation

$$\frac{\partial F}{\partial t}(t,r) + \mu(t,r_t)\frac{\partial F}{\partial r}(t,r) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r) - rF(t,r) = 0 \tag{7}$$

$$F(T,T) = 1 \tag{8}$$

where $F = p(t,T)$, with μ and σ as in Equation (4).

3.2. On the Measures P, Q, and T

3.2.1. On the Measures P and Q and Utility Functions

When moving from the risk-neutral world Q into the real world P, we need to consider that investors are risk-averse. The common convention is to assume this leads to charging a risk premium λ over the risk-free rate. We also know that changing measures only affect the drift term in Equation (4). Therefore, the new drift under P becomes

$$\mu^*(t, r_t) = \mu(t, r_t) + \lambda \tag{9}$$

where λ is the risk premium. This risk premium, in turn, is usually divided by “the market price of risk ϵ” and “the units of risk”, as measured by the variance in the market,

$$\lambda = \epsilon\sigma^2(t, r_t)\,, \tag{10}$$

where ϵ is the coefficient of relative risk aversion and σ is as in Equation (4).

The notion of the “price of risk” comes from the expected utility theory of [43], where the concept of utility represents an individual’s subjective valuation of money, taking into account both their risk attitude and their valuation of money as defined in a parametric utility function. For instance, when valuing an asset, the agent’s risk preference, i.e., whether they are risk lovers or risk averse, impacts the price they are willing to pay for it. The utility function plays a critical role in agent representative theory, where a representative agent acts in a way that reflects the cumulative preferences and actions of all agents with the goal of maximising their expected utility. One of the key assumptions in the representative agent framework is that the market is complete. Additionally, it is assumed that individuals have homogeneous beliefs and time-additive, state-independent utility functions that are strictly concave, increasing, and differentiable.

In this paper, we look into a particular type of utility functions that guarantee constant relative risk aversion (CRRA),

$$U(x) = \begin{cases} \frac{x^{1-\epsilon}}{1-\epsilon} & \text{if } \epsilon \neq 1 \\ \ln x & \text{if } \epsilon = 1 \end{cases} \tag{11}$$

$$CRRA = -\frac{U''(x)x}{U'(x)} = \begin{cases} \epsilon & \text{if } \epsilon \neq 1 \\ 1 & \text{if } \epsilon = 1 \end{cases} \tag{12}$$

where ϵ is the coefficient of relative risk aversion and can be interpreted as our "price of risk". The constant ϵ is positive for risk-averse investors, zero for risk-neutral investors, and negative for risk lovers. [44] also applied CRRA preferences, noting that these preferences sustain the Black–Scholes model in equilibrium. In this direction, we refer to the works of Ait-Sahalia and Lo [45] and He and Leland [46].

Thus, under P, the short-rate dynamics become

$$dr_t = \mu^*(t, r_t)dt + \sigma(t, r_t)dW_t^P \tag{13}$$

where μ^* is as defined in Equation (9) and σ is as in Equation (4). Given these assumptions about investors' preferences, the term structure equation under Q can be re-written as

$$\frac{\partial F}{\partial t}(t,r) + \left\{\mu^*(t,r_t) - \epsilon\sigma^2(t,T)\right\}\frac{\partial F}{\partial r}(t,r) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r) - rF(t,r) = 0 \tag{14}$$

where ϵ is as in Equation (10) and μ^* and σ are as in Equation (13). Recall that from Equations (9) and (10) follow $\mu(t,T) = \mu^*(t,T) - \epsilon\sigma^2(t,T)$. If we recall the Girsanov theorem and take a look at the change of the instantaneous spot rate model from the Q measure to the P measure, we can identify the Girsanov kernel as $\varphi = \epsilon\sigma(t,T)$.

3.2.2. On the Measures Q and T and the Stochasticity of Interest Rates

The risk-neutral measure uses the bank account B as *numeraire*, but one can also choose to use any tradable asset as *numeraire*. If we choose to use T-ZCB, we obtain the so-called T-forward measures, one for each possible maturity.

Using Bayes theorem, we know

$$\Pi(t;\mathcal{X}) = p(t,T)E^Q\left[\frac{\mathcal{X}}{p(T,T)} \cdot L^T(T)\right] \tag{15}$$

where L^T is the Radon–Nikodym derivative $L^T(t) = \dfrac{dT}{dQ}$, on F_t and a Q martingale on $\mathcal{F}_t$.

Now, considering $p(\cdot,T)$ as the numeraire process and applying the technique proposed by Geman et al. [47], we obtain

$$\Pi(t;\mathcal{X}) = p(t,T)E^T_{t,r}\left[\frac{\mathcal{X}}{p(T,T)}\right] = p(t,T)E^T_{t,r}[\mathcal{X}] \tag{16}$$

where $E^T_{t,r}[\cdot]$ denotes a conditional expectation under the T-forward measure that is different for each maturity T.

The expectation hypothesis result follows.

Lemma 1. *Assume that for all $T > 0$, we have $r(T)/B(T)$ in $L^1(Q)$, where B is the bank account and r is the instantaneous spot rate. Then, for every T, the process $f(t,T)$ is a Q^T martingale for $0 \le t \le T$, and in particular, we have*

$$f(t,T) = E^T_t[r(T)]\,, \tag{17}$$

where T is the forward measure.

The result of the above lemma is fundamental as it tells us the T instantaneous forward rate is the expected instantaneous spot rate at time T under the T-forward measure. From there, we can derive the implicit biases following from the measure changes:

$$\begin{aligned} E_t^P[r(T)] =& f(t,T) + bias \\ =& f(t,T) + RA(t,r_t) + SA(t,r_t) \end{aligned}$$

where we have the following risk adjustment (RA) and stochastic adjustment (SA):

$$RA(t,r_t) = E_t^P[r(T)] - E_t^Q[r(T)] \tag{18}$$

$$SA(t,r_t) = E_t^Q[r(T)] - E_t^T[r(T)] \tag{19}$$

are responsible to the bias observed in the real life.

It follows that part of the bias has to do with risk aversion, as considered in the literature, but another part has to do with a stochastic adjustment related to the fact we live in a world with stochastic interest rates. As we will show, the size of both biases play an important role in explaining the expectation hypothesis puzzle.

3.3. ATS Interest Rate Models

In this paper, we focus on the class of affine term structure models (ATS), for which it is possible to obtain bond prices in closed form,

$$p(t,T) = e^{A(t,T) - B(t,T) r_t}, \tag{20}$$

where A and B are deterministic functions that depend only on specific model parameters.

ATS models are a special class of models that assumes linearity for μ and σ^2 in the Q dynamics of the spot rate in Equation (4). It turns out that, for ATS models, the SDE in Equation (14) can be written as a system of ODEs in A and B,

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t} + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) &= -1 \\ B(T,T) &= 0 \\ \frac{\partial A(t,T)}{\partial t} - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) &= 0 \\ A(T,T) &= 0\,, \end{aligned}$$

where μ and σ^2 in Equation (4) have the special form $\mu(t,r_t) = \alpha(t) + \beta(t)r_t$, and $\sigma^2(t,r_t) = \gamma(t) + \delta(t)r_t$ with α, β, γ. and δ is deterministic.

By the definition of the instantaneous forward rate in Equation (1), we also obtain

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r_t\,. \tag{21}$$

ATS have gained widespread popularity in the field of finance due to their versatility and analytical tractability. These models, rooted in the work of Jarrow and Protter [48], are essential tools for valuing fixed-income securities and interest rate derivatives. The appeal of affine models lies in their ability to capture the dynamics of interest rates through a relatively simple set of equations, making them accessible for both academics and practitioners. The Vasicek [18] and Cox–Ingersoll–Ross [19] (CIR) models are the most prominent examples of affine models. Their popularity stems from their capacity to generate closed-form solutions and their compatibility with empirical data, enabling more precise pricing and risk assessment in fixed-income markets. For the same reason, we are able to determine closed-form solutions for risk and stochastic adjustments of both models.

4. Results

In this paper, we study the expectation hypothesis bias for two popular particular instances of the ATS class of models.

4.1. Vasicek Model

The Vasicek instantaneous spot rate model is defined by

$$dr_t = k(\theta - r_t)dt + \sigma dW_t^Q, \tag{22}$$

where r_t is the instantaneous spot interest rate at time t, k is the speed of mean reversion, θ is the long-term or equilibrium level of the interest rate, σ is the volatility of interest rates, and W_t^Q is a Wiener process under the risk-neutral measure Q representing the random shock to interest rates at time t. It captures the stochastic nature of interest rate movements in the risk-neutral world.

The Vasicek model belong to the class of ATS models and thus has a closed-form solution for bond prices, making it analytically tractable. Its simplicity facilitates its ease of use and interpretation. Bond prices are given by (20) with

$$A(t,T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B^2(t,T) \tag{23}$$

$$B(t,T) = \frac{1}{k}[1 - e^{-k(T-t)}]\,. \tag{24}$$

From the above expressions, we know that only the parameter k representing how quickly interest rates revert to the mean θ affects $B(t,T)$, which is what multiplies r_t in the bond price expression. $A(t,T)$ also depends on θ, the level to which interest rates revert, and σ, which represents the random fluctuations in interest rates. This is a well-known result. For further details, we refer to [49].

From (23) and (24) and the forward rate result in (21), we have the following result.

Lemma 2. *Under the Vasicek model, with spot rate Q-dynamics as in Equation* (22)*, we have*

$$f(t,T)^{Vasicek} = \theta\left(1 - e^{-k(T-t)}\right) - \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)^2 + e^{-k(T-t)}r_t\,. \tag{25}$$

What is particularly striking about the above result is the fact that only the last term is stochastic (as it multiplies r_t). The first two terms are deterministic. The first term established is a deterministic mean-reverting term, where $\left(1 - e^{-k(T-t)}\right)$ is a factor that decreases exponentially with time, capturing the tendency of interest rates to revert to the long-term mean. The second deterministic term accounts for the stochastic or random component in the interest rate process. Its negative sign indicates that the volatility adjustment is subtracted from the mean-reverting term.

Here, we are primarily interested in finding closed-form expressions for the adjustments defined in Equations (18) and (19). Proposition 1 states our main Vasicek result.

Proposition 1. *Under the Vasicek model with spot rate Q-dynamics as in Equation* (22)*, and given the definitions of risk adjustment and stochastic adjustment in Equations (18) and (19), respectively, we find*

$$RA^{Vasicek}(t,T) = \frac{\epsilon\sigma^2}{k}\left(1 - e^{-k(T-t)}\right), \tag{26}$$

$$SA^{Vasicek}(t,T) = \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)^2. \tag{27}$$

Notably, both adjustments are purely deterministic, as they do not even depend on r_t. As it turns out, this is a feature that is specific to the Vasicek model.

The risk adjustment (RA) closed form tell us it depends on investors' aversion to risk (ϵ). In particular, if we were to live in a risk-neutral world ($\epsilon = 0$), there would be no RA. In addition, RA increases with both risk aversion and volatility and thus can be interpreted as the compensation required for holding a risky asset. The term $\frac{\epsilon\sigma^2}{k}$ can be seen as the risk premium per unit of time, and $(1 - e^{-k(T-t)})$ is a factor that adjusts this premium over the investment horizon.

The stochastic adjustment (SA) does not depend on investors' risk aversion (as we would expect), and it increases with the volatility σ and reduces with the speed of mean reversion k. It is a pure stochastic term because if interest rates were not stochastic, $\sigma = 0$, it would be zero. The term $\left(1 - e^{-k(T-t)}\right)^2$ adjusts the correction over the investment horizon, reflecting how the stochastic adjustment varies with time.

In Table 1 we quantify both adjustments for various levels of risk aversion ϵ and maturities T. The Vasicek model parameters, $\sigma = 0.01, \theta = 0.1$ and $k = 0.25$, are from [50]. The results allow us to get a sense of the values of both RA and SA and how much they are responsible for the expectation hypothesis bias. As it would be expected, for the case of risk loving investors ($\epsilon = -1$) RA is negative, for risk neutral ($\epsilon = 0$) RA is zero and for risk averse ($\epsilon > 0$), the higher the relative risk aversion coefficient ϵ the higher is the RA, and its weight on the overall bias. SA, on the other hand, does not depend on ϵ, consistently with the formulas in Equations (26) and (27), increasing with maturity T, for fixed σ and k. For $\epsilon = -1$, we observe the total adjustment starts by being negative (for $T = 1, 2$), RA dominates SA but as maturity increases ($T = 5, 10$) it becomes positive with SA dominating RA. Naturally for $\epsilon = 0$ there is no risk adjustment, as investors are risk neutral and the full bias can be attributed to SA. For any fixed $\epsilon > 0$, in absolute terms both adjustments tend to increase with maturity T, but in relative terms the RA tends to decrease with maturity and SA tends to increase. For the chosen parameter values the significance of the bias is relative low overall, when measured as percentage of $E^P[r(T)]$, it does not go beyond 2.62% of the level of interest rates.

In Table 2, we present the same statistics for the case when we allow for higher interest rate volatility—$\sigma = 0.05$ (instead of 0.01 in Table 1). This increase in the "stochasticity" of interest rates had the expected effect, resulting in considerably higher SA and, thus, higher bias. This time, the bias size was as high as 44.9% when measured as a percentage of $E^P[r(T)]$ and for $\epsilon = 5$. However, even for lower relative risk aversion coefficients between 1 and 2, we obtain relative sizes of bias that range from 25.27% to 31.38%. In terms of RA, we also observed increased values for risk-loving investors (higher negative values) and risk-averse investors (higher positive values). Notice that σ shows up in both formulas in Equations (26) and (27), i, but for RA, it always comes multiplied by ϵ, so zero RA for risk-neutral investors is guaranteed. The relative importance of SA over the full bias is similar to that shown in Table 1.

An interesting question we aim at answering is if the new stochastic adjustment can help explain the implicit high levels of relative risk aversion found in the literature. For that, we need to consider the full bias (RA + SA) and implicitly derive what would be the risk-aversion coefficient if one would think that only RA explains the full bias; i.e., we would like to solve

$$RA^*(t,T) = RA(t,T) + SA(t,T) \tag{28}$$

where on the l.h.s. RA^* is as in (26) but depends on an implicit ϵ^*, and on the r.h.s. RA and SA are as in (26) and (27). Corollary 1 tell us the answer in the case of the Vasicek model.

Table 1. Vasicek calculations using Equations (25)–(27), as well as Equation (A2), for various ϵ and T and fixed $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.01$, and $k = 0.25$. All values are given in percentual terms.

	$\epsilon = -1$				$\epsilon = 0$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.155	5.439	7.810	9.317	4.155	5.439	7.810	9.317
$E^P[r(T)]$	4.150	5.435	7.823	9.348	4.159	5.451	7.851	9.384
$SA(t,T)$	0.004	0.012	0.041	0.067	0.004	0.012	0.041	0.067
$RA(t,T)$	−0.009	−0.016	−0.029	−0.037	0.000	0.000	0.000	0.000
bias weight in $E^P[r(T)]$	−0.119	−0.062	0.156	0.328	0.094	0.227	0.519	0.718
SA weight in bias	−79.340	−369.348	334.197	219.642	100.000	100.000	100.000	100.000
RA weight in bias	179.340	469.348	−234.197	−119.642	0.000	0.000	0.000	0.000
	$\epsilon = 1$				$\epsilon = 2$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.155	5.439	7.810	9.317	4.155	5.439	7.810	9.317
$E^P[r(T)]$	4.168	5.467	7.880	9.421	4.177	5.482	7.908	9.458
$SA(t,T)$	0.004	0.012	0.041	0.067	0.004	0.012	0.041	0.067
$RA(t,T)$	0.009	0.016	0.029	0.037	0.018	0.031	0.057	0.073
Bias weight in $E^P[r(T)]$	0.306	0.514	0.879	1.105	0.517	0.800	1.237	1.489
SA weight in bias	30.671	44.038	58.797	64.737	18.113	28.237	41.640	47.860
RA weight in bias	69.329	55.962	41.203	35.263	81.887	71.763	58.360	52.140
	$\epsilon = 3$				$\epsilon = 5$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.155	5.439	7.810	9.317	4.155	5.439	5.439	9.317
$E^P[r(T)]$	4.186	5.498	7.937	9.495	4.203	5.530	5.530	9.568
$SA(t,T)$	0.004	0.012	0.041	0.067	0.004	0.012	0.012	0.067
$RA(t,T)$	0.027	0.047	0.086	0.110	0.044	0.079	0.079	0.184
Bias weight in $E^P[r(T)]$	0.728	1.084	1.592	1.870	1.146	1.647	1.647	2.623
SA weight in bias	12.851	20.780	32.234	37.963	8.129	13.599	13.599	26.856
RA weight in bias	87.149	79.220	67.766	62.037	91.871	86.401	86.401	73.144

Table 2. Vasicek calculations using Equations (25)–(27), as well as Equation (A2), for various ϵ and T and fixed $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, $k = 0.25$. All values in percentual terms.

	$\epsilon = -1$				$\epsilon = 0$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.061	5.141	6.833	7.699	4.061	5.141	6.833	7.699
$E^P[r(T)]$	3.938	5.058	7.138	8.466	4.159	5.451	7.851	9.384
$SA(t,T)$	0.098	0.310	1.018	1.685	0.098	0.310	1.018	1.685
$RA(t,T)$	−0.221	−0.393	−0.713	−0.918	0.000	0.000	0.000	0.000
Bias weight in $E^P[r(T)]$	−3.132	−1.658	4.268	9.062	2.353	5.680	12.968	17.957
SA weight in bias	−79.340	−369.348	334.197	219.642	100.000	100.000	100.000	100.000
RA weight in bias	179.340	469.348	−234.197	−119.642	0.000	0.000	0.000	0.000
	$\epsilon = 1$				$\epsilon = 2$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.061	5.141	6.833	7.699	4.061	5.141	6.833	7.699
$E^P[r(T)]$	4.380	5.844	8.565	10.302	4.601	6.238	9.278	11.220
$SA(t,T)$	0.098	0.310	1.018	1.685	0.098	0.310	1.018	1.685
$RA(t,T)$	0.221	0.393	0.713	0.918	0.442	0.787	1.427	1.836
Bias weight in $E^P[r(T)]$	7.284	12.030	20.218	25.267	11.741	17.579	26.354	31.381
SA weight in bias	30.671	44.038	58.797	64.737	18.113	28.237	41.640	47.860
RA weight in bias	69.329	55.962	41.203	35.263	81.887	71.763	58.360	52.140
	$\epsilon = 3$				$\epsilon = 5$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.061	5.141	6.833	7.699	4.061	5.141	6.833	7.699
$E^P[r(T)]$	4.823	6.631	9.992	12.138	5.265	7.418	11.419	13.974
$SA(t,T)$	0.098	0.310	1.018	1.685	0.098	0.310	1.018	1.685
$RA(t,T)$	0.664	1.180	2.140	2.754	1.106	1.967	3.567	4.590
Bias weight in $E^P[r(T)]$	15.789	22.469	31.613	36.570	22.865	30.694	40.159	44.903
SA weight in bias	12.851	20.780	32.234	37.963	8.129	13.599	22.203	26.856
RA weight in bias	87.149	79.220	67.766	62.037	91.871	86.401	77.797	73.144

Corollary 1. *For the Vasicek with model parameters k, ϵ and implicit risk aversion ϵ^* defined as in (28), we are given*

$$\epsilon^* = \epsilon + \frac{1}{2k}\left(1 - e^{-k(T-t)}\right) \tag{29}$$

From Equation (29), it is clear the implicit risk aversion ϵ^* is always bigger than the true risk aversion ϵ, and the difference between them depends only on k, the speed of mean reversion.

From Table 3, we see that not considering a stochastic adjustment (SA) would lead to an increase in the implicit risk aversion coefficient ϵ^* and the increase is bigger the larger the maturity we consider. In particular, for risk-loving investors ($\epsilon = -1$), we observe that the increase in the implicit risk aversion ϵ^* is such that for maturities higher than 2, they could be perceived as risk-averse instead of risk-lovers. Likewise, risk-neutral investors ($\epsilon = 0$) are perceived as risk-averse whenever one does not take into account SA. The implicit risk-aversion coefficient for the risk averse ($\epsilon > 0$) is higher for the lower mean reversion parameter k, and we can clearly observe that this can lead to perceive investors as much more risk-averse than they actually are.

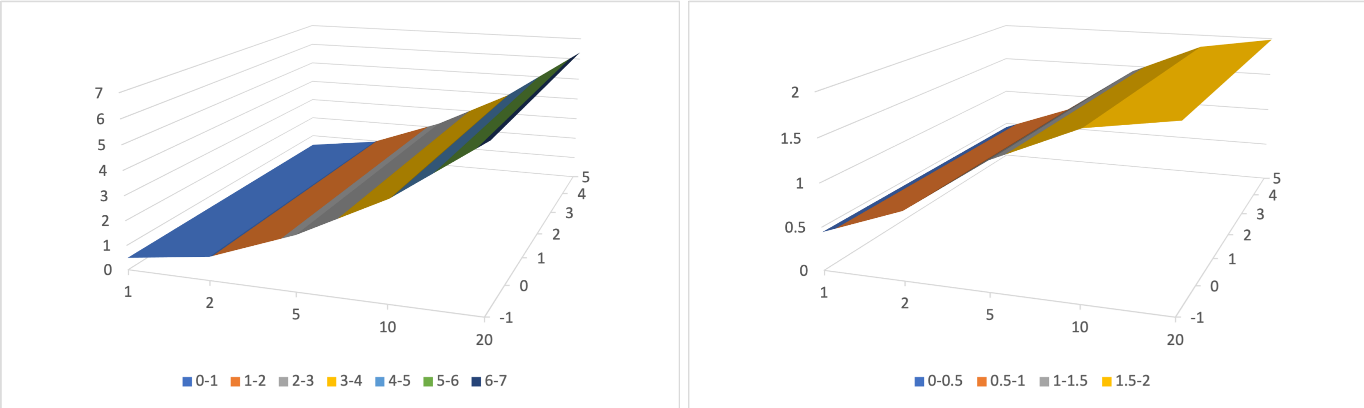
Table 3. Vasicek: implicit risk aversion ϵ^*, as in Equation (29), as a function of maturity T and risk aversion ϵ.

			$k = 0.05$						**$k = 0.1$**		
$T \mid \epsilon$	−1	0	1	2	5	$T \mid \epsilon$	−1	0	1	2	5
1	−0.512	0.488	1.488	2.488	5.488	1	−0.524	0.476	1.476	2.476	5.476
2	−0.048	0.952	1.952	2.952	5.952	2	−0.094	0.906	1.906	2.906	5.906
5	1.212	2.212	3.212	4.212	7.212	5	0.967	1.967	2.967	3.967	6.967
10	2.935	3.935	4.935	5.935	8.935	10	2.161	3.161	4.161	5.161	8.161
20	5.321	6.321	7.321	8.321	11.321	20	3.323	4.323	5.323	6.323	9.323
			$k = 0.25$						**$k = 0.5$**		
$T \mid \epsilon$	−1	0	1	2	5	$T \mid \epsilon$	−1	0	1	2	5
1	−0.558	0.442	1.442	2.442	5.442	1	−0.607	0.393	1.393	2.393	5.393
2	−0.213	0.787	1.787	2.787	5.787	2	−0.368	0.632	1.632	2.632	5.632
5	0.427	1.427	2.427	3.427	6.427	5	−0.082	0.918	1.918	2.918	5.918
10	0.836	1.836	2.836	3.836	6.836	10	−0.007	0.993	1.993	2.993	5.993
20	0.987	1.987	2.987	3.987	6.987	20	0.000	1.000	2.000	3.000	6.000

The difference between the implicit risk aversion and the true risk aversion, $\epsilon^* - \epsilon$, is, however, constant across the various levels of risk aversion as it only depends on k and T. Figure 1 presents this difference for $k = 0.05$ and $k = 0.25$.

In Figure 1b we see the increase in implicit risk aversion in not higher than 2, while empirical studies suggest it to be bigger. On the other hand, from Figure 1a, which takes the case of $k = 0.05$, we see that even with the simple Vasicek model, it is possible to have implicit risk aversion ϵ^* much higher than the true relative risk aversion coefficient ϵ and with values in line with the empirical literature.

The model we have studied up until now has its simplicity as an advantage, but it also has some disadvantages. The Vasicek model assumes a constant volatility, which may not capture the time-varying nature of interest rate volatility observed in the real world. Market volatility tends to change over time, and models that incorporate this feature may provide a better fit. The model also assumes normally distributed interest rate changes. In reality, interest rates often exhibit fat tails and other deviations from normality, particularly during periods of financial stress. These limitations may impact the accuracy of the model in extreme market conditions. Finally, the model's stationarity depends on the chosen parameter values. If not carefully calibrated, the model may fail to exhibit stationary behaviour, leading to unrealistic interest rate paths.



(**a**) $k = 0.05$ (**b**) $k = 0.25$

Figure 1. Increase in implicit risk aversion, $\epsilon^* - \epsilon$, for the Vasicek model.

The next model we look at is the CIR model of [19], which takes into consideration the limitations of the Vasicek model, improving the dynamics of the spot interest rate r_t to take them into account. This extra realism is a plus of the CIR model but comes at the cost of not so nice/harder to interpret formulas.

4.2. CIR Model

The [19] (CIR) instantaneous spot rate model is defined by

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^Q \tag{30}$$

where r_t is the instantaneous short-term interest rate at time t, k is the speed of mean reversion, θ is the long-term or equilibrium mean of the interest rate to which rates revert, σ is the volatility of interest rates, and W_t^Q is a Wiener process under the risk-neutral measure Q representing the random shock to interest rates at time t.

Similarly to the Vasicek model, the CIR model also implies that interest rates mean-revert to the level θ with a speed determined by k. But it incorporates a volatility term, $\sigma\sqrt{r_t}$, allowing it to capture the time-varying nature of volatility in interest rates. This provides more flexibility than the constant volatility assumption of the Vasicek model. The CIR model is commonly used in interest rate modelling due to its ability to capture the observed behaviour of interest rates, such as mean reversion and volatility clustering. It also ensures that interest rates remain non-negative and belong to the ATS class, allowing for closed-form solutions for bond prices and other related financial derivatives.

Bond prices are given by $p(t,T) = A_0(T-t)e^{-B(T-t)r_t}$, where

$$A_0(x) = \left[\frac{2\eta e^{\frac{x}{2}(k+\eta)}}{(\eta+k)(e^{x\eta}-1)+2\eta}\right]^{\frac{2k\theta}{\sigma^2}} \tag{31}$$

$$B(x) = \frac{2(e^{x\eta}-1)}{(\eta+k)(e^{x\eta}-1)+2\eta} \tag{32}$$

with the notation

$$\eta = \sqrt{k^2 + 2\sigma^2} \qquad x = T - t\,. \tag{33}$$

Note that CIR is affine as in Equation (20) with $A(t,T) = \ln A_0(T-t)$. Under the CIR model, we also obtain a closed-form expression for forward rates that depends only on model parameters and the spot rate r_t.

Lemma 3. *Under the CIR model, with spot rate Q-dynamics as in Equation* (30)*, the forward rate is given by*

$$f(t,T)^{CIR} = \frac{2k\theta(e^{\eta x}-1)}{(\eta+k)(e^{\eta x}-1)+2\eta} + \frac{4\eta^2 e^{\eta x}}{((\eta+k)(e^{\eta x}-1)+2\eta))^2} r_t \tag{34}$$

where θ and k are as defined in Equation (30) *and η and x are as defined in Equation* (33)*.*

The deterministic term grows with k, θ, and σ (through η), and the stochastic term depends only on k and σ (through η).

Proposition 2. *Under the CIR model, with spot rate Q-dynamics as in Equation* (30) *and given the definitions of risk adjustment and stochastic adjustment in Equations (18) and (19), respectively, we find*

$$RA^{CIR}(t,T) = \frac{k\theta}{k-\epsilon\sigma^2}\left(1-e^{-(k-\epsilon\sigma^2)x}\right) - \theta\left(1-e^{-kx}\right) + e^{-kx}\left(e^{\epsilon\sigma^2 x}-1\right) r_t\,, \tag{35}$$

$$SA^{CIR}(t,T) = \theta\left(1-e^{-kx}\right) + \frac{2k\theta(e^{\eta x}-1)}{(\eta+k)(e^{\eta x}-1)+2\eta} + \left[e^{-kx} - \frac{4\eta^2 e^{\eta x}}{((\eta+k)(e^{\eta x}-1)+2\eta)^2}\right] r_t\,. \tag{36}$$

The first thing to notice concerning both adjustments is that they are no longer deterministic (as was the case for Vasicek) as they depend on r_t. That is, both adjustments have a deterministic component and a stochastic component.

As would be expected, RA depends on the level of risk aversion, ϵ, and in particular, if we have risk neutrality, we obtain zero RA. Recall that the risk adjustment RA formula captures the difference between the expectations of the instantaneous short-term interest rate r_T under the objective measure P and the risk-neutral measure Q in the CIR model. It involves terms related to mean reversion, risk aversion, and volatility, providing a comprehensive adjustment over the investment horizon $x = T - t$.

The stochastic adjustment SA, on the other hand, does not depend on the risk aversion ϵ; instead, it depends on the volatility through η. Note that when we have $\sigma = 0$, $\eta = k$. Unfortunately, it is not possible to simplify the SA expression any further.

Tables 4 and 5 present the computations for the CIR model when we consider value parameters that are similar to those considered for the Vasicek model. The main difference is in the volatility term because now, the volatility is given by $\sigma\sqrt{r_t}$, so we consider $\sigma = 0.05$ (Table 4) and $\sigma = 0.25$ (Table 5), which, when multiplied by $r_t = 2.5\%$, give volatilities of about 0.79% and 3.95% with the same order of magnitude as the Vasicek volatilities of 1% and 5%.

Comparing both tables, the first conclusion is that, in the higher volatility scenario, the biases are considerably more relevant when measured as a percentage of $E[r_T]$. In Table 4, the values range from -0.107% to 5.340%, while in Table 5, the values range from -2.710% to 81.596%.

The negative values are associated only with the risk-loving investor case ($\epsilon = -1$) and for maturities lower than five years ($T \leq 5$). For risk lovers, just as in the case of the Vasicek model, SA is always positive, while RA is always negative. For lower maturities, RA dominates, while for longer maturities, SA dominates.

For risk-neutral and risk-averse investors, the bias is always positive, and its relevance increases with risk aversion ϵ and maturity T. The relative weight of SA over the full bias, on the other hand, is more relevant for $\sigma = 0.05$ than $\sigma = 0.25$. This happens because relative risk aversion in the model always shows up together with volatility, so the higher the volatility, the higher is RA.

Table 4. CIR calculations using Equations (34)–(36), as well as Equation (A4), for various ϵ and T parameters and fixed $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, and $k = 0.25$. All values in percentual terms.

	$\epsilon = -1$				$\epsilon = 0$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.156	5.439	7.796	9.258	4.156	5.439	7.796	9.258
$E^P[r(T)]$	4.151	5.434	7.807	9.308	4.159	5.451	7.851	9.384
$SA(t,T)$	0.003	0.012	0.055	0.127	0.003	0.012	0.055	0.127
$RA(t,T)$	−0.008	−0.017	−0.044	−0.076	0.000	0.000	0.000	0.000
Bias weight in $E^P[r(T)]$	−0.107	−0.092	0.139	0.548	0.073	0.212	0.703	1.352
SA weight in bias	−68.577	−230.338	506.989	248.881	100.000	100.000	100.000	100.000
RA weight in bias	168.577	330.338	−406.989	−148.881	0.000	0.000	0.000	0.000
	$\epsilon = 1$				$\epsilon = 2$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.156	5.439	7.796	9.258	4.156	5.439	7.796	9.258
$E^P[r(T)]$	4.167	5.468	7.896	9.461	4.174	5.484	7.941	9.539
$SA(t,T)$	0.003	0.012	0.055	0.127	0.003	0.012	0.055	0.127
$RA(t,T)$	0.008	0.017	0.045	0.077	0.015	0.033	0.090	0.155
Bias weight in $E^P[r(T)]$	0.254	0.516	1.265	2.154	0.434	0.819	1.825	2.954
SA weight in bias	28.872	40.985	55.262	62.251	16.856	25.736	38.081	45.022
RA weight in bias	71.128	59.015	44.738	37.749	83.144	74.264	61.919	54.978
	$\epsilon = 3$				$\epsilon = 5$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.156	5.439	7.796	9.258	4.156	5.439	7.796	9.258
$E^P[r(T)]$	4.182	5.501	7.986	9.618	4.197	5.535	8.078	9.780
$SA(t,T)$	0.003	0.012	0.055	0.127	0.003	0.012	0.055	0.127
$RA(t,T)$	0.023	0.050	0.135	0.234	0.038	0.084	0.227	0.395
Bias weight in $E^P[r(T)]$	0.614	1.121	2.384	3.751	0.973	1.724	3.496	5.340
SA weight in bias	11.895	18.737	28.991	35.157	7.478	12.110	19.542	24.290
RA weight in bias	88.105	81.263	71.009	64.843	92.522	87.890	80.458	75.710

Table 5. CIR calculations using Equations (34)–(36), as well as Equation (A4), for various ϵ and T parameters and fixed $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.25$, and $k = 0.25$. All values in percentual terms.

	$\epsilon = -1$				$\epsilon = 0$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.084	5.177	6.721	7.251	4.084	5.177	6.721	7.251
$E^P[r(T)]$	3.976	5.056	6.847	7.758	4.159	5.451	7.851	9.384
$SA(t,T)$	0.075	0.274	1.130	2.133	0.075	0.274	1.130	2.133
$RA(t,T)$	−0.183	−0.395	−1.004	−1.626	0.000	0.000	0.000	0.000
Bias weight in $E^P[r(T)]$	−2.710	−2.396	1.838	6.536	1.807	5.023	14.391	22.730
SA weight in bias	−69.746	−226.050	897.964	420.669	100.000	100.000	100.000	100.000
RA weight in bias	169.746	326.050	−797.964	−320.669	0.000	0.000	0.000	0.000
	$\epsilon = 1$				$\epsilon = 2$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.084	5.177	6.721	7.251	4.084	5.177	6.721	7.251
$E^P[r(T)]$	4.352	5.888	9.091	11.672	4.556	6.371	10.633	14.986
$SA(t,T)$	0.075	0.274	1.130	2.133	0.075	0.274	1.130	2.133
$RA(t,T)$	0.193	0.437	1.240	2.288	0.397	0.920	2.782	5.602
Bias weight in $E^P[r(T)]$	6.165	12.068	26.066	37.875	10.369	18.738	36.788	51.614
SA weight in bias	28.003	38.539	47.683	48.252	15.905	22.937	28.886	27.577
RA weight in bias	71.997	61.461	52.317	51.748	84.095	77.063	71.114	72.423
	$\epsilon = 3$				$\epsilon = 5$			
T	1	2	5	10	1	2	5	10
$f(t,T)$	4.084	5.177	6.721	7.251	4.084	5.177	6.721	7.251
$E^P[r(T)]$	4.772	6.906	12.564	19.928	5.241	8.159	18.091	39.400
$SA(t,T)$	0.075	0.274	1.130	2.133	0.075	0.274	1.130	2.133
$RA(t,T)$	0.613	1.455	4.713	10.543	1.082	2.708	10.239	30.016
Bias weight in $E^P[r(T)]$	14.421	25.037	46.505	63.612	22.079	36.545	62.846	81.596
SA weight in bias	10.919	15.836	19.337	16.827	6.494	9.184	9.938	6.635
RA weight in bias	89.081	84.164	80.663	83.173	93.506	90.816	90.062	93.365

For the exercise of determining the implicit risk aversion, if we would consider the bias to be explained only by risk aversion, as before, we have

$$RA^*(t,T) = RA(t,T) + SA(t,T)$$

but given the formulas in Equations (35) and (36), we need to find the implicit risk aversion ϵ^* numerically.

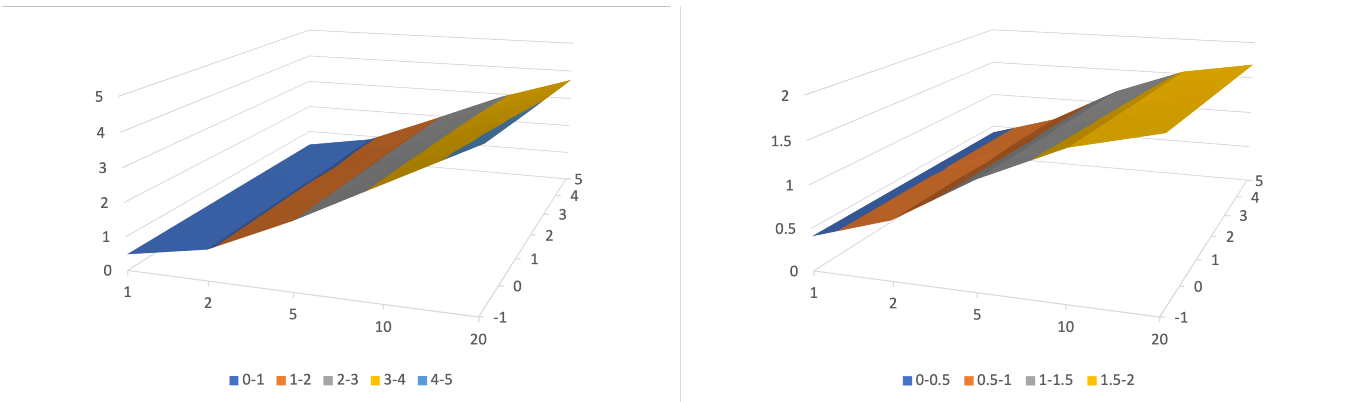
Table 6 presents the implicit risk aversion for combinations of $k = 0.05, 0.25$ and $\sigma = 0.05, 0.25$. For a low speed of mean reversion $k = 0.05$ and low volatility $\sigma = 0.065$, it seems the difference between the implicit and true risk aversion coefficient, $\epsilon^* - \epsilon$, is almost constant across risk aversion coefficients and increasing with maturity. If we however consider $k = 0.05$ but $\sigma = 0.25$, we note that the implicit risk aversion is no longer increasing with maturity, having a humped-shape with higher values in between the maturities $T = 2$ and $T = 5$ depending on the true risk aversion.

Table 6. CIR: implicit risk aversion ϵ^* as a function of maturity T and risk aversion ϵ for $\theta = 0.1$ and $r_t = 2.5\%$

		$k = 0.05$ $\sigma = 0.05$						$k = 0.05$ $\sigma = 0.25$			
$T \mid \epsilon$	−1	0	1	2	5	$T \mid \epsilon$	−1	0	1	2	5
1	−0.523	0.476	1.475	2.474	5.470	1	−0.511	0.461	1.434	2.409	5.341
2	−0.088	0.908	1.903	2.899	5.886	2	−0.097	0.807	1.721	2.643	5.455
5	0.999	1.979	2.958	3.936	6.873	5	0.501	1.193	1.937	2.728	5.326
10	2.257	3.200	4.134	5.079	7.895	10	0.532	1.056	1.693	2.435	5.088
20	3.506	4.361	5.216	6.077	8.682	20	0.342	0.748	1.350	2.139	5.005
		$k = 0.25$ $\sigma = 0.05$						$k = 0.25$ $\sigma = 0.25$			
$T \mid \epsilon$	−1	0	1	2	5	$T \mid \epsilon$	−1	0	1	2	5
1	−0.593	0.406	1.405	2.404	5.402	1	−0.582	0.396	1.375	2.355	5.301
2	−0.302	0.695	1.692	2.669	5.681	2	−0.296	0.640	1.580	2.526	5.387
5	0.244	1.234	2.217	3.214	6.184	5	0.112	0.920	1.753	2.612	5.313
10	0.664	1.642	2.620	3.598	6.534	10	0.254	0.943	1.686	2.483	5.139
20	0.890	1.856	2.823	3.789	6.691	20	0.260	0.871	1.555	2.319	5.030

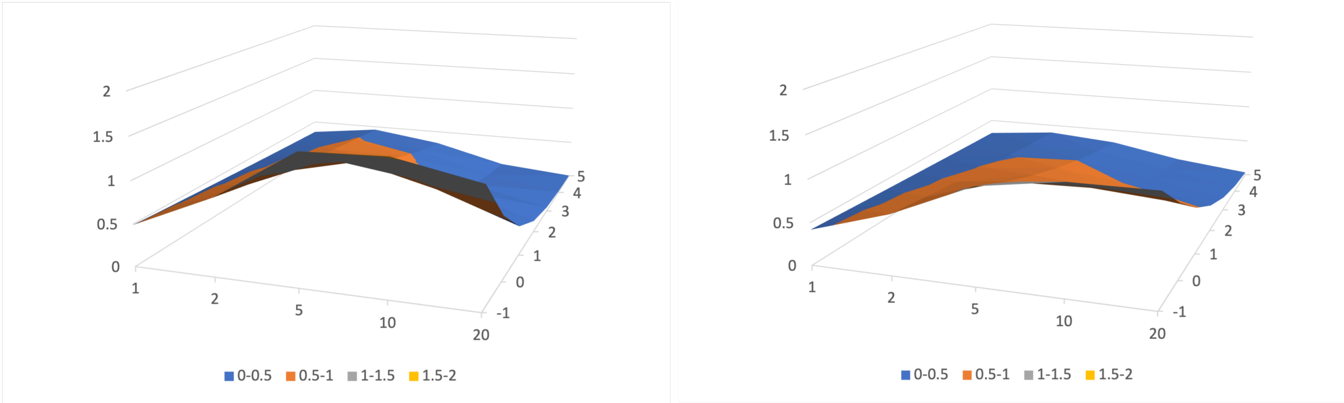
Even when considering a higher speed of mean reversion $k = 0.25$, the same humped shape result happens in the case of $\sigma = 0.25$. A surprising result is that the difference does not increase with maturity; instead, it increases up to $T = 5$ and it then decreases.

Figures 2 and 3 allow for a visualisation of the difference $\epsilon^* - \epsilon$. As in the Vasicek model, the lower the speed of mean reversion, the higher the difference level tends to be.



(**a**) $k = 0.05$ (**b**) $k = 0.25$

Figure 2. Increase in implicit risk aversion, $\epsilon^* - \epsilon$, for CIR with $\sigma = 0.05$, $\theta = 0.1$, and $r_t = 2.5\%$.



(**a**) $k = 0.05$ (**b**) $k = 0.25$

Figure 3. Increase in implicit risk aversion, $\epsilon^* - \epsilon$, for CIR with $\sigma = 0.25$, $\theta = 0.1$, and $r_t = 2.5\%$.

For $\sigma = 0.05$, the differences increase with maturities, but it is rather stable across risk aversion coefficients. For lower mean reversion, one can explain an increase in implicit risk aversion as high as 4 from long maturities, while for high mean reversion, the distortion does not go beyond 2.

For $\sigma = 0.25$, we always obtain humped-shape differences with maximal points between $T = 2$ and $T = 5$ and risk-aversion coefficients close to $\epsilon = 2$. The curve is at a slightly higher level for $k = 0.05$ than for $k = 0.25$, but the difference between implicit risk aversion and true risk aversion does not go beyond 1.

5. Discussion

In the framework of the classical Vasicek and CIR interest rate models, we have successfully computed and evaluated the UET bias, revealing two distinct components stemming from separate measure changes. While the previous literature acknowledged the impact of risk aversion on the UET, our emphasis lies in recognising that, in a world characterised by stochastic interest rates—the only context in which discussing expectations of future short rates makes sense—forward rates indeed represent the expectation of future short rates under T-forward measures. It is imperative to consider both adjustments comprehensively to avoid implicitly assuming excessive risk aversion coefficients.

Our findings extend beyond the realm of financial economic theory to have tangible implications for interest rate modelling. The capacity to quantify and distinguish between risk and stochastic adjustments empowers modellers to make more informed decisions, leading to a more accurate understanding of interest rate dynamics over time. This nuanced approach not only enhances the theoretical foundations of financial economic theory but also directly contributes to the practical aspects of interest rate modelling. It equips practitioners with a refined toolkit for capturing the intricacies of risk and stochasticity, thereby improving the accuracy of predictions and supporting more effective decision-making in the dynamic landscape of financial markets.

6. Conclusions

In summary, our research offers novel theoretical insights into the observed limitations of forward rates as reliable predictors of future spot rates, thereby challenging the unbiased expectation hypothesis.

For the well-established Vasicek and Cox–Ingersoll–Ross (CIR) models, we have successfully derived and quantified what we term the "risk adjustment" (RA) and "stochastic adjustment" (SA). The joint consideration of these components allows us to precisely define the bias evident in the empirical literature, employing both no arbitrage principles and measure changes.

We acknowledge the limitations inherent in relying solely on these models. This could potentially lead to an oversimplification of real-life adjustments. On the other hand, the use of simple, very well-known models allow for closed-form results that are also interpretable.

Looking ahead, our results pave the way for future research avenues within the affine class or short-rate models. Furthermore, beyond this class, one can numerically determine the necessary expected values and compute their differences, enabling the measurement of adjustments in other model types, including those of the Heath–Jarrow–Morton (HJM) type. Lastly, there is potential to explore the direct modelling of SA and RA, i.e., directly capturing measure changes, and to evaluate their implications in terms of interest rate dynamics. This avenue could provide a more nuanced understanding of the underlying mechanisms driving adjustments in different interest rate models, offering valuable insights for future advancements in the field.

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Appendix A

Proof of Lemma 1. Setting $\mathcal{X} = r(T)$ in (15) and (16), we can work out the expected value of $r(T)$ under the T-forward measure,

$$\begin{aligned} E^T_{t,r}[r(T)] &= \frac{1}{p(t,T)} E^Q_{t,r}\left[r(T)\cdot e^{-\int_t^T r(s)\,ds}\right] = -\frac{1}{p(t,T)} E^Q_{t,r}\left[\frac{\partial}{\partial T}\cdot e^{-\int_t^T r(s)\,ds}\right] \\ &= -\frac{1}{p(t,T)}\frac{\partial}{\partial T} E^Q_{t,r}\left[\cdot e^{-\int_t^T r(s)\,ds}\right] = -\frac{p_T(t,T)}{p(t,T)} = -\frac{\partial \ln p(t,T)}{\partial T} = f(t,T) \end{aligned}$$

where $p_T(t,T)$ is the $p(t,T)$ derivative on the order of maturity T. □

Proof of Lemma 2. For the A and B Equations (23) and (24), we obtain

$$\begin{aligned} \frac{\partial B(t,T)}{\partial T} &= e^{-k(T-t)} \\ -\frac{\partial A(t,T)}{\partial T} &= -\left(\theta - \frac{\sigma^2}{2k^2}\right)\left[\frac{\partial B(t,T)}{\partial T} - 1\right] + \frac{\sigma^2}{2k} B(t,T)\frac{\partial B(t,T)}{\partial T} \\ &= \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(1 - e^{-k(T-t)}\right) + \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)e^{-k(T-t)} \\ &= \theta\left(1 - e^{-k(T-t)} - 1\right) + \frac{\sigma^2}{2k^2}\left(e^{-k(T-t)} - 1\right)\left(1 - e^{-k(T-t}\right) \\ &= \theta\left(1 - e^{-k(T-t)}\right) - \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)^2. \end{aligned}$$

The results follow from Equation (21). □

Proof of Proposition 1. We start by determining the expected value of the instantaneous spot rate under the Q measure. Taking the dynamics in (22) and using as an integrating factor $u(s) = e^{ks}$, we obtain

$$E_t^Q[r(T)] = e^{-kT}e^{kt}r(t) + e^{-kT}\int_t^T e^{ks}k\theta ds = r(t)e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right). \tag{A1}$$

Proceeding similarly under the P measure, we get

$$E_t^P[r(T)] = e^{-kT}e^{kt}r(t) + e^{-kT}\int_t^T e^{ks}\left(k\theta + \epsilon\sigma^2\right)ds = r(t)e^{-k(T-t)} + \frac{k\theta + \epsilon\sigma^2}{k}\left(1 - e^{-k(T-t)}\right) \tag{A2}$$

From the definition of risk adjustment, $RA(t, r_t) = E_t^P[r(T)] - E_t^Q[r(T)]$, and the result follows. For the stochastic adjustment, we use $SA(t, r_t) = E_t^Q[r(T)] - E_t^T[r(T)]$ and the fact $E_t^T(r(T)) = f(t, T)$ with $f(t, T)$ from (25). $\square$

Proof of Corollary 1. It follows from definitions of RA and SA in Equations (26) and (27) and solving the equation below w.r.t. ϵ^* that

$$\frac{\epsilon^*\sigma^2}{k}\left(1 - e^{-k(T-t)}\right) = \frac{\epsilon\sigma^2}{k}\left(1 - e^{-k(T-t)}\right) + \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)^2. \quad \square$$

Proof of Lemma 3. Using $\eta = \sqrt{k^2 + 2\sigma^2}$ and $x = T - t$ and defining $\tau_1(x) = 2\eta e^{\frac{x}{2}(k+\eta)}$ and $\tau_2(x) = (n + k)(e^{\eta x} - 1) + 2\eta$, we can rewrite A_0 and B in Equations (31) and (32) as

$$A(x) = \ln A_0(X) \quad \text{where} \quad A_0(x) = \left(\frac{\tau_1(x)}{\tau_2(x)}\right)^{\frac{2k\theta}{\sigma^2}}, \quad \text{and} \quad B(x) = \frac{2(e^{\eta x} - 1)}{\tau_2(x)}.$$

Note that $\frac{\partial A}{\partial T} = \frac{\partial A}{\partial x}$ and $\frac{\partial B}{\partial T} = \frac{\partial B}{\partial x}$; we then have

$$\begin{aligned}
\frac{\partial B(x)}{\partial x} &= \frac{2\eta e^{\eta x}\tau_2(x) - \eta(\eta + k)e^{\eta x}2(e^{\eta x} - 1)}{\tau_2(x)^2} \\
&= \frac{2\eta e^{\eta x}[(\eta + k)(e^{\eta x} - 1) + 2\eta] - 2\eta(\eta + k)e^{\eta x}(e^{\eta x} - 1)}{\tau_2(x)^2} \\
&= \frac{2\eta[(\eta + k)e^{\eta x}(e^{\eta x} - 1) + 2\eta e^{\eta x}] - 2\eta[(\eta + k)e^{\eta x}(e^{\eta x} - 1)]}{\tau_2(x)^2} \\
&= \frac{4\eta^2 e^{\eta x}}{\tau_2(x)^2}
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial A(x)}{\partial x} = -\frac{\frac{\partial A_0(x)}{\partial x}}{A_0(x)} &= \frac{\frac{2k\theta}{\sigma^2}\left(\frac{\tau_1(x)}{\tau_2(x)}\right)^{\frac{2k\theta}{\sigma^2}-1}\left[\frac{\frac{\partial\tau_1(s)}{\partial x}\tau_2(x) - \frac{\partial\tau_2(x)}{\partial x}\tau_1(x)}{\tau_2(x)^2}\right]}{\left(\frac{\tau_1(x)}{\tau_2(x)}\right)^{\frac{2k\theta}{\sigma^2}}} \\
&= -\frac{2k\theta}{\sigma^2}\left(\frac{\tau_2(x)}{\tau_1(x)}\right)\left[\frac{\eta(\eta + k)e^{\frac{x}{2}(k+n)}\tau_2(x) - \eta(\eta + k)e^{\eta x}\tau_1(x)}{\tau_2(x)^2}\right] \\
&= -\frac{2k\theta}{\sigma^2}\left(\frac{\tau_2(x)}{\tau_1(x)}\right)\left[\frac{\frac{1}{2}(\eta + k)\tau_1(x)\tau_2(x) - \eta(\eta + k)e^{\eta x}\tau_1(x)}{\tau_2(x)^2}\right] \\
&= -\frac{2k\theta}{\sigma^2}\frac{\frac{1}{2}(\eta + k)((\eta + k)(e^{\eta x} - 1) + 2\eta) - \eta(\eta + k)e^{\eta x}}{\tau_2(x)} \\
&= -\frac{2k\theta}{\sigma^2}\frac{1}{2}\frac{(k^2 - \eta^2)(e^{\eta x} - 1)}{\tau_2(x)} = \frac{2k\theta}{\sigma^2}\frac{1}{2}\frac{2\sigma^2(e^{\eta x} - 1)}{\tau_2(x)} = \frac{2k\theta(e^{\eta x} - 1)}{\tau_2(x)}
\end{aligned}$$

The result follows from Equation (21). □

Proof of Proposition 2.

$$E_t^Q[r(T)] = e^{-kT}e^{kt}r(t) + e^{-kT}\int_t^T e^{ks}k\theta ds = r_t e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right) \tag{A3}$$

$$\begin{aligned} E_t^P[r(T)] &= e^{-(k-\epsilon\sigma^2)T}e^{(k-\epsilon\sigma^2)t}r(t) + e^{-(k-\epsilon\sigma^2)T}\int_t^T k\theta e^{(k-\epsilon\sigma^2)s}ds \\ &= r(t)e^{-(k-\epsilon\sigma^2)(T-t)} + \frac{k\theta}{(k-\epsilon\sigma^2)}\left(1 - e^{-(k-\epsilon\sigma^2)(T-t)}\right) \end{aligned} \tag{A4}$$

$$\begin{aligned} RA(t, r_t) &= E_t^P[r(T)] - E_t^Q[r(T)] \\ &= \left[r_t e^{-(k-\epsilon\sigma^2)x} + \frac{k\theta}{k-\epsilon\sigma^2}\left(1 - e^{-(k-\epsilon\sigma^2)x}\right)\right] - \left[r_t e^{-kx} + \theta\left(1 - e^{-kx}\right)\right] \\ &= -r_t e^{-kx} + r_t e^{-kx}e^{\epsilon\sigma^2 x} - \theta\left(1 - e^{-kx}\right) + \frac{k\theta}{k-\epsilon\sigma^2}\left(1 - e^{-(k-\epsilon\sigma^2)x}\right) \end{aligned}$$

$$SA(t, r_t) = E_t^Q[r(T)] - E_t^T[r(T)] = \left(r_t e^{-kx} + \theta\left(1 - e^{-kx}\right)\right) - \left(\frac{2k\theta(e^{\eta x} - 1)}{\tau_2(x)} + \frac{4\eta^2 e^{\eta x}}{\tau_2(x)^2}r_t\right) \qquad \square$$

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