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Delta Calculus on Time Scale Formulas That Are Similar to Hilbert-Type Inequalities

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Abstract: In this article, we establish some new generalized inequalities of the Hilbert-type on time scales' delta calculus, which can be considered similar to formulas for inequalities of Hilbert type. The major innovation point is to establish some dynamic inequalities of the Hilbert-type on time scales' delta calculus for delta differentiable functions of one variable and two variables. In this paper, we use the condition $a_j(s_j) = 0$ and $a_j(s_j, z_j) = a_j(w_j, n_j) = 0$, $\forall j = 1, 2, \ldots, n$. These inequalities will be proved by applying Hölder's inequality, the chain rule on time scales, and the mean inequality. As special cases of our results (when $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$), we obtain the discrete and continuous inequalities. Also, we can obtain other inequalities in different time scales, like $\mathbb{T} = q^{\bar{\mathbb{Z}}}$, $q > 1$.

Keywords: Hilbert-type inequalities; Hölder's inequality; mean inequality; kernels; delta integrals; time scales

MSC: 26D10; 26D15; 34N05; 47B38; 39A12

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1. Introduction

During the early 1900s, Hilbert made the discovery of this inequality (refer to [1])

$$\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\frac{a_s c_n}{s+n} \leq \pi\left(\sum_{s=1}^{\infty}a_s^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}c_n^2\right)^{\frac{1}{2}}. \tag{1}$$

Here, $\{a_s\}_{s=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are real sequences satisfying $0 < \sum_{s=1}^{\infty} a_s^2 < \infty$ and $0 < \sum_{n=1}^{\infty} c_n^2 < \infty$. This particular expression is known as Hilbert's double series inequality.

In [2], Schur demonstrated that π in (1) is the most optimal constant achievable. Additionally, he unveiled the integral counterpart of (1), which later became recognized as the Hilbert integral inequality, taking the form

$$\int_0^{\infty}\int_0^{\infty}\frac{f(\eta)g(\tau)}{\eta+\tau}d\eta d\tau \leq \pi\left(\int_0^{\infty}f^2(\eta)d\eta\right)^{\frac{1}{2}}\left(\int_0^{\infty}g^2(\tau)d\tau\right)^{\frac{1}{2}}, \tag{2}$$

where f, g are measurable functions satisfying $0 < \int_0^{\infty} f^2(\eta)d\eta < \infty$ and $0 < \int_0^{\infty} g^2(\tau)d\tau < \infty$.

In [3], an extension of (1) is presented as follows: suppose $l, r > 1$ with $1/l + 1/r = 1$, $\{a_s\}_{s=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are real sequences satisfying $0 < \sum_{s=1}^{\infty} a_s^r < \infty$ and $0 < \sum_{n=1}^{\infty} c_n^l < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\frac{a_s c_n}{s+n}\le\frac{\pi}{\sin\frac{\pi}{r}}\left(\sum_{s=1}^{\infty}a_s^r\right)^{\frac{1}{r}}\left(\sum_{n=1}^{\infty}c_n^l\right)^{\frac{1}{l}}. \tag{3}$$

Here, $\pi/\sin(\pi/r)$ is the optimal constant.

In [4], the authors derived the integral counterpart of (3) as

$$\int_0^{\infty}\int_0^{\infty}\frac{f(\eta)g(\tau)}{\eta+\tau}d\eta d\tau\le\frac{\pi}{\sin\frac{\pi}{r}}\left(\int_0^{\infty}f^r(\eta)d\eta\right)^{\frac{1}{r}}\left(\int_0^{\infty}g^l(\tau)d\tau\right)^{\frac{1}{l}}. \tag{4}$$

Here, $f,g\ge 0$ are measurable functions satisfying $0<\int_0^{\infty}f^r(\eta)d\eta<\infty$ and $0<\int_0^{\infty}g^l(\tau)d\tau<\infty$.

In [5], new inequalities akin to the ones presented in (3) and (4) were established as follows: let $l,r>1$ with $1/l+1/r=1$. Consider sequences $a_w:\{0,1,2,\dots,s\}\subset\mathbb{N}\to\mathbb{R}$ and $c_\vartheta:\{0,1,2,\dots,n\}\subset\mathbb{N}\to\mathbb{R}$ where $a(0)=c(0)=0$. Then

$$\begin{aligned}\sum_{w=1}^{s}\sum_{\vartheta=1}^{n}\frac{|a_w||c_\vartheta|}{lw^{r-1}+r\vartheta^{l-1}} \quad&\le\quad D(l,r,s,n)\left(\sum_{w=1}^{s}(s-w+1)|\nabla a_w|^r\right)^{\frac{1}{r}}\\&\quad\times\left(\sum_{\vartheta=1}^{n}(n-\vartheta+1)|\nabla c_\vartheta|^l\right)^{\frac{1}{l}}.\end{aligned} \tag{5}$$

Here, $\nabla a_w=a_w-a_{w-1}$, $\nabla c_\vartheta=c_\vartheta-c_{\vartheta-1}$ and

$$D(l,r,s,n)=\frac{1}{lr}s^{\frac{r-1}{r}}n^{\frac{l-1}{l}}.$$

Moreover, if $l,r>1$ with $1/l+1/r=1$, $f(w)$ and $g(\vartheta)$ are real-valued continuous functions with $f(0)=g(0)=0$, then

$$\begin{aligned}\int_0^{\eta}\int_0^{\tau}\frac{|f(w)||g(\vartheta)|}{lw^{r-1}+r\vartheta^{l-1}}dwd\vartheta \quad&\le\quad M(l,r,\eta,\tau)\left(\int_0^{\eta}(\eta-w)\left|f'(w)\right|^r dw\right)^{\frac{1}{r}}\\&\quad\times\left(\int_0^{\tau}(\tau-\vartheta)\left|g'(\vartheta)\right|^l d\vartheta\right)^{\frac{1}{l}}.\end{aligned} \tag{6}$$

Here,

$$M(l,r,\eta,\tau)=\frac{1}{lr}\eta^{\frac{r-1}{r}}\tau^{\frac{l-1}{l}}.$$

In [6], Chang-Jian et al. proved some new inequalities of Hilbert type in the difference calculus with "n-dimension" and derived their integral analogues. These inequalities are outlined as follows: let $r_j>1$ such that $1/l_j+1/r_j=1$ and $a_j(w_j)$ are real sequences defined for $w_j=0,1,2,\dots,s_j$, where $s_j\in\mathbb{N}$ and $a_j(0)=0$; $j=1,2,\dots,n$. Define the operator ∇ as $\nabla a_j(w_j)=a_j(w_j)-a_j(w_j-1)$. Then

$$\sum_{w_1=1}^{s_1}\sum_{w_2=1}^{s_2}\cdots\sum_{w_n=1}^{s_n}\frac{\prod_{j=1}^{n}|a_j(w_j)|}{\left(\sum\limits_{j=1}^{n}\frac{w_j}{l_j}\right)^{\sum\limits_{j=1}^{n}\frac{1}{l_j}}}\le K\prod_{j=1}^{n}\left(\sum_{w_j=1}^{s_j}\left(s_j-w_j+1\right)\left|\Delta a_j(w_j)\right|^{r_j}\right)^{\frac{1}{r_j}}. \tag{7}$$

Here,

$$K=\left(n-\sum_{j=1}^{n}\frac{1}{r_j}\right)^{\sum_{j=1}^{n}\frac{1}{r_j}-n}\prod_{j=1}^{n}s_j^{\frac{1}{l_j}}.$$

Also, they proved that if $h_j \geq 1$, $l_j, r_j > 1$ are constants with $1/r_j + 1/l_j = 1$, $f_j(w_j)$ are real valued differentiable functions defined on $[0, \eta_j)$, where $\eta_j \in (0, \infty)$ and $f_j(0) = 0$; $j = 1, 2, \ldots, n$, then

$$\int_0^{\eta_1} \cdots \int_0^{\eta_n} \frac{\prod_{j=1}^n \left| f_j^{h_j}(w_j) \right|}{\left(\sum_{j=1}^n \frac{w_j}{l_j} \right)^{\sum_{j=1}^n \frac{1}{l_j}}} dw_n \ldots dw_1 \leq L \prod_{j=1}^n \left(\int_0^{\eta_j} (\eta_j - w_j) \left| f_j^{h_j - 1}(w_j) . f_j'(w_j) \right|^{r_j} dw_j \right)^{\frac{1}{r_j}}, \tag{8}$$

where

$$L = \left(n - \sum_{j=1}^n \frac{1}{r_j} \right)^{\sum_{j=1}^n \frac{1}{r_j} - n} \prod_{j=1}^n h_j \eta_j^{\frac{1}{l_j}}.$$

Furthermore, they established that if $l_j, r_j > 1$ such that $1/r_j + 1/l_j = 1$, $a_j(w_j, z_j)$ are real sequences defined for (w_j, z_j) where $w_j = 0, 1, 2, \ldots, s_j$, $z_j = 0, 1, 2, \ldots, n_j$; $s_j, n_j \in \mathbb{N}$ and $a_j(0, z_j) = a_j(w_j, 0) = 0 \; \forall j = 1, 2, \ldots, n$. Define the operators ∇_1 and ∇_2 by

$$\nabla_1 a_j(w_j, z_j) = a_j(w_j, z_j) - a_j(w_j - 1, z_j),$$

$$\nabla_2 a_j(w_j, z_j) = a_j(w_j, z_j) - a_j(w_j, z_j - 1).$$

Then

$$\sum_{w_1=1}^{s_1} \sum_{z_1=1}^{t_1} \cdots \sum_{w_n=1}^{s_n} \sum_{z_n=1}^{t_n} \frac{\prod_{j=1}^n \left| a_j(w_j, z_j) \right|}{\left(\sum_{j=1}^n w_j z_j / l_j \right)^{\sum_{j=1}^n \frac{1}{l_j}}}$$

$$\leq R \prod_{j=1}^n \left(\sum_{w_j=1}^{s_j} \sum_{z_j=1}^{t_j} (s_j - w_j + 1)(t_j - z_j + 1) \left| \nabla_2 \nabla_1 a_j(w_j, z_j) \right|^{r_j} \right)^{\frac{1}{r_j}}. \tag{9}$$

Here,

$$R = \left(n - \sum_{j=1}^n \frac{1}{r_j} \right)^{\sum_{j=1}^n \frac{1}{r_j} - n} . \prod_{j=1}^n (s_j n_j)^{\frac{1}{l_j}}.$$

For more details about Hilbert type inequalities, see the papers [5–8]. As applications of our work, we refer to the papers [9,10]. In recent decades, a novel theory, known as time scale theory, has emerged, aimed at unifying continuous calculus and discrete calculus. The results presented in this paper encompass classical continuous and discrete inequalities as special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, respectively. Moreover, these inequalities can be extended to analogous inequalities on various time scales, such as $\mathbb{T} = q^{\overline{\mathbb{Z}}}$ for $q > 1$. Many researchers have delved into dynamic inequalities on time scales, and for a more comprehensive understanding of these dynamic inequalities on time scales, readers are referred to papers [11–17].

The primary objective of this paper is to establish analogous formulas for Hilbert-type inequalities (7) and (8) within the framework of time scales in delta calculus. It is important to note that these formulas are derived under specific conditions, which are $a_j(s_j) = 0$ and $a_j(s_j, z_j) = a_j(w_j, n_j) = 0 \; \forall j = 1, 2, \ldots, n$. These conditions differ from those utilized in a previous work [6]. The outcomes of our research provide novel insights and estimations for these specific categories of inequalities. In particular, we have introduced multivariate summation inequalities for extensions of the Hilbert inequality, which were previously unproven. Additionally, we have obtained their corresponding integral expressions. The proofs of these results are based on the application of Hölder's inequality on time scales and the mean inequality.

The paper is structured as follows: After this introductory section, the subsequent section offers an overview of fundamental concepts in time scale calculus, which serve as the basis for our proofs. The final section is dedicated to presenting our main findings.

2. Basic Principles

In what follows, the time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, and it could be an interval, a union of intervals, or even a set of isolated points. The real numbers (continuous case), integers (discrete case), and various amalgamations of the two constitute the most prevalent instances of time scales. Given $\nu \in \mathbb{T}$, we establish $\sigma : \mathbb{T} \to \mathbb{T}$ and $\mu : \mathbb{T} \to \mathbb{R}$ as $\sigma(\nu) := \inf\{\alpha \in \mathbb{T} : \alpha > \nu\}$ and $\mu(\nu) := \sigma(\nu) - \nu \geq 0$. These components are referred to as the forward jump operator and the forward graininess function, correspondingly. Considering a function $\Im : \mathbb{T} \to \mathbb{R}$, we introduce the notation:

$$\Im^{\sigma}(\nu) = \Im(\sigma(\nu)) \quad \forall \nu \in \mathbb{T}.$$

Additionally, we establish the interval ℓ within the context of $\mathbb{T}$ as:

$$\ell_{\mathbb{T}} := \ell \cap \mathbb{T} \quad \ell \subset \mathbb{R}.$$

Below, we present the concept of the delta derivative along with its properties. We also delve into the chain rule, integration by parts, Fubini's theorem, and the mean inequality, which are discussed and analyzed in the references [4,18–21] and others.

Definition 1 ([20])**.** *We use the term "Δ differentiable" to describe a function $\Im$ being differentiable at $v \in \mathbb{T}$, if $\forall \varepsilon > 0$, there is a neighborhood W of v such that for some β the inequality*

$$|\Im(\sigma(v)) - \Im(w) - \beta(\sigma(v) - w)| \leq \varepsilon |\sigma(v) - w|, \quad w \in W$$

is true and, in this case, we write $\Im^{\Delta}(v) = \beta$.

Theorem 1 (Properties of delta-derivatives [20])**.** *Assume $\Im$ is a function and let $v \in \mathbb{T}^k$, then*

1. *If $\Im$ is differentiable at v, then $\Im$ is continuous at v.*
2. *If $\Im$ is continuous at v and v is right-scattered (i.e., $\sigma(v) > v$), then $\Im$ is differentiable at v with*
$$\Im^{\Delta}(v) = \frac{\Im(\sigma(v)) - \Im(v)}{\mu(v)}.$$
3. *If v is right-dense (i.e., $\sigma(v) = v$), then $\Im$ is differentiable if the limit*
$$\lim_{w \to v} \frac{\Im(v) - \Im(w)}{v - w},$$
exists as a finite number. In this case,
$$\Im^{\Delta}(v) = \lim_{w \to v} \frac{\Im(v) - \Im(w)}{v - w}.$$

Example 1.

1. *If $\mathbb{T} = \mathbb{R}$, then $\sigma(v) = v$, $\mu(v) = 0$ and*
$$\Im^{\Delta}(v) = \lim_{w \to v} \frac{\Im(v) - \Im(w)}{v - w} = \Im'(v) \quad \forall v \in \mathbb{T},$$
where $\Im'$ is the usual derivative.

2. *If* $\mathbb{T} = \mathbb{Z}$, *then* $\sigma(v) = v + 1$, $\mu(v) = 1$ *and*

$$\Im^{\Delta}(v) = \frac{\Im(\sigma(\Im)) - \Im(v)}{\mu(v)} = \Im(v+1) - \Im(v) = \Delta\Im(v),$$

where Δ *is the usual forward difference operator.*

3. *If* $\mathbb{T} = q^{\bar{\mathbb{Z}}} := \{v : v = q^k, k \in \mathbb{Z}, q > 1\} \cup \{0\}$, *then* $\sigma(v) = qv$, $\mu(v) = (q-1)v$ *and*

$$\Im^{\Delta}(v) = \Delta_q \Im(v) = \frac{\Im(qv) - \Im(v)}{(q-1)v} \quad \forall v \in \mathbb{T} \setminus \{0\}.$$

Theorem 2 (Chain Rule [20]). *Given that* $\Upsilon : \mathbb{T} \to \mathbb{R}$ *is a continuous and* Δ *differentiable and* $\Im : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable, then*

$$(\Im \circ \Upsilon)^{\Delta}(v) = \Im^{'}(\Upsilon(\tau))\Upsilon^{\Delta}(v) \quad \text{for} \quad \tau \in [v, \sigma(v)]. \tag{10}$$

Definition 2 ([20]). *A function* $\Im$ *is characterized as rd*$-$*continuous when it exhibits continuity at every right-dense point within* $\mathbb{T}$ *and possesses finite left-sided limits at left-dense points in* $\mathbb{T}$. *We use the symbol* $C_{rd}(\mathbb{T}, \mathbb{R})$ *to represent the sets of all rd-continuous functions, and the symbol* $C(\mathbb{T}, \mathbb{R})$ *to represent the set of all continuous functions.*

The following is a description of the concept of an integral on time scales.

Definition 3 ([20]). $\Re$ *is* Δ *antiderivative of* $\Im$ *if*

$$\Re^{\Delta}(v) = \Im(v) \quad \text{holds} \quad \forall v \in \mathbb{T}^k.$$

As a result, for $a, c \in \mathbb{T}$, *we deduce the integral of* $\Im$ *as*

$$\int_a^c \Im(v)\Delta v = \Re(c) - \Re(a).$$

It is widely acknowledged that any rd-continuous function possesses an antiderivative. As a result, we can deduce the following outcomes.

Theorem 3 ([20]). *If* $v_0, v \in \mathbb{T}$, *then*

$$\left(\int_{v_0}^{v} \Im(\vartheta)\Delta\vartheta\right)^{\Delta} = \Im(v). \tag{11}$$

Theorem 4 ([20]). *If* $a, c, \tau \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ *and* $\Im, \Upsilon \in C_{rd}([a, c]_{\mathbb{T}}, \mathbb{R})$, *then*

1. $\int_a^c [\alpha\Im(\delta) + \beta\Upsilon(\delta)]\Delta\delta = \alpha \int_a^c \Im(\delta)\Delta\delta + \beta \int_a^c \Upsilon(\delta)\Delta\delta$;
2. $\int_a^a \Im(\delta)\Delta\delta = 0$;
3. $\int_a^c \Im(\delta)\Delta\delta = \int_a^{\tau} \Im(\delta)\Delta\delta + \int_{\tau}^c \Im(\delta)\Delta\delta$;
4. *If* $\Im(\delta) \geq 0; \forall \delta \in [a, c]_{\mathbb{T}}$, *then* $\int_a^c \Im(\delta)\Delta\delta \geq 0$.
5. $\left|\int_a^c \Im(\delta)\Delta\delta\right| \leq \int_a^c |\Im(\delta)|\Delta\delta$.

Lemma 1 (Integration by parts [19]). *If* $a, c \in \mathbb{T}$ *and* $\omega, \kappa \in C_{rd}([a, c]_{\mathbb{T}}, \mathbb{R})$, *then*

$$\int_a^c \omega(\delta)\kappa^{\Delta}(\delta)\Delta\delta = [\omega(\delta)\kappa(\delta)]_a^c - \int_a^c \omega^{\Delta}(\delta)\kappa^{\sigma}(\delta)\Delta\delta. \tag{12}$$

Theorem 5 ([19]). *Let* $a, c \in \mathbb{T}$ *and* $\Im \in C_{rd}(\mathbb{T}, \mathbb{R})$. *Then*

(i) *If* $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^c \Im(\delta)\Delta\delta = \int_a^c \Im(\delta)d\delta.$$

(*ii*) *If* $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^c \Im(\delta)\Delta\delta = \sum_{\delta=a}^{c-1} \Im(\delta).$$

(*iii*) *If* $\mathbb{T} = q^{\bar{\mathbb{Z}}}$, *then*

$$\int_a^c \Im(\delta)\Delta\delta = (q-1)\sum_{k=\log_q a}^{\log_q c-1} q^k \Im(q^k).$$

Lemma 2 (Hölder's Inequality [19]). *If* $a, c \in \mathbb{T}$ *and* $\Im, \Upsilon \in C_{rd}([a,c]_{\mathbb{T}}, \mathbb{R}^+)$, *then*

$$\int_a^c \Im(\delta)\Upsilon(\delta)\Delta\delta \le \left[\int_a^c \omega(\delta)\Im^{\eta}(\delta)\Delta\delta\right]^{\frac{1}{\eta}} \left[\int_a^c \omega(\delta)\Upsilon^{\lambda}(\delta)\Delta\delta\right]^{\frac{1}{\lambda}}, \quad (13)$$

where $\eta > 1$ *and* $1/\eta + 1/\lambda = 1$.

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be time scales, CC_{rd} denote the set of functions $\Im(\tau,\xi)$ on $\mathbb{T}_1 \times \mathbb{T}_2$, where $\Im$ is $rd-$continuous in τ, ξ and CC'_{rd} denote the set of all functions CC_{rd}, for which both the Δ_1 partial derivative with respect to τ and Δ_2 partial derivative with respect to ξ exist, and are in CC_{rd}.

Lemma 3 ([18], Theorem 3.3). *Let* $\eta, \lambda \in \mathbb{T}$ *with* $\eta < \lambda$, $f, g \in CC_{rd}([\eta,\lambda]_{\mathbb{T}} \times [\eta,\lambda]_{\mathbb{T}}, \mathbb{R})$ *and* $\gamma, \nu > 1$ *such that* $1/\gamma + 1/\nu = 1$. *Then,*

$$\begin{aligned} &\int_\eta^\lambda \int_\eta^\lambda |f(\tau,\xi)g(\tau,\xi)|\Delta_1\tau\Delta_2\xi \\ \le\ & \left[\int_\eta^\lambda \int_\eta^\lambda |f(\tau,\xi)|^{\gamma}\Delta_1\tau\Delta_2\xi\right]^{\frac{1}{\gamma}} \left[\int_\eta^\lambda \int_\eta^\lambda |g(\tau,\xi)|^{\nu}\Delta_1\tau\Delta_2\xi\right]^{\frac{1}{\nu}}. \quad (14) \end{aligned}$$

Lemma 4 (Fubini's theorem [21]). *If* $\eta, \lambda, c, d \in \mathbb{T}$ *and* $\Im \in CC_{rd}([\eta,\lambda]_{\mathbb{T}} \times [c,d]_{\mathbb{T}}, \mathbb{R})$ *is* $\Delta-$*integrable, then*

$$\int_\eta^\lambda \left(\int_c^d \Im(\tau,\xi)\Delta_2\xi\right)\Delta_1\tau = \int_c^d \left(\int_\eta^\lambda \Im(\tau,\xi)\Delta_1\tau\right)\Delta_2\xi.$$

Lemma 5 (Mean inequality [4]). *If* $\alpha_j, \beta_j > 0$ *for* $j = 1, 2, \ldots, s$, *then*

$$\prod_{j=1}^{s} \alpha_j^{\beta_j} \le \frac{\left(\sum_{j=1}^{s} \alpha_j\beta_j\right)^{\sum_{j=1}^{s}\beta_j}}{\left(\sum_{j=1}^{s} \beta_j\right)^{\sum_{j=1}^{s}\beta_j}}. \quad (15)$$

3. Main Results

Throughout this paper, we will operate under the assumption that the functions are rd-continuous, and we will also consider the existence of the integrals. To substantiate our results, it is necessary to prove the following lemma.

Lemma 6. *Let* $l_j, r_j > 1$ *with* $1/l_j + 1/r_j = 1$ *and* $w_j > 0$, *where* $j = 1, 2, \ldots, n$. *Then*

$$\prod_{j=1}^{n} w_j^{\frac{1}{l_j}} \le \frac{\left(\sum_{j=1}^{n} \frac{w_j}{l_j}\right)^{\sum_{j=1}^{m}\frac{1}{l_j}}}{\left(s - \sum_{j=1}^{s}\frac{1}{r_j}\right)^{\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)}}. \quad (16)$$

Proof. By utilizing Lemma 5 with $\alpha_j = w_j$ and $\beta_j = 1/l_j$, we deduce that

$$\prod_{j=1}^{s} w_j^{\frac{1}{l_j}} \leq \frac{\left(\sum_{j=1}^{s} \frac{w_j}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}}{\left(\sum_{j=1}^{s} \frac{1}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}}. \tag{17}$$

Since $\sum_{j=1}^{s}(1/l_j) = \sum_{j=1}^{s}(1-(1/r_j)) = s - \sum_{j=1}^{s}(1/r_j)$, then (17) becomes

$$\prod_{j=1}^{s} w_j^{\frac{1}{l_j}} \leq \frac{\left(\sum_{j=1}^{s} \frac{w_j}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}}{\left(s - \sum_{j=1}^{s} \frac{1}{r_j}\right)^{s-\sum_{j=1}^{s} \frac{1}{r_j}}},$$

which is (16). □

Theorem 6. *Let $a_j, \varepsilon_j \in \mathbb{T}$, $l_j, r_j > 1$ such that $1/l_j + 1/r_j = 1$ and $\lambda_j \in C_{rd}([a_j, \varepsilon_j]_{\mathbb{T}}, \mathbb{R})$ with $\lambda_j(\varepsilon_j) = 0$; $j = 1, 2, \ldots, s$. Then*

$$\int_{a_s}^{\varepsilon_s} \cdots \int_{a_1}^{\varepsilon_1} \frac{\prod_{j=1}^{s} |\lambda_j(\xi_j)|}{\left(\sum_{j=1}^{s} \frac{\varepsilon_j - \xi_j}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}} \Delta\xi_1 \ldots \Delta\xi_s$$
$$\leq A \prod_{j=1}^{s} \left(\int_{a_j}^{\varepsilon_j} (\sigma(\xi_j) - a_j) \left|\lambda_j^{\Delta}(\xi_j)\right|^{r_j} \Delta\xi_j \right)^{\frac{1}{r_j}}, \tag{18}$$

where

$$A = \left(s - \sum_{j=1}^{s} \frac{1}{r_j}\right)^{\sum_{j=1}^{s} \frac{1}{r_j} - s} \prod_{j=1}^{s} (\varepsilon_j - a_j)^{\frac{1}{l_j}}. \tag{19}$$

Proof. By utilizing the property (5) of Theorem 4, we deduce that

$$\left| \int_{\xi_j}^{\varepsilon_j} \lambda_j^{\Delta}(z_j) \Delta z_j \right| \leq \int_{\xi_j}^{\varepsilon_j} \left|\lambda_j^{\Delta}(z_j)\right| \Delta z_j. \tag{20}$$

Since $\lambda_j(\varepsilon_j) = 0$, then

$$\int_{\xi_j}^{\varepsilon_j} \lambda_j^{\Delta}(z_j) \Delta z_j = \lambda_j(z_j)\Big|_{\xi_j}^{\varepsilon_j} = \lambda_j(\varepsilon_j) - \lambda_j(\xi_j) = -\lambda_j(\xi_j),$$

and then

$$\left| \int_{\xi_j}^{\varepsilon_j} \lambda_j^{\Delta}(z_j) \Delta z_j \right| = |\lambda_j(\xi_j)|. \tag{21}$$

Substituting (21) into (20), we observe that

$$|\lambda_j(\xi_j)| \leq \int_{\xi_j}^{\varepsilon_j} \left|\lambda_j^{\Delta}(z_j)\right| \Delta z_j \quad \text{for } j = 1, 2, \ldots, s,$$

therefore

$$\prod_{j=1}^{s} |\lambda_j(\xi_j)| \leq \prod_{j=1}^{s} \int_{\xi_j}^{\varepsilon_j} \left|\lambda_j^{\Delta}(z_j)\right| \Delta z_j. \tag{22}$$

Applying (13) on $\int_{\xi_j}^{\varepsilon_j} \left|\lambda_j^{\Delta}(z_j)\right| \Delta z_j$ with $l_j, r_j > 1$, $\Im(z_j) = \left|\lambda_j^{\Delta}(z_j)\right|$ and $\Upsilon(z_j) = 1$, we have

$$\begin{aligned}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|\Delta z_j &\le \left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}\left(\int_{\xi_j}^{\varepsilon_j}\Delta z_j\right)^{\frac{1}{l_j}} \\ &= (\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}},\end{aligned}$$

and then

$$\begin{aligned}\prod_{j=1}^{s}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|\Delta z_j &\le \prod_{j=1}^{s}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}} \\ &= \prod_{j=1}^{s}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\prod_{j=1}^{s}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}. \qquad (23)\end{aligned}$$

By substituting (23) into (22) and applying (16) on $\prod_{j=1}^{s}(\varepsilon_j-\xi_j)^{(1/l_j)}$ with $w_j=\varepsilon_j-\xi_j$, we acquire

$$\begin{aligned}\prod_{j=1}^{s}\left|\lambda_j(\xi_j)\right| &\le \prod_{j=1}^{s}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\prod_{j=1}^{s}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}} \\ &\le \frac{\left(\sum_{j=1}^{s}\frac{\varepsilon_j-\xi_j}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}{\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{s-\sum_{j=1}^{s}\frac{1}{r_j}}}\prod_{j=1}^{s}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}. \qquad (24)\end{aligned}$$

Dividing (24) on $\left(\sum_{j=1}^{s}\frac{\varepsilon_j-\xi_j}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}$ and integrating over ξ_j from a_j to ε_j, $j=1,2,\ldots,s$, we conclude that

$$\begin{aligned}&\int_{a_s}^{\varepsilon_s}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{j=1}^{s}\left|\lambda_j(\xi_j)\right|}{\left(\sum_{j=1}^{s}\frac{\varepsilon_j-\xi_j}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\Delta\xi_1\ldots\Delta\xi_s \\ &\le \left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\int_{a_s}^{\varepsilon_s}\cdots\int_{a_1}^{\varepsilon_1}\prod_{j=1}^{s}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}\Delta\xi_1\ldots\Delta\xi_s \\ &= \left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}\int_{a_j}^{\varepsilon_j}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}\Delta\xi_j. \qquad (25)\end{aligned}$$

Again, using (13) on $\int_{a_j}^{\varepsilon_j}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}\Delta\xi_j$ with $l_j,r_j>1$, $\Im(\xi_j)=\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}$ and $\Upsilon(\xi_j)=1$, we obtain

$$\begin{aligned}\int_{a_j}^{\varepsilon_j}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}\Delta\xi_j &\le \left(\int_{a_j}^{\varepsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\Delta\xi_j\right)^{\frac{1}{r_j}}\left(\int_{a_j}^{\varepsilon_j}\Delta\xi_j\right)^{\frac{1}{l_j}} \\ &= (\varepsilon_j-a_j)^{\frac{1}{l_j}}\left(\int_{a_j}^{\varepsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\Delta\xi_j\right)^{\frac{1}{r_j}},\end{aligned}$$

and then

$$\begin{aligned}
&\prod_{j=1}^{s}\int_{a_j}^{\varepsilon_j}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)^{\frac{1}{r_j}}\Delta\xi_j \\
\leq\ & \prod_{j=1}^{s}(\varepsilon_j-a_j)^{\frac{1}{l_j}}\left(\int_{a_j}^{\varepsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\Delta\xi_j\right)^{\frac{1}{r_j}} \\
=\ & \prod_{j=1}^{s}(\varepsilon_j-a_j)^{\frac{1}{l_j}}\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\Delta\xi_j\right)^{\frac{1}{r_j}}. \qquad (26)
\end{aligned}$$

Substituting (26) into (25), we obtain

$$\begin{aligned}
&\int_{a_s}^{\varepsilon_s}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{j=1}^{s}|\lambda_j(\xi_j)|}{\left(\sum_{j=1}^{s}\frac{\varepsilon_j-\xi_j}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\Delta\xi_1\ldots\Delta\xi_s \\
&\leq\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}(\varepsilon_j-a_j)^{\frac{1}{l_j}}\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\Delta\xi_j\right)^{\frac{1}{r_j}}. \qquad (27)
\end{aligned}$$

Now, using (12) on $\int_{a_j}^{\varepsilon_j}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)\Delta\xi_j$ with $\omega(\xi_j)=\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j$ and $\kappa^{\Delta}(\xi_j)=1$, we find that

$$\begin{aligned}
&\int_{a_j}^{\varepsilon_j}\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)\Delta\xi_j \\
=\ & \left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(z_j)\right|^{r_j}\Delta z_j\right)\kappa(\xi_j)\Bigg|_{a_j}^{\varepsilon_j}+\int_{a_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(\xi_j)\right|^{r_j}\kappa^{\sigma}(\xi_j)\Delta\xi_j \\
=\ & \int_{a_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(\xi_j)\right|^{r_j}(\sigma(\xi_j)-a_j)\Delta\xi_j, \qquad (28)
\end{aligned}$$

where $\kappa(\xi_j)=\xi_j-a_j$. Combining (28) with (27), we obtain

$$\begin{aligned}
&\int_{a_s}^{\varepsilon_s}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{j=1}^{s}|\lambda_j(\xi_j)|}{\left(\sum_{j=1}^{s}\frac{\varepsilon_j-\xi_j}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\Delta\xi_1\ldots\Delta\xi_s \\
\leq\ & \left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}(\varepsilon_j-a_j)^{\frac{1}{l_j}} \\
&\times\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(\xi_j)\right|^{r_j}(\sigma(\xi_j)-a_j)\Delta\xi_j\right)^{\frac{1}{r_j}} \\
=\ & A\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\left|\lambda_j^{\Delta}(\xi_j)\right|^{r_j}(\sigma(\xi_j)-a_j)\Delta\xi_j\right)^{\frac{1}{r_j}}.
\end{aligned}$$

Hence, (24) is proved. □

Corollary 1. *Let $\mathbb{T}=\mathbb{Z}$ in Theorem 6, $a_j,\varepsilon_j\in\mathbb{N}$, $l_j,r_j>1$ such that $1/r_j+1/l_j=1$ and λ_j be real sequences with $\lambda_j(\varepsilon_j)=0$; $j=1,2,\ldots,s$. Then, $\sigma(\xi_j)=\xi_j+1$ and*

$$\sum_{\xi_1=a_1}^{\varepsilon_1-1}\sum_{\xi_2=a_2}^{\varepsilon_2-1}\cdots\sum_{\xi_s=a_s}^{\varepsilon_s-1}\frac{\prod_{j=1}^{s}|\lambda_j(\xi_j)|}{\left(\sum_{j=1}^{s}\frac{\varepsilon_j-\xi_j}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\leq A\prod_{j=1}^{s}\left(\sum_{\xi_j=a_j}^{\varepsilon_j-1}(\xi_j-a_j+1)|\Delta\lambda_j(\xi_j)|^{r_j}\right)^{\frac{1}{r_j}}.$$

Here, Δ is the forward difference operator and A is specified as in (19).

Corollary 2. *Let $\mathbb{T} = \mathbb{R}$ in Theorem 6, $a_j, \varepsilon_j \in \mathbb{R}$, $l_j, r_j > 1$ such that $1/r_j + 1/l_j = 1$ and $\lambda_j \in C([a_j, \varepsilon_j], \mathbb{R})$ with $\lambda_j(\varepsilon_j) = 0$; $j = 1, 2, \ldots, s$. Then, $\sigma(\xi_j) = \xi_j$ and*

$$\int_{a_s}^{\varepsilon_s} \cdots \int_{a_1}^{\varepsilon_1} \frac{\prod_{j=1}^{s} |\lambda_j(\xi_j)|}{\left(\sum_{j=1}^{s} \frac{\varepsilon_j - \xi_j}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}} d\xi_1 \ldots d\xi_s \leq A \prod_{j=1}^{s} \left(\int_{a_j}^{\varepsilon_j} \left|\lambda_j'(\xi_j)\right|^{r_j} (\xi_j - a_j) d\xi_j \right)^{\frac{1}{r_j}},$$

where A is given by (19).

Corollary 3. *Let $\mathbb{T} = q^{\overline{\mathbb{Z}}}$ for $q > 1$, $l_j, r_j > 1$ such that $1/r_j + 1/l_j = 1$ and λ_j be real sequences with $\lambda_j(\varepsilon_j) = 0$; $j = 1, 2, \ldots, s$. Then, $\sigma(\xi_j) = q\xi_j$ and*

$$\begin{aligned} &\sum_{\xi_s = \log_q a_s}^{\log_q \varepsilon_s - 1} \cdots \sum_{\xi_1 = \log_q a_1}^{\log_q \varepsilon_1 - 1} \frac{(q-1)^n \prod_{j=1}^{s} \xi_j |\lambda_j(\xi_j)|}{\left(\sum_{j=1}^{s} \frac{\varepsilon_j - \xi_j}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}} \\ \leq\ & A \prod_{j=1}^{s} \left(\sum_{\xi_j = \log_q a_j}^{\log_q \varepsilon_j - 1} (q-1)(q\xi_j - a_j)\xi_j |\Delta_q \lambda_j(\xi_j)|^{r_j} \right)^{\frac{1}{r_j}}, \end{aligned}$$

where A is given by (19) and

$$\Delta_q \lambda_j(\xi_j) = \frac{\lambda_j(q\xi_j) - \lambda_j(\xi_j)}{(q-1)\xi_j} \quad \forall \xi_j \in \mathbb{T} \setminus \{0\}.$$

In the following, we generalize the last theorem for two variables.

Theorem 7. *Let $a_j, \varepsilon_j, \epsilon_j \in \mathbb{T}$, $l_j, r_j > 1$ such that $1/l_j + 1/r_j = 1$, $\lambda_j \in CC'_{rd}([a_j, \varepsilon_j]_{\mathbb{T}} \times [a_j, \epsilon_j]_{\mathbb{T}}, \mathbb{R})$ with $\lambda_j(\tau_j, \varepsilon_j) = \lambda_j(\epsilon_j, \xi_j) = 0$ for $\xi_j \in [a_j, \varepsilon_j]_{\mathbb{T}}$ and $\tau_j \in [a_j, \epsilon_j]_{\mathbb{T}}$; $j = 1, 2, \ldots, s$. Then*

$$\begin{aligned} &\int_{a_s}^{\epsilon_s} \int_{a_1}^{\epsilon_1} \cdots \int_{a_s}^{\varepsilon_s} \int_{a_1}^{\varepsilon_1} \frac{\prod_{j=1}^{s} |\lambda_j(\tau_j, \xi_j)|}{\left(\sum_{j=1}^{s} \frac{(\epsilon_j - \tau_j)(\varepsilon_j - \xi_j)}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}} \Delta_2 \xi_1 \ldots \Delta_2 \xi_s \Delta_1 \tau_1 \ldots \Delta_1 \tau_s \\ &\leq B \prod_{j=1}^{s} \left(\int_{a_j}^{\epsilon_j} \int_{a_j}^{\varepsilon_j} (\sigma(\tau_j) - a_j)(\sigma(\xi_j) - a_j) \left|\lambda_j^{\Delta_2 \Delta_1}(\tau_j, \xi_j)\right|^{r_j} \Delta_2 \xi_j \Delta_1 \tau_j \right)^{\frac{1}{r_j}}, \end{aligned} \tag{29}$$

where

$$B = \left(s - \sum_{j=1}^{s} \frac{1}{r_j} \right)^{\sum_{j=1}^{s} \frac{1}{r_j} - s} \prod_{j=1}^{s} (\epsilon_j - a_j)^{\frac{1}{l_j}} (\varepsilon_j - a_j)^{\frac{1}{l_j}}. \tag{30}$$

Here, the Δ_1-derivative of $\lambda(\tau, \xi)$ is the $\Delta-$derivative with respect to the first variable τ and the Δ_2-derivative of $\lambda(\tau, \xi)$ is the $\Delta-$derivative with respect to the second variable ξ.

Proof. Applying the property (5) of Theorem 4, Fubini's theorem and using the hypothesis $\lambda_j(\tau_j, \varepsilon_j) = \lambda_j(\epsilon_j, \xi_j) = 0$, we obtain

$$\begin{aligned}
\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(t_j,\vartheta_j)\right|\Delta_2\vartheta_j\Delta_1 z_j &\geq \left|\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\Delta_2\vartheta_j\Delta_1 z_j\right| \\
&= \left|\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left[\lambda_j^{\Delta_2}(z_j,\vartheta_j)\right]^{\Delta_1}\Delta_2\vartheta_j\Delta_1 z_j\right| \\
&= \left|\int_{\xi_j}^{\varepsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\left[\lambda_j^{\Delta_2}(z_j,\vartheta_j)\right]^{\Delta_1}\Delta_1 z_j\right)\Delta_2\vartheta_j\right| \\
&= \left|\int_{\xi_j}^{\varepsilon_j}\left(\lambda_j^{\Delta_2}(\epsilon_j,\vartheta_j)-\lambda_j^{\Delta_2}(\tau_j,\vartheta_j)\right)\Delta_2\vartheta_j\right| \\
&= \left|\lambda_j(\epsilon_j,\varepsilon_j)-\lambda_j(\epsilon_j,\xi_j)+\lambda_j(\tau_j,\xi_j)-\lambda_j(\tau_j,\varepsilon_j)\right| \\
&= \left|\lambda_j(\tau_j,\xi_j)\right|,
\end{aligned}$$

and then

$$\prod_{j=1}^{s}\left|\lambda_j(\tau_j,\xi_j)\right| \leq \prod_{j=1}^{s}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|\Delta_2\vartheta_j\Delta_1 z_j. \tag{31}$$

Applying (14) on $\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|\Delta_2\vartheta_j\Delta_1 z_j$ with $l_j, r_j > 1$, $f(z_j,\vartheta_j) = 1$ and $g(z_j,\vartheta_j) = \left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|$, we see that

$$\begin{aligned}
&\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|\Delta_2\vartheta_j\Delta_1 z_j \\
\leq\ &\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{l_j}} \\
=\ &(\epsilon_j-\tau_j)^{\frac{1}{l_j}}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}},
\end{aligned}$$

and then

$$\begin{aligned}
&\prod_{j=1}^{s}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|\Delta_2\vartheta_j\Delta_1 z_j \\
\leq\ &\prod_{j=1}^{s}(\epsilon_j-\tau_j)^{\frac{1}{l_j}}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}} \\
=\ &\prod_{j=1}^{s}(\epsilon_j-\tau_j)^{\frac{1}{l_j}}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\prod_{j=1}^{s}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}.
\end{aligned} \tag{32}$$

Substituting (32) into (31) and applying (16) on $w_j = (\epsilon_j-\tau_j)(\varepsilon_j-\xi_j)$, we obtain

$$\begin{aligned}
\prod_{j=1}^{s}\left|\lambda_j(\tau_j,\xi_j)\right| &\leq \prod_{j=1}^{s}(\epsilon_j-\tau_j)^{\frac{1}{l_j}}(\varepsilon_j-\xi_j)^{\frac{1}{l_j}}\prod_{j=1}^{s}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}} \\
&\leq \frac{\left(\sum_{j=1}^{s}\frac{(\epsilon_j-\tau_j)(\varepsilon_j-\xi_j)}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}{\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{s-\sum_{j=1}^{s}\frac{1}{r_j}}}\prod_{j=1}^{s}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}.
\end{aligned} \tag{33}$$

Note that

$$\begin{aligned}
&\prod_{j=1}^{s}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\\
=&\left(\int_{\tau_1}^{\epsilon_1}\int_{\xi_1}^{\varepsilon_1}\left|\lambda_1^{\Delta_2\Delta_1}(z_1,\vartheta_1)\right|^{r_1}\Delta_2\vartheta_1\Delta_1 z_1\right)^{\frac{1}{r_1}}\ldots\left(\int_{\tau_s}^{\epsilon_s}\int_{\xi_s}^{\varepsilon_s}\left|\lambda_s^{\Delta_2\Delta_1}(z_s,\vartheta_s)\right|^{r_s}\Delta_2\vartheta_s\Delta_1 z_s\right)^{\frac{1}{r_s}},
\end{aligned}$$

and then

$$\begin{aligned}
&\int_{a_s}^{\epsilon_s}\int_{a_1}^{\epsilon_1}\ldots\int_{a_s}^{\varepsilon_s}\int_{a_1}^{\varepsilon_1}\prod_{j=1}^{s}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_1\ldots\Delta_2\xi_s\Delta_1\tau_1\ldots\Delta_1\tau_s\\
=&\int_{a_1}^{\epsilon_1}\int_{a_1}^{\varepsilon_1}\left(\int_{\tau_1}^{\epsilon_1}\int_{\xi_1}^{\varepsilon_1}\left|\lambda_1^{\Delta_2\Delta_1}(z_1,\vartheta_1)\right|^{r_1}\Delta_2\vartheta_1\Delta_1 z_1\right)^{\frac{1}{r_1}}\Delta_2\xi_1\Delta_1\tau_1\\
&\ldots\times\int_{a_s}^{\epsilon_s}\int_{a_s}^{\varepsilon_s}\left(\int_{\tau_s}^{\epsilon_s}\int_{\xi_s}^{\varepsilon_s}\left|\lambda_s^{\Delta_2\Delta_1}(z_s,\vartheta_s)\right|^{r_s}\Delta_2\vartheta_s\Delta_1 z_s\right)^{\frac{1}{r_s}}\Delta_2\xi_s\Delta_1\tau_s\\
=&\prod_{j=1}^{s}\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_j\Delta_1\tau_j. \qquad (34)
\end{aligned}$$

Dividing (33) on $\left(\sum_{j=1}^{s}\frac{(\epsilon_j-\tau_j)(\varepsilon_j-\xi_j)}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}$, integrating over ξ_j from a_j to ε_j and over τ_j from a_j to ϵ_j for $j=1,2,\ldots,s$ and using (34), we conclude that

$$\begin{aligned}
&\int_{a_s}^{\epsilon_s}\int_{a_1}^{\epsilon_1}\ldots\int_{a_s}^{\varepsilon_s}\int_{a_1}^{\varepsilon_1}\frac{\prod_{j=1}^{s}\left|\lambda_j(\tau_j,\xi_j)\right|}{\left(\sum_{j=1}^{s}\frac{(\epsilon_j-\tau_j)(\varepsilon_j-\xi_j)}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\Delta_2\xi_1\ldots\Delta_2\xi_s\Delta_1\tau_1\ldots\Delta_1\tau_s\\
\leq&\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\\
&\times\int_{a_s}^{\epsilon_s}\int_{a_1}^{\epsilon_1}\ldots\int_{a_s}^{\varepsilon_s}\int_{a_1}^{\varepsilon_1}\prod_{j=1}^{s}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_1\ldots\Delta_2\xi_s\Delta_1\tau_1\ldots\Delta_1\tau_s\\
=&\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_j\Delta_1\tau_j. \qquad (35)
\end{aligned}$$

Again, using (14) on $\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_j\Delta_1\tau_j$ with exponents $r_j, l_j>1$ and $f(\xi_j,\tau_j)=1$,

$$g(\xi_j,\tau_j)=\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}},$$

we observe that

$$\begin{aligned}
&\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_j\Delta_1\tau_j\\
\leq&\left(\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}}\left(\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{l_j}}\\
=&(\epsilon_j-a_j)^{\frac{1}{l_j}}(\varepsilon_j-a_j)^{\frac{1}{l_j}}\left(\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}},
\end{aligned}$$

and then

$$\prod_{j=1}^{s}\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right)^{\frac{1}{r_j}}\Delta_2\xi_j\Delta_1\tau_j$$
$$\leq \prod_{j=1}^{s}(\epsilon_j-a_j)^{\frac{1}{l_j}}(\varepsilon_j-a_j)^{\frac{1}{l_j}}\left(\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}}$$
$$= \prod_{j=1}^{s}(\epsilon_j-a_j)^{\frac{1}{l_j}}(\varepsilon_j-a_j)^{\frac{1}{l_j}}$$
$$\times\prod_{j=1}^{s}\left(\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}}. \tag{36}$$

Substituting (36) into (35) and applying the Fubini theorem, we see that

$$\int_{a_s}^{\epsilon_s}\int_{a_1}^{\epsilon_1}\cdots\int_{a_s}^{\varepsilon_s}\int_{a_1}^{\varepsilon_1}\frac{\prod_{j=1}^{s}\left|\lambda_j(\tau_j,\xi_j)\right|}{\left(\sum_{j=1}^{s}\frac{(\epsilon_j-\tau_j)(\varepsilon_j-\xi_j)}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\Delta_2\xi_1\ldots\Delta_2\xi_s\Delta_1\tau_1\ldots\Delta_1\tau_s$$
$$\leq\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}(\epsilon_j-a_j)^{\frac{1}{l_j}}(\varepsilon_j-a_j)^{\frac{1}{l_j}}$$
$$\times\prod_{j=1}^{s}\left(\int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}}$$
$$=\left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}(\epsilon_j-a_j)^{\frac{1}{l_j}}(\varepsilon_j-a_j)^{\frac{1}{l_j}}$$
$$\times\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\left(\int_{a_j}^{\epsilon_j}\left[\int_{\tau_j}^{\epsilon_j}\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\Delta_1 z_j\right]\Delta_1\tau_j\right)\Delta_2\xi_j\right)^{\frac{1}{r_j}}. \tag{37}$$

Now, by applying (12) on $\int_{a_j}^{\epsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1 z_j\right)\Delta_1\tau_j$ with

$$\omega(\tau_j)=\int_{\tau_j}^{\epsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1 z_j \quad\text{and}\quad \kappa^{\Delta}(\tau_j)=1,$$

we find that

$$\begin{aligned}
&\int_{a_j}^{\epsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1 z_j\right)\Delta_1\tau_j\\
=\;&\kappa(\tau_j)\int_{\tau_j}^{\epsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1 z_j\Big|_{a_j}^{\epsilon_j}\\
&+\int_{a_j}^{\epsilon_j}\kappa^{\sigma}(\tau_j)\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1\tau_j\\
=\;&\int_{a_j}^{\epsilon_j}\left[\sigma(\tau_j)-a_j\right]\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1\tau_j,
\end{aligned} \tag{38}$$

where $\kappa(\tau_j)=\tau_j-a_j$. By integrating (38) over ξ_j from a_j to ε_j and using the Fubini theorem, we have

$$\begin{aligned}
& \int_{a_j}^{\varepsilon_j}\int_{a_j}^{\epsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1 z_j\right)\Delta_1\tau_j\Delta_2\xi_j \\
= & \int_{a_j}^{\varepsilon_j}\int_{a_j}^{\epsilon_j}[\sigma(\tau_j)-a_j]\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1\tau_j\Delta_2\xi_j \\
= & \int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}[\sigma(\tau_j)-a_j]\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_2\xi_j\Delta_1\tau_j \\
= & \int_{a_j}^{\epsilon_j}[\sigma(\tau_j)-a_j]\left(\int_{a_j}^{\varepsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_2\xi_j\right)\Delta_1\tau_j. \qquad (39)
\end{aligned}$$

Again, using (12) on the term $\int_{a_j}^{\varepsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_2\xi_j$ with

$$\omega(\tau_j)=\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j \text{ and } \kappa^{\Delta}(\xi_j)=1,$$

we see that

$$\begin{aligned}
& \int_{a_j}^{\varepsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_2\xi_j \\
= & \ \kappa(\xi_j)\left(\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right)\Bigg|_{a_j}^{\varepsilon_j} \\
& +\int_{a_j}^{\varepsilon_j}\kappa^{\sigma}(\xi_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_2\xi_j \\
= & \int_{a_j}^{\varepsilon_j}(\sigma(\xi_j)-a_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_2\xi_j, \qquad (40)
\end{aligned}$$

where $\kappa(\xi_j)=\xi_j-a_j$. Substituting (40) into (39) and applying Fubini's theorem, we obtain

$$\begin{aligned}
& \int_{a_j}^{\varepsilon_j}\int_{a_j}^{\epsilon_j}\left(\int_{\tau_j}^{\epsilon_j}\left[\int_{\xi_j}^{\varepsilon_j}\left|\lambda_j^{\Delta_2\Delta_1}(z_j,\vartheta_j)\right|^{r_j}\Delta_2\vartheta_j\right]\Delta_1 z_j\right)\Delta_1\tau_j\Delta_2\xi_j \\
= & \int_{a_j}^{\epsilon_j}[\sigma(\tau_j)-a_j]\left(\int_{a_j}^{\varepsilon_j}(\sigma(\xi_j)-a_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_2\xi_j\right)\Delta_1\tau_j \\
= & \int_{a_j}^{\epsilon_j}\int_{a_j}^{\varepsilon_j}[\sigma(\tau_j)-a_j](\sigma(\xi_j)-a_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_2\xi_j\Delta_1\tau_j \\
= & \int_{a_j}^{\varepsilon_j}\int_{a_j}^{\epsilon_j}[\sigma(\tau_j)-a_j](\sigma(\xi_j)-a_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_1\tau_j\Delta_2\xi_j. \qquad (41)
\end{aligned}$$

Substituting (41) into (37), we obtain

$$\begin{aligned}
& \int_{a_s}^{\epsilon_s}\int_{a_1}^{\epsilon_1}\cdots\int_{a_s}^{\varepsilon_s}\int_{a_1}^{\varepsilon_1}\frac{\prod_{j=1}^{s}|\lambda_j(\tau_j,\xi_j)|}{\left(\sum_{j=1}^{s}\frac{(\epsilon_j-\tau_j)(\varepsilon_j-\xi_j)}{l_j}\right)^{\sum_{j=1}^{s}\frac{1}{l_j}}}\Delta_2\xi_1\ldots\Delta_2\xi_s\Delta_1\tau_1\ldots\Delta_1\tau_s \\
\leq & \left(s-\sum_{j=1}^{s}\frac{1}{r_j}\right)^{\sum_{j=1}^{s}\frac{1}{r_j}-s}\prod_{j=1}^{s}(\epsilon_j-a_j)^{\frac{1}{l_j}}(\varepsilon_j-a_j)^{\frac{1}{l_j}} \\
& \times\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\int_{a_j}^{\epsilon_j}(\sigma(\tau_j)-a_j)(\sigma(\xi_j)-a_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}} \\
= & \ B\prod_{j=1}^{s}\left(\int_{a_j}^{\varepsilon_j}\int_{a_j}^{\epsilon_j}(\sigma(\tau_j)-a_j)(\sigma(\xi_j)-a_j)\left|\lambda_j^{\Delta_2\Delta_1}(\tau_j,\xi_j)\right|^{r_j}\Delta_2\xi_j\Delta_1\tau_j\right)^{\frac{1}{r_j}}.
\end{aligned}$$

Hence, (29) is proved. □

Corollary 4. *Let $\mathbb{T} = \mathbb{Z}$ in Theorem 7, $a_j, \varepsilon_j, \epsilon_j \in \mathbb{Z}$, $r_j, l_j > 1$ such that $1/r_j + 1/l_j = 1$ and λ_j be real sequences with $\lambda_j(\tau_j, \varepsilon_j) = \lambda_j(\epsilon_j, \xi_j) = 0$ for $\xi_j \in [a_j, \varepsilon_j]$ and $\tau_j \in [a_j, \epsilon_j]$, where $j = 1, 2, \ldots, s$. Then, $\sigma(\tau_j) = \tau_j + 1$, $\sigma(\xi_j) = \xi_j + 1$ and*

$$\sum_{\tau_s=a_s}^{\epsilon_s-1} \sum_{\tau_1=a_1}^{\epsilon_1-1} \cdots \sum_{\xi_s=a_s}^{\varepsilon_s-1} \sum_{\xi_1=a_1}^{\varepsilon_1-1} \frac{\prod_{j=1}^{s} |\lambda_j(\tau_j, \xi_j)|}{\left(\sum_{j=1}^{s} \frac{(\epsilon_j - \tau_j)(\varepsilon_j - \xi_j)}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}}$$
$$\leq B \prod_{j=1}^{s} \left(\sum_{\xi_j=a_j}^{\varepsilon_j-1} \sum_{\tau_j=a_j}^{\epsilon_j-1} (\tau_j - a_j + 1)(\xi_j - a_j + 1) |\Delta_2 \Delta_1 \lambda_j(\tau_j, \xi_j)|^{r_j} \right)^{\frac{1}{r_j}},$$

where B is given by (30).

Corollary 5. *Let $\mathbb{T} = \mathbb{R}$ in Theorem 7, $a_j, \varepsilon_j, \epsilon_j \in \mathbb{R}$, $r_j, l_j > 1$ such that $1/r_j + 1/l_j = 1$ and $\lambda_j \in CC'([a_j, \varepsilon_j] \times [a_j, \epsilon_j], \mathbb{R})$ with $\lambda_j(\tau_j, \varepsilon_j) = \lambda_j(\epsilon_j, \xi_j) = 0$ for $\xi_j \in [a_j, \varepsilon_j]_{\mathbb{T}}$ and $\tau_j \in [a_j, \epsilon_j]_{\mathbb{T}}$, where $j = 1, 2, \ldots, s$. Then, $\sigma(\tau_j) = \tau_j$, $\sigma(\xi_j) = \xi_j$ and*

$$\int_{a_s}^{\epsilon_s} \int_{a_1}^{\epsilon_1} \cdots \int_{a_s}^{\varepsilon_s} \int_{a_1}^{\varepsilon_1} \frac{\prod_{j=1}^{s} |\lambda_j(\tau_j, \xi_j)|}{\left(\sum_{j=1}^{s} \frac{(\epsilon_j - \tau_j)(\varepsilon_j - \xi_j)}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}} d\xi_1 \ldots d\xi_s d\tau_1 \ldots d\tau_s$$
$$\leq B \prod_{j=1}^{s} \left(\int_{a_j}^{\varepsilon_j} \int_{a_j}^{\epsilon_j} (\tau_j - a_j)(\xi_j - a_j) \left| \frac{\partial^2 \lambda_j(\tau_j, \xi_j)}{\partial \xi_j \partial \tau_j} \right|^{r_j} d\xi_j d\tau_j \right)^{\frac{1}{r_j}},$$

where B is given by (30).

Corollary 6. *Let $\mathbb{T} = q^{\overline{\mathbb{Z}}}$ for $q > 1$, $a_j, \varepsilon_j, \epsilon_j \in \mathbb{T}$, $r_j, l_j > 1$ such that $1/r_j + 1/l_j = 1$ and λ_j are real sequences with $\lambda_j(\tau_j, \varepsilon_j) = \lambda_j(\epsilon_j, \xi_j) = 0$ for $\xi_j \in [a_j, \varepsilon_j]$ and $\tau_j \in [a_j, \epsilon_j]$, where $j = 1, 2, \ldots, s$. Then, $\sigma(\tau_j) = q\tau_j$, $\sigma(\xi_j) = q\xi_j$ and*

$$\sum_{\tau_s=\log_q a_s}^{\log_q \epsilon_s - 1} \sum_{\tau_1=\log_q a_1}^{\log_q \epsilon_1 - 1} \cdots \sum_{\xi_s=\log_q a_s}^{\log_q \varepsilon_{s-1}} \sum_{\xi_1=\log_q a_1}^{\log_q \varepsilon_{1-1}} \frac{(q-1)^{2n} \prod_{j=1}^{s} \tau_j \xi_j |\lambda_j(\tau_j, \xi_j)|}{\left(\sum_{j=1}^{s} \frac{(\epsilon_j - \tau_j)(\varepsilon_j - \xi_j)}{l_j}\right)^{\sum_{j=1}^{s} \frac{1}{l_j}}}$$
$$\leq B \prod_{j=1}^{s} \left(\sum_{\tau_j=\log_q a_j}^{\log_q \epsilon_j - 1} \sum_{\xi_j=\log_q a_j}^{\log_q \varepsilon_{j-1}} (q\tau_j - a_j)(q\xi_j - a_j)(q-1)^2 \tau_j \xi_j \left| \Delta_q^2 \Delta_q^1 \lambda_j(\tau_j, \xi_j) \right|^{r_j} \right)^{\frac{1}{r_j}}.$$

Here, B is given by (30) and the Δ_q^1-derivative of $\lambda(\tau, \xi)$ is the Δ_q-derivative with respect to the first variable τ and the Δ_q^2-derivative of $\lambda(\tau, \xi)$ is the Δ_q-derivative with respect to the second variable ξ.

4. Conclusions

In this study, a generalization of the Hilbert-type inequalities within the framework of time scales in delta calculus. We should note that we used different conditions from some previous results; thus, various refinements of the classic Hilbert-type inequalities are obtained. Throughout the work, it is shown that some known results from the literature are obtained as particular cases of ours. In future research, we aim to showcase these inequalities by utilizing nabla calculus on time scales.

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