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Saturated (n, m)-Regular Semigroups

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Abstract: The aim of this paper is to determine several saturated classes of structurally regular semigroups. First, we show that structurally (n, m)-regular semigroups are saturated in a subclass of semigroups for any pair (n, m) of positive integers. We also demonstrate that, for all positive integers n and k with $1 \leq k \leq n$, the variety of structurally $(0, n)$-left seminormal bands is saturated in the variety of structurally $(0, k)$-bands. As a result, in the category of structurally $(0, k)$-bands, epis from structurally $(0, n)$-left seminormal bands is onto.

Keywords: dominions; epimorphisms; zigzag; saturated; structurally regular

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1. Introduction and Preliminaries

The morphism $\Theta : S \to T$ is known as an *epimorphism* (*epi* for short) in the category of all semigroups if for all morphisms ϕ, ψ with $\Theta\phi = \Theta\psi$ implies $\phi = \psi$, where throughout this article we write mappings to the right of their arguments. The *morphic image* of a morphism Θ is the subset of codomain T that is the image of the morphism. It is simple to confirm that all surjective morphisms are epi. Depending on the category under examination, the reverse may or may not be true. It holds true for some categories, such as sets and groups. However, in the category of semigroups, there are non-surjective epimorphisms. For instance, the inclusion $i : (\mathbb{Z}, \cdot) \to (\mathbb{Q}, \cdot)$, is an epimorphism in the category of semigroups. Therefore, it is worthwhile to investigate the classes of semigroups in which epis are onto or otherwise not onto. Epimorphisms in the category of semigroups are investigated using dominions and zigzags. The systematic study of epimorphisms and dominion in semigroups was initiated by Isbell [1] and Howie and Isbell [2].

Assuming that U is a subsemigroup of a semigroup S, we say that U dominates an element $d \in S$ if for every semigroup Q and all morphisms $\phi, \psi : S \to Q$, $\phi|_U = \psi|_U$ implies $d\phi = d\psi$. The set containing all elements of such type is said to be the *dominion* of U in S and is denoted by $Dom(U, S)$. We say that U is *closed* in S if $Dom(U, S) = U$ and *absolutely closed* if it is closed in every enclosing semigroup S. If $Dom(U, S) = S$, a semigroup U is said to be *epimorphically embedded* in a semigroup S. If $Dom(U, S) \neq S$ for any properly containing semigroup S, the semigroup U is said to be *saturated*. It is clear that $i : S\alpha \to T$ is the inclusion map if, and only if, $Dom(S\alpha, T) = T$, and that $\alpha : S \to T$ is epi.

Let $\mathcal{C}$ be the class of semigroups. If $\mathcal{C}$ is closed under morphic images and each member of $\mathcal{C}$ is saturated, then every epi from a member of $\mathcal{C}$ is onto. If $Dom(U, S) \neq S$ for any properly containing semigroup S inside $\mathcal{C}$, a semigroup U is said to be $\mathcal{C}$-*saturated*. If all members of a class $\mathcal{C}$ of semigroups are saturated, the class is said to be saturated. We say that $\mathcal{C}_1$ is $\mathcal{C}_2$-*saturated* if every member of $\mathcal{C}_1$ is $\mathcal{C}_2$-*saturated*. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of semigroups with $\mathcal{C}_1 \subseteq \mathcal{C}_2$, we say that $\mathcal{C}_1$ is $\mathcal{C}_2$-*saturated* if every member of $\mathcal{C}_1$ is $\mathcal{C}_2$-*saturated*.

Isbell provided the useful characterization of semigroup dominions, known as the Isbell's Zigzag Theorem which is the main tool to prove the main results of this paper (Theorems 5 and 7). The theorem is stated as:

Theorem 1 ([3], Theorem 8.3.5)**.** *Let U be a subsemigroup of a semigroup S and $d \in S$. Then $d \in Dom(U, S)$ if, and only if, $d \in U$ or there exists a system of equalities for d as under:*

$$\begin{aligned} d &= a_0 y_1 & a_0 &= x_1 a_1 \\ a_1 y_1 &= a_2 y_2 & x_1 a_2 &= x_2 a_3 \\ &\vdots & &\vdots \\ a_{2i-1} y_i &= a_{2i} y_{i+1} & x_i a_{2i} &= x_{i+1} a_{2i+1} \\ a_{2m-1} y_m &= a_{2m} & x_m a_{2m} &= d \end{aligned} \tag{1}$$

where $a_i \in U(0 \leq i \leq 2m)$ and $x_i, y_i \in S(1 \leq i \leq m)$.

The above system of equalities (1) is said to be the *zigzag of length m* in S over U with value d. In whatever follows, by zigzag equations, we shall mean a system of equations of type (1). Further, we mention that the bracketed statements shall mean statements dual to each other.

The following results due to Khan are also useful for our investigation:

Theorem 2 ([4], Result 3)**.** *Let U be a subsemigroup of a semigroup S. Take any $d \in S \setminus U$, such that $d \in Dom(U, S)$, and let (1) be a zigzag of minimal length m over U with value d. Then $x_i, y_i \in S \setminus U(1 \leq i \leq m)$.*

Theorem 3 ([4], Result 4)**.** *Let U be a subsemigroup of a semigroup S and $Dom(U, S) = S$. Then, for any $d \in S \setminus U$ and any positive integer k, there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$, such that $d = b_1 b_2 \cdots b_k d_k$ $[d = d_k b_k b_{k-1} \cdots b_1]$. In particular, $d \in S^k$ for every positive integer k.*

Definition 1. *An element a of a semigroup S is said to be regular if there exists an element b in S, such that $aba = a$ and $bab = b$ (b is called an inverse element) and semigroup consisting entirely of such type of elements is called* regular*.*

The set of all inverses of a regular element a is denoted by V_a.

Definition 2. *An element a of S is said to be* idempotent *if $a^2 = a$ and the set of all idempotent elements of a semigroup S is denoted by $E(S)$.*

Definition 3. *A semigroup consisting entirely of idempotent elements is called a* band*.*

Definition 4. *A band is said to be*

(i) *left [right] regular if it satisfies the identity $axa = ax[axa = xa]$,*
(ii) *left [right] seminormal if it satisfies the identity $axy = axyay$ $[yxa = yayxa]$.*

The following countable family of congruences on a semigroup S was introduced by Samuel J. L. Kopamu in [5]. For each ordered pair (n, m) of non-negative integers, the congruence $\theta(n, m)$ is defined as

$$\theta(n, m) = \{(a, b) : zaw = zbw, \text{ for all } z \in S^n \text{ and } w \in S^m\},$$

where $S^1 = S$ and S^0 denotes the set containing the empty word. In particular,

$$\theta(0, m) = \{(a, b) : av = bv, \text{for all } v \in S^m\},$$

while $\theta(0,0)$ is the identity relation on S.

The notion of structurally regular semigroups was introduced by Kopamu in [6]. He provided its characterization, listed some examples, and examined its relationship with various known generalizations of the class of regular semigroups.

Definition 5. *A semigroup S is said to be structurally regular if there exists some ordered pair (n,m) of non-negative integers, such that $S/\theta(n,m)$ is regular.*

The class of structurally regular semigroups is larger than the class of regular semigroups. Indeed, it is distinct from each of the following well-known extensions of the class of regular semigroups, locally regular semigroups, weakly regular semigroups, eventually regular semigroups and nilpotent extensions of regular semigroups (see [6], for more details). Clearly, every regular semigroup is structurally (structurally $(0,0)$) regular.

For any class $\mathcal{V}$ of regular semigroups, we say that a semigroup S is a *structurally (n,m)-$\mathcal{V}$* semigroup if $S/\theta(n,m)$ belongs to $\mathcal{V}$. In particular, a semigroup S is said to be *structurally (n,m)- inverse [or band]* if $S/\theta(n,m)$ is a generalised inverse [or band]. More precisely, for any class $\mathcal{V}$ of semigroups and any $(n,m)\in\mathbb{N}^{\{0\}}\times\mathbb{N}^{\{0\}}$, we define a class of semigroups

$$\mathcal{V}^{(n,m)} = \{S : S/\theta(n,m) \in \mathcal{V}\}.$$

According to ([5], Theorem 4.2) , $\mathcal{V}^{(n,m)}$ is a variety of semigroups, if so is $\mathcal{V}$.

Definition 6. *An element a of a semigroup S is said to be an (n,m)-idempotent if it is $\theta(n,m)$ related to a^2; that is, if $za^2w = zaw$ for all $z\in S^n$ and $w\in S^m$.*

We denote the set of all (n,m)-idempotents of S by

$$E_{(n,m)}(S) = \{x\in S : (x,x^2)\in\theta(n,m)\} = \{x\in S : zxw = zx^2w\ \forall\, z\in S^n, w\in S^m\}.$$

The statement that x is an (n,m)-idempotent in S is equivalent to that of $x\theta(n,m)$ is idempotent in $S/\theta(n,m)$, so $E_{(n,m)}(S) = E(S/\theta(n,m))$. Even $E(S)\subseteq E_{(n,m)}(S)$ as every idempotent of S is truly an (n,m)-idempotent of S.

The next result provides the useful characterization of structurally regular semigroups.

Theorem 4 ([6], Theorem 2.1)**.** *Let (n,m) be an ordered pair of non-negative integers. For any semigroup S, $S/\theta(n,m)$ is regular (and hence, S is structurally regular) if, and only if, for each element a of S, there exists x' in S such that*

$$zxx'xw = zxw \text{ and } zx'xx'w = zx'w, \text{for all } z\in S^n \text{ and } w\in S^m.$$

The condition that for each element x there exists y such that $zxw = zxyxw$ for all z in S^n and w in S^m implies that there exists an element $x^* = yxy$, such that $zxw = zxx^*xw$ and $zx^*w = zx^*xx^*w$. Therefore, the set

$$V_S(x;n,m) = \{x^*\in S : (xx^*x, x), (x^*xx^*, x^*)\in\theta(n,m)\}$$

is non-empty. We refer to each element of the set $V_S(x;n,m)$ as an $(n,m) - \textit{inverse}$ of x. Clearly, $V(x)\subseteq V_S(x;n,m)$ and S is structurally (n,m)-regular if every element of S has an (n,m)-inverse in S. Note that, if x^* is an (n,m) inverse of x in a semigroup S, then xx^* and x^*x are in $E_{(n,m)}(S)$.

In 1975, Gardner [7] proved that any epimorphism from a regular ring is onto, in the category of rings. Therefore, it is natural to ask the same question for semigroups, and indeed Hall [8] has posed the question, does there exist a regular semigroup which is not saturated? This is equivalent to asking the question, does there exist an epimorphism from a regular semigroup which is not onto (in the category of semigroups)? In this direction Hall [9] had shown that epimorphisms are onto for finite regular semigroups. Higgins [10,11] had shown that epimorphisms are onto for generalised inverse semigroups and epimorphisms are onto for locally inverse semigroups, respectively. Recently, Shah et al. [12] have shown that epis from a structurally (n, m) generalised inverse semigroup is surjective.

2. Epis and Structurally (n, m)-Regular Semigroups

Epis are not onto for structurally regular semigroups in general, as they are not onto for regular semigroups. Since there exists a regular semigroup which is not saturated (Ref. [13] [Example 7.15]). Thus, the problem of finding saturated varieties of semigroups is an open problem. Therefore, it becomes natural to ask that under what conditions epis are onto for structurally regular semigroups. In this section, we show that structurally regular semigroups are saturated in a subclass of semigroups.

Let U and S be any semigroups. Then

$$\theta^S(n,m) = \{(x,y) \in S \times S : zxw = zyw \ \forall z \in S^n, w \in S^m\},$$

$$\theta^U(n,m) = \{(x,y) \in U \times U : zxw = zyw \ \forall z \in U^n, w \in U^m\}.$$

Next lemma shows that the class of structurally (n, m)-regular semigroups is closed under morphic images.

Lemma 1 ([12], Corollary C.2)**.** *Any morphic image of structurally (n, m)-regular semigroup is structurally (n, m)-regular.*

To prove the main result of this section, we shall need the following lemma in which U is a structurally (n, m)-regular semigroup and S is any semigroup with U as a proper subsemigroup, such that $Dom(U, S) = S$. For any semigroup A, $A^{(1)}$ denotes the semigroup A with identity adjoined.

Lemma 2 ([12], Lemma 2.5)**.** *For any $x, y \in S \setminus U$ and $u, v \in U^{(1)}$*

$$xuavy = xuaa^*avy, \text{and } xua^*vy = xua^*aa^*vy \text{ for all } a \in U.$$

Let $\mathcal{C}_e$ $[\mathcal{C}^e]$ be the class of semigroups, such that for any $U, S \in \mathcal{C}_e$ with $U \subseteq S$, $se = ses$ $[es = ses]$ for all $s \in S$ and $e \in E_{(n,m)}(U)$.

Theorem 5. *Let U be a structurally (n, m)-regular semigroup. Then, U is $\mathcal{C}_e$-saturated.*

Proof. Suppose, on the contrary, that U is not $\mathcal{C}_e$-*saturated*. Then, there exists a semigroup S in $\mathcal{C}_e$ containing U properly, such that $Dom(U, S) = S$. Let $d \in S \setminus U$, then by Theorem 1 there exists a zigzag equation of type (1) in S over U with value d of minimum length m. Now, by using $se = ses$ for all $s \in S$ and $e \in E_{(n,m)}(U)$, we have

$$\begin{aligned}
d &= x_1a_1y_1 \text{ (by zigzag equations)} \\
&= x_1a_1a_1^*a_1y_1 \text{ (by Lemma 2)} \\
&= x_1a_1a_1^*x_1a_1y_1 \text{ (since } se = ses) \\
&= x_1a_1a_1^*x_2a_3y_2 \text{ (by zigzag equations)} \\
&= x_1a_1a_1^*x_2a_3a_3^*a_3y_2 \text{ (by Lemma 2)} \\
&= x_1a_1a_1^*x_2a_3a_3^*x_2a_3y_2 \text{ (since } se = ses) \\
&= \left(\prod_{i=1}^{2} x_ia_{2i-1}a_{2i-1}^*\right)x_2a_3y_2 \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{m} x_i a_{2i-1} a_{2i-1}^*\right) x_m a_{2m-1} y_m \\
&= \left(\prod_{i=1}^{m-1} x_i a_{2i-1} a_{2i-1}^*\right) x_m a_{2m-1} a_{2m-1}^* a_{2m-1} y_m \ \text{ (by Lemma 2)} \\
&= \left(\prod_{i=1}^{m-1} x_i a_{2i-1} a_{2i-1}^*\right) x_{m-1} a_{2m-2} a_{2m-1}^* a_{2m} \ \text{ (by zigzag equations)} \\
&= \left(\prod_{i=1}^{m-2} x_i a_{2i-1} a_{2i-1}^*\right) x_{m-1} a_{2m-3} a_{2m-3}^* x_{m-1} a_{2m-2} a_{2m-1}^* a_{2m}) \ \text{ (since } se = ses\text{)} \\
&= \left(\prod_{i=1}^{m-2} x_i a_{2i-1} a_{2i-1}^*\right) x_{m-1} a_{2m-3} a_{2m-3}^* a_{2m-2} a_{2m-1}^* a_{2m}) \\
&= \left(\prod_{i=1}^{m-2} x_i a_{2i-1} a_{2i-1}^*\right) x_{m-2} a_{2m-4} a_{2m-3}^* a_{2m-2} a_{2m-1}^* a_{2m} \ \text{(by zigzag equations)} \\
&\vdots \\
&= \left(\prod_{i=1}^{2} x_i a_{2i-1} a_{2i-1}^*\right) x_2 a_4 \left(\prod_{i=3}^{m} a_{2i-1}^* a_{2i}\right) \\
&= x_1 a_1 a_1^* x_2 a_3 a_3^* a_4 \left(\prod_{i=3}^{m} a_{2i-1}^* a_{2i}\right) \\
&= x_1 a_1 a_1^* x_1 a_2 \left(\prod_{i=2}^{m} a_{2i-1}^* a_{2i}\right) \ \text{ (by zigzag equations)} \\
&= x_1 a_1 a_1^* a_2 \left(\prod_{i=2}^{m} a_{2i-1}^* a_{2i}\right) \ \text{ (since } se = ses\text{)} \\
&= a_0 \left(\prod_{i=1}^{m} a_{2i-1}^* a_{2i}\right).
\end{aligned}
$$

Hence, $d \in U$, a contradiction as required. □

Dually, we can prove the following theorem.

Theorem 6. *Let U be a structurally (n, m)-regular semigroup. Then, U is $\mathcal{C}^e$-saturated.*

Thus, we have the following immediate corollary.

Corollary 1. *In class $\mathcal{C}_e$ $[\mathcal{C}^e]$ of semigroups, for each pair (n, m) of positive integers, any epi from a structurally (n, m)-regular semigroup is onto.*

Example 1. Let $S = \{0, u\}$ be two element semi-lattice. Define the Cartesian product $T = S^1 \times S = \{(s_1, s_2) : s_1 \in S^1 \text{ and } s_2 \in S\}$, where S^1 is the semigroup obtained by adjoining an identity element to S. Define a binary operation $*$ by $(s_1, s_2) * (s_1', s_2') = (s_1 s_2', s_2 s_2')$. It can been easily shown that $(T, *)$ is a semigroup. Now take any $\theta(1, 0)$-related elements, say $(s_1, s_2)$and (s_1', s_2'). Then, for all $(a, b) \in T$, we have

$$
\begin{aligned}
(a, b) * (s_1, s_2) &= (a, b) * (s_1', s_2') \\
\Rightarrow (as_2, bs_2) &= (as_2', bs_2') \\
\Rightarrow as_2 &= as_2',
\end{aligned}
$$

for all $a \in S^1$. Since S^1 is monoid, it follows that $s_2 = s_2'$ and hence quotient $T/\theta(1, 0)$ is isomorphic to the semi-lattice S. Therefore, T is structurally regular.

3. Epis and Structurally $(0, n)$-Bands

In [14], Ahanger and Shah proved that in the variety of all bands any epi from the left [right] seminormal band is surjective and thus extending the result of Alam and Khan [15], that the variety of left [right] seminormal bands is closed. Moreover in [12], Shah and Bano proved that the varieties of structurally $(0, n)$-left regular bands are saturated in the varieties of structurally $(0, k)$ left regular bands for any k and n with $1 \leq k \leq n$. In this section, we generalize the above results by proving that the variety of structurally $(0, n)$-left seminormal bands is saturated in the variety of structurally $(0, k)$-bands for any k and n with $1 \leq k \leq n$. In particular, we show that, in the category of structurally $(0, k)$-bands, any epi from a structurally $(0, n)$-left seminormal band is onto.

It can be easily verified that for each positive integer n and k with $1 \leq k \leq n$, the class of structurally $(0, n)$ semigroups is contained in the class of structurally $(0, k)$ semigroups.

Definition 7. *A structurally $(0, k)$-band B is said to be structurally $(0, k)$-left regular band, if $B/\theta(0, k)$ is a left regular band; that is, for any $a, x \in S$, we have*

$$xaw = xaxw \text{ for all } w \in B^k.$$

Definition 8. *A structurally $(0, k)$-band B is said to be structurally $(0, k)$-left seminormal band, if $B/\theta(0, k)$ is left seminormal band; that is, for any a, x, y in S, we have*

$$axyw = axyayw \text{ for all } w \in B^k.$$

Dually, a structurally $(k, 0)$-right seminormal band or a structurally $(k, 0)$-right regular band can be defined.

Remark 1 ([5], Theorem 4.2)**.** *The class $\mathcal{V}^{(0,n)}$ of a structurally $(0, n)$-left seminormal bands is a variety for each positive integer n. Furthermore, for each positive integers k and n with $1 \leq k \leq n$, $\mathcal{V}^{(0,n)} \subseteq \mathcal{V}^{(0,k)}$.*

In order to prove the main result of this section, we first prove the following lemmas in which U is a structurally $(0, n)$-left seminormal band and S is any structurally $(0, k)$-band containing U as a proper subband, such that $Dom(U, S) = S$.

Lemma 3. *If any $d \in S \setminus U$ has zigzag equations of type (1) in S over U of the shortest length m, then for all $w \in S^k$ we have,*

$$a_0 a_2 w = a_0 a_2 x_2 a_3 a_0 a_2 w.$$

Proof. From (1), we have

$$\begin{aligned}
a_0 a_2 w &= x_1 a_1 a_2 w \ \text{(by zigzag equations)} \\
&= x_1 a_1 (a_2 x_1) a_1 a_2 w \ \text{(since } S \text{ is } (0,k)\text{-band)} \\
&= a_0 (a_2 x_1 a_2 x_1) a_1 a_2 w \ \text{(since } S \text{ is } (0,k)\text{-band)} \\
&= a_0 a_2 x_2 a_3 x_1 a_1 a_2 w \ \text{(by zigzag equations)} \\
&= a_0 a_2 x_2 a_3 a_0 a_2 w,
\end{aligned}$$

as required. □

Lemma 4. *If any $d \in S \setminus U$ has zigzag equations of type (1), then for all $w \in S^k$*

$$a_0 a_2 u_2 a_4 w = a_0 a_2 u_2 a_4 a_0 a_2 x_3 a_5 a_0 a_2 u_2 a_4 w,$$

where $y_2 = u_2 v_2 \bar{y_2}$ for some $u_2 \in U, v_2 \in U^k$ and $\bar{y_2} \in S \setminus U$.

Proof. Since (1) is the zigzag of shortest length, so by Theorems 2 and 3, we can factorize y_2 as $y_2 = u_2 v_2 \bar{y_2}$, where $u_2 \in U, v_2 \in U^k$ and $\bar{y_2} \in S \setminus U$. Now

$$\begin{aligned}
a_0a_2u_2a_4w &= (a_0a_2u_2a_4(a_0a_2)u_2a_4)w \ \text{ (since } S \text{ is } (0,k)\text{-band)} \\
&= a_0a_2u_2a_4(a_0a_2x_2a_3a_0a_2)u_2a_4w \ \text{ (by Lemma 3 as } u_2a_4w \in S^k) \\
&= a_0a_2u_2(a_4a_0a_2x_2)a_3a_0a_2u_2a_4w \\
&= a_0a_2u_2(a_4a_0a_2(x_2a_4)a_0a_2x_2)a_3a_0a_2u_2a_4w \ \text{ (since } S \text{ is } (0,k)\text{-band)} \\
&= a_0a_2u_2a_4a_0a_2(x_3a_5)a_0a_2x_2a_3a_0a_2u_2a_4w \ \text{ (by zigzag equations)} \\
&= a_0a_2u_2a_4a_0a_2(x_3a_5)a_0a_2u_2a_4w, \ \text{ (by Lemma 3 as } u_2a_4w \in S^k)
\end{aligned}$$

as required. □

Lemma 5. *If any $d \in S \setminus U$ has zigzag equations of type (1) in S over U of shortest length m, then for all $w \in S^k$*

$$s_jw = s_js_{j-1}\cdots s_2a_0a_2x_{j+1}a_{2j+1}s_{j-1}a_{2j}w \text{ with } 3 \le j \le m,$$

where $s_i = a_0a_2u_2a_4u_3a_6\cdots u_ia_{2i}$ and $y_i = u_iv_i\bar{y_i}$, $u_i \in U, v_i \in U^k$ and $\bar{y_i} \in S \setminus U$ with $2 \le i \le m$.

Proof. Since (1) is the zigzag of shortest length, so by Theorems 2 and 3, we can factorize y_i as $y_i = u_iv_i\bar{y_i}$ with $u_i \in U, v_i \in U^k$ and $\bar{y_i} \in S \setminus U$ for $i = 1, 2, \cdots m$. We now prove the lemma by induction on j. For $j = 3$, we have

$$\begin{aligned}
s_3w &= (a_0a_2u_2a_4)(u_3)(a_6)w \\
&= a_0a_2u_2a_4u_3a_6(a_0a_2u_2a_4)a_6w \\
&\qquad \text{(since } U \text{ is structurally } (0,n)\text{-left seminormal band)} \\
&= a_0a_2u_2a_4u_3(a_6a_0a_2u_2a_4a_0a_2x_3)a_5a_0a_2u_2a_4a_6w \ \text{ (by Lemma 4 as } a_6w \in S^k) \\
&= a_0a_2u_2a_4u_3a_6a_0a_2u_2a_4a_0a_2(x_3a_6)(a_0a_2u_2a_4a_0a_2x_3a_5a_0a_2u_2a_4)a_6w \\
&\qquad \text{(since } S \text{ is structurally } (0,k)\text{-band)} \\
&= (a_0a_2u_2a_4u_3a_6)(a_0a_2u_2a_4)a_0a_2(x_4a_7)(a_0a_2u_2a_4a_0a_2x_3a_5a_0a_2u_2a_4)a_6w \\
&\qquad \text{(by zigzag equations)} \\
&= (a_0a_2u_2a_4u_3a_6)(a_0a_2u_2a_4)a_0a_2(x_4a_7)(a_0a_2u_2a_4)a_6w \ \text{ (by Lemma 4 as } a_6w \in S^k) \\
&= s_3s_2a_0a_2x_4a_7s_2a_6w.
\end{aligned}$$

Thus, the lemma holds for $j = 3$. Assume for the sake of induction that the lemma holds for $j = r$ $(r \ge 3)$. Then, we have

$$s_rw = s_rs_{r-1}\cdots s_2a_0a_2x_{r+1}a_{2r+1}s_{r-1}a_{2r}w.$$

We now show that it also holds for $j = r + 1$. Now

$$\begin{aligned}
s_{r+1}w &= a_0a_2u_2a_4\cdots u_ra_{2r}u_{r+1}a_{2r+2}w \\
&= s_ru_{r+1}a_{2r+2}w \\
&= s_ru_{r+1}a_{2r+2}s_ra_{2r+2}w \ \text{ (since } U \text{ is structurally } (0,n)\text{-left seminormal band)} \\
&= s_ru_{r+1}(a_{2r+2}s_rs_{r-1}\cdots s_2a_0a_2x_{r+1})a_{2r+1}s_{r-1}a_{2r}a_{2r+2}w \\
&\qquad \text{(by inductive hypothesis, as } a_{2r+2}w \in U^k \text{)} \\
&= s_ru_{r+1}(a_{2r+2}s_r\cdots s_2a_0a_2(x_{r+1}a_{2r+2})s_rs_{r-1}\cdots s_2a_0a_2x_{r+1})a_{2r+1}s_{r-1}a_{2r}a_{2r+2}w \\
&\qquad \text{(since } S \text{ is structurally } (0,k)\text{-band)} \\
&= s_ru_{r+1}a_{2r+2}s_rs_{r-1}\cdots s_2a_0a_2(x_{r+2}a_{2r+3})(s_rs_{r-1}\cdots s_2a_0a_2x_{r+1}a_{2r+1}s_{r-1}a_{2r})a_{2r+2}w
\end{aligned}$$

$$\text{(by zigzag equations)}$$
$$= (s_r u_{r+1} a_{2r+2}) s_r s_{r-1} \cdots s_2 a_0 a_2 (x_{r+2} a_{2r+3}) s_r a_{2r+2} w$$
$$\text{(by inductive hypothesis, as } a_{2r+2} w \in U^k \text{)}$$
$$= s_{r+1} s_r s_{r-1} \cdots s_2 a_0 a_2 (x_{r+2} a_{2r+3}) s_r a_{2r+2} w,$$

as required. □

Theorem 7. *For each positive integer n and k with $1 \leq k \leq n$, the variety $\mathcal{V}^{(0,n)}$ of structurally $(0,n)$-left seminormal bands is saturated in the variety $\mathcal{V}^{(0,k)}$ of structurally $(0,k)$-bands.*

Proof. Assume, on the contrary, that the variety $\mathcal{V}^{(0,n)}$ of structurally left $(0,n)$-seminormal bands is not saturated in the variety of structurally $(0,k)$-bands for $k \geq n$. Then, there exists a structurally left $(0,n)$-seminormal band U and a structurally $(0,k)$-band containing U properly, such that $Dom(U,S) = S$. Take any $d \in S \setminus U$, then by Theorem 1, d has a zigzag of type (1) in S over U of minimum length m. Since the zigzag is of minimum length, so by Theorem 2, $y_i \in S \setminus U$ for all $i, 1 \leq i \leq m$. Therefore, by Theorem 3, we can write

$$y_i = u_i w_i \bar{y}_i \tag{2}$$

with $u_i \in U, w_i \in U^k$ and $\bar{y}_i \in S \setminus U$ for $i = 1, 2, \cdots m$. Now, we have

$$\begin{aligned}
d &= x_1 a_1 y_1 \quad \text{(by zigzag equations)} \\
&= x_1 a_1 u_1 w_1 \bar{y}_1 \quad \text{(by Equation (2))} \\
&= x_1 a_1 a_1 u_1 w_1 \bar{y}_1 \quad \text{(since } S \text{ is structurally } (0,k)\text{-band)} \\
&= x_1 a_1 a_1 y_1 \quad \text{(by Equation (2))} \\
&= a_0 a_2 y_2 \quad \text{(by zigzag equations)} \\
&= a_0 a_2 u_2 w_2 \bar{y}_2 \quad \text{(by Equation (2))} \\
&= a_0 a_2 x_2 (a_3)(a_0 a_2)(u_2) w_2 \bar{y}_2 \quad \text{(by Lemma 3)} \\
&= a_0 a_2 x_2 a_3 a_0 a_2 u_2 a_3 u_2 w_2 \bar{y}_2 \text{ (since } U \text{ is structurally left } (0,n)\text{-seminormal band)} \\
&= a_0 a_2 x_2 a_3 a_0 a_2 u_2 a_3 y_2 \quad \text{(by Equation (2))} \\
&= a_0 a_2 x_2 a_3 a_0 a_2 u_2 a_4 y_3 \quad \text{(by zigzag equations)} \\
&= (a_0 a_2 x_2 a_3 a_0 a_2) u_2 a_4 u_3 w_3 \bar{y}_3 \qquad \text{(by Equation (2))} \\
&= (a_0 a_2 u_2 a_4) u_3 w_3 \bar{y}_3 \quad \text{(by Lemma 3)} \\
&= a_0 a_2 u_2 a_4 a_0 a_2 x_3 (a_5)(a_0 a_2 u_2 a_4)(u_3) w_3 \bar{y}_3 \quad \text{(by Lemma 4)} \\
&= a_0 a_2 u_2 a_4 a_0 a_2 x_3 (a_5 a_0 a_2 u_2 a_4 u_3 a_5 u_3) w_3 \bar{y}_3 \\
&\qquad \text{(since } U \text{ is structurally left } (0,n)\text{-seminormal band)} \\
&= a_0 a_2 u_2 a_4 a_0 a_2 x_3 a_5 a_0 a_2 u_2 a_4 u_3 a_5 y_3 \quad \text{(by Equation (2))} \\
&= a_0 a_2 u_2 a_4 a_0 a_2 x_3 a_5 a_0 a_2 u_2 a_4 u_3 a_6 y_4 \quad \text{(by zigzag equations)} \\
&= (a_0 a_2 u_2 a_4 a_0 a_2 x_3 a_5 a_0 a_2 u_2 a_4) u_3 a_6 u_4 w_4 \bar{y}_4 \quad \text{(by Equation (2))} \\
&= (a_0 a_2 u_2 a_4 u_3 a_6) u_4 w_4 \bar{y}_4 \quad \text{(by Lemma 4)} \\
&= s_3 u_4 w_4 \bar{y}_4 \\
&= s_3 s_2 a_0 a_2 x_4 a_7 s_2 a_6 u_4 w_4 \bar{y}_4 \quad \text{(by Lemma 5)} \\
&= s_3 s_2 a_0 a_2 x_4 (a_7)(a_0 a_2 u_2 a_4 a_6)(u_4) w_4 \bar{y}_4 \quad \text{(by Lemma 5)} \\
&= s_3 s_2 a_0 a_2 x_4 a_7 a_0 a_2 u_2 a_4 a_6 u_4 a_7 u_4 w_4 \bar{y}_4 \\
&\qquad \text{(since } U \text{ is structurally left } (0,n)\text{-seminormal band)} \\
&= s_3 s_2 a_0 a_2 x_4 a_7 a_0 a_2 u_2 a_4 a_6 u_4 a_7 y_4 \quad \text{(by Equation (2))} \\
&= s_3 s_2 a_0 a_2 x_4 a_7 a_0 a_2 u_2 a_4 a_6 u_4 a_8 y_5 \quad \text{(by zigzag equations)} \\
&= (s_3 s_2 a_0 a_2 x_4 a_7 a_0 a_2 u_2 a_4 a_6) u_4 a_8 u_5 w_5 \bar{y}_5 \quad \text{(by Equation (2))}
\end{aligned}$$

$$= s_3u_4a_8u_5w_5\bar{y}_5 \quad \text{(by Lemma 5)}$$
$$= s_4u_5w_5\bar{y}_5$$

Continuing as above, we obtain

$$\begin{aligned}
d &= s_{m-1}u_mw_m\bar{y}_m \\
&= s_{m-1}s_{m-2}\cdots s_2a_0a_2x_m(a_{2m-1}s_{m-2}a_{2m-2}u_m)w_m\bar{y}_m \quad \text{(by Lemma 5)} \\
&= s_{m-1}s_{m-2}\cdots s_2a_0a_2x_m(a_{2m-1}s_{m-2}a_{2m-2}u_ma_{2m-1}u_m)w_m\bar{y}_m \\
&\qquad \text{(since } U \text{ is structurally left } (0,n)\text{-seminormal band)} \\
&= (s_{m-1}s_{m-2}\cdots s_2a_0a_2x_ma_{2m-1}s_{m-2}a_{2m-2})u_ma_{2m-1}u_mw_m\bar{y}_m \\
&= s_{m-1}u_ma_{2m-1}u_mw_m\bar{y}_m \quad \text{(by Lemma 5)} \\
&= s_{m-1}u_ma_{2m-1}y_m \quad \text{(by Equation (2))} \\
&= s_{m-1}u_ma_{2m} \quad \text{(by zigzag equations)}.
\end{aligned}$$

Thus, $d \in U$, which is a contradiction. □

Dually, we can prove the following:

Theorem 8. *For each positive integers n and k with $1 \le k \le n$ the variety $\mathcal{V}^{(n,0)}$ of structurally $(n,0)$-right seminormal bands is saturated in the variety $\mathcal{V}^{(k,0)}$ of structurally $(k,0)$-bands.*

Corollary 2. *For each positive integers n and k with $1 \le k \le n$ the variety $\mathcal{V}^{(0,n)}$ [$\mathcal{V}^{(n,0)}$] of structurally $(0,n)$-left [$(n,0)$-right] regular bands is saturated in the variety $\mathcal{V}^{(0,k)}$ [$\mathcal{V}^{(k,0)}$] of structurally (o,k)-bands [$(k,0)$-bands].*

Corollary 3. *In the category of structurally $(0,k)$-bands [$(k,0)$-bands] any epi from a structurally $(0,n)$-left [$(n,0)$-right] seminormal bands is surjective for each positive integers k and n with $1 \le k \le n$.*

Corollary 4. *In the category of structurally $(0,k)$-bands [$(k,0)$-bands] any epi from a structurally $(0,n)$-left [$(n,0)$-right] regular bands is surjective for each positive integers k and n with $1 \le k \le n$.*

Example 2. Let $S = \{a_1, a_2, a_3, a_4\}$ be a four element semigroup. The Cayley's table for S is given below:

.	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_1	a_4
a_2	a_1	a_2	a_2	a_1
a_3	a_1	a_2	a_3	a_4
a_4	a_4	a_1	a_4	a_4.

It can be easily verified that S is a regular band. Let $U = \{a_1, a_2, a_3\}$ be a subsemigroup of S. Thus, $a_4 \in S \setminus U$. It is clear that $a_4 \in Dom(U,S)$, since we have the following zigzag equation for a_4,

$$a_4 = a_1a_4 \qquad a_1 = a_4a_2$$
$$a_2a_4 = a_1 \qquad a_4a_1 = a_4.$$

Since $U \subseteq Dom(U,S) \subseteq S$. Therefore, $Dom(U,S) = S$. Thus, it is worth interesting to finding those varieties of regular semigroup and regular bands for which $Dom(U,S) \neq S$.

4. Conclusions

In the present paper, authors have determined several saturated varieties of structurally regular semigroups. It has been shown that structurally (n,m)-regular semigroups are saturated in a subclass of semigroups for any pair (n,m) of positive integers. Then it has been shown that, the variety of structurally $(0,n)$-left seminormal bands is saturated in the variety of structurally $(0,k)$-bands. As a result, in the category of structurally $(0,k)$-bands, epis from structurally $(0,n)$-left seminormal bands is onto.

The results obtained in the paper have their immense utility as they imply that epis from these classes are onto. We hope to explore further classes of semigroups which are more general for which epis are onto; for example we list some open problems in this direction:

(i) As the determination of all saturated classes of bands has not been settled yet, an effort may be made in this direction.
(ii) Is epi from a structurally locally inverse semigroup onto or not.

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