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Global Stability of Traveling Waves for the Lotka–Volterra Competition System with Three Species

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Abstract: The stability of traveling waves for the Lotka–Volterra competition system with three species is investigated in this paper. Specifically, we first show the asymptotic behavior of traveling wave solutions and then establish the local stability and the global stability under the weighted functional space. For local stability, the spectrum approach is used, while for global stability, the comparison principle and squeezing theorem are combined.

Keywords: asymptotic behavior; Lotka–Volterra model; three species; global stability; weighted functional space

MSC: 35K57; 35B35; 92D25

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1. Introduction

The aim of this paper is to study the stability of traveling waves for the Lotka–Volterra competition system with three species as follows:

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + \tilde{u}(1 - \tilde{u} - b_{12}\tilde{v} - b_{13}\tilde{w}), \\ \tilde{v}_t = d_1\tilde{v}_{xx} + \alpha\tilde{v}(1 - b_{21}\tilde{u} - \tilde{v}), \\ \tilde{w}_t = d_2\tilde{w}_{xx} + \beta\tilde{w}(1 - b_{31}\tilde{u} - \tilde{w}). \end{cases} \tag{1}$$

To proceed, we first transform the variables so that $\tilde{u} = u$, $\tilde{v} = 1 - v$, $\tilde{w} = 1 - w$ and the system (1) is converted into the following cooperative system:

$$\begin{cases} u_t = u_{xx} + u(1 - u - b_{12} + b_{12}v - b_{13} + b_{13}w), \\ v_t = d_1 v_{xx} + \alpha(1 - v)(b_{21}u - v), \\ w_t = d_2 w_{xx} + \beta(1 - w)(b_{31}u - w), \end{cases} \tag{2}$$

with the initial value $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ and $w(x,0) = w_0(x)$ for $x \in \mathbb{R}$. In this system, u, v and w are the population densities of three species, respectively; $d_i (i = 1,2)$ is the diffusion coefficient of species i; b_{1j} and $b_{j1} (j = 2,3)$ denote the competition coefficients between the other two species j and the first species; and α and β stand for the growth rates of the two species of v, w, respectively. All the coefficients are positive. Further, we can understand that there are three species u, v and w living together, and species u is a predator, while species v, w are both prey. However, v, w do not directly affect each other, and the predator u acts as a mediator for v and w.

The Lotka–Volterra model is well-known for better describing changes in biological populations, and many mathematicians are interested in its dynamics. In particular, many studies on the existence, stability, and invasion speed of traveling wave solutions have been generated on the two species competitive model, see [1–9]. For the three-species competition model, the studies on the dynamical behaviors are also receiving increased attention. The existence of traveling wave solutions for the three-species system has been extensively studied in [10–14]. In addition, many scholars [15–18] investigated the speed selection, and for more studies on other aspects of the three-species system, please see [19–21]. Among them, Pan et al. [15] converted the competitive system into a cooperative system and investigated the determinism of the invasion velocity by the upper and lower solution method. We shall directly employ some results in [15] for this study.

For a competitive system, understanding the conditions under which a species survives or dies is always an important and interesting topic in dynamics, and traveling wave solutions can be used to help us answer this question. By a simple calculation, we can find that (2) admits at least five equilibrium points in the range $\{(u,v,w)|0\le u\le 1, 0\le v\le 1, 0\le w\le 1\}$, i.e., (0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1) and (1, 1, 1). Then, this paper focuses on the traveling waves connecting the equilibrium points $e_0=(0,0,0)$ and $e_1=(1,1,1)$ in the form

$$(u,v,w)(x,t)=(\bar{U},\bar{V},\bar{W})(z), \quad z=x-ct, \tag{3}$$

where c is called the wave speed and $(\bar{U},\bar{V},\bar{W})$ is called the wave profile. For the convenience of discussion, we always assume that

$$\frac{1}{2}<b_{12}+b_{13}<1, b_{21}>1, b_{31}>1. \tag{4}$$

A Similar assumption has been made in other papers studying the three-species model, such as [15,17,22], and the assumption is essential for stability properties in this paper. This condition means that v,w are weak competitors of u and it makes the point $(0,0,0)$ unstable and the point $(1,1,1)$ is stable. By substituting (3) into (2), we have

$$\begin{cases} \bar{U}_{zz}+c\bar{U}_z+\bar{U}(1-b_{12}-b_{13}-\bar{U}+b_{12}\bar{V}+b_{13}W)=0, \\ d_1\bar{V}_{zz}+c\bar{V}_z+\alpha(1-\bar{V})(b_{21}\bar{U}-\bar{V})=0, \\ d_2\bar{W}_{zz}+c\bar{W}_z+\beta(1-\bar{W})(b_{31}\bar{U}-\bar{W})=0, \\ (\bar{U},\bar{V},\bar{W})(-\infty)=e_1, \quad (\bar{U},\bar{V},\bar{W})(+\infty)=e_0. \end{cases} \tag{5}$$

The existence of the traveling wave has been given in other related literature. Pan et al. [15] gave the existence of the traveling wave when $c\ge c^*$ and the minimal wave speed is linearly determined for $c^*=c_0=2\sqrt{1-b_{12}-b_{13}}$. Apart from that, the asymptotic behavior of $(\bar{U},\bar{V},\bar{W})$ near the equilibrium point $(0,0,0)$ is also given in [15], see the following lemma.

Lemma 1. *For any $c>c_0$ and constants $C_1>0, C_3>0, C_4>0$, or $C_1=0$ with $C_2>0, C_3>0, C_4>0$, when $z\to\infty$, $(\bar{U},\bar{V},\bar{W})$ has the following asymptotic behavior:*

$$\begin{aligned}\begin{pmatrix}\bar{U}(z)\\ \bar{V}(z)\\ \bar{W}(z)\end{pmatrix} \sim C_1\begin{pmatrix}1\\ -\frac{\alpha b_{21}}{\Gamma_1(\mu_1)}\\ -\frac{\beta b_{31}}{\Gamma_2(\mu_1)}\end{pmatrix}e^{-\mu_1 z}+C_2\begin{pmatrix}1\\ -\frac{\alpha b_{21}}{\Gamma_1(\mu_2)}\\ -\frac{\beta b_{31}}{\Gamma_2(\mu_2)}\end{pmatrix}e^{-\mu_2 z}\\ +C_3\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-\mu_3 z}+C_4\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-\mu_4 z},\end{aligned} \tag{6}$$

where $\Gamma_1(\mu)=d_1\mu^2-c\mu-\alpha$, $\Gamma_2(\mu)=d_2\mu^2-c\mu-\beta$ and

$$\begin{aligned} \mu_1(c)&=\tfrac{1}{2}[c-\sqrt{c^2-4(1-b_{12}-b_{13})}],\\ \mu_2(c)&=\tfrac{1}{2}[c+\sqrt{c^2-4(1-b_{12}-b_{13})}],\\ \mu_3(c)&=\tfrac{1}{2d_1}(c+\sqrt{c^2+4d_1\alpha}),\\ \mu_4(c)&=\tfrac{1}{2d_2}(c+\sqrt{c^2+4d_2\beta}). \end{aligned} \tag{7}$$

Throughout this article, for better determining the weight function later, we always assume that μ_1 is the minimum between $\mu_i (i=1,2,3,4)$. To make the assumption true, we summarize the required parameter conditions and we can find that restrictions are only proposed for c and $d_i(i = 1, 2)$. It is not contrary with other assumptions in our paper.

In order to study the stability of the traveling wave, we need to determine the solution with $(u_0(x), v_0(x), w_0(x))$ as the initial value whether converges to $(\bar{U}, \bar{V}, \bar{W})$. Hence, a change of variables $(u, v, w)(x, t) = (U, V, W)(z, t)$ further transforms (2) into a partial differential model

$$\begin{cases} U_t = U_{zz} + cU_z + U(1 - b_{12} - b_{13} - U + b_{12}V + b_{13}W), \\ V_t = d_1 V_{zz} + cV_z + \alpha(1 - V)(b_{21}U - V), \\ W_t = d_2 W_{zz} + cW_z + \beta(1 - W)(b_{31}U - W), \\ U(z, 0) = u_0(z), V(z, 0) = v_0(z), W(z, 0) = w_0(z), \quad \forall z \in \mathbb{R}. \end{cases} \tag{8}$$

We know that $(\bar{U}, \bar{V}, \bar{W})$ is the steady-state to the above new system. We need to add the following extra assumption about the steady-state in order to obtain global stability:

$$\bar{U} \geq \max\left\{ b_{12}\bar{V} + b_{13}\bar{W}, \frac{1}{b_{21}}\bar{V}, \frac{1}{b_{31}}\bar{W} \right\}. \tag{9}$$

It is not difficult to find that we can demonstrate that the condition (9) is not empty by using the linear selection condition in Theorem 4.1 in [15]. By choosing $\bar{V} = b_{21}\bar{U}$, $\bar{W} = b_{31}\bar{U}$ and combining the condition (4), we have $b_{12}\bar{V} + b_{13}\bar{W} = (b_{12}b_{21} + b_{13}b_{31})\bar{U} \leq \bar{U}$ because of the linear selection condition $-2(1 - b_{12} - b13) + b_{12}b_{21} + b_{13}b_{31} \leq 0$.

The attention, which focused on the stability of traveling waves, has increased and various methods have been shed light on, where the weighted energy method and the spectral analysis were widely used. In terms of local stability, Hou and Li [23] demonstrated the local stability of traveling waves of nonlinear reaction-diffusion equations in different weighted Banach spaces by employing a new method to analyze the location of the spectra. To investigate the stability of the traveling wave solutions with non-critical wave speeds, Leung et al. [24] similarly analyzed the spectrum of the linearization operator in the exponentially weighted Banach space. In terms of global stability, Wu and Xing [25] proved that traveling front solutions with critical speeds are globally exponentially stable in some exponentially weighted spaces. By using a combination of the weighted energy method and the Green's function technique, the global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations was given in [26]. For additional research on stability by using the weighted energy method, see also [27–30].

More specifically, in recent years, there have been numerous investigations on the stability of the Lotka–Volterra diffusion model. Chen et al. [31] applied the weighted energy method to study the nonlinear stability of a discrete three-species Lotka–Volterra competitive diffusion system with monostable traveling wavefronts. The global asymptotic stability of a diffuse multispecies Lotka–Volterra interaction model for the non-homogeneous coexistence equilibrium state was established by using the Lyapunov function method in [32]. Ma and Guo [33] combined the monotonic dynamical systems theory, the sub-super solutions method, master spectrum theory to study the global asymptotic stability of the coexisting steady state of a competitive Lotka–Volterra reaction-diffusion model with an advection term arising. Alhasanat and Ou [2] showed the global stability of the traveling waves of the Lotka–Volterra diffusion model by using the upper–lower solution method together with the squeezing technique. Further reading on the stability of the Lotka–Volterra diffusion model may be found at [28,34–38].

Research on the existence of traveling waves and the choice of linear and nonlinear minimal wave speeds for the three-species competition model has been successful. However, the stability of traveling waves has received less attention. In light of this, we investigate both the local and global stability of the steady-state $(\bar{U}, \bar{V}, \bar{W})$ under the weighted functional space in this research.

Theorem 1. *For any $c > c^*$ and the weight function $w(z)$*

$$w(z) = \begin{cases} e^{a(z-z_0)}, & z > z_0, \\ 1, & z \le z_0 \end{cases} \tag{10}$$

with some constants z_0, $a \in (\mu_1, min\{\mu_2, \mu_3, \mu_4\})$, the traveling wave solution $(\bar{U}, \bar{V}, \bar{W})(z)$ is locally stable in the weighted functional space L_w^p, which is defined in Definition 1.

Theorem 2. *Suppose $c > c^*$, conditions (4)–(9) hold true and the initial data of the solution $(U, V, W)(z, t)$ to (8) are*

$$U_0(z) = U(z, 0), V_0(z) = V(z, 0), W_0(z) = W(z, 0), \tag{11}$$

which satisfy

$$\begin{aligned} &(0, 0, 0) \le (U_0, V_0, W_0)(z) \le (1, 1, 1), \forall z \in \mathbb{R}, \\ &\liminf_{z \to -\infty} (U_0, V_0, W_0)(z) > (0, 0, 0) \end{aligned} \tag{12}$$

and

$$\begin{aligned} &|U_0(z) - \bar{U}(z)| \in L_w^\infty(\mathbb{R}), \\ &|V_0(z) - \bar{V}(z)| \in L_w^\infty(\mathbb{R}), \\ &|W_0(z) - \bar{W}(z)| \in L_w^\infty(\mathbb{R}), \end{aligned} \tag{13}$$

then the traveling wave solution exists globally with

$$(0, 0, 0) \le (U, V, W)(z, t) \le (1, 1, 1), \forall (z, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{14}$$

and for positive constants k and η, there are

$$\begin{aligned} &\sup_{z \in \mathbb{R}} |U(z, t) - \bar{U}(z)| \le k e^{-\eta t}, \quad t > 0, \\ &\sup_{z \in \mathbb{R}} |V(z, t) - \bar{V}(z)| \le k e^{-\eta t}, \quad t > 0, \\ &\sup_{z \in \mathbb{R}} |W(z, t) - \bar{W}(z)| \le k e^{-\eta t}, \quad t > 0, \end{aligned} \tag{15}$$

i.e., any solution satisfying the conditions converges exponentially to the equilibrium solution $(\bar{U}, \bar{V}, \bar{W})(z)$.

Despite the fact that the local stability of the Lotka–Volterra competition system with three species has been demonstrated in [28], we refer to its methodology for the verification of global stability before introducing a new weighted functional space to prove our Theorem 1. The spectral problem is explored in the weighted functional space to determine the sign of the real part of the eigenvalues and further obtain the result of local stability. For global stability, to prove our Theorem 2, we construct the upper solution based on the assumptions (4)–(9), and then the comparison principle is utilized for global stability.

The rest of this paper is organized as follows. In Section 2, we linearize the model and perform a spectral analysis on it in the suitable weighted functional space, which led to the conclusion of local stability. Then, also under the weighted functional space, the global stability is proved by combining the upper-lower solution method and the squeezing theorem in Section 3. Conclusions are shown in Section 4.

2. The Local Stability

We first introduce a weighted functional space L_w^p different from [28] before studying the local stability for the subsequent proof of global stability.

Definition 1. *$L^p(\mathbb{R})$ is the well-known Lebesgue space of integrable functions. Define a weighted functional space L_w^p as follows:*

$$L_w^p = \{f(z) : w(z)f(z) \in L^p(\mathbb{R}), p \ge 1, z \in \mathbb{R}\}. \tag{16}$$

The norm is

$$\|f(z)\|_{L^p_w} = \left(\int_{-\infty}^{\infty} w(z)|f(z)|^p dz\right)^{1/p}, \tag{17}$$

and the weight function is

$$w(z) = \left(\frac{1}{w_1(z)}, \frac{1}{w_2(z)}, \frac{1}{w_3(z)}\right), \tag{18}$$

where

$$\begin{aligned} w_1(z) &= \begin{cases} e^{-\bar{p}(z-z_0)}, & z > z_0, \\ 1, & z \le z_0, \end{cases} \\ w_2(z) &= \begin{cases} e^{-\bar{q}(z-z_0)}, & z > z_0, \\ 1, & z \le z_0, \end{cases} \\ w_3(z) &= \begin{cases} e^{-\bar{r}(z-z_0)}, & z > z_0, \\ 1, & z \le z_0, \end{cases} \end{aligned} \tag{19}$$

with some constants $z_0, \bar{p}, \bar{q}$ and $\bar{r}$, where $\bar{p}, \bar{q}, \bar{r}$ are positive.

In this paper, we study the local stability in the presence of perturbations. By analyzing the behavior of the traveling waves under this small perturbation over a long period of time, the solution can be considered as locally stable if it converges to the steady-state solution.

Let

$$\begin{aligned} U(z,t) &= \bar{U}(z) + \delta\phi_1(z)e^{\lambda t}, \\ V(z,t) &= \bar{V}(z) + \delta\phi_2(z)e^{\lambda t}, \\ W(z,t) &= \bar{W}(z) + \delta\phi_3(z)e^{\lambda t}, \end{aligned} \tag{20}$$

where $\delta \ll 1$, $\phi_1(z), \phi_2(z), \phi_3(z)$ are real functions and λ is a parameter.

Let $\Phi = (\phi_1, \phi_2, \phi_3)^T$ and in order to facilitate the exploration of the spectrum of the operator $\mathscr{L}$ on the space L^p_w, we write Φ in the following form:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} w_1\psi_1 \\ w_2\psi_2 \\ w_3\psi_3 \end{pmatrix}, \tag{21}$$

where $\psi_i (i = 1, 2, 3)$ belong to L^p.

By substituting (20) into (8) and linearizing it at $(\bar{U}, \bar{V}, \bar{W})$, we can obtain the following spectral problem:

$$\lambda\Phi = \mathscr{L}\Phi := D\Phi'' + c\Phi' + J(z)\Phi, \tag{22}$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix}, \tag{23}$$

$$J(z) = \begin{pmatrix} 1 - b_{12} - b_{13} - 2\bar{U} + b_{12}\bar{V} + b_{13}\bar{W} & b_{12}\bar{U} & b_{13}\bar{U} \\ \alpha b_{21}(1 - \bar{V}) & \alpha(-1 - b_{21}\bar{U} + 2\bar{V}) & 0 \\ \beta b_{31}(1 - \bar{W}) & 0 & \beta(-1 - b_{31}\bar{U} + 2\bar{W}) \end{pmatrix}. \tag{24}$$

By examining the maximum real part sign of the spectrum λ of the operator $\mathscr{L}$, we can now evaluate the local stability of the traveling wave solution.

Combine (19) and substitute (21) into (22) to obtain

$$\lambda\Psi = \mathscr{L}_w\Psi := D\Psi'' + M(z)\Psi' + N(z)\Psi, \tag{25}$$

where $\Psi = (\psi_1, \psi_2, \psi_3)^T$,

$$M(z) = \begin{pmatrix} c + 2\frac{w_1'}{w_1} & 0 & 0 \\ 0 & c + 2d_1\frac{w_2'}{w_2} & 0 \\ 0 & 0 & c + 2d_2\frac{w_3'}{w_3} \end{pmatrix}, \tag{26}$$

and

$$\begin{aligned} N(z) = &\begin{pmatrix} \frac{w_1''}{w_1} + c\frac{w_1'}{w_1} & 0 & 0 \\ 0 & d_1\frac{w_2''}{w_2} + c\frac{w_2'}{w_2} & 0 \\ 0 & 0 & d_2\frac{w_3''}{w_3} + c\frac{w_3'}{w_3} \end{pmatrix} \\ &+ \begin{pmatrix} 1 - b_{12} - b_{13} - 2\bar{U} + b_{12}\bar{V} + b_{13}\bar{W} & b_{12}\bar{U}\frac{w_2}{w_1} & b_{13}\bar{U}\frac{w_3}{w_1} \\ \alpha b_{21}(1-\bar{V})\frac{w_1}{w_2} & \alpha(-1 - b_{21}\bar{U} + 2\bar{V}) & 0 \\ \beta b_{31}(1-\bar{W})\frac{w_1}{w_3} & 0 & \beta(-1 - b_{31}\bar{U} + 2\bar{W}) \end{pmatrix}. \end{aligned} \tag{27}$$

Then, we can use the following details from Theorem A.2 in [39] to determine the essential spectrum of the operator $\mathscr{L}_w$. After choosing the weight function to compel the essential spectrum to locate in the left-half complex plane, we may determine the sign of the maximum real part of the point spectrum in the weighted space. We choose

$$\bar{p} \in (\mu_1, \mu_2), \quad \bar{q} \in (0, \mu_3), \quad \bar{r} \in (0, \mu_4) \tag{28}$$

such that

$$\begin{aligned} \bar{p} - \mu_1 < \bar{q} \le \bar{p}, \\ \bar{p} - \mu_1 < \bar{r} \le \bar{p}, \end{aligned} \tag{29}$$

where $\mu_i (i = 1, 2, 3, 4)$ are defined in (7). $M(z)$ and $N(z)$ are bound by the preconditions mentioned above. Therefore, we define

$$\lim_{z \to \pm\infty} M(z) = M_\pm, \lim_{z \to \pm\infty} N(z) = N_\pm, \tag{30}$$

and an algebraic curves $S_\pm$,

$$S_\pm := \{\lambda \mid \det(-\tau^2 D + i\tau M_\pm + N_\pm - \lambda I) = 0, -\infty < \tau < \infty\}. \tag{31}$$

The union of areas within or on the curves S_+ and S_- contains the essential spectrum of the operator $\mathscr{L}_w$. If we prove that $\max(Re(\lambda)) < 0$ for $z \to \pm\infty$, respectively, then $S_\pm$ are on the left-half complex plane, which implies that the essential spectrum of $\mathscr{L}_w$ lies on the left-half complex plane, for further details, see [28].

Because μ_1 is the smallest parameter, we choose $\bar{p} = \bar{q} = \bar{r} = a$, where a is a constant and $a \in (\mu_1, min\{\mu_2, \mu_3, \mu_4\})$. Then, the weight function is as follows:

$$w(z) = \begin{cases} e^{a(z-z_0)}, & z > z_0, \\ 1, & z \le z_0, \end{cases} \tag{32}$$

where z_0 is defined in (19).

In order to obtain the local stability for (22), we next determine the sign of the major eigenvalue in the point spectrum.

Lemma 2. *For $\Phi \in L_w^p$, the real part of the eigenvalue λ of (22) is all negative.*

Proof. Consider an associated linear partial differential system

$$f_t = Df_{zz} + cf_z + J(z)f, \tag{33}$$

where $f(z,t) = (f_1(z,t), f_2(z,t), f_3(z,t))$. By comparing with (22), we know that $e^{\lambda t}\Phi$ is a solution of the above system with the same λ and Φ as in (22). According to the well-known Krein–Rutman theorem in [40], a compact linear operator which is strongly positive has a simple principal eigenvalue with a strongly positive eigenvector. For each given initial data set $\phi \in L^p$, let $R_t = f(t,z,\phi)$ indicate the solution semiflow of (33) and we can find that R_t satisfies the requirements of the theorem. So, we have

$$|e^{\lambda t}| < e^{\lambda_{max} t}, \tag{34}$$

where λ_{max} is the simple principal eigenvalue. To proceed, we can prove $\lambda < 0$ by contradiction for two cases.

Case 1. $\lambda = 0$.

For any $c > c^*$, obviously we have $\bar{U}(z) \sim C_1 e^{-\mu_1 z}, C_1 > 0$, as $z \to \infty$ and $\bar{V}(z), \bar{W}(z)$ are the same asymptotic behaviors as $\bar{U}(z)$. So, the operator $\mathscr{L}$ defined in (22) has an eigenvalue $\lambda = 0$ with the one-sign eigenvector $(-\bar{U}', -\bar{V}', -\bar{W}')(z)$, which is strongly positive. Because of (32) and $a > \mu_1$, we can check that $(-\bar{U}', -\bar{V}', -\bar{W}')(z)$ is not inside the weighted functional space L^p_w.

Case 2. $\lambda > 0$.

For $z \to +\infty$ and $\Phi \in L^p_w$, there is obviously that $\bar{\Phi}(z) = (-\bar{U}', -\bar{V}', -\bar{W}')(z) > \Phi(z)$ except for sets of measure zero.

For $z \to -\infty$, assume that the eigenfunction of (22) possesses the asymptotic behavior as $ke^{\mu z}$ for some positive constants k, μ. Thus, we can obtain the corresponding characteristic equation as follows:

$$\begin{vmatrix} \mu^2 + c\mu - 1 - \lambda & b_{12} & b_{13} \\ 0 & d_1\mu^2 + c\mu + \alpha(1 - b_{21}) - \lambda & 0 \\ 0 & 0 & d_2\mu^2 + c\mu + \beta(1 - b_{31}) - \lambda \end{vmatrix} = 0, \tag{35}$$

which exist three positive roots,

$$\begin{aligned} \hat{\mu}_1 &= \frac{-c + \sqrt{c^2 + 4(1+\lambda)}}{2}, \\ \hat{\mu}_2 &= \frac{-c + \sqrt{c^2 + 4d_1(\lambda + \alpha(b_{21} - 1))}}{2}, \\ \hat{\mu}_3 &= \frac{-c + \sqrt{c^2 + 4d_2(\lambda + \beta(b_{31} - 1))}}{2}. \end{aligned} \tag{36}$$

These statements demonstrate that μ increases with λ, which also implies that $\bar{\Phi}(z) > \Phi(z)$.

Hence, we can choose a sufficiently large $\bar{k}$ such that $\bar{k}\bar{\Phi} \geq |\Phi|$. Both $\bar{\Phi}$ and $|\Phi|e^{\lambda t}$ are solutions of (33), so by using the comparison principle, we have $\bar{k}\bar{\Phi} \geq |\Phi|e^{\lambda t}$, which is not correct for a sufficiently large t. Thus, we complete the proof. □

3. The Global Stability

This section will analyze the global stability of the equilibrium solution $(\bar{U}, \bar{V}, \bar{W})(z)$ in the weighted functional space L^p_w with $p = \infty$, where the norm is defined as $\|f\|_{L^\infty_w} = esssup_{z\in\mathbb{R}}|w(z)f(z)|$ with a special weight function $w(z)$. Based on Theorem 1, we choose the weight function $w(z)$, which is defined in (32), and let

$$a = \mu_1 + \epsilon, \tag{37}$$

where ϵ is a sufficiently small positive number.

First of all, assume the initial data of the solution of (8) as

$$U_0(z) = U(z,0), V_0(z) = V(z,0), W_0(z) = W(z,0), \tag{38}$$

which satisfy

$$\begin{aligned}&(0,0,0) \le (U_0, V_0, W_0)(z) \le (1,1,1), \forall z \in \mathbb{R},\\&\liminf_{z\to-\infty}(U_0, V_0, W_0)(z) > (0,0,0)\end{aligned} \tag{39}$$

and

$$\begin{aligned}&|U_0(z) - \bar{U}(z)| \in L^\infty_w(\mathbb{R}),\\&|V_0(z) - \bar{V}(z)| \in L^\infty_w(\mathbb{R}),\\&|W_0(z) - \bar{W}(z)| \in L^\infty_w(\mathbb{R}).\end{aligned} \tag{40}$$

Based on the above conditions, for $z \in \mathbb{R}$, we define

$$\begin{aligned}U_0^+(z) &= \max\{U_0(z), \bar{U}(z)\}, & U_0^-(z) &= \min\{U_0(z), \bar{U}(z)\},\\V_0^+(z) &= \max\{V_0(z), \bar{V}(z)\}, & V_0^-(z) &= \min\{V_0(z), \bar{V}(z)\},\\W_0^+(z) &= \max\{W_0(z), \bar{W}(z)\}, & W_0^-(z) &= \min\{W_0(z), \bar{W}(z)\}\end{aligned} \tag{41}$$

which can be viewed as the initial value of the solutions (U^+, V^+, W^+) and (U^-, V^-, W^-) for (8). That is to say that (U^+, V^+, W^+) and (U^-, V^-, W^-) satisfy

$$\begin{cases}U_t^\pm = U_{zz}^\pm + cU_z^\pm + U^\pm(1 - b_{12} - b_{13} - U^\pm + b_{12}V^\pm + b_{13}W^\pm),\\V_t^\pm = d_1V_{zz}^\pm + cV_z^\pm + \alpha(1 - V^\pm)(b_{21}U^\pm - V^\pm),\\W_t^\pm = d_2W_{zz}^\pm + cW_z^\pm + \beta(1 - W^\pm)(b_{31}U^\pm - W^\pm),\\(U_0^\pm, V_0^\pm, W_0^\pm)(z) = (U^\pm, V^\pm, W^\pm)(z,0).\end{cases} \tag{42}$$

By using the comparison principle, we have

$$\begin{aligned}(0,0,0) &\le (U^-, V^-, W^-)(z,t) \le (U, V, W)(z,t)\\&\le (U^+, V^+, W^+)(z,t) \le (1,1,1), \quad \forall (z,t) \in \mathbb{R}\times\mathbb{R}^+,\\(0,0,0) &\le (U^-, V^-, W^-)(z,t) \le (\bar{U}, \bar{V}, \bar{W})(z)\\&\le (U^+, V^+, W^+)(z,t) \le (1,1,1), \quad \forall (z,t) \in \mathbb{R}\times\mathbb{R}^+.\end{aligned} \tag{43}$$

Next, we need to demonstrate the convergence of $(U^\pm, V^\pm, W^\pm)(z,t)$ to the wavefront $(\bar{U}, \bar{V}, \bar{W})(z)$ in the subsequent lemmas, respectively.

Lemma 3. *Under the conditions (38)–(42), $(U^+, V^+, W^+)(z,t)$ converges to $(\bar{U}, \bar{V}, \bar{W})(z)$.*

Proof. We define

$$\begin{aligned}F(z,t) &= U^+(z,t) - \bar{U}(z),\\G(z,t) &= V^+(z,t) - \bar{V}(z),\\H(z,t) &= W^+(z,t) - \bar{W}(z),\end{aligned} \tag{44}$$

with the initial value

$$\begin{aligned}F(z,0) &= U_0^+(z) - \bar{U}(z),\\G(z,0) &= V_0^+(z) - \bar{V}(z),\\H(z,0) &= W_0^+(z) - \bar{W}(z).\end{aligned} \tag{45}$$

It is simple to see from inequality (43) that

$$(0,0,0) \le (F, G, H)(z,t) \le (1,1,1), \quad \forall z \in \mathbb{R}, \tag{46}$$

for $t \ge 0$. Afterwards, by combining (5) and (42) and performing some transformations, we obtain

$$\begin{cases}F_t = F_{zz} + cF_z + (1 - b_{12} - b_{13})F + (F + \bar{U})(-F + b_{12}G + b_{13}H) + (-\bar{U} + b_{12}\bar{V} + b_{13}\bar{W})F,\\G_t = d_1G_{zz} + cG_z + \alpha(b_{21}F - G) + \alpha(G + \bar{V})(-b_{21}F + G) + \alpha(-b_{21}\bar{U} + \bar{V})G,\\H_t = d_2H_{zz} + cH_z + \beta(b_{31}F - H) + \beta(H + \bar{W})(-b_{31}F + H) + \beta(-b_{31}\bar{U} + \bar{W})H.\end{cases} \tag{47}$$

Let

$$\begin{pmatrix} F \\ G \\ H \end{pmatrix}(z,t) = e^{-a(z-z_0)} \begin{pmatrix} \bar{F} \\ \bar{G} \\ \bar{H} \end{pmatrix}(z,t), \forall (z,t) \in (\mathbb{R}, \mathbb{R}^+), \tag{48}$$

where $\bar{F}, \bar{G}, \bar{H} \in L^\infty(\mathbb{R})$ and z_0 is defined in (32). We will then demonstrate this lemma in two scenarios.

Case 1. Assume $z \in [z_0, +\infty)$ for any fixed z_0.

Substituting (48) into (47), we have

$$\begin{aligned} \begin{pmatrix} \bar{F} \\ \bar{G} \\ \bar{H} \end{pmatrix}_t &= D \begin{pmatrix} \bar{F} \\ \bar{G} \\ \bar{H} \end{pmatrix}_{zz} + Q \begin{pmatrix} \bar{F} \\ \bar{G} \\ \bar{H} \end{pmatrix}_z + A(a) \begin{pmatrix} \bar{F} \\ \bar{G} \\ \bar{H} \end{pmatrix} + \begin{pmatrix} (-\bar{U} + b_{12}\bar{V} + b_{13}\bar{W})\bar{F} \\ \alpha(-b_{21}\bar{U} + \bar{V})\bar{G} \\ \beta(-b_{31}\bar{U} + \bar{W})\bar{H} \end{pmatrix} \\ &+ \begin{pmatrix} (e^{-a(z-z_0)}\bar{F} + \bar{U})(-\bar{F} + b_{12}\bar{G} + b_{13}\bar{H}) \\ \alpha(e^{-a(z-z_0)\bar{G}} + \bar{V})(-b_{21}\bar{F} + \bar{G}) \\ \beta(e^{-a(z-z_0)\bar{H}} + \bar{W})(-b_{31}\bar{F} + \bar{H}) \end{pmatrix} \\ &:= \begin{pmatrix} \mathscr{L}_1(\bar{F}, \bar{G}, \bar{H}) \\ \mathscr{L}_2(\bar{F}, \bar{G}, \bar{H}) \\ \mathscr{L}_3(\bar{F}, \bar{G}, \bar{H}) \end{pmatrix}, \end{aligned} \tag{49}$$

where D is defined in (23),

$$Q = \begin{pmatrix} c-2a & 0 & 0 \\ 0 & c-2d_1a & 0 \\ 0 & 0 & c-2d_2a \end{pmatrix} \tag{50}$$

and

$$A(a) = \begin{pmatrix} \Gamma_3(a) & 0 & 0 \\ \alpha b_{21} & \Gamma_1(a) & 0 \\ \beta b_{31} & 0 & \Gamma_2(a) \end{pmatrix}, \tag{51}$$

where $\Gamma_3(a) = a^2 - ca + 1 - b_{12} - b_{13}$ and Γ_1, Γ_2 are given in Lemma 1. Assume that $(\xi_1, \xi_2, \xi_3) = (\xi_1(a), \xi_2(a), \xi_3(a))$ is the eigenvector of the matrix $A(a)$ at eigenvalue $a^2 - ca + 1 - b_{12} - b_{13}$ and a direct calculation gives

$$\begin{aligned} \xi_1 &= \Gamma_3(a) - \Gamma_1(a) = (1-d_1)(\mu_1^2 + \epsilon) + 1 - b_{12} - b_{13} + \alpha, \\ \xi_2 &= \alpha b_{21}, \\ \xi_3 &= \frac{\beta b_{31}(\Gamma_3(a) - \Gamma_1(a))}{\Gamma_3(a) - \Gamma_2(a)} = \frac{\beta b_{31}[(1-d_1)(\mu_1^2+\epsilon) + 1 - b_{12} - b_{13} + \alpha]}{(1-d_2)(\mu_1^2+\epsilon) + 1 - b_{12} - b_{13} + \beta}. \end{aligned} \tag{52}$$

Then, we also define

$$\begin{aligned} \bar{F}_1(z,t) &= k_1\xi_1 e^{-\eta_1 t}, \\ \bar{G}_1(z,t) &= k_1\xi_2 e^{-\eta_1 t}, \\ \bar{H}_1(z,t) &= k_1\xi_3 e^{-\eta_1 t}, \quad \forall (z,t) \in (\mathbb{R}, \mathbb{R}^+), \end{aligned} \tag{53}$$

where k_1, η_1 are positive. Since $\bar{F}(z,0), \bar{G}(z,0), \bar{H}(z,0) \in L_w^\infty$, thus we can choose

$$k_1 \geq max_{z\in\mathbb{R}}\{\frac{\bar{F}(z,0)}{\xi_1}, \frac{\bar{G}(z,0)}{\xi_2}, \frac{\bar{H}(z,0)}{\xi_3}\}. \tag{54}$$

For $z \to +\infty$, by using (9), substituting (54) into the right side of (49) and performing the calculation, we find

$$\begin{aligned} &\mathscr{L}_1(\bar{F}_1, \bar{G}_1, \bar{H}_1) < 0, \\ &\mathscr{L}_2(\bar{F}_1, \bar{G}_1, \bar{H}_1) < 0, \\ &\mathscr{L}_3(\bar{F}_1, \bar{G}_1, \bar{H}_1) < 0. \end{aligned} \tag{55}$$

This means that we can find a suitable η_1 such that the inequality

$$\begin{pmatrix} \bar{F}_1 \\ \bar{G}_1 \\ \bar{H}_1 \end{pmatrix}_t = -\eta_1 k_1 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_2 \end{pmatrix} e^{-\eta_1 t} \geq \begin{pmatrix} \mathscr{L}_1(\bar{F}_1, \bar{G}_1, \bar{H}_1) \\ \mathscr{L}_2(\bar{F}_1, \bar{G}_1, \bar{H}_1) \\ \mathscr{L}_3(\bar{F}_1, \bar{G}_1, \bar{H}_1) \end{pmatrix} \tag{56}$$

holds.

Hence, $(\bar{F}_1, \bar{G}_1, \bar{H}_1)$ is equivalent to an upper solution. Then, by using the comparison principle on an unbounded domain, see [41], we have

$$\begin{aligned} (F, G, H)(z, t) &= (\bar{F}, \bar{G}, \bar{H}) e^{-a(z - z_0)} \\ &\leq (\bar{F}_1, \bar{G}_1, \bar{H}_1) e^{-a(z - z_0)} \\ &= k_1(\xi_1, \xi_2, \xi_3) e^{-a(z - z_0) - \eta_1 t}, \quad \forall (z, t) \in (z_0, +\infty] \times \mathbb{R}^+. \end{aligned} \tag{57}$$

Now, we also need to verify the convergence of (F, G, H) to (0,0,0) at $z \in (-\infty, z_0]$.

Case 2. Assume $z \in (-\infty, z_0]$ for any fixed z_0.

System (47) can be represented in another form:

$$\begin{pmatrix} F \\ G \\ H \end{pmatrix}_t = D \begin{pmatrix} F \\ G \\ H \end{pmatrix}_{zz} + c \begin{pmatrix} F \\ G \\ H \end{pmatrix}_z + J(z) \begin{pmatrix} F \\ G \\ H \end{pmatrix} + \begin{pmatrix} (-F + b_{12} G + b_{13} H) F \\ \alpha(-b_{21} F + G) G \\ \beta(-b_{31} F + H) H \end{pmatrix} \tag{58}$$

where $J(z)$ is defined in (24) and we write $J(z)$ as $J(z) = (J_{ij})_{3\times 3}$. Now, we present a new 3×3 matrix B_{ϵ_1},

$$B_{\epsilon_1} = \begin{pmatrix} -1 + \epsilon_1 & b_{12} + \epsilon_1 & b_{13} + \epsilon_1 \\ \epsilon_1 & \alpha(1 - b_{21}) + \epsilon_1 & 0 \\ \epsilon_1 & 0 & \beta(1 - b_{31}) + \epsilon_1 \end{pmatrix} = (B_{ij})_{3\times 3}, \tag{59}$$

for some given small $\epsilon_1 > 0$. When $z \in (-\infty, z_0]$, due to the fact that $(\bar{U}, \bar{V}, \bar{W})$ is nearing (1, 1, 1) for any z in this range, the inequality $J_{ij} < B_{ij} (i, j = 1, 2, 3)$ holds.

If we build an autonomous system related to B_{ϵ_1} with $(\hat{F}, \hat{G}, \hat{H})(t)$ as the solution:

$$\begin{pmatrix} \hat{F} \\ \hat{G} \\ \hat{H} \end{pmatrix}_t = B \begin{pmatrix} \hat{F} \\ \hat{G} \\ \hat{H} \end{pmatrix} + \begin{pmatrix} (-\hat{F} + b_{12} \hat{G} + b_{13} \hat{H}) \hat{F} \\ \alpha(-b_{21} \hat{F} + \hat{G}) \hat{G} \\ \beta(-b_{31} \hat{F} + \hat{H}) \hat{H} \end{pmatrix}, \tag{60}$$

and the initial value satisfies

$$\begin{aligned} &\hat{F}(0) \geq \bar{F}(z, 0), \\ &\hat{G}(0) \geq \bar{G}(z, 0), \\ &\hat{H}(0) \geq \bar{H}(z, 0), \quad \forall z \in \mathbb{R}, \end{aligned} \tag{61}$$

then we can verify that $(\hat{F}, \hat{G}, \hat{H})(t)$ is an upper solution to the system (58).

We must now determine if $(\hat{F}, \hat{G}, \hat{H})(t)$ converges to $(0, 0, 0)$ as $t \to \infty$. We can use the Jacobi matrix $J(0, 0, 0)$ to examine the behavior close to $(0, 0, 0)$, which is one of its fixed points. By using (60), the equation $J(0, 0, 0) = B_{\epsilon_1}$ has three eigenvalues denoted as $\hat{\lambda}_3 < \hat{\lambda}_2 < \hat{\lambda}_1 < 0$. As a result, the point at $(0, 0, 0)$ is stable, meaning that the flow in the $\hat{F}\hat{G}\hat{H}$-space converges to the origin for every $(\hat{F}, \hat{G}, \hat{H})(0)$ in the range

$[0,1]\times[0,\delta_1]\times[0,\delta_2]$ with $0<\delta_i\le 1(i=1,2)$. The maximum possible value of $\delta_i(i=1,2)$ depends on the position of the nonconstant fixed point to the system (3.24) near or inside the box $[0,1]\times[0,1]\times[0,1]$. If the point is far away from the box, then $\delta_i(i=1,2)$ can be 1; If the point is near the boundary of the box, then the maximum possible value of δ_1 in $(b_{21}-1-\frac{\epsilon_1}{\alpha},1)$ and δ_2 in $(b_{31}-1-\frac{\epsilon_1}{\beta},1)$; if the point is inside the box, then δ_1 is close to $b_{21}-1-\frac{\epsilon_1}{\alpha}$ and δ_2 is close to $b_{31}-1-\frac{\epsilon_1}{\beta}$. Thus, we find that

$$(\hat{F},\hat{G},\hat{H})=\hat{k}_1(\hat{C}_1,\hat{C}_2,\hat{C}_3)e^{\hat{\lambda}_1 t},t\to\infty. \tag{62}$$

Here, $\hat{k}_1>0$ and $(\hat{C}_1,\hat{C}_2,\hat{C}_3)$ is the eigenvector of B_{ϵ_1} with the eigenvalue $\hat{\lambda}_1$.

Finally, we have

$$(F,G,H)(z_0,t)\le k_1(\xi_1,\xi_2,\xi_3)e^{-\eta_1 t}\le\hat{k}_1(\xi_1,\xi_2,\xi_3)e^{-\bar{\lambda}_1 t} \tag{63}$$

at $z=z_0$ by choosing a large enough $\hat{k}_1$ and $\bar{\lambda}_1=\min\{\eta_1,-\hat{\lambda}_1\}$. And by comparison on the domain$(-\infty,z_0]\times[0,\infty)$, see [42], we find that

$$(F,G,H)(z,t)\le\hat{k}_1(\xi_1,\xi_2,\xi_3)e^{-\bar{\lambda}_1 t},\quad\forall(z,t)\in(-\infty,z_0]\times\mathbb{R}^+. \tag{64}$$

Up to here, the proof is complete. □

Lemma 4. *Under the above conditions (38)–(42), $(U^-,V^-,W^-)(z,t)$ converges to $(\bar{U},\bar{V},\bar{W})(z)$.*

Proof. We define

$$\begin{aligned} I(z,t)&=\bar{U}(z)-U^-(z,t),\\ K(z,t)&=\bar{V}(z)-V^-(z,t),\\ S(z,t)&=\bar{W}(z)-W^-(z,t), \end{aligned} \tag{65}$$

with the initial value

$$\begin{aligned} I(z,0)&=\bar{U}(z)-U_0^-(z),\\ K(z,0)&=\bar{V}(z)-V_0^-(z),\\ S(z,0)&=\bar{V}(z)-W_0^-(z). \end{aligned} \tag{66}$$

By inequalities (43), it is easy to see that

$$(0,0,0)\le(I,K,S)(z,t)\le(1,1,1),\quad\forall z\in\mathbb{R},\quad t\ge 0. \tag{67}$$

Then, repeat the steps above, and I, K and S satisfy the system

$$\begin{pmatrix} I\\ K\\ S\end{pmatrix}_t=D\begin{pmatrix} I\\ K\\ S\end{pmatrix}_{zz}+c\begin{pmatrix} I\\ K\\ S\end{pmatrix}_z+J(z)\begin{pmatrix} I\\ K\\ S\end{pmatrix}-\begin{pmatrix}(-I+b_{12}K+b_{13}S)I\\ \alpha(-b_{21}I+K)K\\ \beta(-b_{31}I+S)S\end{pmatrix}, \tag{68}$$

where $J(z)$ is defined in (24). Similarly, we analyze it in two cases.

Case 1. Let $(z,t)\in(z_0,+\infty]\times\mathbb{R}^+$.

By using an approach similar to the proof of Lemma 3 with (9) and the facts $I<\bar{U},K<\bar{V},S<\bar{W}$. There exist $\eta_2>0$ and

$$k_2\ge e^{a(z-z_0)}max_{z\in\mathbb{R}}\{\frac{I(z,0)}{\xi_1},\frac{K(z,0)}{\xi_2},\frac{S(z,0)}{\xi_3}\} \tag{69}$$

such that

$$(I,K,S)(z,t)\le k_2(\xi_1,\xi_2,\xi_3)e^{-\eta_2 t},\quad\forall(z,t)\in(z_0,+\infty]\times\mathbb{R}^+. \tag{70}$$

Case 2. Let $(z,t)\in(-\infty,z_0]\times\mathbb{R}^+$.

Now, we need to introduce $w(z)$ defined in (32) with $a = \mu_1 + \epsilon$ to study the stability under the weighted functional space L^p_w. Defined

$$\begin{pmatrix} \hat{I} \\ \hat{K} \\ \hat{S} \end{pmatrix}_t = B_{\epsilon_1} \begin{pmatrix} \hat{I} \\ \hat{K} \\ \hat{S} \end{pmatrix} - \frac{1}{w(z)} \begin{pmatrix} (-\hat{I} + b_{12}\hat{K} + b_{13}\hat{S})\hat{I} \\ \alpha(-b_{21}\hat{I} + \hat{K})\hat{K} \\ \beta(-b_{31}\hat{I} + \hat{S})\hat{S} \end{pmatrix}, \tag{71}$$

and the initial date satisfies

$$\begin{aligned} &\hat{I}(0) \geq I(z,0), \\ &\hat{K}(0) \geq K(z,0), \\ &\hat{S}(0) \geq S(z,0), \quad \forall z \in \mathbb{R}. \end{aligned} \tag{72}$$

We can check that $(\hat{I}, \hat{K}, \hat{S})(t)$ is an upper solution to the system (68). As in Lemma 3, $(\hat{I}, \hat{K}, \hat{S})$ also converges to $(0,0,0)$ when all initial value on the space $[0,1] \times [0,1] \times [0,1]$ except $(1,1,1)$ by analyzing the phase plane. Finally, for some $\hat{k}_2, \bar{\lambda}_2 > 0$, we have

$$(I,K,S)(z,t) \leq \hat{k}_2(\xi_1, \xi_2, \xi_3)e^{-\bar{\lambda}_2 t}, \quad \forall (z,t) \in (-\infty, z_0] \times \mathbb{R}^+. \tag{73}$$

This completes the proof. □

In the end, we can prove Theorem 2 on the global stability.

Proof of Theorem 2. From (43), we have

$$\begin{aligned} |I(z,t)| \leq |U(z,t) - \bar{U}(z)| \leq |F(z,t)|, \\ |K(z,t)| \leq |V(z,t) - \bar{V}(z)| \leq |G(z,t)|, \\ |S(z,t)| \leq |W(z,t) - \bar{W}(z)| \leq |H(z,t)|, \end{aligned} \tag{74}$$

for $\forall (z,t) \in \mathbb{R} \times \mathbb{R}^+$. Combining Lemmas 3 and 4 and the squeezing theorem, it is easy to find that, for all $(z,t) \in \mathbb{R} \times \mathbb{R}^+$,

$$\begin{aligned} &|U(z,t) - \bar{U}(z)| \leq ke^{-\eta t}, \quad t > 0, \\ &|V(z,t) - \bar{V}(z)| \leq ke^{-\eta t}, \quad t > 0, \\ &|W(z,t) - \bar{W}(z)| \leq ke^{-\eta t}, \quad t > 0, \end{aligned} \tag{75}$$

where $k, \eta > 0$. Hence, the proof is done. □

4. Conclusions

We examined if traveling waves in the Lotka–Volterra competition model with three species (2) display both local and global stability under the condition (4). Theorem 1 demonstrates, utilizing linearization and the crucial spectrum analysis, that the traveling wave solution is locally stable in a weighted functional space. Additionally, Theorem 2 demonstrates that all solutions converge to the wavefront solution using the upper-and-lower solution method and the squeezing theorem under the added constraint (9).

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