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Dirichlet and Neumann Boundary Value Problems for Dunkl Polyharmonic Equations

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Abstract: Dunkl operators are a family of commuting differential–difference operators associated with a finite reflection group. These operators play a key role in the area of harmonic analysis and theory of spherical functions. We study the solution of the inhomogeneous Dunkl polyharmonic equation based on the solutions of Dunkl–Possion equations. Furthermore, we construct the solutions of Dirichlet and Neumann boundary value problems for Dunkl polyharmonic equations without invoking the Green's function.

Keywords: neumann problem; dirichlet problem; dunkl polyharmonic equation

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1. Introduction

Dunkl operators were introduced by Dunkl [1,2]. These operators are first-order differential–difference operators which generalize partial derivatives. Moreover, they are commuting. Their most important property is invariant under reflections. Based on Dunkl operators, people can construct Dunkl Laplace operators. The Dunkl Laplace operator is the sum of a second-order differential operator, which is used to study models of mechanics [3,4]. In fact, the study of the theory of Dunkl Laplacian operators is a very difficult task. The main reason for this difficulty is that Dunkl Laplace is not invariant under the whole orthogonal group. However, it is the intertwining operator that allows interchange in the Dunkl derivatives with the usual partial derivatives. The property of the operator allows us to establish the structure of the Lie algebra [5–8]. Based on the Lie algebra structure, we study Dirichlet and Neumann boundary value problems via the framework of Dunkl analysis in this paper.

The Dirichlet problem (see [9]) is a very important boundary value problem for polyharmonic equations. The solutions of Dirichlet problem and its related problems for polyharmonic equations are given via Green function. Furthermore, the solvability conditions for these problems were also studied in past studies [10–13]. Neumann boundary value problems, unlike Dirichlet problems, require more restrictions on the boundary conditions and are more complicated [14]. The solutions of the Neumann problem for polyharmonic equations are given via the well-known Almansi formula without invoking the Green's function [15]. It is the aim of the present paper to extend this idea to study Dirichlet and Neumann boundary value problems related to Dunkl polyhamonic equations in a different way.

In this paper, we begin with an introduction to Dunkl operators and Dunkl Laplace operators. In the next section, we construct solutions for inhomogeneous Dunkl polyharmonic equations based on the solutions of Dunkl–Possion equation [16]. In Section 4, we investigate Dirichlet problems for Dunkl biharmonic equations. Moreover, we study

Dirichlet problems for Dunkl polyharmonic equations in Section 5. In Section 6, we consider solutions of Neumann problem for a non-homogeneous Dunkl polyharmonic equation.

2. Preliminaries

The purpose of this section is to introduce Dunkl and Dunkl Laplace operators. For these details, readers can refer to [1–3,5–8].

Let R^m be the Euclidean space. Let $\{e_1, e_2, \cdots e_m\}$ be the standard basis of R^m. Let $R_{0,m}$ be the associated real Clifford algebra in which $e_i e_j + e_j e_i = -2\delta_{i,j}$. In fact, $\delta_{i,j} = 0$, if $i \neq j$; thus, $\delta_{i,j} = 1$, if $i = j$. The vector space $R_{0,m}$ is generated via $e_A = e_{l_1} e_{l_2} \cdots e_{l_k}$, where $1 \leq l_1 < l_2 \cdots < l_k \leq m$, $e_0 = 1$. Each $a \in R_{0,m}$ can be written as $a = \sum\limits_A a_A e_A$, where $a_A \in R$. Let $x = (x_1, \cdots, x_m) \in R^m$. Thus, we have $x = \sum\limits_{i=1}^{m} x_i e_i$. Furthermore, it is easy to obtain $x^2 = -|x|^2$.

For $\xi \in R^m \backslash \{0\}$, the reflection σ_ξ is defined through

$$\sigma_\xi(x) = x - 2\frac{\langle x,\ \xi\rangle}{|\xi|^2}\xi,\ x \in R^m.$$

Let $\Re$ be a finite subset of R^m. If $\sigma_\xi(\Re) = \Re$, the set $\Re$ is called a root system. Let $\Re_+$ be a hyperplane through the origin. Thus, we have $\Re = \Re_+ \cup (-\Re_+)$. The subgroup $W \subset O(m, \Re)$, generated via the reflections $\{\sigma_\xi | \xi \in \Re\}$, is called the finite reflection group.

If a function $\kappa : \Re \to C$ is invariant under the group W, the function κ is called a multiplicity function. Setting $\kappa_\xi := \kappa(\xi)$, for $\xi \in \Re$. We will denote $\gamma = \sum\limits_{\xi \in \Re_+} \kappa_\xi$.

For $g(x) \in C^1(R^m)$, the Dunkl operators T_i are given as

$$T_i g(x) = \frac{\partial g(x)}{\partial x_i} + \sum_{\xi \in \Re_+} \kappa_\xi \frac{g(x) - g(\sigma_\xi x)}{\langle x,\ \xi\rangle}\xi_i,$$

where $i = 1, \ldots, m$.

The Dunkl Laplace operator Δ_h is given as

$$\Delta_h g(x) = \sum_{i=1}^{m} T_i^2 g(x) = \Delta g(x) + 2\sum_{\xi \in R_+} \kappa_\xi \left(\frac{\langle \nabla g(x),\ \xi\rangle}{\langle \xi,\ x\rangle} - \frac{g(x) - g(\sigma_\xi x)}{\langle \xi, x\rangle^2} \right),$$

where Δ is the Laplace operator and ∇ is the gradient operator. For $\kappa_\xi = 0$, the Dunkl Laplace operator Δ_h is the Laplace operator. Let $g(x) \in C^2(R^m)$. If the function $g(x)$ satisfies $\Delta_h g(x) = 0$, it is called a Dunkl harmonic function.

If we allow Δ_h to act on x^2, we have $\Delta_h x^2 = -\Delta_h |x^2| = -(4m + 2\gamma) = -2\mu$, where $\mathrm{Re}\gamma \geq 0$, and μ is considered as the Dunkl version of the dimension.

3. Solutions of Inhomogeneous Dunkl Polyharmonic Equations

Definition 1. *Let $\Omega \subset R^m$. Let E be the Euler operator. Therefore, the generalized Euler operator E_λ is given as*

$$E_\lambda = \lambda + E = \lambda + \sum_{i=1}^{m} x_i \partial_{x_i},$$

where $\lambda \in R$. It allows us to obtain the property: if the function $g(x)$ satisfies $\Delta_h g(x) = 0$, it can be said that $E_\lambda g(x)$ also satisfies $\Delta_h g(x) = 0$. Morever, we have $EP_l(x) = lP_l(x)$, where $P_l(x)$ is a homogeneous Clifford-valued polynomial of degree l.

Lemma 1. **[5].** *The operators x^2, Δ_h, E_μ generate the lie algebra*

$$E_{\frac{\mu}{2}}x^2 - x^2E_{\frac{\mu}{2}} = 2x^2, E_{\frac{\mu}{2}}\Delta_h - \Delta_h E_{\frac{\mu}{2}} = 2\Delta_h, x^2\Delta_h - \Delta_h x^2 = 4E_{\frac{\mu}{2}}.$$

Let Ω^* be a star domain. Let $f(x) \in C^\infty(\Omega^*) \otimes R_{0,m}$. Therefore, we study a solution of the Dunkl–Possion equation via Clifford analysis

$$\Delta_h g(x) = f(x). \tag{1}$$

Theorem 1. **[16].** *Let $f(x) \in C^\infty(\Omega^*) \otimes R_{0,m}$. A solution of Equation (1) can be found in the form*

$$g(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \Delta_h^s f(\alpha x) d\alpha. \tag{2}$$

We assume all infinite series in this paper converge absolutely and uniformly in the unit ball $S = \{x \in R^m : |x| < 1\}$.

In this section, we consider the inhomogeneous Dunkl polyharmonic equation

$$\Delta_h^k g(x) = f(x) \tag{3}$$

where $f(x) \in C^\infty(S) \otimes R_{0,m}$.

Using Theorem 1, we obtain the following result.

Theorem 2. *Let $f(x) \in C^\infty(S) \otimes R_{0,m}$. Therefore,*

$$g(x) = \frac{x^{2k}}{2^{k-1}(k-1)!} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{4^{s+k}(s+k)!s!} \int_0^1 (1-\alpha)^{s+k-1} \alpha^{\mu+s-1} \Delta_h^s f(\alpha x) d\alpha \tag{4}$$

is a solution of the Equation (3).

Proof of Theorem 2. We prove via induction. For $k = 1$,

$$g(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \Delta_h^s f(\alpha x) d\alpha, \tag{5}$$

which is a solution of Equation (3) for $k = 1$. We suppose that formula (4) holds for $k = p$. Therefore, for $k = p + 1$, we will prove that this formula is also valid.

Let $\Delta_h^p g(x) = u(x)$. Next, using (3), we have $\Delta_h u(x) = f(x)$. Through applying Theorem 1, we have

$$u(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!} \int_0^1 (1-\alpha)^s \alpha^{\mu+s-1} \Delta_h^s f(\alpha x) d\alpha. \tag{6}$$

In addition, using the inductive assumption, $g(x)$ can be written as

$$g(x) = \frac{x^{2p}}{2^{p-1}(p-1)!} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{4^{s+p}(s+p)!s!} \int_0^1 (1-\alpha)^{s+p-1} \alpha^{\mu+s-1} \Delta_h^s u(\alpha x) d\alpha. \tag{7}$$

Using $\Delta_h^s u(x) = \Delta_h^{s-1} f(x)$, from (7), we obtain the representation

$$\begin{aligned} g(x) = &\tfrac{x^{2p}}{2^{p-1}(p-1)!} \tfrac{1}{4^p p!} \textstyle\int_0^1 (1-\alpha)^{p-1} \alpha^{\mu-1} u(\alpha x) d\alpha \\ &+ \tfrac{x^{2p}}{2^{p-1}(p-1)!} \textstyle\sum_{s=1}^{\infty} \tfrac{(-1)^s x^{2s}}{4^{s+p}(s+p)!s!} \int_0^1 (1-\alpha)^{s+p-1} \alpha^{\mu+s-1} \Delta_h^{s-1} f(\alpha x) d\alpha. \end{aligned} \tag{8}$$

Let us transform the first integral in relation (8) with the use of the representation (6):

$$\begin{aligned}
&\tfrac{1}{4^p p!}\int_0^1 (1-\alpha)^{p-1}\alpha^{\mu-1}u(\alpha x)d\alpha \\
&= \tfrac{1}{4^p p!}\int_0^1 (1-\alpha)^{p-1}\alpha^{\mu-1}\sum_{s=0}^{\infty}\tfrac{(-1)^s(\alpha x)^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 (1-\beta)^s\beta^{\mu+s-1}\Delta_h^s f(\alpha\beta x)d\beta d\alpha \\
&= \tfrac{1}{4^p p!}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1\int_0^1 (1-\alpha)^{p-1}\alpha^{2s+\mu+1}(1-\beta)^s\beta^{\mu+s-1}\Delta_h^s f(\alpha\beta x)d\beta d\alpha \\
&= \tfrac{1}{4^p p!}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1\int_0^1 (1-\alpha)^{p-1}\alpha(\alpha-\alpha\beta)^s(\alpha\beta)^{\mu+s-1}\Delta_h^s f(\alpha\beta x)d(\alpha\beta)d\alpha.
\end{aligned}$$

Let $\alpha\beta = t$. Therefore, through changing the integration order, we have

$$\begin{aligned}
&\tfrac{1}{4^p p!}\int_0^1 (1-\alpha)^{p-1}\alpha^{\mu-1}u(\alpha x)d\alpha \\
&= \tfrac{1}{4^p p!}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1\int_0^{\alpha} (1-\alpha)^{p-1}\alpha(\alpha-t)^s t^{\mu+s-1}\Delta_h^s f(tx)dtd\alpha
\end{aligned}$$

$$= \frac{1}{4^p p!}\sum_{s=0}^{\infty}\frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 t^{\mu+s-1}\Delta_h^s f(tx)\int_t^1 (1-\alpha)^{p-1}\alpha(\alpha-t)^s d\alpha dt. \tag{9}$$

Let $\alpha = t + (1-t)\beta$. Therefore,

$$\begin{aligned}
&\int_t^1 \alpha(1-\alpha)^{p-1}(\alpha-t)^s d\alpha \\
&= \int_0^1 (1-\beta)^{p-1}\beta^s(1-t)^{p+s}[t+(1-t)\beta]d\beta \\
&= (1-t)^{p+s}t\int_0^1 (1-\beta)^{p-1}\beta^s d\beta + (1-t)^{p+s+1}\int_0^1 (1-\beta)^{p-1}\beta^{s+1}d\beta.
\end{aligned}$$

We note that

$$\int_0^1 (1-\alpha)^{k-1}\alpha^{s-1}d\alpha = \frac{(s-1)!(k-1)!}{(s+k-1)!}. \tag{10}$$

Therefore

$$\begin{aligned}
&\int_t^1 (1-\alpha)^{p-1}\alpha(\alpha-t)^s d\alpha \\
&= \tfrac{(p-1)!s!}{(p+s)!}t(1-t)^{p+s} + \tfrac{(s+1)!(p-1)!}{(p+s+1)!}(1-t)^{p+s+1} \\
&= \tfrac{(p-1)!s!(1-t)^{p+s}}{(p+s)!}\left(\tfrac{(s+1)(1-t)}{s+p+1} + t\right) \\
&= \tfrac{(p-1)!s!(s+1+pt)}{(p+s+1)!}(1-t)^{p+s}.
\end{aligned}$$

Through substituting the value of the integral into (9), and through making the change in variables $t \to \alpha$, we reduce relation (9) to the form

$$\begin{aligned}
&\tfrac{1}{4^p p!}\int_0^1 (1-\alpha)^{k-1}\alpha^{\mu-1}u(\alpha x)d\alpha \\
&= \tfrac{1}{4^p p!}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{p+s}\tfrac{(p-1)!s!(s+1+p\alpha)}{(p+s+1)!}\Delta_h^s f(\alpha x)d\alpha \\
&= \tfrac{1}{4^p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{p+s}\tfrac{s!(s+1+p\alpha)}{p(p+s+1)!}\Delta_h^s f(\alpha x)d\alpha.
\end{aligned} \tag{11}$$

Now, we consider Formula (8) and transform the second integral through replacing $p \to p+1:$

$$\begin{aligned}
&\sum_{s=1}^{\infty}\tfrac{(-1)^s x^{2s}}{4^{s+p}(s+p)!s!}\int_0^1 (1-\alpha)^{s+p-1}\alpha^{\mu+s-1}\Delta_h^{s-1}f(\alpha x)d\alpha \\
&= \sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+p+1}(s+p+1)!(s+1)!}\int_0^1 (1-\alpha)^{s+p}\alpha^{\mu+s}\Delta_h^s f(\alpha x)d\alpha \\
&= \tfrac{1}{4^p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 (1-\alpha)^{s+p}\alpha^{\mu+s-1}\left(\tfrac{s!(-\alpha)}{(p+s+1)!}\right)\Delta_h^s f(\alpha x)d\alpha.
\end{aligned} \tag{12}$$

Through applying the sum of the resulting expressions in (11) and (12), we rewrite formula (8) in the form

$$\begin{aligned}
&\tfrac{1}{4^p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{p+s}\tfrac{s!(s+1+p\alpha)}{p(p+s+1)!}\Delta_h^s f(\alpha x)d\alpha \\
&+\tfrac{1}{4^p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{s+p}\left(\tfrac{s!(-\alpha)}{(p+s+1)!}\right)\Delta_h^s f(\alpha x)d\alpha \\
&=\tfrac{1}{4^p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{s+p}\left(\tfrac{s!(s+1+p\alpha)}{p(p+s+1)!}+\tfrac{s!(-\alpha)}{(p+s+1)!}\right)\Delta_h^s f(\alpha x)d\alpha \\
&=\tfrac{1}{4^p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{p+s}\tfrac{(s+1)!}{p(p+s+1)!}\Delta_h^s f(\alpha x)d\alpha \\
&=\tfrac{1}{4^p p}\sum_{s=0}^{\infty}\tfrac{(-1)^s x^{2(s+1)}}{4^{s+1}(p+s+1)!s!}\int_0^1 \alpha^{\mu+s-1}(1-\alpha)^{p+s}\Delta_h^s f(\alpha x)d\alpha,
\end{aligned}$$

which implies formula (4) holds for $k = p+1$. The proof is complete. □

Corollary 1. *Let $P_l(x)$ be as stated in Definition 1. Therefore, the solution to the equation $\Delta_h^k g(x) = P_l(x)$ can be represented in the form*

$$g(x) = \frac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{[\frac{l}{2}]}\frac{(-1)^s x^{2s}\Delta_h^s P_l(x)}{4^{s+k}(s+k)s!(l+\mu-s)\cdots(l+\mu+k-1)}, \tag{13}$$

where $\left[\frac{l}{2}\right]$ is the integer part of $\frac{l}{2}$.

Proof of Corollary 1. Using Definition 1, we obtain $\Delta_h^k P_l(\alpha x) = \alpha^{l-2k}\Delta_h^k P_l(x)$. Therefore, (4) can be transformed into

$$\begin{aligned}
g(x) &= \tfrac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{[\frac{l}{2}]}\tfrac{(-1)^s x^{2s}}{4^{s+k}(s+k)!s!}\int_0^1 (1-\alpha)^{s+k-1}\alpha^{\mu+s-1}\Delta_h^s P_l(\alpha x)d\alpha \\
&= \tfrac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{[\frac{l}{2}]}\tfrac{(-1)^s x^{2s}\Delta_h^s P_l(x)}{4^{s+k}(s+k)!s!}\int_0^1 (1-\alpha)^{s+k-1}\alpha^{l+\mu-2s+s-1}d\alpha.
\end{aligned}$$

Using formula (10), we have

$$g(x) = \frac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{[\frac{l}{2}]}\frac{(-1)^s x^{2s}\Delta_h^s P_l(x)\Gamma(s+k)\Gamma(l+\mu-s)}{4^{s+k}(s+k)!s!\Gamma(l+\mu+k)}.$$

Using $\Gamma(s) = (s-1)!$, we find that

$$g(x) = \frac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{[\frac{l}{2}]}\frac{(-1)^s x^{2s}\Delta_h^s P_l(x)\Gamma(l+\mu-s)}{4^{s+k}(s+k)s!\Gamma(l+\mu+k)}.$$

It follows that

$$g(x) = \frac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{[\frac{l}{2}]}\frac{(-1)^s x^{2s}\Delta_h^s P_l(x)}{4^{s+k}(s+k)s!(l+\mu-s)\cdots(l+\mu+k-1)},$$

which completes the proof. □

Lemma 2. **[16].** *Let $f(x)$ be as stated in Theorem 1. If Re $\mu \geq 0$, we can state that*

$$\Delta_h\left[x^{2k}f(x)\right] = x^{2k}\Delta_h f(x) + 4kx^{2k-2}\mathrm{E}_{\mu+k-1}f(x). \tag{14}$$

Corollary 2. *Let $P_l(x)$ be as stated in Definition 1. Thus, the function*

$$g(x) = \frac{x^{2k+2i}P_l(x)}{2^{k-1}(k-1)!}\sum_{s=0}^{i}\frac{(-1)^s(2i-2s+2,2)_s(2l+2\mu+2i-2s,2)_s}{4^{s+k}(s+k)s!(l+\mu+2i+k-1)!},$$

is a solution of the equation $\Delta_h^k g(x) = x^{2i}P_l(x)$. For $s=0$, $(c,d)_0 = 1$. For $s = 1,2,\cdots$, $(c,d)_s = c(c+d)\cdots(c+sd-d)$.

Proof of Corollary 2. Let $f(x) = x^{2i}P_l(x)$. We calculate this solution using Formula (4) to obtain

$$g(x) = \frac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{4^{s+k}(s+k)!s!}\int_0^1 (1-\alpha)^{s+k-1}\alpha^{\mu+s-1}\Delta_h^s\left[(\alpha x)^{2i}P_l(\alpha x)\right]d\alpha.$$

Let us derive an expression for $\Delta_h^s\left[x^{2i}P_l(x)\right]$. Using Lemma 2, we have

$$\Delta_h\left[x^{2i}P_l(x)\right] = 4ix^{2i-2}(l+\mu+i-1)P_l(x).$$

Therefore, for $2s \le 2i$, we have

$$\begin{aligned}
&\Delta_h^s\left[x^{2i}P_l(x)\right] \\
&= 2i(2i-2)\cdots(2i-2s+2)(2l+2\mu+2i-2)\cdots(2l+2\mu+2i-2s)x^{2i-2s}P_l(x) \\
&= (2i-2s+2,\ 2)_s(2l+2\mu+2i-2s,2)_s x^{2i-2s}P_l(x).
\end{aligned}$$

Thus, as $P_l(\alpha x) = \alpha^l P_l(x)$, we have

$$\begin{aligned}
&\int_0^1 (1-\alpha)^{s+k-1}\alpha^{\mu+s-1}\Delta_h^s\left[(\alpha x)^{2i}P_l(\alpha x)\right]d\alpha \\
&= \int_0^1 (1-\alpha)^{s+k-1}\alpha^{\mu+s-1}\alpha^{2i-2s}(2i+2-2s,\ 2)_s(2l+2i+2\mu-2s,2)_s x^{2i-2s}P_l(\alpha x)d\alpha \\
&= x^{2i-2s}P_l(x)(2i-2s+2,\ 2)_s(2l+2\mu+2i-2s,2)_s\int_0^1 (1-\alpha)^{s+k-1}\alpha^{\mu+2i+l-s-1}d\alpha \\
&= x^{2i-2s}P_l(x)(2i-2s+2,\ 2)_s(2l+2\mu+2i-2s,2)_s\tfrac{(s+k-1)!(\mu+2i+l-s-1)!}{(\mu+2i+k+l-1)!}.
\end{aligned}$$

Thus, $g(x)$ is transformed into

$$\begin{aligned}
g(x) &= \tfrac{x^{2k}}{2^{k-1}(k-1)!}\sum_{s=0}^{i}\tfrac{(-1)^s x^{2s}}{4^{s+k}(s+k)!s!}\int_0^1 (1-\alpha)^{s+k-1}\alpha^{\mu+s-1}\Delta_h^s\left[(\alpha x)^{2i}P_l(\alpha x)\right]d\alpha \\
&= \tfrac{x^{2k}P_l(x)}{2^{k-1}(k-1)!}\sum_{s=0}^{i}\tfrac{(-1)^s x^{2s}x^{2i-2s}(2i+2-2s,\ 2)_s(2l+2i-2s+2\mu,2)_s}{4^{s+k}(s+k)!s!(\mu+2i+k+l-1)!} \\
&= \tfrac{x^{2k+2i}P_l(x)}{2^{k-1}(k-1)!}\sum_{s=0}^{i}\tfrac{(-1)^s(2i-2s+2,\ 2)_s(2l+2\mu+2i-2s,2)_s}{4^{s+k}(s+k)s!(\mu+2i+k+l-1)!}.
\end{aligned}$$

Thus, we complete the proof. □

4. Dirichlet Boundary Value Problems for Dunkl Biharmonic Equations

4.1. Homogeneous Dirichlet Problems for Inhomogeneous Dunkl Biharmonic Equations

In this section, we study the homogeneous Dirichlet problem for the inhomogeneous Dunkl biharmonic equation in S: find a function $u(x)$, such that

$$\begin{cases} \Delta_h^2 u(x) = f(x), \\ u|_{\partial S} = 0,\ \frac{\partial u}{\partial n}\Big|_{\partial s} = 0, \end{cases} \tag{15}$$

with a polynomial function $f(x)$. Here, n is the unit outward normal to the vector.

Lemma 3. [16]. *Let $f_l(x)$ be a homogeneous Clifford-valued polynomial of degree l. Thus,*

$$f_l(x) = R_l(x) + x^2 R_{l-2}(x) + \cdots + x^{2i} R_{l-2i}(x),$$

where $R_{l-2i}(x)$, we find homogeneous Dunkl harmonic polynomials and

$$R_{l-2i}(x) = \frac{(2l-4i+2\mu-2)}{(2,2)_i} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^{s+i} f_l(x)}{(2,2)_s (2l-4i-2s+2\mu-2,2)_{s+i+1}} \tag{16}$$

where $(c,d)_s$ is as stated in Corollary 2.

Theorem 3. *Let $f(x)$ be as stated in Problem (15). Thus, the function*

$$g(x) = \left(\frac{x^2+1}{4}\right)^2 \int_0^1 \sum_{k=0}^{\infty} \frac{(1-\alpha)^{k+1}}{(k+2)!k!} \left(\frac{1+\alpha x^2}{4}\right)^k \Delta_h^k f(\alpha x) \alpha^{\mu-1} d\alpha \tag{17}$$

is a solution of Problem (15).

Proof of Theorem 3. Step 1: Firstly, we study the homogeneous Dirichlet problem to find a function for the Dunkl biharmonic equation satisfying

$$\begin{cases} \Delta_h^2 u(x) = x^{2i} R_{l-2i}(x), \\ u|_{\partial S} = 0, \ \frac{\partial u}{\partial n}\Big|_{\partial s} = 0. \end{cases} \tag{18}$$

Using Corollary 1, we have

$$\frac{x^{2i+4} R_{l-2i}(x)}{(2i+4)(2i+2)(2l-2i+2\mu+2)(2l-2i+2\mu)}$$

as a solution to the equation $\Delta_h^2 u(x) = x^{2i} R_{l-2i}(x)$, while we also have

$$u_i(x) = \frac{\left[x^{2i+4} + (-1)^i (i+1) + (-1)^i (i+2) x^2\right] R_{l-2i}(x)}{(2i+4)(2i+2)(2l-2i+2\mu+2)(2l-2i+2\mu)}. \tag{19}$$

as a solution of the Dirichlet problem (18).

Note that $R_{l-2i}(x)$ are the homogenous Dunkl harmonic polynomials and $x^2 = -|x|^2$. It is easy to check if Formula (19) is correct. We expand the polynomial $f_l(x)$ with the use of the Almansi Formula (19) into terms of the form $x^{2i} R_{l-2i}(x)$,

$$f_l(x) = R_l(x) + x^2 R_{l-2}(x) + \cdots + x^{2i} R_{l-2i}(x), \ l-2i \geq 0.$$

Let us apply Formula (13) to both sides. Thus, using Lemma 3, the solution of the equation $\Delta_h^2 u(x) = x^{2i} R_{l-2i}(x)$ has the form

$$\frac{x^{2i+4} R_{l-2i}(x)}{2} \sum_{s=0}^{i} \frac{(-1)^s (2i-2s+2,\,2)_s (2l+2\mu-2i-2s,2)_s}{4^{s+2}(s+2)s!(\mu+l+1)!},$$

where the homogeneous polynomials $R_{l-2i}(x)$ have the form

$$R_{l-2i}(x) = \frac{(2l-4i+2\mu-2)}{(2,2)_i} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} \Delta_h^{s+i} f_l(x)}{(2,2)_s (2l-4i-2s+2\mu-2,2)_{s+i+1}}.$$

Considering the Dirichlet problem (17), we have the solution

$$\frac{\left[x^{2i+4}+(i+1)-(i+2)x^{2}\right]R_{l-2i}(x)}{2}\sum_{s=0}^{i}\frac{(-1)^{s}(2i-2s+2,\,2)_{s}(2l+2\mu-2i-2s,2)_{s}}{4^{s+2}(s+2)s!(\mu+l+1)!}.$$

Secondly, we consider the following homogeneous boundary value problem for the inhomogeneous polyharmonic equation in S:

$$\begin{cases} \Delta_h^2 u(x)=f_l(x), \\ u|_{\partial S}=0,\ \left.\frac{\partial u}{\partial n}\right|_{\partial s}=0, \end{cases}$$

where $f_l(x)$ is a Clifford-valued polynomial of degree l, and n is the unit outward normal to the unit sphere ∂s.

Using Formula (19), we have

$$\begin{aligned} u_l(x)&=\sum_{i=0}^{[\frac{l}{2}]}u_i(x)\\ &=\sum_{i=0}^{[\frac{l}{2}]}\frac{\left[x^{2i+4}+(-1)^i(i+1)+(-1)^i(i+2)x^2\right]R_{l-2i}(x)}{(2i+4)(2i+2)(2l-2i+2\mu+2)(2l-2i+2\mu)}\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i(i+1)x^{2i+4}\Delta_h^i f_l(x)}{(2,2)_{i+2}(2l-2i+2\mu,\,2)_{i+2}}+\sum_{i=0}^{[\frac{l}{2}]}\frac{\left[(-1)^i(i+1)+(-1)^i(i+2)x^2\right]R_{l-2i}(x)}{(2i+4)(2i+2)(2l-2i+2\mu+2)(2l-2i+2\mu)}\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i(i+1)x^{2i+4}\Delta_h^i f_l(x)}{(2,2)_{i+2}(2l-2i+2\mu,\,2)_{i+2}}\\ &\quad+\sum_{i=0}^{[\frac{l}{2}]}\frac{\left[(-1)^i(i+1)+(-1)^i(i+2)x^2\right]}{(2,2)_{i+2}}\sum_{2i+2j\le l}^{\infty}\frac{(-1)^j(2l+\mu-4i-2)x^{2j}\Delta_h^{i+j}f_l(x)}{(2,2)_j(2l+\mu-4i-2j-2,\,2)_{i+j+3}}. \end{aligned}$$

Let $i+j=k$. Therefore, the last equality becomes

$$\begin{aligned} &\sum_{i=0}^{\infty}(-1)^i\frac{(i+1)x^{2i+4}\Delta_h^i f_l(x)}{4^{i+2}(i+2)!(l+\mu-i)\cdots(l+\mu+1)}\\ &+\sum_{k=0}^{\infty}\frac{\Delta_h^k f_l(x)}{4^{k+2}}\sum_{i=0}^{k}\frac{(-1)^i(l+\mu-2k+2i-1)\left[(k-i+1)\left(x^{2i}-x^{2i+2}\right)\right](2l+\mu-4i-2)}{(l+\mu-2k+2i-1)!i!(l+\mu-2k+i-1)\cdots(l+\mu-k+i+1)}\\ &=\sum_{k=0}^{\infty}\frac{\Delta_h^k f_l(x)}{4^{k+2}}\sum_{i=0}^{k+2}\frac{(-1)^i x^{2i}i(l+\mu-2k+2i-3)(l+\mu-k+i+1)}{i!(k-i+2)!(l+\mu-2k+i-2)\cdots(l+\mu-k+i+1)}\\ &+\sum_{k=0}^{\infty}\frac{\Delta_h^k f_l(x)}{4^{k+2}}\sum_{i=0}^{k+2}\frac{(-1)^i x^{2i}(k-i+1)(l+\mu-2k+2i-1)(l+\mu-2k+i-2)}{i!(k-i+2)!(l+\mu-2k+i-2)\cdots(l+\mu-k+i+1)}\\ &=\left(x^2+1\right)^2\sum_{k=0}^{\infty}\frac{(k+1)\Delta_h^k f_l(x)}{4^{k+2}(k+2)!}\sum_{i=0}^{k}\frac{(-1)^i k(k-1)\cdots(k-i+1)x^{2i}}{i!(l+\mu-2k+i)\cdots(l+\mu-k+i+1)}. \end{aligned}$$

Applying Formula (10), we have

$$\frac{1}{(l+\mu-2k+i)\cdots(l+\mu-k+i+1)}=\frac{1}{(k+1)!}\int_0^1(1-\alpha)^{k+1}\alpha^{l+\mu+i-2k-1}d\alpha.$$

Thus,

$$\begin{aligned} &\int_0^1(1-\alpha)^{k+1}\alpha^{l+\mu-2k-1}\sum_{i=0}^{k}\frac{k(k-1)\cdots(k-i+1)\alpha^i x^{2i}}{i!(l+\mu-2k+i)\cdots(l+\mu-k+i+1)}d\alpha\\ &=\int_0^1(1-\alpha)^{k+1}\left(1+\alpha x^2\right)^k\alpha^{l+\mu-2k-1}d\alpha. \end{aligned}$$

Therefore, we have

$$u_l(x)=\frac{\left(x^2+1\right)^2}{4^2}\int_0^1\sum_{k=0}^{\infty}\frac{\left(1+\alpha x^2\right)^k(1-\alpha)^{k+1}}{4^k(k+2)!k!}\Delta_h^k f_l(\alpha x)\alpha^{\mu-1}d\alpha.$$

Thirdly, we consider the boundary value problem (15).

Since $u(x)$ is an arbitrary polynomial, let $u(x) = \sum\limits_{l} u_l(x)$. and let $u_l(x)$ denote the polynomial solution of Dirichlet problem (15). Thus,

$$u(x) = \sum_{l} u_l(x) = \frac{\left(x^2+1\right)^2}{4^2} \sum_{k=0}^{\infty} \int_0^1 \frac{\left(1+\alpha x^2\right)^k (1-\alpha)^{k+1}}{4^k (k+2)! k!} \Delta_h^k f(\alpha x) \alpha^{\mu-1} d\alpha.$$

We complete the proof. □

4.2. Inhomogeneous Dirichlet Problems for Homogeneous Dunkl Biharmonic Equations

In this section, we investigate the inhomogeneous Dirichlet problem for the homogeneous Dunkl biharmonic equation to find a function $u(x)$, such that

$$\begin{cases} \Delta_h^2 u(x) = 0, \\ u|_{\partial S} = f_1(x), \ \left.\frac{\partial u}{\partial n}\right|_{\partial s} = 0, \end{cases} \tag{20}$$

with a Clifford-valued polynomial right-hand side $f_1(x)$ for $m \geq 2$. Here, n is the unit outward normal to the unit sphere ∂S.

Theorem 4. *If $u(x) \in C^2(S) \otimes R_{0,m}$, it is true that*

$$u(x) = f_1(x) + \tfrac{1+x^2}{2} E f_1(x)$$
$$+ \tfrac{\left(x^2+1\right)^2}{2} \int_0^1 \sum_{k=0}^{\infty} \tfrac{\left(1+\alpha x^2\right)^k (1-\alpha)^k}{4^{k+1} (k+1)! k!} \Delta_h^{k+1} \left(E f_1 - \tfrac{1-\alpha}{2k+4} \Delta_h f_1 \right) (\alpha x) \alpha^{\mu-1} d\alpha.$$

is a solution of Problem (20).

Proof of Theorem 4. With the help of Formula (17), we will find the function

$$g(x) = \frac{\left(x^2+1\right)^2}{4^2} \int_0^1 \sum_{k=0}^{\infty} \frac{\left(1+\alpha x^2\right)^k (1-\alpha)^{k+1}}{4^k (k+2)! k!} \Delta_h^{k+2} f_1(\alpha x) \alpha^{\mu-1} d\alpha$$

as the solution to the following problem

$$\begin{cases} \Delta_h^2 g(x) = \Delta_h^2 f_1(x), \ x \in \Omega, \\ g|_{\partial S} = 0, \ \left.\frac{\partial g}{\partial n}\right|_{\partial s} = 0. \end{cases} \tag{21}$$

Let the Dunkl harmonic polynomial $h(x)$ satisfies the condition $h(x)|_{\partial S} = E f_1(x)|_{\partial S}$.

Therefore, the function $u(x) = f_1(x) + \frac{1+x^2}{2} h(x) - g(x)$. We can check if that the function satisfies the equation $\Delta_h^2 u(x) = \Delta_h^2 f_1(x) - \Delta_h^2 g(x) = 0$. Through applying the properties of the operator E, we have $u|_{\partial S} = f_1(x)$ and

$$\left.\frac{\partial u}{\partial n}\right|_{\partial s} = E\left(f_1(x) + \frac{1+x^2}{2} h(x) \right)\Bigg|_{\partial S} = \left(E f_1(x) - h(x)\right)|_{\partial S} = 0.$$

The polynomial $h(x)$ is written as

$$h(x) = E f_1(x) - \frac{x^2+1}{4^2} \int_0^1 \sum_{k=0}^{\infty} \frac{\left(1+\alpha x^2\right)^k (1-\alpha)^{k+1}}{4^k (k+2)! k!} \Delta_h^{k+2} f_1(\alpha x) \alpha^{\mu-1} d\alpha.$$

Thus, the solution $u(x)$ is written as

$$u(x) = f_1(x) + \tfrac{1+x^2}{2} E f_1(x) + \tfrac{\left(x^2+1\right)^2}{2} \int_0^1 \sum_{k=0}^{\infty} \tfrac{\left(1+\alpha x^2\right)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^{k+1} \left(E f_1 - \tfrac{1-\alpha}{2k+4} \Delta_h f_1 \right) (\alpha x) \alpha^{\mu-1} d\alpha.$$

□

4.3. Inhomogeneous Dirichlet Problems for Homogeneous Dunkl Biharmonic Equations

In this section, we consider the inhomogeneous Dirichlet problem for the homogeneous Dunkl biharmonic equation to find a function $u(x)$, such that

$$\begin{cases} \Delta_h^2 u(x) = 0, \\ u|_{\partial S} = 0, \ \frac{\partial u}{\partial n}\Big|_{\partial s} = f_2(x), \end{cases} \tag{22}$$

with a right-hand side Clifford-valued polynomial $f_2(x)$ for $m \geq 2$. Here, n is the unit outward normal to the unit sphere ∂S.

Theorem 5. *If $u(x) \in C^2(S) \otimes R_{0,m}$, it is also true that*

$$u(x) = \tfrac{1+x^2}{2} f_2(x) - \tfrac{\left(x^2+1\right)^2}{2} \int_0^1 \sum_{k=0}^{\infty} \tfrac{\left(1+\alpha x^2\right)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^{k+1} f_2(\alpha x) \alpha^{\mu-1} d\alpha$$

is a solution to Problem (22).

Proof of Theorem 5. Supposing that $g(x)$ satisfies the condition $g(x)|_{\partial S} = f_2(x)|_{\partial S}$. Let $u(x) = \frac{1+x^2}{2} g(x)$. Thus, the function $u(x)$ satisfies $u(x)|_{\partial S} = 0$, and

$$\frac{\partial u(x)}{\partial n}\bigg|_{\partial s} = Eu|_{\partial S} = \left(g(x) E\left(\frac{x^2+1}{2}\right) + \frac{x^2+1}{2} E g(x) \right)\bigg|_{\partial S} = f_2(x)|_{\partial S}.$$

The function $g(x)$ can be written as

$$g(x) = f_2(x) - \frac{x^2+1}{4^2} \int_0^1 \sum_{k=0}^{\infty} \frac{\left(1+\alpha x^2\right)^k (1-\alpha)^{k+1}}{4^k (k+2)!k!} \Delta_h^{k+1} f_2(\alpha x) \alpha^{\mu-1} d\alpha.$$

Therefore, we have

$$u(x) = \tfrac{1+x^2}{2} f_2(x) - \tfrac{\left(x^2+1\right)^2}{2} \int_0^1 \sum_{k=0}^{\infty} \tfrac{\left(1+\alpha x^2\right)^k (1-\alpha)^k}{4^{k+1}(k+1)!k!} \Delta_h^{k+1} f_2(\alpha x) \alpha^{\mu-1} d\alpha.$$

□

4.4. Inhomogeneous Dirichlet Problems for Inhomogeneous Dunkl Biharmonic Equations

In this section, we study another mathematical problem. Assuming that $f(x)$, $f_1(x)$, and $f_2(x)$ are Clifford-valued polynomial functions, we find a function $u(x)$, such that

$$\begin{cases} \Delta_h^2 u(x) = f(x), \\ u|_{\partial S} = f_1(x), \ \frac{\partial u}{\partial n}\Big|_{\partial s} = f_2(x). \end{cases} \tag{23}$$

Theorem 6. *If $u(x) \in C^2(S) \otimes R_{0,m}$, it is also true that*

$$\begin{aligned} u(x) &= f_1(x) + \tfrac{1+x^2}{2}[f_1(x) - Ef_2(x)] \\ &+ \tfrac{1+x^2}{2}\int_0^1 \sum_{k=0}^{\infty} \tfrac{(1+\alpha x^2)^k (1-\alpha)^{k+1}}{4^k (k+1)! k!} \Delta_h^k \Big[\Delta_h (Ef_1 - f_2) + \tfrac{1-\alpha}{2k+4}\big(f - \Delta_h^2 f_1\big)\Big](\alpha x) d\alpha \end{aligned}$$

is a solution of Dirichlet problem (23).

Proof of Theorem 6. Problem (23) is a combination of three problems: Equations (15), (20), and (22). Applying Theorems 3, 4, and 5, we find the result. □

5. Dirichlet Boundary Value Problems for Dunkl Polyharmonic Equations

5.1. Homogeneous Dirichlet Problems for Inhomogeneous Dunkl Polyharmonic Equations

In this section, we consider the homogeneous Dirichlet problem in the unit ball. We aim to find a function $u(x)$, such that

$$\begin{cases} \Delta_h^k u(x) = f(x), \\ u|_{\partial S} = 0,\ \frac{\partial^i u}{\partial n^i}\Big|_{\partial s} = 0,\ i = 1, \cdots k-1, \end{cases} \tag{24}$$

with a right-hand side Clifford-valued polynomial $f(x)$ for $m \geq 2$.

Next, we give the important property of the Euler operator E as follows.

Lemma 4. **[12].** *Let $\Omega \subset R^m$. Then*

$$\frac{\partial^k u}{\partial n^k}\Big|_{\partial\Omega} = E^{[k]} u\Big|_{\partial\Omega},$$

where the factorial power operator $E^{[k]} = E(E-1)\cdots(E-k+1)$.

Theorem 7. *The solution of Dirichlet problem (24) can be written as*

$$G(x) = \frac{(x^2+1)^k}{2(2k-2)!!}\int_0^1 \sum_{s=0}^{\infty} \frac{(1+\alpha x^2)^s (1-\alpha)^{s+k-1} \alpha^{\mu+s-1}}{(2s+2k)!!(2s)!!} \Delta_h^s f(\alpha x) d\alpha, \tag{25}$$

where $s \leq \left[deg \frac{f(x)}{2}\right]$.

Proof of Theorem 7. Using Theorem 2, we obtain the solution to the equation $\Delta_h^k u(x) = f(x)$. Let $f_l(x)$ be a homogeneous polynomial of degree l. Using Lemmas 2 and 3, we have the solution of the equation $\Delta_h^k u(x) = f_l(x)$, given as

$$u_l(x) = \sum_{i=0}^{[\frac{l}{2}]} \frac{x^{2i+2k} f_{l-2i}(x)}{(2i+2,2)_k (2l-2i+2\mu,\ 2)_k}.$$

Step 1: we consider the following homogeneous boundary value problem for inhomogeneous polyharmonic equations in the unit ball $S = \{x \in R^m : |x| < 1\}$

$$\begin{cases} \Delta_h^k u(x) = x^{2i} f_{l-2i}(x), \\ u|_{\partial S} = 0,\ \frac{\partial^i u}{\partial n^i}\Big|_{\partial s} = 0,\ \ i = 1, \cdots, k-1. \end{cases} \tag{26}$$

The solution of the homogeneous Dirichlet problem (26) is given as

$$u_i(x) = \frac{\left(x^2+1\right)^k f_{l-2i}(x)}{(2i+2,2)_k(2l-2i+2\mu,\ 2)_k}\sum_{j=0}^{i}\frac{(i+k)(i+k-1)\cdots(i-j+1)}{(j+k)!}\left(x^2+1\right)^j.$$

Step 2: we consider the following homogeneous boundary value problem for inhomogeneous polyharmonic equations in the unit ball $S = \{x \in R^m : |x| < 1\}$

$$\begin{cases} \Delta_h^k u(x) = f_l(x), \\ u|_{\partial S} = 0,\ \left.\frac{\partial^i u}{\partial n^i}\right|_{\partial s} = 0,\ \ i = 1, \cdots, k-1. \end{cases} \tag{27}$$

The solution of the homogeneous Dirichlet problem (27) is given as

$$u_l(x) = \sum_{i=0}^{[\frac{l}{2}]} u_i(x) = \sum_{i=0}^{[\frac{l}{2}]} \frac{\left(x^2+1\right)^k f_{l-2i}(x)}{(2i+2,2)_k(2l-2i+2\mu,\ 2)_k}\sum_{j=0}^{i}\frac{(i+k)\cdots(i-j+1)}{(j+k)!}\left(x^2+1\right)^j.$$

Let $(c)_s = c(c+1)\cdots(c+s-1)$. Thus, we obtain

$$u_l(x) = \left(x^2+1\right)^k \sum_{i=0}^{[\frac{l}{2}]} C_{i+k-1}^{k-1} \frac{\Delta^i f_l(x)}{4^{i+k}(i+k)!}\sum_{j=0}^{i} C_i^j \frac{x^{2j}}{(l-2i+j-\mu)_{i+k}}.$$

Applying the properties of the Euler gamma and beta functions, we have

$$\frac{1}{(l-2i+j-\mu)_{i+k}} = \frac{1}{(i+k-1)!}\int_0^1 (1-\alpha)^{i+k-1}\alpha^{l-2i+j-\mu-1}d\alpha.$$

Thus, we have

$$\frac{1}{i!}\int_0^1 (1-\alpha)^{i+k-1}\alpha^{l-2i+j-\mu-1}\sum_{j=0}^{i} C_i^j x^{2j} d\alpha = \frac{1}{i!}\int_0^1 (1-\alpha)^{i+j-1}\left(1+\alpha x^2\right)^i \alpha^{l-2i-\mu-1}d\alpha.$$

Therefore, we have

$$u_l(x) = \frac{\left(x^2+1\right)^k}{2(2k-2)!!}\int_0^1 \sum_{s=0}^{[\frac{l}{2}]} \frac{(1+\alpha x^2)^s(1-\alpha)^{s+k-1}}{(2s+2k)!!(2s)!!}\Delta_h^s f(\alpha x)\alpha^{\mu+s-1}d\alpha.$$

where the sum over s is finite, and the upper summation index is $\left[\frac{l}{2}\right]$.

Step 3: we consider the problem (24). We note that the function $f(x)$ can be written as $f(x) = \sum_l f_l(x)$. Therefore, the solution to problem (24) is given as $u(x) = \sum_l u_l(x)$, where $u_l(x)$ is a solution of Dirichlet problem (27). It follows from Theorem 3 that

$$\begin{aligned} u(x) &= \frac{\left(x^2+1\right)^k}{2(2k-2)!!}\int_0^1 \sum_{s=0}^{\infty} \frac{\left(1+\alpha x^2\right)^s(1-\alpha)^{s+k-1}\alpha^{\mu+s-1}}{(2s+2k)!!(2s)!!}\Delta_h^s \sum_l f_l(\alpha x)d\alpha \\ &= \frac{\left(x^2+1\right)^k}{2(2k-2)!!}\int_0^1 \sum_{s=0}^{\infty} \frac{\left(1+\alpha x^2\right)^s(1-\alpha)^{s+k-1}\alpha^{\mu+s-1}}{(2s+2k)!!(2s)!!}\Delta_h^s f(\alpha x)d\alpha. \end{aligned}$$

□

5.2. Inhomogeneous Dirichlet Problems for Homogeneous Dunkl Polyharmonic Equations

In this section, we consider the inhomogeneous Dirichlet problems for the homogeneous Dunkl polyharmonic equation in S.

$$\begin{cases} \Delta_h^k u(x) = 0, \\ u|_{\partial S} = f_0(x), \ \frac{\partial^i u}{\partial n^i}\Big|_{\partial s} = f_i(x), \ i = 1, \cdots k-1, \end{cases} \tag{28}$$

where n is the outward normal to ∂S, and $f_i(x)$ are Clifford-valued polynomials.

Theorem 8. *The solution of Dirichlet problem (28) is given as*

$$\begin{aligned} u(x) = &\sum_{l=0}^{k-1} \left(x^2+1\right)^l \frac{1}{(2l)!!} \sum_{j=0}^{l} \frac{1}{j!} \sum_{i=0}^{j} (-1)^i E \cdots (E - 2(i-1)) f_i(x) \\ &- \sum_{i=0}^{k-i} \frac{\left(x^2+1\right)^i}{2(2(k-i)-2)!!} \int_0^1 \sum_{s=0}^{\infty} \frac{\left(1+\alpha x^2\right)^s (1-\alpha)^{s+k-i-1} \alpha^{\mu-1}}{(2s+2(k-i))!!(2s)!!} \Delta_h^{s+k-1} f_i(\alpha x) d\alpha. \end{aligned}$$

Proof of Theorem 8. Using Lemma 4, we can check out this result directly. □

5.3. Inhomogeneous Dirichlet Problem for a Inhomogeneous Dunkl Polyharmonic Equation

In this section, we construct a solution to the inhomogeneous Dirichlet problem for the inhomogeneous Dunkl polyharmonic equation in S. That is, finding a function $u(x)$ satisfying

$$\begin{cases} \Delta_h^k u(x) = f(x), \\ u|_{\partial S} = f_0(x), \ \frac{\partial^i u}{\partial n^i}\Big|_{\partial s} = f_i(x), \ i = 1, \cdots k-1, \end{cases} \tag{29}$$

where n is the outward normal to ∂S, and $f(x)$, $f_i(x)$ are Clifford-valued polynomial boundary data.

Theorem 9. *If $u(x) \in C^k(S) \otimes R_{0,m}$, then the function*

$$\begin{aligned} u(x) = &\sum_{i=0}^{k-1} \left(x^2+1\right)^i \frac{1}{(2i)!!} \sum_{j=0}^{i} \frac{1}{j!} \sum_{s=0}^{j} \frac{1}{s!} (-1)^s E \cdots (E - 2(s-1)) f_s(x) \\ &- \sum_{i=0}^{k-i} \frac{\left(x^2+1\right)^i}{2(2(k-i)-2)!!} \int_0^1 \sum_{s=0}^{\infty} \frac{\left(1+\alpha x^2\right)^s (1-\alpha)^{s+k-i-1} \alpha^{\mu-1}}{(2s+2(k-i))!!(2s)!!} \Delta_h^{s+k-1} f_i(\alpha x) d\alpha \\ &+ \frac{\left(x^2+1\right)^k}{2(2k-2)!!} \int_0^1 \sum_{s=0}^{\infty} \frac{\left(1+\alpha x^2\right)^s (1-\alpha)^{s+k-1} \alpha^{\mu-1}}{(2s+2k)!!(2s)!!} \Delta_h^s f(\alpha x) d\alpha. \end{aligned}$$

is a solution of Problem (29).

Proof of Theorem 9. This result follows directly from Theorems 7 and 8. □

6. Neumann Problem for a Nonhomogeneous Dunkl Polyharmonic Equation

Consider the Neumann problem for a non-homogeneous Dunkl polyharmonic equation in S :

$$\begin{cases} \Delta_h^k u(x) = Q(x), \\ \frac{\partial^i u}{\partial n^i}\Big|_{\partial s} = f_i(s), \ s \in \partial S, \ i = 1, \cdots k. \end{cases} \tag{30}$$

where n is the outward normal to ∂S, and $Q(x)$, $f_i(x)$ are Clifford-valued polynomial boundary data.

Theorem 10. *A solution to the Neumann problem (30) can be written as*

$$u(x) = \int_0^1 g(tx)t^{\mu-1}dt,$$

where Re $\mu > 0$ *and* $g(x)$ *the solution to the Dirichlet problem*

$$\begin{cases} \Delta_h^k g(x) = \big(E_\mu + 2k\big)Q(x),\ x \in S, \\ g|_{\partial S} = f_1(s),\ \frac{\partial^i g}{\partial n^i}\Big|_{\partial s} = i f_i(s) + f_{i+1}(s),\ s \in \partial S,\ i = 2, \cdots k-1. \end{cases} \tag{31}$$

Proof of Theorem 10. Using Lemma 1, we have

$$\begin{aligned} \Delta_h^k E_\mu u(x) &= \Delta_h^{k-1} E_\mu \Delta_h u(x) + 2\Delta_h^k u(x) \\ &= \Delta_h^{k-2} E_\mu \Delta_h^2 u(x) + 4\Delta_h^k u(x) = \cdots = (E_\mu \Delta_h^k + 2k\Delta_h^k)u(x). \end{aligned}$$

If we apply $E_\mu + 2k$ to both sides of the equation $\Delta_h^k u(x) = Q(x)$, then

$$\big(E_\mu + 2k\big)\Delta_h^k u(x) = \Delta_h^k E_\mu u(x) = \big(E_\mu + 2k\big)Q(x).$$

which implies that $g(x) = E_\mu u(x)$ satisfies the equation $\Delta_h^k g(x) = \big(E_\mu + 2k\big)Q(x)$.

For $i = 1$,

$$\frac{\partial u}{\partial n}\Big|_{\partial s} = E_\mu\, u(x)|_{\partial S} = g(x)|_{\partial s} = f_1(s).$$

Using Lemma 4, we have

$$\frac{\partial^i u}{\partial n^i}\Big|_{\partial s} = E^{[i]}\, u|_{\partial S} = (E-1)\cdots(E-i+1)\, f|_{\partial S} = f_i(s),\ i = 2, \cdots, k.$$

Noting that

$$(E-1)\cdots(E-i+1) = E(E-1)\cdots(E-i+2) - (i-1)(E-1)\cdots(E-i+2).$$

Therefore

$$\frac{\partial^i g}{\partial n^i}\Big|_{\partial s} - (i-1)\,\frac{\partial^{i-1} u}{\partial n^{i-1}}\Big|_{\partial s} = f_i(s),\ i = 2, \cdots, k.$$

It follows that the function $g(x)$ satisfies the boundary condition of problem (31)

$$\frac{\partial^i g}{\partial n^i}\Big|_{\partial s} = i f_i(s) + f_{i+1}(s),\ s \in \partial S,\ i = 2, \cdots k-1.$$

Supposing that

$$u(x) = \int_0^1 g(tx)t^{\mu-1}dt.$$

Therefore

$$\begin{aligned} g(x) &= \int_0^1 \tfrac{d}{dt}\big[g(tx)t^{\mu-1}\big]dt = \int_0^1 \big[\mu t^{\mu-1} g(tx) + t^{\mu-1} E g(tx)\big]dt \\ &= \int_0^1 E_\mu g(tx)\, t^{\mu-1}dt = E_\mu u(x). \end{aligned}$$

Since $g(x)$ is a solution of the Dirichlet problem, it follows that $u(x)$ is a solution of the Neumann problem (30). □

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References

1. Dunkl, C.F. Differential-difference operators associated to reflection groups. *Trans. Am. Math. Soc.* **1989**, *311*, 167–183. [CrossRef]
2. Dunkl, C.F.; Xu, Y. *Orthogonal Polynomials of Several Variables*; Cambridge Univ. Press: Cambridge, UK, 2001.
3. Olshanetsky, M.A.; Perelomov, A.M. Quantum integrable systems related to Lie algebras. *Phys. Rep.* **1983**, *94*, 313–404. [CrossRef]
4. Dunkl, C.F. Integral kernels with reflection group invariance. *Can. J. Math.* **1991**, *43*, 1213–1227. [CrossRef]
5. Heckman, G.J. A remark on the Dunkl differential-difference operators. *Prog. Math.* **1991**, *101*, 181–191.
6. De Bie, H.; De Schepper, N. Clifford-Gegenbauer polynomials related to the Dunkl Dirac operator. *Bull. Belg. Math. Soc. Simon Stevin* **2011**, *18*, 193–214. [CrossRef]
7. De Bie, H.; Orsted, B.; Somberg, P.; Soucek, V. Dunkl operators and a family realzations of osp(1|2). *Trans. Am. Math. Soc.* **2012**, *364*, 3875–3902. [CrossRef]
8. Fei, M.G.; Cerejeiras, P.; Kähler, U. Spherical Dunkl-monogenics and a factorization of the Dunkl-Laplacian. *J. Phys. A Math. Theor.* **2010**, *43*, 445202. [CrossRef]
9. Courant, R.; Hilbert, D. *Methods of Mathematical Physics: Partial Differential Equations*; Wiley: New York, NY, USA, 1989; Volume 1, pp. 179–240.
10. Kalmenov, T.S.; Suragan, D. On a new method for constructing the Green function of the Dirichlet problem for the polyharmonic equation. *Differ. Equ.* **2012**, *48*, 435–438. [CrossRef]
11. Kanguzhin, B.E.; Koshanov, B.D. Necessary and sufficient conditions for the solvability of boundary value problems for an inhomogeneous polyharmonic equation in a ball. *Ufim. Mat. Zhurnal* **2010**, *2*, 41–52.
12. Karachik, V.V. Polynomial solutions of the Dirichlet problem for the 3-harmonic equation in a ball. *Zh. Fed. Univ. Mat. Fiz.* **2012**, *4*, 527–546.
13. Karachik, V.V. Solution of the Dirichlet problem with polynomial data for the polyharmonic equation in a ball. *Differ. Equ.* **2015**, *51*, 1033–1042. [CrossRef]
14. Karachik, V.V. Dirichlet and Neumann boundary value problems for the polyharmonic equation in the unit ball. *Mathematics* **2021**, *9*, 1907. [CrossRef]
15. Karachik, V.V.; Turmetov, B.; Yuan, H.F. Four boundary value problems for a nonlocal biharmonic equation in the unit ball. *Mathematics* **2022**, *10*, 1158. [CrossRef]
16. Yuan, H.F. Dirichlet type problems for Dunkl-Poisson equations. *Bound. Value Probl.* **2016**, *2016*, 222. [CrossRef]