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On Weak Variational Control Inequalities via Interval Analysis

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Abstract: This paper deals with the connections between the interval-valued optimal control problem and the associated weak variational control inequality. More precisely, by considering the (strictly) LU-convexity and path independence properties of the involved curvilinear integral functionals, we establish a result on the existence of LU-optimal solutions for the interval-valued optimal control problem under study, and a result on the existence of solutions for the associated weak variational control inequality.

Keywords: *LU*-convexity; path independence; *LU*-optimal solution; weak variational control inequality

MSC: 26B25; 49J40; 90C30

1. Introduction

This paper deals with the interval-valued optimal control problem, which plays an important role in studying uncertainty in optimization problems. The errors due to data uncertainty or imprecision have created the need to investigate certain real-world problems. Various scientists have contributed to this research direction. Among the techniques used, interval-valued optimization is an emerging branch dealing with the uncertainty of optimization problems. In this regard, variational inequalities, first introduced by Hartman and Stampacchia [1], have been observed to be useful mathematical objects for studying optimization problems. Giannessi [2] stated remarkable results on variational inequalities and complementarity problems. Moore [3,4] suggested interval analysis to study optimization problems determined by interval-valued functions. Stefanini and Bede [5] continued by defining the generalized Hukuhara differentiability associated with interval-valued functions. In addition, some sufficiency and duality results for interval-valued programming problems have been established by Jayswal et al. [6]. Additionally, Liu [7] studied variational inequalities and optimization problems, and Treanţă [8] contributed to the study of vector variational inequalities and multiobjective optimization problems. Jayswal et al. [9] formulated and proved some results for multiple objective optimization problems and vector variational inequalities. Connections between the solutions of some interval-valued multiple objective optimization problems and vector variational inequalities have been derived by Zhang et al. [10]. Jha et al. [11], via the associated modified problems and saddle point criteria, presented several results for interval-valued variational problems. Treanţă [12,13] provided important connections between the notions of optimal solution, KT-pseudoinvex point, and a saddle-point of an interval-valued functional of the Lagrange type. Recently, Treanţă [14–16] formulated optimality conditions for some multi-dimensional interval-valued variational problems. Additionally, Guo et al. [17] established optimality conditions and duality results for a class of generalized convex interval-valued optimization problems. In [18], Guo et al. provided a complete study on

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the properties of symmetric gH-derivative. More precisely, a necessary and sufficient condition for the symmetric gH-differentiability of interval-valued functions has been presented. Further, the authors clarified the relationship between the symmetric gH-differentiability and gH-differentiability. For more information and connected results on this topic, we direct the reader to the following research papers: Antczak [19], Hanson [1], Lodwick [20], Myskova [21], Wu [22], Zhang et al. [23], Zhang et al. [24], Jayswal and Baranwal [25], and references therein.

In this paper, we continue and improve the research mentioned above. Concretely, we establish some equivalence relations between LU-optimal solutions of the considered interval-valued optimal control problem and solutions of the associated weak variational control inequality. The present paper has several merits, as follows: (i) defining, by using the *LU*-order relation, the notion of *LU*-optimal solution for functionals determined by path-independent curvilinear integrals, (ii) formulating original and innovative proofs associated with the main results, and (iii) providing a mathematical context determined by infinite-dimensional function spaces and curvilinear integral-type functionals. These elements are new in the area of interval-valued optimal control problems. The limitations of the study: (1) the concept of "LU-convexity" is strongly used in our arguments; consequently, in our next research works, we will try to improve this aspect and replace it with a general one; (2) also, the well-posedness study of the considered problem is still an open problem that should be investigated.

The paper continues as follows: Section 2 presents notations, preliminary ingredients, and definitions on interval-valued functional of the curvilinear integral type, (strictly) convex real-valued curvilinear integral type functional, and (strictly) LU-convex interval-valued curvilinear integral type functional; in Section 3, we state some existence results of LU-optimal solutions for the considered interval-valued control problem, and of solutions for the corresponding weak variational control inequalities; Section 4 formulates the conclusions of the paper.

2. Preliminaries

In this paper, we consider $\mathbb{R}^m, \mathbb{R}^n$ and $\mathbb{R}^k$ denote the standard Euclidean spaces, Θ is a domain in $\mathbb{R}^m$ and $C \subset \Theta$ is a piecewise differentiable curve joining the following two multiple variables of evolution $t_0 = (t_0^\gamma), t_1 = (t_1^\gamma), \gamma = \overline{1,m}$, included in Θ, and $t = (t^\gamma), \gamma = \overline{1,m}$, is the current point in Θ. Denote by $D_\gamma, \gamma = \overline{1,m}$, the operator associated with the total derivative, and let M represent the space of all piecewise smooth state functions $\sigma : \Theta \mapsto \mathbb{R}^n$, with $\frac{\partial \sigma}{\partial t^\gamma}(t) := \sigma_\gamma(t)$ as the first-order partial derivative of σ with respect to $t^\gamma, \gamma = \overline{1,m}$. Additionally, let N be the space of all continuous control functions $\tau : \Theta \mapsto \mathbb{R}^k$, and K denotes the set of all closed and bounded intervals in $\mathbb{R}$. For $A = [a^L, a^U], B = [b^L, b^U] \in K$, the real numbers a^L, b^L indicate the lower bounds, and a^U, b^U indicate the upper bounds of A and B, respectively. The interval operations are performed as follows:

- $A = B$ is equivalent with $a^L = b^L$ and $a^U = b^U$
- if $a^L = a^U = a$, then $A = [a, a] = a$
- $A + B = [a^L + b^L, a^U + b^U]$
- $-A = -[a^L, a^U] = [-a^U, -a^L]$
- $A - B = [a^L - b^U, a^U - b^L]$
- $d + A = [d + a^L, d + a^U], \quad d \in \mathbb{R}$
- $dA = [da^L, da^U], \quad d \in \mathbb{R}, \ d \geq 0$
- $dA = [da^U, da^L], \quad d \in \mathbb{R}, \ d < 0.$

In addition, the following conventions for any two intervals $A, B \in K$ will be used:

- $A \preceq_{LU} B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$
- $A \prec_{LU} B$ if and only if $A \preceq_{LU} B$ and $A \neq B$.

Next, on the line of Treanţă [12,13,15], we introduce the interval-valued functionals determined by curvilinear integrals, (strictly) convexity associated with real-valued curvi-

linear integral type functionals, and (strictly) LU-convexity for interval-valued curvilinear integral type functionals. Additionally, we define the path independence of the involved curvilinear integrals.

Definition 1. *A curvilinear integral type functional*

$$\Psi : M \times N \mapsto K, \quad \Psi(\sigma, \tau) = \int_C \psi_\alpha(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha$$

is named an interval-valued functional if it is formulated as

$$\Psi(\sigma, \tau) = \left[\int_C \psi_\alpha^L(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha, \int_C \psi_\alpha^U(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha\right],$$

where

$$\Psi^L(\sigma, \tau) := \int_C \psi_\alpha^L(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha,$$

$$\Psi^U(\sigma, \tau) := \int_C \psi_\alpha^U(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha,$$

$$\Psi^L(\sigma, \tau),\ \Psi^U(\sigma, \tau) : M \times N \mapsto \mathbb{R},$$

are real-valued curvilinear integral type functionals, with $\psi_\alpha : \Theta \times M \times M \times N \to K,\ \psi_\alpha = [\psi_\alpha^L, \psi_\alpha^U],\ \alpha = \overline{1, m}$*, satisfying the condition*

$$\int_C \psi_\alpha^L(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha \leq \int_C \psi_\alpha^U(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha.$$

Definition 2. *An interval-valued curvilinear integral type functional* Ψ *is called path-independent if the real-valued curvilinear integral type functionals* Ψ^L *and* Ψ^U *are path-independent, that is, the following equalities* $D_\gamma\psi_\alpha^L = D_\alpha\psi_\gamma^L$ *and* $D_\gamma\psi_\alpha^U = D_\alpha\psi_\gamma^U$ *are satisfied, for* $\alpha \neq \gamma$*.*

Definition 3. *A real-valued curvilinear integral type functional*

$$H : M \times N \mapsto \mathbb{R}, \quad H(\sigma, \tau) = \int_C h_\alpha(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha$$

is named (strictly) convex at $(\sigma^0, \tau^0) \in M \times N$ *if*

$$H(\sigma, \tau) - H(\sigma^0, \tau^0) \geq (>) \int_C \frac{\partial h_\alpha}{\partial \sigma}\Big(t, \sigma^0(t), \sigma_\gamma^0(t), \tau(t)\Big)\Big(\sigma(t) - \sigma^0(t)\Big)dt^\alpha$$

$$+ \int_C \frac{\partial h_\alpha}{\partial \sigma_\gamma}\Big(t, \sigma^0(t), \sigma_\gamma^0(t), \tau(t)\Big)D_\gamma\Big(\sigma(t) - \sigma^0(t)\Big)dt^\alpha$$

$$+ \int_C \frac{\partial h_\alpha}{\partial \tau}\Big(t, \sigma^0(t), \sigma_\gamma^0(t), \tau(t)\Big)\Big(\tau(t) - \tau^0(t)\Big)dt^\alpha,$$

for all $(\sigma, \tau) \in M \times N$*.*

If the above inequality is valid for each $(\sigma^0, \tau^0) \in M \times N$*, then the real-valued curvilinear integral type functional* $H : M \times N \mapsto \mathbb{R}$ *is named (strictly) convex on* $M \times N$*.*

Definition 4. *An interval-valued curvilinear integral type functional*

$$\Psi : M \times N \mapsto K, \quad \Psi(\sigma, \tau) = \int_C \psi_\alpha(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha = [\Psi^L(\sigma, \tau), \Psi^U(\sigma, \tau)]$$

is called LU-convex at $(\sigma^0, \tau^0) \in M \times N$ *if both the real-valued curvilinear integral type functionals* $\Psi^L(\sigma, \tau) = \int_C \psi_\alpha^L(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha$ *and* $\Psi^U(\sigma, \tau) = \int_C \psi_\alpha^U(t, \sigma(t), \sigma_\gamma(t), \tau(t))dt^\alpha$ *are convex at* $(\sigma^0, \tau^0) \in M \times N$*.*

Definition 5. *If the real-valued curvilinear integral type functionals $\Psi^L(\sigma,\tau)$ and $\Psi^U(\sigma,\tau)$ are convex at $(\sigma^0,\tau^0) \in M \times N$ and $\Psi^L(\sigma,\tau)$ or/and $\Psi^U(\sigma,\tau)$ is strictly convex at $(\sigma^0,\tau^0) \in M \times N$, then the interval-valued curvilinear integral type functional*

$$\Psi : M \times N \mapsto K, \quad \Psi(\sigma,\tau) = \int_C \psi_\alpha(t,\sigma(t),\sigma_\gamma(t),\tau(t))dt^\alpha$$

is named strictly LU-convex at $(\sigma^0,\tau^0) \in M \times N$.

Taking into account the above-mentioned mathematical framework, we state the following interval-valued optimization problem:

$$(IVP) \qquad \min_{(\sigma(\cdot),\tau(\cdot))} \int_C \psi_\alpha(t,\sigma(t),\sigma_\gamma(t),\tau(t))dt^\alpha$$

$$\text{subject to}$$

$$X_\delta(t,\sigma(t),\sigma_\gamma(t),\tau(t)) \leq 0, \quad \delta = \overline{1,l}$$

$$Y_\eta(t,\sigma(t),\sigma_\gamma(t),\tau(t)) := \frac{\partial\sigma}{\partial t^\gamma}(t) - Q_\eta(t,\sigma(t),\tau(t)) = 0, \quad \eta = \overline{1,r}$$

$$\sigma|_{t=t_0,t_1} = given,$$

where $t \in \Theta, \psi_\alpha : \Theta \times M \times M \times N \mapsto K$, $X_\delta : \Theta \times M \times M \times N \mapsto \mathbb{R}$ and $Y_\eta : \Theta \times M \times M \times N \mapsto \mathbb{R}$ are C^1-class functions. Let us denote the feasible solution set to (IVP) as

$$\Omega = \{(\sigma,\tau) \in M \times N \mid X_\delta(t,\sigma(t),\sigma_\gamma(t),\tau(t)) \leq 0,$$

$$Y_\eta(t,\sigma(t),\sigma_\gamma(t),\tau(t)) = 0,\ \sigma|_{t=t_0,t_1} = given\}$$

and consider that Ω is a convex subset of $M \times N$.

For simplicity, we use the following notations: $\sigma = \sigma(t), \tau = \tau(t), \zeta = (t,\sigma(t),\sigma_\gamma(t),\tau(t))$, $\zeta^0 = \left(t,\sigma^0(t),\sigma^0_\gamma(t),\tau^0(t)\right), \psi_{\alpha,\sigma} = \frac{\partial\psi_\alpha}{\partial\sigma}, \psi_{\alpha,\sigma_\gamma} = \frac{\partial\psi_\alpha}{\partial\sigma_\gamma}$ and $\psi_{\alpha,\tau} = \frac{\partial\psi_\alpha}{\partial\tau}$.

Definition 6. *A point $(\sigma^0,\tau^0) \in \Omega$ is named a (strong) LU-optimal solution to (IVP) if*

$$\int_C \psi_\alpha(\zeta^0)dt^\alpha \preceq_{LU} (\prec_{LU}) \int_C \psi_\alpha(\zeta)dt^\alpha$$

for all $(\sigma,\tau) \in \Omega$.

Now, on the line of Treanţă [8], in order to provide some characterizations of the solution set associated with the interval-valued optimal control problem (IVP), we formulate the following variational control inequalities:

** find $(\sigma^0,\tau^0) \in \Omega$ in such a way there exists no $(\sigma,\tau) \in \Omega$, fulfilling the following variational control inequality:*

$$(VC) \qquad \int_C \left\{\psi^L_{\alpha,\sigma}\left(\zeta^0\right) + \psi^U_{\alpha,\sigma}\left(\zeta^0\right)\right\}\left(\sigma - \sigma^0\right)dt^\alpha$$

$$+ \int_C \left\{\psi^L_{\alpha,\tau}\left(\zeta^0\right) + \psi^U_{\alpha,\tau}\left(\zeta^0\right)\right\}\left(\tau - \tau^0\right)dt^\alpha$$

$$+ \int_C \left\{\psi^L_{\alpha,\sigma_\gamma}\left(\zeta^0\right) + \psi^U_{\alpha,\sigma_\gamma}\left(\zeta^0\right)\right\}D_\gamma\left(\sigma - \sigma^0\right)dt^\alpha \leq 0;$$

** find $(\sigma^0,\tau^0) \in \Omega$ in such a way that there exists no $(\sigma,\tau) \in \Omega$, fulfilling the following weak variational control inequality:*

$$(WVC) \qquad \int_C \left\{\psi^L_{\alpha,\sigma}\left(\zeta^0\right) + \psi^U_{\alpha,\sigma}\left(\zeta^0\right)\right\}\left(\sigma - \sigma^0\right)dt^\alpha$$

$$+\int_C \left\{\psi^L_{\alpha,\tau}\left(\zeta^0\right)+\psi^U_{\alpha,\tau}\left(\zeta^0\right)\right\}\left(\tau-\tau^0\right)dt^\alpha$$
$$+\int_C \left\{\psi^L_{\alpha,\sigma_\gamma}\left(\zeta^0\right)+\psi^U_{\alpha,\sigma_\gamma}\left(\zeta^0\right)\right\}D_\gamma\left(\sigma-\sigma^0\right)dt^\alpha<0;$$

* *find* $(\sigma^0,\tau^0)\in\Omega$ *in such a way that, for all* $(\sigma,\tau)\in\Omega$*, the following split variational control inequalities*

$$(SVC)\qquad \int_C \psi^L_{\alpha,\sigma}\left(\zeta^0\right)\left(\sigma-\sigma^0\right)dt^\alpha+\int_C \psi^L_{\alpha,\tau}\left(\zeta^0\right)\left(\tau-\tau^0\right)dt^\alpha$$
$$+\int_C \psi^L_{\alpha,\sigma_\gamma}\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)dt^\alpha>0,$$
$$\int_C \psi^U_{\alpha,\sigma}\left(\zeta^0\right)\left(\sigma-\sigma^0\right)dt^\alpha+\int_C \psi^U_{\alpha,\tau}\left(\zeta^0\right)\left(\tau-\tau^0\right)dt^\alpha$$
$$+\int_C \psi^U_{\alpha,\sigma_\gamma}\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)dt^\alpha>0$$

are satisfied.

3. Main Results

This section, via solutions of the weak variational control inequality (WVC), formulates and proves an existence result for the LU-optimal solutions of the interval-valued optimal control problem (IVP).

For a better understanding of the mathematical context associated with the control problems under study, we state some recent auxiliary results provided by Tareq [26] (see Theorems 1–4).

The next result, by considering the solution associated with the variational control inequality (VC), provides a sufficient condition for a pair $(\sigma^0,\tau^0)\in\Omega$ to become an LU-optimal solution to (IVP).

Theorem 1. *Consider* $(\sigma^0,\tau^0)\in\Omega$ *is a solution to* (VC) *and* $\int_C \psi_\alpha(\zeta)dt^\alpha$ *is LU-convex at* $(\sigma^0,\tau^0)\in\Omega$*. Then* $(\sigma^0,\tau^0)\in\Omega$ *is an LU-optimal solution to* (IVP)*.*

The following theorem represents the reciprocal of the previous result.

Theorem 2. *Consider* $(\sigma^0,\tau^0)\in\Omega$ *is an LU-optimal solution to* (IVP) *and* $\int_C -\psi_\alpha(\zeta)dt^\alpha$ *is strictly LU-convex at* $(\sigma^0,\tau^0)\in\Omega$*. Then,* $(\sigma^0,\tau^0)\in\Omega$ *is a solution to* (VC)*.*

In the following, by using the solution associated with the split variational control inequality (SVC), the next theorem provides a sufficient condition for a pair $(\sigma^0,\tau^0)\in\Omega$ to become a strong LU-optimal solution to (IVP).

Theorem 3. *Consider* $(\sigma^0,\tau^0)\in\Omega$ *fulfills* (SVC) *and* $\int_C \psi_\alpha(\zeta)dt^\alpha$ *is LU-convex at* $(\sigma^0,\tau^0)\in\Omega$*. Then,* $(\sigma^0,\tau^0)\in\Omega$ *is a strong LU-optimal solution to* (IVP)*.*

The following result provides the conditions such that the reciprocal of the previous result is satisfied.

Theorem 4. *Consider* $(\sigma^0,\tau^0)\in\Omega$ *is a strong LU-optimal solution to* (IVP) *and* $\int_C -\psi_\alpha(\zeta)dt^\alpha$ *is LU-convex at* $(\sigma^0,\tau^0)\in\Omega$*. Then,* $(\sigma^0,\tau^0)\in\Omega$ *fulfills* (SVC)*.*

In the following, we state and prove the main results of the present paper. The next result, by considering the solution associated with the variational control inequality (VC), provides a sufficient condition for a pair $(\sigma^0, \tau^0) \in \Omega$ to become an LU-optimal solution to (IVP).

Theorem 5. *Consider $(\sigma^0, \tau^0) \in \Omega$ is an LU-optimal solution to (IVP) and $\int_C -\psi_\alpha(\zeta)dt^\alpha$ is LU-convex at $(\sigma^0, \tau^0) \in \Omega$. Then, $(\sigma^0, \tau^0) \in \Omega$ fulfills the weak variational control inequality (WVC).*

Proof. By hypothesis, $(\sigma^0, \tau^0) \in \Omega$ is an LU-optimal solution to (IVP). Thus, there exists no $(\sigma, \tau) \in \Omega$ satisfying

$$\int_C \psi_\alpha(\zeta)dt^\alpha \prec_{LU} \int_C \psi_\alpha\left(\zeta^0\right)dt^\alpha,$$

equivalent with

$$\begin{aligned}
&\int_C \psi_\alpha^L(\zeta)dt^\alpha < \int_C \psi_\alpha^L\left(\zeta^0\right)dt^\alpha \text{ and } \int_C \psi_\alpha^U(\zeta)dt^\alpha \leq \int_C \psi_\alpha^U\left(\zeta^0\right)dt^\alpha,\\
&\text{or } \int_C \psi_\alpha^L(\zeta)dt^\alpha \leq \int_C \psi_\alpha^L\left(\zeta^0\right)dt^\alpha \text{ and } \int_C \psi_\alpha^U(\zeta)dt^\alpha < \int_C \psi_\alpha^U\left(\zeta^0\right)dt^\alpha,\\
&\text{or } \int_C \psi_\alpha^L(\zeta)dt^\alpha < \int_C \psi_\alpha^L\left(\zeta^0\right)dt^\alpha \text{ and } \int_C \psi_\alpha^U(\zeta)dt^\alpha < \int_C \psi_\alpha^U\left(\zeta^0\right)dt^\alpha.
\end{aligned}$$

Thus, there exists no $(\sigma, \tau) \in \Omega$ fulfilling

$$\int_C \left\{\psi_\alpha^L(\zeta) + \psi_\alpha^U(\zeta)\right\}dt^\alpha < \int_C \left\{\psi_\alpha^L\left(\zeta^0\right) + \psi_\alpha^U\left(\zeta^0\right)\right\}dt^\alpha. \tag{1}$$

Now, we proceed by contradiction and assume that $(\sigma^0, \tau^0) \in \Omega$ is not a solution to the weak variational control inequality (WVC). In consequence, there exists $(\sigma, \tau) \in \Omega$ so that

$$\begin{aligned}
&\int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma}^U\left(\zeta^0\right)\right\}\left(\sigma - \sigma^0\right)dt^\alpha + \int_C \left\{\psi_{\alpha,\tau}^L\left(\zeta^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\right\}\left(\tau - \tau^0\right)dt^\alpha\\
&\quad + \int_C \left\{\psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)\right\}D_\gamma\left(\sigma - \sigma^0\right)dt^\alpha < 0.
\end{aligned} \tag{2}$$

Since the functional $\int_C -\psi_\alpha(\zeta)dt^\alpha$ is LU-convex at $(\sigma^0, \tau^0) \in \Omega$, we obtain

$$\begin{aligned}
&\int_C \psi_\alpha^L(\zeta)dt^\alpha - \int_C \psi_\alpha^L\left(\zeta^0\right)dt^\alpha\\
&\leq \int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma - \sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau - \tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right)D_\gamma\left(\sigma - \sigma^0\right)\right\}dt^\alpha
\end{aligned}$$

and

$$\begin{aligned}
&\int_C \psi_\alpha^U(\zeta)dt^\alpha - \int_C \psi_\alpha^U\left(\zeta^0\right)dt^\alpha\\
&\leq \int_C \left\{\psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma - \sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau - \tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)D_\gamma\left(\sigma - \sigma^0\right)\right\}dt^\alpha
\end{aligned}$$

for all $(\sigma, \tau) \in \Omega$. From the above inequalities, we obtain

$$\begin{aligned}
&\int_C \left\{\psi_\alpha^L(\zeta) + \psi_\alpha^U(\zeta)\right\}dt^\alpha - \int_C \left\{\psi_\alpha^L\left(\zeta^0\right) + \psi_\alpha^U\left(\zeta^0\right)\right\}dt^\alpha\\
&\leq \int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma}^U\left(\zeta^0\right)\right\}\left(\sigma - \sigma^0\right)dt^\alpha + \int_C \left\{\psi_{\alpha,\tau}^L\left(\zeta^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\right\}\left(\tau - \tau^0\right)dt^\alpha\\
&\quad + \int_C \left\{\psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)\right\}D_\gamma\left(\sigma - \sigma^0\right)dt^\alpha,
\end{aligned}$$

for all $(\sigma, \tau) \in \Omega$, which, together with the inequality (2), yields the following inequality:

$$\int_C \left\{ \psi_\alpha^L(\zeta) + \psi_\alpha^U(\zeta) \right\} dt^\alpha < \int_C \left\{ \psi_\alpha^L\left(\zeta^0\right) + \psi_\alpha^U\left(\zeta^0\right) \right\} dt^\alpha,$$

for a point $(\sigma, \tau) \in \Omega$, which contradicts the inequality (1).

□

The following theorem represents the converse result of the previous one.

Theorem 6. *Consider $(\sigma^0, \tau^0) \in \Omega$ fulfills the weak variational control inequality (WVC) and the interval-valued functional $\int_C \psi_\alpha(\zeta) dt^\alpha$ is strictly LU-convex at $(\sigma^0, \tau^0) \in \Omega$. Then, $(\sigma^0, \tau^0) \in \Omega$ is an LU-optimal solution to (IVP).*

Proof. By hypothesis, $(\sigma^0, \tau^0) \in \Omega$ is a solution of (WVC). Therefore, there exists no $(\sigma, \tau) \in \Omega$, fulfilling

$$\int_C \left\{ \psi_{\alpha,\sigma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma}^U\left(\zeta^0\right) \right\} \left(\sigma - \sigma^0\right) dt^\alpha + \int_C \left\{ \psi_{\alpha,\tau}^L\left(\zeta^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right) \right\} \left(\tau - \tau^0\right) dt^\alpha$$
$$+ \int_C \left\{ \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right) \right\} D_\gamma\left(\sigma - \sigma^0\right) dt^\alpha < 0. \tag{3}$$

Now, we proceed by contradiction and assume that $(\sigma^0, \tau^0) \in \Omega$ is not an LU-optimal solution to (IVP). Thus, there exists $(\sigma, \tau) \in \Omega$, fulfilling

$$\int_C \psi_\alpha(\zeta) dt^\alpha \prec_{LU} \int_C \psi_\alpha\left(\zeta^0\right) dt^\alpha,$$

equivalent with

$$\int_C \psi_\alpha^L(\zeta) dt^\alpha < \int_C \psi_\alpha^L\left(\zeta^0\right) dt^\alpha \text{ and } \int_C \psi_\alpha^U(\zeta) dt^\alpha \leq \int_C \psi_\alpha^U\left(\zeta^0\right) dt^\alpha, \tag{4}$$

or

$$\int_C \psi_\alpha^L(\zeta) dt^\alpha \leq \int_C \psi_\alpha^L\left(\zeta^0\right) dt^\alpha \text{ and } \int_C \psi_\alpha^U(\zeta) dt^\alpha < \int_C \psi_\alpha^U\left(\zeta^0\right) dt^\alpha, \tag{5}$$

or

$$\int_C \psi_\alpha^L(\zeta) dt^\alpha < \int_C \psi_\alpha^L\left(\zeta^0\right) dt^\alpha \text{ and } \int_C \psi_\alpha^U(\zeta) dt^\alpha < \int_C \psi_\alpha^U\left(\zeta^0\right) dt^\alpha. \tag{6}$$

From strict LU-convexity property of $\int_C \psi_\alpha(\zeta) dt^\alpha$, it results

$$\int_C \psi_\alpha^L(\zeta) dt^\alpha - \int_C \psi_\alpha^L\left(\zeta^0\right) dt^\alpha$$
$$> \int_C \left\{ \psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma - \sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau - \tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right) D_\gamma\left(\sigma - \sigma^0\right) \right\} dt^\alpha$$

and

$$\int_C \psi_\alpha^U(\zeta) dt^\alpha - \int_C \psi_\alpha^U\left(\zeta^0\right) dt^\alpha$$
$$\geq \int_C \left\{ \psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma - \sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau - \tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right) D_\gamma\left(\sigma - \sigma^0\right) \right\} dt^\alpha,$$

or

$$\int_C \psi_\alpha^L(\zeta) dt^\alpha - \int_C \psi_\alpha^L\left(\zeta^0\right) dt^\alpha$$
$$\geq \int_C \left\{ \psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma - \sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau - \tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right) D_\gamma\left(\sigma - \sigma^0\right) \right\} dt^\alpha$$

and

$$\int_C \psi_\alpha^U(\zeta)dt^\alpha - \int_C \psi_\alpha^U\left(\zeta^0\right)dt^\alpha$$
$$> \int_C \left\{\psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha,$$

or

$$\int_C \psi_\alpha^L(\zeta)dt^\alpha - \int_C \psi_\alpha^L\left(\zeta^0\right)dt^\alpha$$
$$> \int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha,$$

and

$$\int_C \psi_\alpha^U(\zeta)dt^\alpha - \int_C \psi_\alpha^U\left(\zeta^0\right)dt^\alpha$$
$$> \int_C \left\{\psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha,$$

for all $(\sigma,\tau)\in\Omega$. On combining the above three inequalities with the inequalities (4)–(6), respectively, we obtain

$$\int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha < 0 \text{ and}$$
$$\int_C \left\{\psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha \le 0,$$
$$\int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha \le 0 \text{ and}$$
$$\int_C \left\{\psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha < 0,$$
$$\int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^L\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha < 0 \text{ and}$$
$$\int_C \left\{\psi_{\alpha,\sigma}^U\left(\zeta^0\right)\left(\sigma-\sigma^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\left(\tau-\tau^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)D_\gamma\left(\sigma-\sigma^0\right)\right\}dt^\alpha < 0.$$

From the above inequalities, it follows that

$$\int_C \left\{\psi_{\alpha,\sigma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma}^U\left(\zeta^0\right)\right\}\left(\sigma-\sigma^0\right)dt^\alpha + \int_C \left\{\psi_{\alpha,\tau}^L\left(\zeta^0\right) + \psi_{\alpha,\tau}^U\left(\zeta^0\right)\right\}\left(\tau-\tau^0\right)dt^\alpha +$$
$$\int_C \left\{\psi_{\alpha,\sigma_\gamma}^L\left(\zeta^0\right) + \psi_{\alpha,\sigma_\gamma}^U\left(\zeta^0\right)\right\}D_\gamma\left(\sigma-\sigma^0\right)dt^\alpha < 0$$

holds, for $(\sigma,\tau)\in\Omega$, which contradicts the inequality (3). □

Application. Let us extremize the interval-valued curvilinear integral cost functional given by

$$(IVP1) \qquad \min_{(\sigma(\cdot),\tau(\cdot))} \int_C \psi(\zeta)(dt^1+dt^2)$$
$$= \left[\int_C \psi^L(\zeta)(dt^1+dt^2), \int_C \psi^U(\zeta)(dt^1+dt^2)\right]$$
$$= \left[\int_C (\tau-4)^2(dt^1+dt^2), \int_C \tau^2(dt^1+dt^2)\right]$$

subject to

$$\frac{\partial\sigma}{\partial t^1} = \frac{\partial\sigma}{\partial t^2} = 3-\tau,$$
$$81-\sigma^2 \le 0,$$
$$\sigma(0,0) = 6, \quad \sigma(3,3) = 8.$$

The feasible solution set of $(IVP1)$ is given by

$$\Omega = \{(\sigma,\tau) \in M \times N : 81 - \sigma^2 \leq 0,\ \frac{\partial \sigma}{\partial t^1} = \frac{\partial \sigma}{\partial t^2} = 3 - \tau,$$

$$\sigma(0,0) = 6,\ \sigma(3,3) = 8\},$$

and, taking into account the notations used in this paper, we have $X(\zeta) = 81 - \sigma^2$, $Y_\eta(\zeta) = \frac{\partial \sigma}{\partial t^\eta} - 3 + \tau,\ \eta = 1,2$. Let us assume that we are only interested in the affine state and control real-valued functions and $t \in \Theta = [0,3]^2 = [0,3] \times [0,3] \subset \mathbb{R}^2$. It can be easily seen that the real-valued functionals $\int_C \psi^L(\zeta)(dt^1 + dt^2)$ and $\int_C \psi^U(\zeta)(dt^1 + dt^2)$ are (strictly) convex at $(\sigma^0,\tau^0) = \left(\frac{1}{3}(t^1 + t^2) + 6, \frac{8}{3}\right)$. Thus, the interval-valued functional

$$\Psi(\sigma,\tau) = \left[\int_C \psi^L(\zeta)(dt^1 + dt^2), \int_C \psi^U(\zeta)(dt^1 + dt^2)\right]$$

is strictly LU-convex at $(\sigma^0,\tau^0) = \left(\frac{1}{3}(t^1 + t^2) + 6, \frac{8}{3}\right)$. By imposing the condition

$$\psi^L(\zeta) \leq \psi^U(\zeta)$$

it results

$$(\tau - 4)^2 \leq \tau^2$$

involving

$$\tau \geq 2.$$

Since the following inequality

$$\int_C \Big[\{\psi^L_\sigma\left(\zeta^0\right) + \psi^U_\sigma\left(\zeta^0\right)\}\left(\sigma - \sigma^0\right) + \{\psi^L_{y_\gamma}\left(\zeta^0\right) + \psi^U_{y_\gamma}\left(\zeta^0\right)\}D_\gamma\left(\sigma - \sigma^0\right)$$

$$+\{\psi^L_\tau\left(\zeta^0\right) + \psi^U_\tau\left(\zeta^0\right)\}\left(\tau - \tau^0\right)\Big](dt^1 + dt^2)$$

$$= \int_C \left(2\tau^0 - 8 + 2z^0\right)\left(\tau - \tau^0\right)(dt^1 + dt^2)$$

$$= \int_C \left(4\tau^0 - 8\right)\left(\tau - \tau^0\right)(dt^1 + dt^2) \not\leq 0,$$

holds at $\left(\sigma^0(t) = \frac{1}{3}\left(t^1 + t^2\right) + 6, \tau^0(t) = \frac{8}{3}\right)$, for all $(\sigma,\tau) \neq (\sigma^0,\tau^0) \in \Omega$, then it is a solution to variational control inequality (WVC). By using Theorem 6, it follows that (σ^0,τ^0) is an LU-optimal solution of $(IVP1)$. Indeed, it can be verified that the inequality

$$\int_C \psi(\zeta^0)(dt^1 + dt^2) \preceq_{LU} \int_C \psi(\zeta)(dt^1 + dt^2)$$

is satisfied.

4. Conclusions

By considering the solutions of a new class of weak variational control inequalities, denoted by (WVC) , in this paper we have established new results on the existence of LU-optimal solutions for the associated interval-valued optimal control problem, denoted by (IVP). The derived results have been generated by the LU-convexity property and path independence of the involved functionals determined by curvilinear integrals. As further developments of the results stated in this paper, we mention the formulation of the associated duality theory, well-posedness results, and saddle-point optimality criteria.

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