mathematics

Article

Determining the Coefficients of the Thermoelastic System from Boundary Information

Xiaoming Tan

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China; xtan@bit.edu.cn

Abstract: Given a compact Riemannian manifold (M, g) with smooth boundary ∂M, we give an explicit expression for the full symbol of the thermoelastic Dirichlet-to-Neumann map Λ_g with variable coefficients $\lambda, \mu, \alpha, \beta \in C^\infty(\bar{M})$. We prove that Λ_g uniquely determines partial derivatives of all orders of these coefficients on the boundary ∂M. Moreover, for a nonempty smooth open subset $\Gamma \subset \partial M$, suppose that the manifold and these coefficients are real analytic up to Γ. We show that Λ_g uniquely determines these coefficients on the whole manifold $\bar{M}$.

Keywords: thermoelastic system; thermoelastic Calderón's problem; thermoelastic Dirichlet-to-Neumann map; inverse problems; pseudodifferential operators

MSC: 35R30; 74F05; 74E05; 58J32; 58J40

check for updates

Citation: Tan, X. Determining the Coefficients of the Thermoelastic System from Boundary Information. *Mathematics* **2023**, *11*, 2147. https://doi.org/10.3390/math11092147

Academic Editor: Alberto Ferrero

Received: 22 March 2023
Revised: 17 April 2023
Accepted: 18 April 2023
Published: 4 May 2023

1. Introduction

In this paper, we will study the thermoelastic Calderón problem, i.e., whether one can uniquely determine the Lamé coefficients λ, μ, and the other two physical coefficients α, β of a thermoelastic body by boundary information. Let (M, g) be a compact Riemannian manifold of dimension n with smooth boundary ∂M. We consider the manifold M as an inhomogeneous, isotropic, thermoelastic body. Assume that the coefficient $\beta \in C^\infty(\bar{M})$, the Lamé coefficients $\lambda, \mu \in C^\infty(\bar{M})$, and the heat conduction coefficient $\alpha \in C^\infty(\bar{M})$ of the thermoelastic body satisfy

$$\mu > 0, \quad \lambda + \mu \geqslant 0, \quad \alpha > 0. \tag{1}$$

1.1. Thermoelastic Operator

We denote by grad, div, Δ_g, Δ_B, and Ric, respectively, the gradient operator, the divergence operator, the Laplace–Beltrami operator, the Bochner Laplacian, and the Ricci tensor with respect to the metric g. For the displacement vector field $\boldsymbol{u} \in [C^\infty(M)]^n$ and the temperature variation $\theta \in C^\infty(M)$, we define the thermoelastic operator T_g with variable coefficients as (cf. [1–4])

$$T_g \begin{bmatrix} \boldsymbol{u} \\ \theta \end{bmatrix} := \begin{bmatrix} L_g + \rho\omega^2 & -\beta \operatorname{grad} \\ i\omega\theta_0\beta \operatorname{div} & \alpha\Delta_g + i\omega\gamma \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \theta \end{bmatrix}, \tag{2}$$

where the Lamé operator L_g with variable coefficients is defined by (see [4])

$$\begin{aligned} L_g \boldsymbol{u} := {} & \mu\Delta_B \boldsymbol{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} + \mu \operatorname{Ric}(\boldsymbol{u}) \\ & + (\operatorname{grad} \lambda) \operatorname{div} \boldsymbol{u} + (S\boldsymbol{u})(\operatorname{grad} \mu). \end{aligned} \tag{3}$$

Here the strain tensor S (also called the deformation tensor) of type $(1, 1)$ is defined by (see [5], p. 562)

$$S\boldsymbol{u} := \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t, \tag{4}$$

where the superscript t denotes the transpose. The coefficient $\beta \in C^\infty(\bar{M})$ depends on the Lamé coefficients and the linear expansion coefficient of the thermoelastic body, γ is the specific heat per unit volume, θ_0 is the reference temperature, ρ is the density of the thermoelastic body, ω is the angular frequency, and $i = \sqrt{-1}$.

In particular, the Lamé operator with constant coefficients has the form $L\boldsymbol{u} = \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})$ in Euclidean bounded domains (see [1,6]).

1.2. Thermoelastic Calderón Problem

We first consider the following Dirichlet boundary value problem for the thermoelastic system

$$\begin{cases} T_g\boldsymbol{U} = 0 & \text{in } M, \\ \boldsymbol{U} = \boldsymbol{V} & \text{on } \partial M, \end{cases} \tag{5}$$

where $\boldsymbol{U} = (\boldsymbol{u}, \theta)^t$. Problem (5) is an extension of the boundary value problem for classical elastic system. Particularly, when M is a bounded Euclidean domain and the temperature is not taken into consideration, problem (5) reduces to the corresponding problem for classical elastic system.

For any vector $\boldsymbol{V} \in [H^{1/2}(\partial M)]^{n+1}$, there is a unique solution $\boldsymbol{U} \in [H^1(M)]^{n+1}$ solving problem (5) by the theory of elliptic operators. Therefore, we define the thermoelastic Dirichlet-to-Neumann map $\Lambda_g : [H^{1/2}(\partial M)]^{n+1} \to [H^{-1/2}(\partial M)]^{n+1}$ associated with the thermoelastic operator T_g as (see [3])

$$\Lambda_g(\boldsymbol{U}|_{\partial M}) := \begin{bmatrix} \lambda\nu \operatorname{div} + \mu\nu S & -\beta\nu \\ 0 & \alpha\partial_\nu \end{bmatrix} \boldsymbol{U} \quad \text{on } \partial M, \tag{6}$$

where ν is the outward unit normal vector to the boundary ∂M. The thermoelastic Dirichlet-to-Neumann map Λ_g is an elliptic, self-adjoint pseudodifferential operator of order one defined on the boundary. For the studies about other types of Dirichlet-to-Neumann map, we also refer the reader to [3,7–9] and references therein.

In this paper, we will study the thermoelastic Calderón problem on a Riemannian manifold, which determines the coefficients $\lambda, \mu, \alpha, \beta \in C^\infty(\bar{M})$ by the thermoelastic Dirichlet-to-Neumann map Λ_g. By giving explicit expressions for Λ_g and its full symbol $\sigma(\Lambda_g)$, we show that Λ_g uniquely determines the coefficients $\lambda, \mu, \alpha, \beta$.

We briefly recall some uniqueness results for the classical Calderón problem and the elastic Calderón problem. The classical Calderón problem [10] is concerned with whether one can uniquely determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This problem has been studied for decades. For a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n \geqslant 2$, Kohn and Vogelius [11] proved a famous uniqueness result on the boundary for C^∞-conductivities, that is, if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then, $\frac{\partial^{|J|}\gamma_1}{\partial x^J}\big|_{\partial\Omega} = \frac{\partial^{|J|}\gamma_2}{\partial x^J}\big|_{\partial\Omega}$ for all multi-indices $J \in \mathbb{N}^n$. This settled the uniqueness problem on the boundary in the real analytic category. They extended the uniqueness result to piecewise real analytic conductivities in [12]. In dimensions $n \geqslant 3$, in a celebrated paper [13], Sylvester and Uhlmann proved the uniqueness of the C^∞-conductivities by constructing the complex geometrical optics solutions. The classical Calderón problem has attracted much attention for decades (see, for example, [14–18] in two dimensional cases, and [19–22] in higher dimensional cases). We also refer the reader to the survey articles [23,24] for the classical Calderón problem and related topics.

For the elastic Calderón problem, partial uniqueness results for determination of Lamé coefficients from boundary measurements were obtained. For a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, Nakamura and Uhlmann [25] proved that one can determine the full Taylor series of Lamé coefficients on the boundary in all dimensions $n \geqslant 2$ and for a generic anisotropic elastic tensor in two dimensions. In [26], Imanuvilov and Yamamoto also proved the global uniqueness of the Lamé coefficients $\lambda, \mu \in C^{10}(\bar{\Omega})$. In three-dimensional Euclidean domains, Nakamura and Uhlmann [27] and Eskin and

Ralston [28] proved the global uniqueness of Lamé coefficients provided that $\nabla\mu$ is small in a suitable norm. However, in dimensions $n \geqslant 3$, the global uniqueness of the Lamé coefficients $\lambda, \mu \in C^\infty(\bar{\Omega})$ without the smallness assumption ($\|\nabla\mu\| < \varepsilon_0$ for some small positive ε_0) remains an open problem (see [29], p. 210). We also refer the reader to [30–33] for the elastic Calderón problem.

Recently, Tan and Liu [4] gave an explicit expression for the full symbol of the elastic Dirichlet-to-Neumann map on a Riemannian manifold M, and showed that the elastic Dirichlet-to-Neumann map uniquely determines partial derivatives of all orders of the Lamé coefficients on the boundary. Moreover, for a nonempty open subset $\Gamma \subset \partial M$, suppose that the manifold and the Lamé coefficients are real analytic up to Γ, they proved that the elastic Dirichlet-to-Neumann map uniquely determines the Lamé coefficients on the whole manifold $\bar{M}$.

In mathematics, physics, and engineering, there are lots of inverse problems have been studied for decades. Here we do not list all the references about these topics. We refer the reader to [34–38] for Maxwell's equations, to [39–49] for incompressible fluid, Schrödinger operator, elastic operator, and the related problems. For the studies about other types of Dirichlet-to-Neumann map, we also refer the reader to [3,7–9,50,51] and references therein.

Before we state the main results of this paper, we recall some basic concepts about boundary normal coordinates, pseudodifferential operators and symbols.

1.3. Boundary Normal Coordinates

We briefly introduce the construction of geodesic coordinates with respect to the boundary ∂M (see [21], [52], p. 532).

For each boundary point $x' \in \partial M$, let $\gamma_{x'} : [0, \varepsilon) \to \bar{M}$ denote the unit-speed geodesic starting at x' and normal to ∂M. If $x' := \{x_1, \dots, x_{n-1}\}$ are any local coordinates for ∂M near $x_0 \in \partial M$, we can extend them smoothly to functions on a neighborhood of x_0 in $\bar{M}$ by letting them be constant along each normal geodesic $\gamma_{x'}$. If we then define x_n to be the parameter along each $\gamma_{x'}$, it follows easily that $\{x_1, \dots, x_n\}$ form coordinates for $\bar{M}$ in some neighborhood of x_0, which we call the boundary normal coordinates determined by $\{x_1, \dots, x_{n-1}\}$. In these coordinates $x_n > 0$ in M, and the boundary ∂M is locally characterized by $x_n = 0$. A standard computation shows that the metric has the form $g = g_{\alpha\beta}\, dx_\alpha\, dx_\beta + dx_n^2$.

1.4. Pseudodifferential Operators and Symbols

We recall some concepts of pseudodifferential operators and their symbols (cf. [52], Chapter 7).

Assuming $U \subset \mathbb{R}^n$ and $m \in \mathbb{R}$, we define $S^m_{1,0} = S^m_{1,0}(U, \mathbb{R}^n)$ to consist of C^∞-functions $p(x,\xi)$ satisfying for every compact set $V \subset U$,

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \leqslant C_{V,\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}, \quad x \in V,\ \xi \in \mathbb{R}^n$$

for all $\alpha, \beta \in \mathbb{N}^n$, where $D^\alpha = D^{\alpha_1} \cdots D^{\alpha_n}$, $D_j = -i\frac{\partial}{\partial x_j}$ and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. The elements of $S^m_{1,0}$ are called symbols of order m. It is clear that $S^m_{1,0}$ is a Fréchet space with semi-norms given by

$$\|p\|_{V,\alpha,\beta} := \sup_{x \in V} \left|\left(D^\beta_x D^\alpha_\xi p(x,\xi)\right)(1 + |\xi|)^{-m+|\alpha|}\right|.$$

Let $p(x,\xi) \in S^m_{1,0}$ and $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} u(y)\, dy$ be the Fourier transform of u. A pseudodifferential operator in an open set U is essentially defined by a Fourier integral operator

$$P(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(x,\xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$$

for $u \in C_0^\infty(U)$. In such a case, we say the associated operator $P(x,D)$ belongs to OPS^m. We denote $OPS^{-\infty} = \bigcap_m OPS^m$. If there are smooth $p_{m-j}(x,\xi)$, homogeneous in ξ of degree $m-j$ for $|\xi| \geqslant 1$, that is, $p_{m-j}(x,r\xi) = r^{m-j}p_{m-j}(x,\xi)$ for $r>0$, and if

$$p(x,\xi) \sim \sum_{j\geqslant 0} p_{m-j}(x,\xi) \tag{7}$$

in the sense that

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S_{1,0}^{m-N-1}$$

for all N, then, we say $p(x,\xi) \in S_{cl}^m$, or just $p(x,\xi) \in S^m$. We denote $S^{-\infty} = \bigcap_m S^m$. We call $p_m(x,\xi)$ the principal symbol of $P(x,D)$. We say $P(x,D) \in OPS^m$ is elliptic of order m if on each compact $V \subset U$ there are constants C_V and $r < \infty$ such that

$$|p(x,\xi)^{-1}| \leqslant C_V \langle \xi \rangle^{-m}, \quad |\xi| \geqslant r.$$

We can now define a pseudodifferential operator on a manifold M. In particular,

$$P : C_0^\infty(M) \to C^\infty(M)$$

belongs to $OPS_{1,0}^m(M)$ if the kernel of P is smooth off the diagonal in $M \times M$ and if for any coordinate neighborhood $U \subset M$ with $\Phi : U \to \mathcal{O}$ a diffeomorphism onto an open subset $\mathcal{O} \subset \mathbb{R}^n$, the map $\tilde{P} : C_0^\infty(\mathcal{O}) \to C^\infty(\mathcal{O})$ given by

$$\tilde{P}u := P(u \circ \Phi) \circ \Phi^{-1}$$

belongs to $OPS_{1,0}^m(\mathcal{O})$. We refer the reader to [53–55] for more details.

1.5. The Main Results of This Paper

For the sake of simplicity, we denote by $i = \sqrt{-1}$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, $\xi^\alpha = g^{\alpha\beta}\xi_\beta$, $|\xi'| = \sqrt{\xi^\alpha \xi_\alpha}$, I_n the $n \times n$ identity matrix,

$$[a^\alpha_\beta] := \begin{bmatrix} a^1_1 & \cdots & a^1_{n-1} \\ \vdots & \ddots & \vdots \\ a^{n-1}_1 & \cdots & a^{n-1}_{n-1} \end{bmatrix}, \quad [a^j_k] := \begin{bmatrix} a^1_1 & \cdots & a^1_n \\ \vdots & \ddots & \vdots \\ a^n_1 & \cdots & a^n_n \end{bmatrix},$$

and

$$\begin{bmatrix} [a^j_k] & [b^j] \\ [c_k] & d \end{bmatrix} := \left[\begin{array}{cc:c} [a^\alpha_\beta] & [a^\alpha_n] & [b^\alpha] \\ [a^n_\beta] & a^n_n & b^n \\ \hdashline [c_\beta] & c_n & d \end{array}\right] = \left[\begin{array}{ccc:c} a^1_1 & \cdots & a^1_n & b^1 \\ \vdots & \ddots & \vdots & \vdots \\ a^n_1 & \cdots & a^n_n & b^n \\ \hdashline c_1 & \cdots & c_n & d \end{array}\right],$$

where $1 \leqslant \alpha, \beta \leqslant n-1$, and $1 \leqslant j,k \leqslant n$.

The main results of this paper are the following three theorems.

Theorem 1. *Let (M,g) be a compact Riemannian manifold of dimension n with smooth boundary ∂M. Assume that the coefficient $\beta \in C^\infty(\bar{M})$, the Lamé coefficients λ, $\mu \in C^\infty(\bar{M})$, and the heat conduction coefficient $\alpha \in C^\infty(\bar{M})$ satisfy $\mu > 0, \lambda + \mu \geqslant 0$, and $\alpha > 0$. Let $\sigma(\Lambda_g) \sim$*

$\sum_{j\leqslant 1} p_j(x,\xi')$ *be the full symbol of the thermoelastic Dirichlet-to-Neumann map* Λ_g*. Then, in boundary normal coordinates,*

$$p_1(x,\xi') = \begin{bmatrix} \mu|\xi'|I_{n-1} + \frac{\mu(\lambda+\mu)}{(\lambda+3\mu)|\xi'|}[\xi^\alpha\xi_\beta] & -\frac{2i\mu^2}{\lambda+3\mu}[\xi^\alpha] & 0 \\ \frac{2i\mu^2}{\lambda+3\mu}[\xi_\beta] & \frac{2\mu(\lambda+2\mu)}{\lambda+3\mu}|\xi'| & 0 \\ 0 & 0 & \alpha|\xi'| \end{bmatrix}, \tag{8}$$

$$p_0(x,\xi') = \begin{bmatrix} \mu I_{n-1} & 0 & 0 \\ 0 & \lambda+2\mu & 0 \\ 0 & 0 & \alpha \end{bmatrix} q_0(x,\xi') - \begin{bmatrix} 0 & 0 & 0 \\ \lambda[\Gamma^\alpha_{\alpha\beta}] & \lambda\Gamma^\alpha_{\alpha n} & -\beta \\ 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

$$p_{-m}(x,\xi') = \begin{bmatrix} \mu I_{n-1} & 0 & 0 \\ 0 & \lambda+2\mu & 0 \\ 0 & 0 & \alpha \end{bmatrix} q_{-m}(x,\xi'), \quad m \geqslant 1, \tag{10}$$

where $q_{-m}(x,\xi')$*,* $m \geqslant 0$*, are the remain symbols of a pseudodifferential operator (see (42) in Section 2).*

By studying the full symbol of the thermoelastic Dirichlet-to-Neumann map Λ_g, we prove the following result:

Theorem 2. *Let* (M,g) *be a compact Riemannian manifold of dimension n with smooth boundary* ∂M*. Assume that the coefficient* $\beta \in C^\infty(\bar{M})$*, the Lamé coefficients* λ*,* $\mu \in C^\infty(\bar{M})$*, and the heat conduction coefficient* $\alpha \in C^\infty(\bar{M})$ *satisfy* $\mu > 0, \lambda+\mu \geqslant 0$*, and* $\alpha > 0$*. Then, the thermoelastic Dirichlet-to-Neumann map* Λ_g *uniquely determines* $\frac{\partial^{|J|}\lambda}{\partial x^J}$*,* $\frac{\partial^{|J|}\mu}{\partial x^J}$*,* $\frac{\partial^{|J|}\alpha}{\partial x^J}$*, and* $\frac{\partial^{|J|}\beta}{\partial x^J}$ *on the boundary* ∂M *for all multi-indices J.*

The uniqueness result in Theorem 2 can be extended to the whole manifold for the real analytic setting.

Theorem 3. *Let* (M,g) *be a compact Riemannian manifold of dimension n with smooth boundary* ∂M*, and let* $\Gamma \subset \partial M$ *be a nonempty open subset. Suppose that the manifold is real analytic up to* Γ*, and the coefficients* $\lambda, \mu, \alpha, \beta$ *are also real analytic up to* Γ *and satisfy* $\mu > 0$*,* $\lambda+\mu \geqslant 0$*, and* $\alpha > 0$*. Then, the thermoelastic Dirichlet-to-Neumann map* Λ_g *uniquely determines* λ*,* μ*,* α*, and* β *on* $\bar{M}$*.*

Theorem 3 shows that the global uniqueness of real analytic coefficients on a real analytic Riemannian manifold. To the best of our knowledge, this is the first global uniqueness result for variable coefficients in thermoelasticity on a Riemannian manifold. It is clear that Theorem 3 also holds for any real analytic bounded Euclidean domain with smooth boundary.

1.6. The Main Ideas of this Paper

The main ideas of this paper are as follows. First, Liu [2] established a method such that one can calculate the full symbol of the elastic Dirichlet-to-Neumann map with constant coefficients. In [4], the full symbol of the elastic Dirichlet-to-Neumann map with variable coefficients was obtained. The full symbol of the thermoelastic Dirichlet-to-Neumann map with constant coefficients was obtained in [3]. Combining the methods and the results in [2–4], we can deal with the case for variable coefficients in thermoelasticity.

In boundary normal coordinates, there is a factorization for the thermoelastic operator T_g as follows:

$$A^{-1}T_g = I_{n+1}\frac{\partial^2}{\partial x_n^2} + B\frac{\partial}{\partial x_n} + C = \Big(I_{n+1}\frac{\partial}{\partial x_n} + B - Q\Big)\Big(I_{n+1}\frac{\partial}{\partial x_n} + Q\Big),$$

where B, C are two differential operators, and $Q = Q(x, \partial_{x'})$ is a pseudodifferential operator. As a result, we obtain the equation

$$Q^2 - BQ - \left[\frac{\partial}{\partial x_n}, Q\right] + C = 0,$$

where $[\frac{\partial}{\partial x_n}, Q]$ is the commutator. The corresponding full symbol equation of the above equation is

$$\sum_J \frac{(-i)^{|J|}}{J!} \partial^J_{\xi'} q \, \partial^J_{x'} q - \sum_J \frac{(-i)^{|J|}}{J!} \partial^J_{\xi'} b \, \partial^J_{x'} q - \frac{\partial q}{\partial x_n} + c = 0, \tag{11}$$

which is an $(n+1) \times (n+1)$ matrix equation, where the sum is over all multi-indices J, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, and $x' = (x_1, \ldots, x_{n-1})$. Here b, c, and q are the full symbols of the operators B, C, and Q, respectively.

Note that the computations of the full symbols of matrix-valued pseudodifferential operators (i.e., solving the above full symbol Equation (11)) are quite difficult tasks. There are two major difficulties:

(i) How to solve the unknown matrix q_1 from the following matrix equation?

$$q_1^2 - b_1 q_1 + c_2 = 0, \tag{12}$$

where q_1, b_1, and c_2 are the principal symbols of Q, B, and C, respectively.

(ii) How to solve the unknown matrix q_{-m-1}, $m \geqslant -1$, from the following Sylvester equation?

$$(q_1 - b_1) q_{-m-1} + q_{-m-1} q_1 = E_{-m}, \quad m \geqslant -1, \tag{13}$$

where q_{-m-1}, $m \geqslant -1$, are the remain symbols, and E_{-m}, $m \geqslant -1$, are given by (38)–(40).

For the first part of the problem, generally, the quadratic matrix equation of the form

$$X^2 + U_1 X + V_1 = 0 \tag{14}$$

can not be solved exactly, where X is an unknown matrix, U_1 and V_1 are given matrices. In other words, there is not a general formula of the solution represented by the coefficients of matrix equation (14). Fortunately, in our setting, b_1 and c_2 can be represented as special block matrices. This implies that the q_1 can also be represented as a block matrix, which is a linear combination of I_{n+1} and a special matrix F_1 with the property $F_1^2 = 0$ (see (56) in Section 2). Then, by solving a system of the coefficients, we can obtain the explicit expression for q_1 (see (41)).

For the second part of the problem, the matrix equation of the form

$$U_2 X + X V_2 = Y \tag{15}$$

is called the Sylvester equation (see [56], Chapter 9), where X is an unknown matrix, U_2, V_2, and Y are given matrices. Let

$$\mathcal{A} = \begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix}.$$

The vectorization $\operatorname{vec}\mathcal{A}$ of the matrix $\mathcal{A}$ is a column vector, which is defined by (see [56], Chapter 9)

$$\operatorname{vec}\mathcal{A} := (a_1^1, a_1^2, \dots, a_1^n, a_2^1, a_2^2, \dots, a_2^n, \dots, a_n^1, a_n^2, \dots, a_n^n)^t. \tag{16}$$

The Kronecker product $\mathcal{A} \otimes \mathcal{B}$ of two matrices $\mathcal{A}$ and $\mathcal{B}$ is defined by (see [56], Chapter 9)

$$\mathcal{A} \otimes \mathcal{B} := \begin{bmatrix} a_1^1\mathcal{B} & a_2^1\mathcal{B} & \cdots & a_n^1\mathcal{B} \\ a_1^2\mathcal{B} & a_2^2\mathcal{B} & \cdots & a_n^2\mathcal{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n\mathcal{B} & a_2^n\mathcal{B} & \cdots & a_n^n\mathcal{B} \end{bmatrix}. \tag{17}$$

There are some properties of Kronecker product and vectorization as follows (see [56], Chapter 9):

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{C} + \mathcal{B} \otimes \mathcal{C}, \tag{18}$$

$$\mathcal{C} \otimes (\mathcal{A} + \mathcal{B}) = \mathcal{C} \otimes \mathcal{A} + \mathcal{C} \otimes \mathcal{B}, \tag{19}$$

$$(\mathcal{A} \otimes \mathcal{B})^{-1} = \mathcal{A}^{-1} \otimes \mathcal{B}^{-1}, \tag{20}$$

$$(\mathcal{A} \otimes \mathcal{B})(\mathcal{C} \otimes \mathcal{D}) = \mathcal{A}\mathcal{C} \otimes \mathcal{B}\mathcal{D}, \tag{21}$$

$$\operatorname{vec}(\mathcal{A}\mathcal{B}) = (I_n \otimes \mathcal{A}) \operatorname{vec}\mathcal{B}, \tag{22}$$

$$\operatorname{vec}(\mathcal{B}\mathcal{C}) = (\mathcal{C}^t \otimes I_n) \operatorname{vec}\mathcal{B}, \tag{23}$$

$$\operatorname{vec}(\mathcal{A}\mathcal{B}\mathcal{C}) = (\mathcal{C}^t \otimes \mathcal{A}) \operatorname{vec}\mathcal{B}. \tag{24}$$

It follows from (15), (22), and (23) that

$$\begin{aligned} \operatorname{vec} Y &= \operatorname{vec}(U_2 X + X V_2) \\ &= \big((I_n \otimes U_2) + (V_2^t \otimes I_n)\big) \operatorname{vec}(X) \\ &:= G \operatorname{vec} X. \end{aligned} \tag{25}$$

Therefore, (15) has a unique solution if and only if G is invertible and

$$\operatorname{vec} X = G^{-1} \operatorname{vec} Y.$$

Thus, we can obtain X from $\operatorname{vec} X$. Finally, we obtain the symbols q_j, $j \leqslant 1$, of the pseudodifferential operator Q. Finally, using this method, we solve (13) and obtain the symbols q_{-m-1} for $m \geqslant -1$. This implies that we obtain $Q(x, \partial_{x'})$ (modulo a smoothing operator) on the boundary.

Next, we flatten the boundary and induce a Riemannian metric in a neighborhood of the boundary, and give a local representation for the thermoelastic Dirichlet-to-Neumann map Λ_g with variable coefficients in boundary normal coordinates, that is,

$$\Lambda_g = A\Big(-\frac{\partial}{\partial x_n}\Big) - K \quad \text{on } \partial M,$$

where A and K are two matrices. Note that, in boundary normal coordinates, the operator $\frac{\partial}{\partial x_n}\big|_{\partial M}$ can be represented as a pseudodifferential operator (modulo a smoothing operator) of order one in $x' = (x_1, \dots, x_{n-1})$ depending smoothly on x_n. Therefore, we have

$$\Lambda_g = (AQ - K)|_{\partial M}$$

modulo a smoothing operator (see (72) in Section 2).

Finally, we obtain the full symbol of the thermoelastic Dirichlet-to-Neumann map Λ_g, which contain the information about the coefficients $\lambda, \mu, \alpha, \beta$, and their derivatives on the boundary. Thus, we can prove that they can be uniquely determined by the thermoelastic

Dirichlet-to-Neumann map. Furthermore, we prove that the coefficients can be uniquely determined on the whole manifold $\bar{M}$ by the thermoelastic Dirichlet-to-Neumann map provided the manifold and these coefficients are real analytic.

This paper is organized as follows. In Section 2, we derive a factorization of thermoelastic operator T_g with variable coefficients, and compute the full symbol of the pseudodifferential operator Q, we then give the explicit expression of the thermoelastic Dirichlet-to-Neumann map Λ_g in boundary normal coordinates. In Section 3, we prove Theorem 1 and Theorem 2 for boundary determination. Finally, Section 4 is devoted to proving Theorem 3 for global uniqueness.

2. Symbols of the Pseudodifferential Operators

Let (M, g) be a compact Riemannian manifold of dimension n with smooth boundary ∂M. In the local coordinates $\{x_j\}_{j=1}^n$, we denote by $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ and $\{dx_j\}_{j=1}^n$, respectively, the natural basis for the tangent space $T_x M$ and the cotangent space $T_x^* M$ at the point $x \in M$. We will use the Einstein summation convention. The Greek indices run from 1 to $n-1$, whereas the Roman indices run from 1 to n, unless otherwise specified. Then, the Riemannian metric g is given by $g = g_{jk}\, dx_j \otimes dx_k$.

Let $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$ be the covariant derivative with respect to $\frac{\partial}{\partial x_j}$ and $\nabla^j = g^{jk}\nabla_k$. Then, for displacement vector field $\boldsymbol{u}$, we denote by div the divergence operator, i.e.,

$$\operatorname{div} \boldsymbol{u} = \nabla_j u^j = \frac{\partial u^j}{\partial x_j} + \Gamma^j_{jk} u^k, \quad \boldsymbol{u} = u^j \frac{\partial}{\partial x_j} \in \mathfrak{X}(M). \tag{26}$$

Here the Christoffel symbols

$$\Gamma^m_{jk} = \frac{1}{2} g^{ml}\Big(\frac{\partial g_{jl}}{\partial x_k} + \frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l}\Big),$$

and $[g^{jk}] = [g_{jk}]^{-1}$. For smooth function $f \in C^\infty(M)$, the gradient operator is given by

$$\operatorname{grad} f = (\nabla^j f)\frac{\partial}{\partial x_j} = g^{jk}\frac{\partial f}{\partial x_k}\frac{\partial}{\partial x_j}, \quad f \in C^\infty(M). \tag{27}$$

The Laplace–Beltrami operator is given by

$$\Delta_g f = g^{jk}\Big(\frac{\partial^2 f}{\partial x_j \partial x_k} - \Gamma^l_{jk}\frac{\partial f}{\partial x_l}\Big), \quad f \in C^\infty(M). \tag{28}$$

The Lamé operator (3) with variable coefficients can be rewritten as (see [4])

$$\begin{aligned}(L_g \boldsymbol{u})^j &= \mu \Delta_g u^j + (\lambda+\mu)\nabla^j \nabla_k u^k + (\nabla^j \lambda)\nabla_k u^k + (\nabla^k \mu)(\nabla_k u^j + \nabla^j u_k) \\ &\quad + \mu g^{kl}\Big(2\Gamma^j_{km}\frac{\partial u^m}{\partial x_l} + \frac{\partial \Gamma^j_{kl}}{\partial x_m} u^m\Big), \quad j = 1, 2, \ldots, n. \end{aligned} \tag{29}$$

In boundary normal coordinates, we write the Laplace–Beltrami operator as

$$\Delta_g = \frac{\partial^2}{\partial x_n^2} + \Gamma^\alpha_{\alpha n}\frac{\partial}{\partial x_n} + g^{\alpha\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta} + \Big(g^{\alpha\beta}\Gamma^\gamma_{\gamma\alpha} + \frac{\partial g^{\alpha\beta}}{\partial x_\alpha}\Big)\frac{\partial}{\partial x_\beta}. \tag{30}$$

Combining this and (2), (3), (26)–(29), we deduce that (cf. [3,4])

$$A^{-1} T_g = I_{n+1}\frac{\partial^2}{\partial x_n^2} + B\frac{\partial}{\partial x_n} + C, \tag{31}$$

where

$$A=\begin{bmatrix}\mu I_{n-1} & 0 & 0\\ 0 & \lambda+2\mu & 0\\ 0 & 0 & \alpha\end{bmatrix}, \tag{32}$$

$B=B_1+B_0$, $C=C_2+C_1+C_0$, and

$$B_1=(\lambda+\mu)\begin{bmatrix}0 & \frac{1}{\mu}\big[g^{\alpha\beta}\frac{\partial}{\partial x_\beta}\big] & 0\\ \frac{1}{\lambda+2\mu}\big[\frac{\partial}{\partial x_\beta}\big] & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$B_0=\begin{bmatrix}\Gamma^\alpha_{\alpha n}I_{n-1}+2[\Gamma^\alpha_{n\beta}] & 0 & 0\\ \frac{\lambda+\mu}{\lambda+2\mu}[\Gamma^\alpha_{\alpha\beta}] & \Gamma^\alpha_{n\alpha} & -\frac{\beta}{\lambda+2\mu}\\ 0 & \frac{i\omega\beta\theta_0}{\alpha} & \Gamma^\alpha_{n\alpha}\end{bmatrix}+\begin{bmatrix}\frac{1}{\mu}\frac{\partial\mu}{\partial x_n}I_{n-1} & \frac{1}{\mu}[\nabla^\alpha\lambda] & 0\\ \frac{1}{\lambda+2\mu}\big[\frac{\partial\mu}{\partial x_\beta}\big] & \frac{1}{\lambda+2\mu}\frac{\partial(\lambda+2\mu)}{\partial x_n} & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$C_2=\begin{bmatrix}(g^{\alpha\beta}\frac{\partial^2}{\partial x_\alpha\partial x_\beta})I_{n-1}+\frac{\lambda+\mu}{\mu}\big[g^{\alpha\gamma}\frac{\partial^2}{\partial x_\gamma\partial x_\beta}\big] & 0 & 0\\ 0 & \frac{\mu}{\lambda+2\mu}g^{\alpha\beta}\frac{\partial^2}{\partial x_\alpha\partial x_\beta} & 0\\ 0 & 0 & g^{\alpha\beta}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\end{bmatrix},$$

$$\begin{aligned}C_1=&\begin{bmatrix}\big((g^{\alpha\beta}\Gamma^\gamma_{\alpha\gamma}+\frac{\partial g^{\alpha\beta}}{\partial x_\alpha})\frac{\partial}{\partial x_\beta}\big)I_{n-1} & 0 & 0\\ 0 & \frac{\mu}{\lambda+2\mu}\big(g^{\alpha\beta}\Gamma^\gamma_{\alpha\gamma}+\frac{\partial g^{\alpha\beta}}{\partial x_\alpha}\big)\frac{\partial}{\partial x_\beta} & 0\\ 0 & 0 & \big(g^{\alpha\beta}\Gamma^\gamma_{\alpha\gamma}+\frac{\partial g^{\alpha\beta}}{\partial x_\alpha}\big)\frac{\partial}{\partial x_\beta}\end{bmatrix}\\
&+\frac{\lambda+\mu}{\mu}\begin{bmatrix}\big[g^{\alpha\gamma}\Gamma^\rho_{\rho\beta}\frac{\partial}{\partial x_\gamma}\big] & \big[g^{\alpha\gamma}\Gamma^\rho_{\rho n}\frac{\partial}{\partial x_\gamma}\big] & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\\
&+\begin{bmatrix}2\big[g^{\gamma\rho}\Gamma^\alpha_{\rho\beta}\frac{\partial}{\partial x_\gamma}\big] & 2\big[g^{\gamma\rho}\Gamma^\alpha_{\rho n}\frac{\partial}{\partial x_\gamma}\big] & -\frac{\beta}{\mu}\big[g^{\alpha\beta}\frac{\partial}{\partial x_\beta}\big]\\ \frac{2\mu}{\lambda+2\mu}\big[g^{\gamma\rho}\Gamma^n_{\rho\beta}\frac{\partial}{\partial x_\gamma}\big] & 0 & 0\\ \frac{i\omega\beta\theta_0}{\alpha}\big[\frac{\partial}{\partial x_\beta}\big] & 0 & 0\end{bmatrix}\\
&+\begin{bmatrix}\frac{1}{\mu}(\nabla^\alpha\mu\frac{\partial}{\partial x_\alpha})I_{n-1}+\frac{1}{\mu}\big[\nabla^\alpha\lambda\frac{\partial}{\partial x_\beta}+g^{\alpha\gamma}\frac{\partial\mu}{\partial x_\beta}\frac{\partial}{\partial x_\gamma}\big] & \frac{1}{\mu}\frac{\partial\mu}{\partial x_n}\big[g^{\alpha\beta}\frac{\partial}{\partial x_\beta}\big] & 0\\ \frac{1}{\lambda+2\mu}\frac{\partial\lambda}{\partial x_n}\big[\frac{\partial}{\partial x_\beta}\big] & \frac{1}{\lambda+2\mu}\nabla^\alpha\mu\frac{\partial}{\partial x_\alpha} & 0\\ 0 & 0 & 0\end{bmatrix},\end{aligned}$$

$$\begin{aligned}C_0=&(\lambda+\mu)\begin{bmatrix}\frac{1}{\mu}\big[g^{\alpha\gamma}\frac{\partial\Gamma^\rho_{\rho\beta}}{\partial x_\gamma}\big] & \frac{1}{\mu}\big[g^{\alpha\gamma}\frac{\partial\Gamma^\rho_{\rho n}}{\partial x_\gamma}\big] & 0\\ \frac{1}{\lambda+2\mu}\big[\frac{\partial\Gamma^\alpha_{\alpha\beta}}{\partial x_n}\big] & \frac{1}{\lambda+2\mu}\frac{\partial\Gamma^\alpha_{\alpha n}}{\partial x_n} & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}\big[g^{ml}\frac{\partial\Gamma^\alpha_{ml}}{\partial x_\beta}\big] & \big[g^{ml}\frac{\partial\Gamma^\alpha_{ml}}{\partial x_n}\big] & 0\\ \frac{\mu}{\lambda+2\mu}\big[g^{ml}\frac{\partial\Gamma^n_{ml}}{\partial x_\beta}\big] & \frac{\mu}{\lambda+2\mu}g^{ml}\frac{\partial\Gamma^n_{ml}}{\partial x_n} & 0\\ 0 & 0 & 0\end{bmatrix}\\
&+\begin{bmatrix}\frac{\rho\omega^2}{\mu}I_{n-1} & 0 & 0\\ 0 & \frac{\rho\omega^2}{\lambda+2\mu} & 0\\ \frac{i\omega\beta\theta_0}{\alpha}[\Gamma^\alpha_{\alpha\beta}] & \frac{i\omega\beta\theta_0}{\alpha}\Gamma^\alpha_{\alpha n} & \frac{i\omega\gamma}{\alpha}\end{bmatrix}\\
&+\begin{bmatrix}\frac{1}{\mu}\big[(\nabla^\alpha\lambda)\Gamma^\gamma_{\beta\gamma}-\frac{\partial\mu}{\partial x_\gamma}\frac{\partial g^{\alpha\gamma}}{\partial x_\beta}\big] & \frac{1}{\mu}\big[(\nabla^\alpha\lambda)\Gamma^\beta_{\beta n}-\frac{\partial\mu}{\partial x_\beta}\frac{\partial g^{\alpha\beta}}{\partial x_n}\big] & 0\\ \frac{1}{\lambda+2\mu}\frac{\partial\lambda}{\partial x_n}[\Gamma^\alpha_{\alpha\beta}] & \frac{1}{\lambda+2\mu}\frac{\partial\lambda}{\partial x_n}\Gamma^\alpha_{\alpha n} & 0\\ 0 & 0 & 0\end{bmatrix}.\end{aligned}$$

Let

$$b(x,\xi')=b_1(x,\xi')+b_0(x,\xi')$$

and

$$c(x,\xi') = c_2(x,\xi') + c_1(x,\xi') + c_0(x,\xi')$$

be the full symbols of B and C, respectively. We denote

$$\xi^\alpha = g^{\alpha\beta}\xi_\beta, \quad |\xi'| = \sqrt{\xi^\alpha \xi_\alpha}.$$

Thus, we have

$$b_1(x,\xi') = i(\lambda+\mu)\begin{bmatrix} 0 & \frac{1}{\mu}[\xi^\alpha] & 0 \\ \frac{1}{\lambda+2\mu}[\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

$$b_0(x,\xi') = B_0, \tag{34}$$

$$c_2(x,\xi') = -\begin{bmatrix} |\xi'|^2 I_{n-1} + \frac{\lambda+\mu}{\mu}[\xi^\alpha\xi_\beta] & 0 & 0 \\ 0 & \frac{\mu}{\lambda+2\mu}|\xi'|^2 & 0 \\ 0 & 0 & |\xi'|^2 \end{bmatrix}, \tag{35}$$

$$\begin{aligned} c_1(x,\xi') = {} & i\begin{bmatrix} \big(\xi^\alpha\Gamma^\beta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x_\alpha}\big) I_{n-1} & 0 & 0 \\ 0 & \frac{\mu}{\lambda+2\mu}\big(\xi^\alpha\Gamma^\beta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x_\alpha}\big) & 0 \\ 0 & 0 & \xi^\alpha\Gamma^\beta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x_\alpha} \end{bmatrix} \\ & + \frac{i(\lambda+\mu)}{\mu}\begin{bmatrix} [\xi^\alpha\Gamma^\gamma_{\gamma\beta}] & \Gamma^\beta_{\beta n}[\xi^\alpha] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2i[\xi^\gamma\Gamma^\alpha_{\gamma\beta}] & 2i[\xi^\gamma\Gamma^\alpha_{\gamma n}] & -\frac{i\beta}{\mu}[\xi^\alpha] \\ \frac{2i\mu}{\lambda+2\mu}[\xi^\gamma\Gamma^n_{\gamma\beta}] & 0 & 0 \\ -\frac{\omega\beta\theta_0}{\alpha}[\xi_\beta] & 0 & 0 \end{bmatrix} \\ & + i\begin{bmatrix} \frac{1}{\mu}(\xi_\alpha\nabla^\alpha\mu)I_{n-1} + \frac{1}{\mu}\big[\xi_\beta\nabla^\alpha\lambda + \xi^\alpha\frac{\partial\mu}{\partial x_\beta}\big] & \frac{1}{\mu}\frac{\partial\mu}{\partial x_n}[\xi^\alpha] & 0 \\ \frac{1}{\lambda+2\mu}\frac{\partial\lambda}{\partial x_n}[\xi_\beta] & \frac{1}{\lambda+2\mu}\xi_\alpha\nabla^\alpha\mu & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{36}$$

$$c_0(x,\xi') = C_0. \tag{37}$$

For the convenience of stating the following proposition, we define

$$E_1 := i\sum_\alpha \frac{\partial(q_1-b_1)}{\partial\xi_\alpha}\frac{\partial q_1}{\partial x_\alpha} + b_0 q_1 + \frac{\partial q_1}{\partial x_n} - c_1, \tag{38}$$

$$\begin{aligned} E_0 := {} & i\sum_\alpha\Big(\frac{\partial(q_1-b_1)}{\partial\xi_\alpha}\frac{\partial q_0}{\partial x_\alpha} + \frac{\partial q_0}{\partial\xi_\alpha}\frac{\partial q_1}{\partial x_\alpha}\Big) + \frac{1}{2}\sum_{\alpha,\beta}\frac{\partial^2 q_1}{\partial\xi_\alpha\partial\xi_\beta}\frac{\partial^2 q_1}{\partial x_\alpha\partial x_\beta} \\ & - q_0^2 + b_0 q_0 + \frac{\partial q_0}{\partial x_n} - c_0, \end{aligned} \tag{39}$$

and

$$E_{-m} := b_0 q_{-m} + \frac{\partial q_{-m}}{\partial x_n} - i\sum_\alpha\frac{\partial b_1}{\partial\xi_\alpha}\frac{\partial q_{-m}}{\partial x_\alpha} - \sum_{\substack{-m\leqslant j,k\leqslant 1 \\ |J|=j+k+m}}\frac{(-i)^{|J|}}{J!}\partial^J_{\xi'}q_j\,\partial^J_{x'}q_k \tag{40}$$

for $m \geqslant 1$.

We then derive the microlocal factorization of the thermoelastic operator T_g.

Proposition 4. *There exists a pseudodifferential operator $Q(x, \partial_{x'})$ of order one in x' depending smoothly on x_n such that*

$$A^{-1}T_g = \Big(I_{n+1}\frac{\partial}{\partial x_n} + B - Q\Big)\Big(I_{n+1}\frac{\partial}{\partial x_n} + Q\Big)$$

modulo a smoothing operator. Moreover, let $q(x,\xi') \sim \sum_{j\leqslant 1} q_j(x,\xi')$ be the full symbol of $Q(x,\partial_{x'})$. Then, in boundary normal coordinates,

$$q_1(x,\xi') = |\xi'|I_{n+1} + \frac{\lambda+\mu}{\lambda+3\mu}F_1, \tag{41}$$

$$\begin{aligned} q_{-m-1}(x,\xi') = {} & \frac{1}{2|\xi'|}E_{-m} - \frac{\lambda+\mu}{4(\lambda+3\mu)|\xi'|^2}(F_2E_{-m} + E_{-m}F_1) \\ & + \frac{(\lambda+\mu)^2}{4(\lambda+3\mu)^2|\xi'|^3}F_2E_{-m}F_1, \quad m \geqslant -1, \end{aligned} \tag{42}$$

where

$$F_1 = \begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & i[\xi^\alpha] & 0 \\ i[\xi_\beta] & -|\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{43}$$

$$F_2 = \begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & -\frac{i(\lambda+2\mu)}{\mu}[\xi^\alpha] & 0 \\ -\frac{i\mu}{\lambda+2\mu}[\xi_\beta] & -|\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

Proof. It follows from (31) that

$$I_{n+1}\frac{\partial^2}{\partial x_n^2} + B\frac{\partial}{\partial x_n} + C = \Big(I_{n+1}\frac{\partial}{\partial x_n} + B - Q\Big)\Big(I_{n+1}\frac{\partial}{\partial x_n} + Q\Big).$$

Equivalently,

$$Q^2 - BQ - \Big[I_{n+1}\frac{\partial}{\partial x_n}, Q\Big] + C = 0, \tag{45}$$

where the commutator $\big[I_{n+1}\frac{\partial}{\partial x_n}, Q\big]$ is defined by, for any $v \in C^\infty(M)$,

$$\begin{aligned} \Big[I_{n+1}\frac{\partial}{\partial x_n}, Q\Big]v &:= I_{n+1}\frac{\partial}{\partial x_n}(Qv) - Q\Big(I_{n+1}\frac{\partial}{\partial x_n}\Big)v \\ &= \frac{\partial Q}{\partial x_n}v. \end{aligned}$$

Recall that if G_1 and G_2 are two pseudodifferential operators with full symbols $g_1 = g_1(x,\xi)$ and $g_1 = g_2(x,\xi)$, respectively, then the full symbol $\sigma(G_1G_2)$ of the operator G_1G_2 is given by (see [52], p. 11, [54], p. 71 and also [53,57])

$$\sigma(G_1G_2) \sim \sum_J \frac{(-i)^{|J|}}{J!}\partial_\xi^J g_1\, \partial_x^J g_2,$$

where the sum is over all multi-indices J. Let $q = q(x,\xi')$ be the full symbol of the operator $Q(x,\partial_{x'})$, we write $q(x,\xi') \sim \sum_{j\leqslant 1} q_j(x,\xi')$ with $q_j(x,\xi')$ homogeneous of degree j in ξ'. Hence, we get the following full symbol equation of (45)

$$\sum_J \frac{(-i)^{|J|}}{J!}\partial_{\xi'}^J q\, \partial_{x'}^J q - \sum_J \frac{(-i)^{|J|}}{J!}\partial_{\xi'}^J b\, \partial_{x'}^J q - \frac{\partial q}{\partial x_n} + c = 0. \tag{46}$$

We shall determine q_j recursively so that (46) holds modulo $S^{-\infty}$. Grouping the homogeneous terms of degree two in (46), we have

$$q_1^2 - b_1 q_1 + c_2 = 0. \tag{47}$$

Note that c_2 can be rewritten as (see (35))

$$c_2 = -|\xi'|^2 I_{n+1} - \begin{bmatrix} \frac{\lambda+\mu}{\mu}[\xi^\alpha \xi_\beta] & 0 & 0 \\ 0 & -\frac{\lambda+\mu}{\lambda+2\mu}|\xi'|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{48}$$

In our notations, $[\xi^\alpha] = (\xi^1, \dots, \xi^{n-1})^t$ is a column vector, $[\xi_\beta] = (\xi_1, \dots, \xi_{n-1})$ is a row vector, and $[\xi^\alpha \xi_\beta]$ is an $(n-1) \times (n-1)$ matrix. Then,

$$\begin{aligned} [\xi^\alpha] \cdot [\xi_\beta] &= [\xi^\alpha \xi_\beta], \\ [\xi_\beta] \cdot [\xi^\alpha] &= |\xi'|^2, \\ [\xi^\alpha \xi_\beta] \cdot [\xi^\alpha] &= |\xi'|^2 [\xi^\alpha], \\ [\xi_\beta] \cdot [\xi^\alpha \xi_\beta] &= |\xi'|^2 [\xi_\beta], \\ [\xi^\alpha \xi_\beta] \cdot [\xi^\alpha \xi_\beta] &= |\xi'|^2 [\xi^\alpha \xi_\beta]. \end{aligned}$$

We find that

$$\begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & 0 & 0 \\ 0 & |\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & [\xi^\alpha] & 0 \\ [\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} [\xi^\alpha \xi_\beta] & 0 & 0 \\ 0 & |\xi'|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & 0 & 0 \\ 0 & |\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & [\xi^\alpha] & 0 \\ [\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & [\xi^\alpha] & 0 \\ [\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & 0 & 0 \\ 0 & |\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= |\xi'| \begin{bmatrix} 0 & [\xi^\alpha] & 0 \\ [\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

In view of the special forms of b_1 and c_2, we set q_1 has the form

$$q_1 = |\xi'| I_{n+1} + \begin{bmatrix} s_1 \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & i s_2 [\xi^\alpha] & 0 \\ i s_3 [\xi_\beta] & -s_4 |\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{49}$$

where s_j, $1 \leqslant j \leqslant 4$, are coefficients to be determined. Substituting (49), (33), and (48) into (47), we get

$$\begin{aligned} 0 = {}& |\xi'|^2 I_{n+1} + \begin{bmatrix} (s_1^2 - s_2 s_3)[\xi^\alpha \xi_\beta] & i s_2 (s_1 - s_4)|\xi'|[\xi^\alpha] & 0 \\ i s_3 (s_1 - s_4)|\xi'|[\xi_\beta] & (s_4^2 - s_2 s_3)|\xi'|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & + 2 \begin{bmatrix} s_1 [\xi^\alpha \xi_\beta] & i s_2 |\xi'|[\xi^\alpha] & 0 \\ i s_3 |\xi'|[\xi_\beta] & -s_4 |\xi'|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - i(\lambda+\mu) \left\{ \begin{bmatrix} 0 & \frac{1}{\mu}|\xi'|[\xi^\alpha] & 0 \\ \frac{1}{\lambda+2\mu}|\xi'|[\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} \frac{i s_3}{\mu}[\xi^\alpha \xi_\beta] & -\frac{s_4}{\mu}|\xi'|[\xi^\alpha] & 0 \\ \frac{s_1}{\lambda+2\mu}|\xi'|[\xi_\beta] & \frac{i s_2}{\lambda+2\mu}|\xi'|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} - |\xi'|^2 I_{n+1} + \begin{bmatrix} -\frac{\lambda+\mu}{\mu}[\xi^\alpha \xi_\beta] & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\lambda+2\mu}|\xi'|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, we have the following equations of coefficients:

$$
\begin{cases}
s_1^2 - s_2 s_3 + 2s_1 + \dfrac{\lambda+\mu}{\mu}(s_3 - 1) = 0, \\
s_2(s_1 - s_4) + 2s_2 + \dfrac{\lambda+\mu}{\mu}(s_4 - 1) = 0, \\
s_3(s_1 - s_4) + 2s_3 - \dfrac{\lambda+\mu}{\lambda+2\mu}(s_1 + 1) = 0, \\
s_4^2 - s_2 s_3 - 2s_4 + \dfrac{\lambda+\mu}{\lambda+2\mu}(s_2 + 1) = 0.
\end{cases} \tag{50}
$$

Because we have chosen the outer normal vector ν on the boundary, we should take

$$
s_1 \geqslant 0, \quad 1 - s_4 > 0. \tag{51}
$$

Such a choice implies that the real part of q_1 is positive definite. Solving the above equations with the conditions (51) and (1), we then get

$$
s_1 = s_2 = s_3 = s_4 = \frac{\lambda+\mu}{\lambda+3\mu}. \tag{52}
$$

Let

$$
F_1 = \begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & i[\xi^\alpha] & 0 \\ i[\xi_\beta] & -|\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{53}
$$

Then, we obtain (41) immediately by combining (49), (52), and (53).

Grouping the homogeneous terms of degree $-m$ $(m \geqslant -1)$ in (46), we get

$$
(q_1 - b_1)q_{-m-1} + q_{-m-1}q_1 = E_{-m}, \tag{54}
$$

where E_{-m}, $m \geqslant -1$, are given by (38)–(40). Equation (54) is called the Sylvester equation (see [56], Chapter 9).

Let

$$
F_2 = \begin{bmatrix} \frac{1}{|\xi'|}[\xi^\alpha \xi_\beta] & -\frac{i(\lambda+2\mu)}{\mu}[\xi^\alpha] & 0 \\ -\frac{i\mu}{\lambda+2\mu}[\xi_\beta] & -|\xi'| & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{55}
$$

Then, from (33), (52), (53), and (55), we have

$$
F_1^2 = F_2^2 = 0, \tag{56}
$$
$$
b_1 = s_1(F_1 - F_2), \tag{57}
$$
$$
b_1 F_1 = -s_1 F_2 F_1, \tag{58}
$$
$$
b_1 F_2 = s_1 F_1 F_2. \tag{59}
$$

By (41) and (57), we get

$$
q_1 - b_1 = |\xi'| I_{n+1} + s_1 F_2. \tag{60}
$$

Recall that, in (17) and (16), $\otimes$ and vec denote the Kronecker product and the vectorization of matrices, respectively. It follows from (25) that

$$\begin{aligned} \operatorname{vec} E_{-m} &= \operatorname{vec}((q_1 - b_1)q_{-m-1} + q_{-m-1}q_1) \\ &= H \operatorname{vec} q_{-m-1}, \quad m \geqslant -1, \end{aligned} \tag{61}$$

where

$$H = (I_{n+1} \otimes (q_1 - b_1)) + (q_1^t \otimes I_{n+1}). \tag{62}$$

Combining (62) and (60), we obtain

$$\begin{aligned} H &= I_{n+1} \otimes (|\xi'| I_{n+1} + s_1 F_2) + ((|\xi'| I_{n+1} + s_1 F_1^t) \otimes I_{n+1}) \\ &= 2|\xi'| I_{n+1} \otimes I_{n+1} + s_1 (I_{n+1} \otimes F_2 + F_1^t \otimes I_{n+1}). \end{aligned} \tag{63}$$

In view of (18)–(21), (56), and H is of order one in ξ', thus, we set H^{-1} has the form

$$H^{-1} = \frac{1}{2|\xi'|} I_{n+1} \otimes I_{n+1} + \frac{s_5}{|\xi'|^2}(I_{n+1} \otimes F_2 + F_1^t \otimes I_{n+1}) + \frac{s_6}{|\xi'|^3}(F_1^t \otimes F_2), \tag{64}$$

where s_5 and s_6 are coefficients to be determined. From (61), we have

$$\operatorname{vec} q_{-m-1} = H^{-1} \operatorname{vec} E_{-m}, \quad m \geqslant -1. \tag{65}$$

Combining (64), (65), and (22)–(24), we obtain, for $m \geqslant -1$,

$$q_{-m-1} = \frac{1}{2|\xi'|} E_{-m} + \frac{s_5}{|\xi'|^2}(F_2 E_{-m} + E_{-m} F_1) + \frac{s_6}{|\xi'|^3} F_2 E_{-m} F_1. \tag{66}$$

It follows from (21), (56), (63), and (64) that

$$\begin{aligned} I_{(n+1)^2} &= H H^{-1} \\ &= I_{n+1} \otimes I_{n+1} + \frac{2s_5}{|\xi'|}(I_{n+1} \otimes F_2 + F_1^t \otimes I_{n+1}) + \frac{2s_6}{|\xi'|^2}(F_1^t \otimes F_2) \\ &\quad + \frac{s_1}{2|\xi'|}(I_{n+1} \otimes F_2 + F_1^t \otimes I_{n+1}) + \frac{s_1 s_5}{|\xi'|^2}(I_{n+1} \otimes F_2^2 + (F_1^t)^2 \otimes I_{n+1} \\ &\quad + 2F_1^t \otimes F_2) + \frac{s_1 s_6}{|\xi'|^3}(F_1^t \otimes F_2^2 + (F_1^t)^2 \otimes F_2) \\ &= I_{n+1} \otimes I_{n+1} + \left(2s_5 + \frac{s_1}{2}\right)\frac{1}{|\xi'|}(I_{n+1} \otimes F_2 + F_1^t \otimes I_{n+1}) \\ &\quad + 2(s_6 + s_1 s_5)\frac{1}{|\xi'|^2}(F_1^t \otimes F_2). \end{aligned}$$

Note that $I_{n+1} \otimes I_{n+1} = I_{(n+1)^2}$. This implies that

$$\begin{cases} 2s_5 + \dfrac{s_1}{2} = 0, \\ s_6 + s_1 s_5 = 0. \end{cases}$$

Recall that $s_1 = \frac{\lambda+\mu}{\lambda+3\mu}$ by (52). Thus, solving the above equations, we get

$$\begin{cases} s_5 = -\dfrac{s_1}{4} = -\dfrac{\lambda+\mu}{4(\lambda+3\mu)}, \\ s_6 = \dfrac{s_1^2}{4} = \dfrac{(\lambda+\mu)^2}{4(\lambda+3\mu)^2}. \end{cases} \tag{67}$$

Substituting (67) into (66), we immediately get (42). □

From Proposition 4, we get the full symbol of the pseudodifferential operator $Q(x, \partial_{x'})$. This implies that we obtain $Q(x, \partial_{x'})$ (modulo a smoothing operator) on the boundary.

Proposition 5. *In boundary normal coordinates, the thermoelastic Dirichlet-to-Neumann map Λ_g can be written as*

$$\Lambda_g = A\Big(-\frac{\partial}{\partial x_n}\Big) - K \quad \text{on } \partial M, \tag{68}$$

where A is given by (32), and

$$K = \begin{bmatrix} 0 & \mu\big[g^{\alpha\beta}\frac{\partial}{\partial x_\beta}\big] & 0 \\ \lambda\big[\frac{\partial}{\partial x_\beta} + \Gamma^{\alpha}_{\alpha\beta}\big] & \lambda\Gamma^{\alpha}_{\alpha n} & -\beta \\ 0 & 0 & 0 \end{bmatrix}. \tag{69}$$

Proof. By (4), we have

$$((S\boldsymbol{u})\nu)^j = (S\boldsymbol{u})^j_k \nu^k = (\nabla^j u_k + \nabla_k u^j)\nu^k.$$

In boundary normal coordinates, we take $\nu = (0, \ldots, 0, -1)^t$ and $\partial_\nu = -\partial_{x_n}$. In particular, $u_n = u^n$ since $g_{jn} = \delta_{jn}$ in boundary normal coordinates. We get

$$((S\boldsymbol{u})\nu)^j = -(\nabla^j u_n + \nabla_n u^j).$$

Note that $\Gamma^n_{nk} = \Gamma^k_{nn} = 0$ and $g^{\alpha\beta}\Gamma^n_{\beta\gamma} + \Gamma^{\alpha}_{n\gamma} = 0$ in boundary normal coordinates. Thus,

$$\begin{aligned} ((S\boldsymbol{u})\nu)^\alpha &= -(\nabla^\alpha u_n + \nabla_n u^\alpha) \\ &= -\Big[g^{\alpha\beta}\Big(\frac{\partial u^n}{\partial x_\beta} + \Gamma^n_{\beta\gamma}u^\gamma\Big) + \frac{\partial u^\alpha}{\partial x_n} + \Gamma^\alpha_{n\gamma}u^\gamma\Big] \\ &= -g^{\alpha\beta}\frac{\partial u^n}{\partial x_\beta} - \frac{\partial u^\alpha}{\partial x_n}, \end{aligned} \tag{70}$$

$$((S\boldsymbol{u})\nu)^n = -(\nabla^n u_n + \nabla_n u^n) = -2\frac{\partial u^n}{\partial x_n}. \tag{71}$$

Hence, we immediately obtain (68) by combining (26), (6), (70), and (71). □

In boundary normal coordinates, the operator $\frac{\partial}{\partial x_n}\big|_{\partial M}$ can be represented as the pseudodifferential operator $Q(x, \partial_{x'})$ (modulo a smoothing operator) of order one in x' depending smoothly on x_n. Hence, we have the following proposition.

Proposition 6. *In boundary normal coordinates, the thermoelastic Dirichlet-to-Neumann map Λ_g can be represented as*

$$\Lambda_g = (AQ - K)|_{\partial M} \tag{72}$$

modulo a smoothing operator, where A and K are given by (32) and (69), respectively.

Proof. We use the boundary normal coordinates (x', x_n) with $x_n \in [0, T]$. Since the principal symbol of the thermoelastic operator T_g is negative definite, the hyperplane $x_n = 0$ is non-characteristic. Hence, T_g is partially hypoelliptic with respect to this boundary (see [58], p. 107). Therefore, the solution to the equation $T_g\boldsymbol{U} = 0$ is smooth in normal variable, that is, $\boldsymbol{U} \in [C^\infty([0,T];\mathfrak{D}'(\mathbb{R}^{n-1}))]^{n+1}$ locally. From Proposition 4, we see that (5) is locally equivalent to the following system of equations for $\boldsymbol{U}, \boldsymbol{W} \in [C^\infty([0,T];\mathfrak{D}'(\mathbb{R}^{n-1}))]^{n+1}$:

$$\Big(I_{n+1}\frac{\partial}{\partial x_n} + Q\Big)\boldsymbol{U} = \boldsymbol{W}, \quad \boldsymbol{U}|_{x_n=0} = \boldsymbol{V},$$
$$\Big(I_{n+1}\frac{\partial}{\partial x_n} + B - Q\Big)\boldsymbol{W} = \boldsymbol{Y} \in [C^\infty([0,T]\times\mathbb{R}^{n-1})]^{n+1}.$$

Inspired by [2] (cf. [21]), if we substitute $t = T - x_n$ into the second equation above, then, we get a backwards generalized heat equation

$$\frac{\partial \boldsymbol{W}}{\partial t} - (B - Q)\boldsymbol{W} = -\boldsymbol{Y}.$$

Since $\boldsymbol{U}$ is smooth in the interior of the manifold M by interior regularity for elliptic operator T_g, it follows that $\boldsymbol{W}$ is also smooth in the interior of M, and so $\boldsymbol{W}|_{x_n=T}$ is smooth. In view of the real part of q_1 (the principal symbol of Q) is positive definite (see (41)), we get that the solution operator for this heat equation is smooth for $t > 0$ (see [57], p. 134). Therefore,

$$\frac{\partial \boldsymbol{U}}{\partial x_n} + Q\boldsymbol{U} = \boldsymbol{W} \in [C^\infty([0,T]\times\mathbb{R}^{n-1})]^{n+1}$$

locally. If we set $\mathcal{R}\boldsymbol{V} = \boldsymbol{W}|_{\partial M}$, then, $\mathcal{R}$ is a smoothing operator and

$$\frac{\partial \boldsymbol{U}}{\partial x_n}\bigg|_{\partial M} = -Q\boldsymbol{U}|_{\partial M} + \mathcal{R}\boldsymbol{V}. \tag{73}$$

Combining (73) and (68), we immediately obtain (72). □

3. Determining Coefficients on the Boundary

In this section we will prove the uniqueness results for the coefficients λ, μ, α, and β on the boundary. We first prove Theorem 1.

Proof of Theorem 1. Let $\sigma(\Lambda_g) \sim \sum_{j\leqslant 1} p_j(x,\xi')$ be the full symbol of the thermoelastic Dirichlet-to-Neumann map Λ_g. According to (72) and (69) we have

$$p_1(x,\xi') = Aq_1(x,\xi') - k_1, \tag{74}$$
$$p_0(x,\xi') = Aq_0(x,\xi') - k_0, \tag{75}$$
$$p_{-m}(x,\xi') = Aq_{-m}(x,\xi'), \quad m \geqslant 1, \tag{76}$$

where A is given by (32), and

$$k_1 = \begin{bmatrix} 0 & i\mu[\xi^\alpha] & 0 \\ i\lambda[\xi_\beta] & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k_0 = \begin{bmatrix} 0 & 0 & 0 \\ \lambda[\Gamma^\alpha_{\alpha\beta}] & \lambda\Gamma^\alpha_{\alpha n} & -\beta \\ 0 & 0 & 0 \end{bmatrix}. \tag{77}$$

Substituting (41), (32), and (77) into (74), we immediately obtain (8). Similarly, (9) and (10) can also be obtained. □

We then prove the uniqueness of the coefficients on the boundary.

Proof of Theorem 2. It follows from (8)–(10) that the Lamé coefficients λ and μ only appear in the $n \times n$ submatrices. In the Lamé system, the uniqueness of $\frac{\partial^{|J|}\lambda}{\partial x^J}$ and $\frac{\partial^{|J|}\mu}{\partial x^J}$ on the boundary for all multi-indices J has been proved in [4]. Clearly, this particular result also holds in thermoelastic system, and the proof is the same as that of [4]. Thus, we only need to prove the uniqueness of the coefficients α and β on the boundary.

From (8), we know that the $(n+1, n+1)$-entry of p_1 is

$$(p_1)^{n+1}_{n+1} = \alpha|\xi'|.$$

This shows that p_1 uniquely determines α on the boundary. Furthermore, the tangential derivatives $\frac{\partial \alpha}{\partial x_\gamma}$ for $1 \leqslant \gamma \leqslant n-1$ can also be uniquely determined by p_1 on the boundary.

Let b_0' and c_1' be the terms that only involve the partial derivatives of λ and μ in the expressions (34) and (36), respectively. That is,

$$b_0' = \begin{bmatrix} \frac{1}{\mu}\frac{\partial \mu}{\partial x_n} I_{n-1} & \frac{1}{\mu}[\nabla^\alpha \lambda] & 0 \\ \frac{1}{\lambda+2\mu}\left[\frac{\partial \mu}{\partial x_\beta}\right] & \frac{1}{\lambda+2\mu}\frac{\partial(\lambda+2\mu)}{\partial x_n} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$c_1' = i\begin{bmatrix} \frac{1}{\mu}(\xi_\alpha \nabla^\alpha \mu) I_{n-1} + \frac{1}{\mu}\left[\xi_\beta \nabla^\alpha \lambda + \xi^\alpha \frac{\partial \mu}{\partial x_\beta}\right] & \frac{1}{\mu}\frac{\partial \mu}{\partial x_n}[\xi^\alpha] & 0 \\ \frac{1}{\lambda+2\mu}\frac{\partial \lambda}{\partial x_n}[\xi_\beta] & \frac{1}{\lambda+2\mu}\xi_\alpha \nabla^\alpha \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows from (42) that

$$q_0 = \tilde{q}_0 + \frac{1}{2|\xi'|}E_1' - \frac{\lambda+\mu}{4(\lambda+3\mu)|\xi'|^2}(F_2E_1' + E_1'F_1) + \frac{(\lambda+\mu)^2}{4(\lambda+3\mu)^2|\xi'|^3}F_2E_1'F_1,$$

where $E_1' = b_0'q_1 - c_1'$, and $\tilde{q}_0$ is the solution of the corresponding equation with constant coefficients (see [3], p. 13). Hence, we see that q_0 has the form (see [3], p. 13)

$$q_0 = \begin{bmatrix} * & * & \frac{i\beta}{(\lambda+3\mu)|\xi'|}[\xi_\alpha] \\ * & * & -\frac{\beta}{\lambda+3\mu} \\ \frac{\mu\omega\beta\theta_0}{\alpha(\lambda+3\mu)|\xi'|}[\xi_\beta] & \frac{i\mu\omega\beta\theta_0}{\alpha(\lambda+3\mu)} & * \end{bmatrix}, \tag{78}$$

where $*$ denotes the terms which we do not care (of course, they can be computed explicitly).

Therefore, combining (78), (75), and (77), we get the $(n, n+1)$-entry $(p_0)^n_{n+1}$, that is,

$$(p_0)^n_{n+1} = \beta - \frac{\beta(\lambda+2\mu)}{\lambda+3\mu} = \frac{\beta\mu}{\lambda+3\mu}.$$

This implies that p_0 uniquely determines β on the boundary and the tangential derivatives $\frac{\partial \beta}{\partial x_\gamma}$ on the boundary for $1 \leqslant \gamma \leqslant n-1$, since λ and μ have been determined on the boundary by the previous arguments.

According to the above discussion, we see from (75) that q_0 is uniquely determined by p_0 since the boundary values of λ, μ, α, and β have been uniquely determined. By (54), we can determine E_1 from the knowledge of q_0. For $k \geqslant 0$, we denote by $\mathcal{T}_{-k} = \mathcal{T}_{-k}(\lambda, \mu, \alpha, \beta)$ the terms that involve only the boundary values of $\lambda, \mu, \alpha, \beta$, and their normal derivatives of order ar most k (which have been uniquely determined). Note that $\mathcal{T}_{-k}$ may be different in different expressions.

From (38), we have

$$E_1 = b_0 q_1 + \frac{\partial q_1}{\partial x_n} - c_1 + \mathcal{T}_0. \tag{79}$$

By (76) and (54), we know that q_{-1} is uniquely determined by p_{-1}, and E_0 can be determined from the knowledge of q_{-1}. From (39), we see that

$$E_0 = \frac{\partial q_0}{\partial x_n} + \mathcal{T}_{-1}. \tag{80}$$

From (78), we find that the $(n, n+1)$-entry $(\frac{\partial q_0}{\partial x_n})^n_{n+1}$ and the $(n+1, n)$-entry $(\frac{\partial q_0}{\partial x_n})^{n+1}_n$ of $\frac{\partial q_0}{\partial x_n}$ are, respectively,

$$\begin{aligned}\Big(\frac{\partial q_0}{\partial x_n}\Big)^n_{n+1} &= -\frac{\frac{\partial \beta}{\partial x_n}(\lambda+3\mu) - \beta(\frac{\partial \lambda}{\partial x_n} + 3\frac{\partial \mu}{\partial x_n})}{(\lambda+3\mu)^2} \\ &= -\frac{1}{\lambda+3\mu}\frac{\partial \beta}{\partial x_n} + \mathcal{T}_{-1}, \qquad (81)\\ \Big(\frac{\partial q_0}{\partial x_n}\Big)^{n+1}_n &= \frac{-\beta\mu(\lambda+3\mu)\frac{\partial \alpha}{\partial x_n} + \alpha\mu(\lambda+3\mu)\frac{\partial \beta}{\partial x_n} + \alpha\beta(\lambda\frac{\partial \mu}{\partial x_n} - \mu\frac{\partial \lambda}{\partial x_n})}{\alpha^2(\lambda+3\mu)^2} \\ &= -\frac{\beta\mu}{\alpha^2(\lambda+3\mu)}\frac{\partial \alpha}{\partial x_n} + \frac{\mu}{\alpha(\lambda+3\mu)}\frac{\partial \beta}{\partial x_n} + \mathcal{T}_{-1}. \qquad (82)\end{aligned}$$

Since $\alpha, \beta, \lambda, \mu, \frac{\partial \lambda}{\partial x_n}$ and $\frac{\partial \mu}{\partial x_n}$ have been determined on the boundary, then, $\frac{\partial \beta}{\partial x_n}$ can be determined by $(\frac{\partial q_0}{\partial x_n})^n_{n+1}$ on the boundary, and $\frac{\partial \alpha}{\partial x_n}$ can be determined by $(\frac{\partial q_0}{\partial x_n})^{n+1}_n$ on the boundary. This implies that p_{-1} uniquely determines $\frac{\partial \alpha}{\partial x_n}$ and $\frac{\partial \beta}{\partial x_n}$ on the boundary.

By (54), we have

$$(q_1 - b_1)\frac{\partial q_0}{\partial x_n} + \frac{\partial q_0}{\partial x_n} q_1 = \frac{\partial E_1}{\partial x_n} + \mathcal{T}_{-1}.$$

This implies that $\frac{\partial E_1}{\partial x_n}$ can be determined from the knowledge of $\frac{\partial q_0}{\partial x_n}$. By (76) and (54), we know that q_{-2} is uniquely determined by p_{-2}, and E_{-1} can be determined from the knowledge of q_{-2}. From (40), we see that

$$E_{-1} = \frac{\partial q_{-1}}{\partial x_n} + \mathcal{T}_{-2}.$$

By (54), we have

$$(q_1 - b_1)\frac{\partial q_{-1}}{\partial x_n} + \frac{\partial q_{-1}}{\partial x_n} q_1 = \frac{\partial E_0}{\partial x_n} + \mathcal{T}_{-2}.$$

This implies that $\frac{\partial E_0}{\partial x_n}$ can be determined from the knowledge of $\frac{\partial q_{-1}}{\partial x_n}$. From (80), we have

$$\frac{\partial E_0}{\partial x_n} = \frac{\partial^2 q_0}{\partial x_n^2} + \mathcal{T}_{-2}.$$

Thus, it follows from (81) and (82) that

$$\begin{aligned}\Big(\frac{\partial^2 q_0}{\partial x_n^2}\Big)^n_{n+1} &= -\frac{1}{\lambda+3\mu}\frac{\partial^2 \beta}{\partial x_n^2} + \mathcal{T}_{-2},\\ \Big(\frac{\partial^2 q_0}{\partial x_n^2}\Big)^{n+1}_n &= -\frac{\beta\mu}{\alpha^2(\lambda+3\mu)}\frac{\partial^2 \alpha}{\partial x_n^2} + \frac{\mu}{\alpha(\lambda+3\mu)}\frac{\partial^2 \beta}{\partial x_n^2} + \mathcal{T}_{-2}.\end{aligned}$$

Since $\lambda, \mu, \alpha, \beta, \frac{\partial \lambda}{\partial x_n}, \frac{\partial \mu}{\partial x_n}, \frac{\partial^2 \lambda}{\partial x_n^2}, \frac{\partial^2 \mu}{\partial x_n^2}, \frac{\partial \alpha}{\partial x_n}$, and $\frac{\partial \beta}{\partial x_n}$ have been determined on the boundary, then, $\frac{\partial^2 \beta}{\partial x_n^2}$ can be determined by $(\frac{\partial^2 q_0}{\partial x_n^2})^n_{n+1}$ on the boundary, and $\frac{\partial^2 \alpha}{\partial x_n^2}$ can be determined by $(\frac{\partial^2 q_0}{\partial x_n^2})^{n+1}_n$ on the boundary. This implies that p_{-2} uniquely determines $\frac{\partial \alpha^2}{\partial x_n^2}$ and $\frac{\partial^2 \beta}{\partial x_n^2}$ on the boundary.

Finally, we consider p_{-m-1} for $m \geqslant 1$. By (76) and (54), we have p_{-m-1} uniquely determines q_{-m-1}, and E_{-m} can be determined from the knowledge of q_{-m-1}. From (40), we obtain

$$E_{-m} = \frac{\partial q_{-m}}{\partial x_n} + \mathcal{T}_{-m-1}.$$

We see from (54) that

$$(q_1 - b_1)\frac{\partial q_{-m}}{\partial x_n} + \frac{\partial q_{-m}}{\partial x_n} q_1 = \frac{\partial E_{-m+1}}{\partial x_n} + \mathcal{T}_{-m-1}.$$

This implies that $\frac{\partial E_{-m+1}}{\partial x_n}$ can be determined from the knowledge of $\frac{\partial q_{-m}}{\partial x_n}$.

We end this proof by induction. Suppose we have shown that, by iteration, E_{-m} uniquely determines

$$\frac{\partial^m E_0}{\partial x_n^m} = \frac{\partial^{m+1} q_0}{\partial x_n^{m+1}} + \mathcal{T}_{-m-1}, \tag{83}$$

which further determines $\frac{\partial^{m+1}\alpha}{\partial x_n^{m+1}}$ and $\frac{\partial^{m+1}\beta}{\partial x_n^{m+1}}$ on the boundary since we have

$$\begin{aligned}
\left(\frac{\partial^{m+1} q_0}{\partial x_n^{m+1}}\right)_{n+1}^{n} &= -\frac{1}{\lambda+3\mu}\frac{\partial^{m+1}\beta}{\partial x_n^{m+1}} + \mathcal{T}_{-m-1},\\
\left(\frac{\partial^{m+1} q_0}{\partial x_n^{m+1}}\right)_{n}^{n+1} &= -\frac{\beta\mu}{\alpha^2(\lambda+3\mu)}\frac{\partial^{m+1}\alpha}{\partial x_n^{m+1}} + \frac{\mu}{\alpha(\lambda+3\mu)}\frac{\partial^{m+1}\beta}{\partial x_n^{m+1}} + \mathcal{T}_{-m-1}.
\end{aligned}$$

By (76) and (54), we know that q_{-m-2} is uniquely determined by p_{-m-2}, and E_{-m-1} can be determined from the knowledge of q_{-m-2}. Hence, E_{-m-1} uniquely determines $\frac{\partial^{m+2} q_0}{\partial x_n^{m+2}}$ by iteration. It follows that

$$\begin{aligned}
\left(\frac{\partial^{m+2} q_0}{\partial x_n^{m+2}}\right)_{n+1}^{n} &= -\frac{1}{\lambda+3\mu}\frac{\partial^{m+2}\beta}{\partial x_n^{m+2}} + \mathcal{T}_{-m-2},\\
\left(\frac{\partial^{m+2} q_0}{\partial x_n^{m+2}}\right)_{n}^{n+1} &= -\frac{\beta\mu}{\alpha^2(\lambda+3\mu)}\frac{\partial^{m+2}\alpha}{\partial x_n^{m+2}} + \frac{\mu}{\alpha(\lambda+3\mu)}\frac{\partial^{m+2}\beta}{\partial x_n^{m+2}} + \mathcal{T}_{-m-2}.
\end{aligned}$$

This implies that p_{-m-2} uniquely determines $\frac{\partial^{m+2}\alpha}{\partial x_n^{m+2}}$ and $\frac{\partial^{m+2}\beta}{\partial x_n^{m+2}}$ on the boundary.

Therefore, by combining the uniqueness result of $\frac{\partial^{|J|}\lambda}{\partial x^J}$, $\frac{\partial^{|J|}\mu}{\partial x^J}$ (see [4]), and the above arguments, we conclude that the thermoelastic Dirichlet-to-Neumann map Λ_g uniquely determines $\frac{\partial^{|J|}\lambda}{\partial x^J}$, $\frac{\partial^{|J|}\mu}{\partial x^J}$, $\frac{\partial^{|J|}\alpha}{\partial x^J}$, and $\frac{\partial^{|J|}\beta}{\partial x^J}$ on the boundary for all multi-indices J. $\square$

4. Global Uniqueness of Real Analytic Coefficients

This section is devoted to proving the global uniqueness of real analytic coefficients λ, μ, α, and β on a real analytic manifold. More precisely, we prove that the thermoelastic Dirichlet-to-Neumann map Λ_g uniquely determines the real analytic coefficients on the whole manifold $\bar{M}$.

We recall that the definitions of real analytic functions and real analytic hypersurfaces of a Riemannian manifold. Let $f(x)$ be a real-valued function defined on an open set $\Omega \subset \mathbb{R}^n$. For $y \in \Omega$, we call $f(x)$ real analytic at y if there exist $a_J \in \mathbb{R}$ and a neighborhood N_y of y such that $f(x) = \sum_J a_J (x-y)^J$ for all $x \in N_y$ and $J \in \mathbb{N}^n$. We say $f(x)$ is real analytic on an open set Ω if $f(x)$ is real analytic at each $y \in \Omega$.

Let (M, g) be a Riemannian manifold. A subset U of M is said to be an $(n-1)$-dimensional real analytic hypersurface if U is nonempty and if for every point $x \in U$, there is a real analytic diffeomorphism of a unit open ball $B(0,1) \subset \mathbb{R}^n$ onto an open neighborhood N_x of x such that $B(0,1) \cap \{x \in \mathbb{R}^n | x_n = 0\}$ maps onto $N_x \cap U$.

In order to prove Theorem 3, we need the following lemma (see [59], p. 65).

Lemma 7. (*Unique continuation of real analytic functions*) *Let $M \subset \mathbb{R}^n$ be a connected open set and $f(x)$ be a real analytic function defined on M. Let $y \in M$. Then, $f(x)$ is uniquely determined in M if we know $\frac{\partial^{|J|} f(y)}{\partial x^J}$ for all $J \in \mathbb{N}^n$. In particular, $f(x)$ is uniquely determined in M by its values in any nonempty open subset of M.*

Note that Lemma 7 still holds for real analytic functions defined on real analytic manifolds. Finally, we prove Theorem 3.

Proof of Theorem 3. According to Theorem 2, it has been proved that the thermoelastic Dirichlet-to-Neumann map Λ_g uniquely determines $\frac{\partial^{|J|}\lambda}{\partial x^J}$, $\frac{\partial^{|J|}\mu}{\partial x^J}$, $\frac{\partial^{|J|}\alpha}{\partial x^J}$, and $\frac{\partial^{|J|}\beta}{\partial x^J}$ on the boundary for all multi-indices J. Hence, for any point $x_0 \in \Gamma$, the coefficients can be uniquely determined in some neighborhood of x_0 by the analyticity of the coefficients on $M \cup \Gamma$. Furthermore, it follows from Lemma 7 that the coefficients can be uniquely determined in M. Therefore, by combining Theorem 2, we conclude that the coefficients λ, μ, α, and β can be uniquely determined on $\bar{M}$ by the thermoelastic Dirichlet-to-Neumann map Λ_g. □

Remark 8. *By applying the method of Kohn and Vogelius [12], we can also prove that the thermoelastic Dirichlet-to-Neumann map Λ_g uniquely determines the coefficients λ, μ, α, and β on $\bar{M}$, provided the manifold and the coefficients are piecewise analytic.*

Funding: National Natural Science Foundation of China (No. 12271031), National Key Research and Development Program of China (No. 2022YFC3310300).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data generated or analyzed during this study are included in this published article.

Acknowledgments: The author would like to sincerely thank the anonymous referees and the editor for their many helpful comments and valuable suggestions on the original manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kupradze, V.; Gegelia, T.; Basheleishvili, M.; Burchuladze, T. *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*; North-Holland Series in Applied Mathematics & Mechanics 25; North-Holland Pub. Co.: Amsterdam, The Netherlands, 1979.
2. Liu, G. Determination of isometric real-analytic metric and spectral invariants for elastic Dirichlet-to-Neumann map on Riemannian manifolds. *arXiv* **2019**, arXiv:1908.05096.
3. Liu, G.; Tan, X. Asymptotic expansion of the heat trace of the thermoelastic Dirichlet-to-Neumann map. *arXiv* **2022**, arXiv:2206.01374.
4. Tan, X.; Liu, G. Determining Lamé coefficients by elastic Dirichlet-to-Neumann map on a Riemannian manifold. *arXiv* **2022**, arXiv:2211.06650.
5. Taylor, M. *Partial Differential Equations III*, 2d ed.; Springer Science & Business Media: New York, NY, USA, 2011.
6. Landau, L.; Lifshitz, E.M. *Theory of Elasticity*, 3rd ed.; Butterworth Heinemann: Oxford, UK, 1986.
7. Liu, G. The Weyl-type asymptotic formula for biharmonic Steklov eigenvalues on Riemannian manifolds. *Adv. Math.* **2011**, *228*, 2162–2217. [CrossRef]
8. Liu, G. Asymptotic expansion of the trace of the heat kernel associated to the Dirichlet-to-Neumann operator. *J. Differ. Equ.* **2015**, *259*, 2499–2545. [CrossRef]
9. Liu, G.; Tan, X. Spectral invariants of the magnetic Dirichlet-to-Neumann map on Riemannian manifolds. *J. Math. Phys.* **2023**, *64*, 041501. [CrossRef]
10. Calderón, A. On an inverse boundary value problem. *Comput. Appl. Math.* **2006**, *25*, 133–138. [CrossRef]

11. Kohn, R.; Vogelius, M. Determining conductivity by boundary measurements. *Comm. Pure Appl. Math.* **1984**, *37*, 289–298. [CrossRef]
12. Kohn, R.; Vogelius, M. Determining conductivity by boundary measurements. II. Interior results. *Comm. Pure Appl. Math.* **1985**, *38*, 643–667. [CrossRef]
13. Sylvester, J.; Uhlmann, G. A global uniqueness theorem for an inverse boundary value problem. *Ann. Math.* **1987**, *125*, 153–169. [CrossRef]
14. Astala, K.; Päivxaxrinta, L. Calderón's inverse conductivity problem in the plane. *Ann. Math.* **2006**, *163*, 265–299. [CrossRef]
15. Astala, K.; Lassas, M.; Päivxaxrinta, L. Calderón's inverse problem for anisotropic conductivity in the plane. *Comm. Partial Differ. Equ.* **2005**, *30*, 207–224. [CrossRef]
16. Imanuvilov, O.; Uhlmann, G.; Yamamoto, M. The Calderón problem with partial data in two dimensions. *J. Am. Math. Soc.* **2010**, *23*, 655–691. [CrossRef]
17. Nachman, A.I. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. Math.* **1996**, *143*, 71–96. [CrossRef]
18. Sun, Z.; Uhlmann, G. Anisotropic inverse problems in two dimensions. *Inverse Probl.* **2003**, *19*, 1001–1010. [CrossRef]
19. Caro, P.; Rogers, K.M. Global uniqueness for the Calderón problem with Lipschitz conductivities. In *Forum of Mathematics, Pi*; Cambridge University Press: Cambridge, UK, 2016; Volume 4, p. e2.
20. Haberman, B.; Tataru, D. Uniqueness in Calderón's problem with Lipschitz conductivities. *Duke Math. J.* **2013**, *162*, 497–516. [CrossRef]
21. Lee, J.; Uhlmann, G. Determining anisotropic real-analytic conductivities by boundary measurements. *Commun. Pure Appl. Math.* **1989**, *42*, 1097–1112. [CrossRef]
22. Sun, Z.; Uhlmann, G. Inverse problems in quasilinear anisotropic media. *Am. J. Math.* **1997**, *119*, 771–797.
23. Uhlmann, G. Electrical impedance tomography and Calderón's problem. *Inverse Probl.* **2009**, *25*, 123011. [CrossRef]
24. Uhlmann, G. Inverse problems: Seeing the unseen. *Bull. Math. Sci.* **2014**, *4*, 209–279. [CrossRef]
25. Nakamura, G.; Uhlmann, G. Inverse problems at the boundary for an elastic medium. *SIAM J. Math. Anal.* **1995**, *26*, 263–279. [CrossRef]
26. Imanuvilov, O.; Yamamoto, M. Global uniqueness in inverse boundary value problems for the Navier–Stokes equations and Lamé system in two dimensions. *Inverse Probl.* **2015**, *31*, 035004. [CrossRef]
27. Nakamura, G.; Uhlmann, G. Erratum: Global uniqueness for an inverse boundary problem arising in elasticity. *Invent. Math.* **2003**, *152*, 205–207. [CrossRef]
28. Eskin, G.; Ralston, J. On the inverse boundary value problem for linear isotropic elasticity. *Inverse Probl.* **2002**, *18*, 907–921. [CrossRef]
29. Isakov, V. *Inverse Problems for Partial Differential Equations*, 3rd ed.; Applied Mathematical Sciences; Springer: New York, NY, USA, 2017; Volume 127.
30. Akamatsu, M.; Nakamura, G.; Steinberg, S. Identification of the Lamé coefficients from boundary observations. *Inverse Probl.* **1991**, *7*, 335–354. [CrossRef]
31. Imanuvilov, O.; Uhlmann, G.; Yamamoto, M. On uniqueness of Lamé coefficients from partial Cauchy data in three dimensions. *Inverse Probl.* **2012**, *28*, 125002. [CrossRef]
32. Imanuvilov, O.; Yamamoto, M. On reconstruction of Lamé coefficients from partial Cauchy data. *J. Inverse-Ill-Posed Probl.* **2011**, *19*, 881–891. [CrossRef]
33. Nakamura, G.; Uhlmann, G. Identification of Lamé parameters by boundary measurements. *Am. J. Math.* **1993**, *115*, 1161–1187. [CrossRef]
34. Liu, G. Determining anisotropic real-analytic metric from boundary electromagnetic information. *arXiv* **2019**, arXiv:1909.12803.
35. McDowall, S.R. Boundary determination of material parameters from electromagnetic boundary information. *Inverse Problems* **1997**, *13*, 153–163. [CrossRef]
36. Joshi, M.S.; McDowall, S.R. Total determination of material parameters from electromagnetic boundary information. *Pac. J. Math.* **2000**, *193*, 107–129. [CrossRef]
37. Caro, P.; Zhou, T. Global uniqueness for an IBVP for the time-harmonic Maxwell equations. *Anal. PDE* **2014**, *7*, 375–405. [CrossRef]
38. Pichler, M. An inverse problem for Maxwell equations with Lipschitz parameters. *Inverse Probl.* **2018**, *34*, 025006. [CrossRef]
39. Liu, G. The geometric invariants for the spectrum of the Stokes operator. *Math. Ann.* **2022**, *382*, 1985–2032. [CrossRef]
40. Liu, G. Geometric invariants of spectrum of the Navier-Lamé operator. *J. Geom. Anal.* **2021**, *31*, 10164–10193. [CrossRef]
41. Liu, G. Determining the viscosity from the boundary information for incompressible fluid. *arXiv* **2020**, arXiv:2006.04310.
42. Heck, H.; Wang, J.-N.; Li, X. Identification of viscosity in an incompressible fluid. *Indiana Univ. Math. J.* **2007**, *56*, 2489–2510.
43. Li, X.; Wang, J.-N. Determination of viscosity in the stationary Navier–Stokes equations. *J. Differ. Equ.* **2007**, *242*, 24–39. [CrossRef]
44. Dos Santos Ferreira, D.; Kenig, C.; Sjöstrand, J.; Uhlmann, G. Determining a Magnetic Schrödinger Operator from Partial Cauchy Data. *Commun. Math. Phys.* **2007**, *271*, 467–488. [CrossRef]
45. Dos Santos Ferreira, D.; Kenig, C.; Salo, M.; Uhlmann, G. Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.* **2009**, *178*, 119–171. [CrossRef]
46. Päivärinta, L.; Salo, M.; Uhlmann, G. Inverse scattering for the magnetic Schrödinger operator. *J. Funct. Anal.* **2010**, *259*, 1771–1798. [CrossRef]

47. Wang, B.; Li, H.; Bao, J. Determination of the insulated inclusion in conductivity problem and related Eshelby conjecture. *J. Differ. Equ.* **2014**, *257*, 4503–4524. [CrossRef]
48. Zhang, Y.; Gong, R. Second order asymptotical regularization methods for inverse problems in partial differential equations. *J. Comput. Appl. Math.* **2020**, *375*, 112798. [CrossRef]
49. Zhang, Y.; Hofmann, B. Two new non-negativity preserving iterative regularization methods for ill-posed inverse problems. *Inverse Probl. Imaging* **2021**, *15*, 229–256. [CrossRef]
50. Liu, G. Spectral invariants of the perturbed polyharmonic Steklov problem. *Calc. Var. Partial Differ. Equ.* **2022**, *61*, 19. [CrossRef]
51. Liu, G. On asymptotic properties of biharmonic Steklov eigenvalues. *J. Differ. Equ.* **2016**, *261*, 4729–4757. [CrossRef]
52. Taylor, M. *Partial Differential Equations II*, 2nd ed.; Springer Science & Business Media: New York, NY, USA, 2011.
53. Grubb, G. *Functional Calculus of Pseudo-Differential Boundary Problems*; Birkhäuser: Boston, MA, USA, 1986.
54. Hörmander, L. *The Analysis of Partial Differential Operators III*; Springer: Berlin/Heidelberg, Germany, 1985.
55. Taylor, M. *Pseudodifferential Operators*; Princeton University Press: Princeton, NJ, USA, 1981.
56. Bernstein, D.S. *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas*; Revised and expanded edition; Princeton University Press: Princeton, NJ, USA, 2018.
57. Treves, F. *Introduction to Pseudodifferential and Fourier Integral Operator*; Plenum Press: New York, NY, USA, 1980.
58. Hörmander, L. *Linear Partial Differential Operators*; Springer: Berlin/Heidelberg, Germany, 1963.
59. John, F. *Partial Differential Equations*, 4th ed.; Springer: New York, NY, USA, 1982.