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CDE' Inequality on Graphs with Unbounded Laplacian

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Abstract: In this paper, we derive the gradient estimates of semigroups in terms of the modified curvature-dimension inequality CDE' for unbounded Laplacians on complete graphs with non-degenerate measures.

Keywords: unbounded Laplacian; heat semigroup; CDE' inequality; gradient estimate

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1. Introduction

It is insightful that curvature-dimension inequality $CD(n,K)$ with the dimension parameter n and the lower bound of Ricci curvature K, from Bochner's identity, can be used as a substitute for the lower Ricci curvature bound on metric spaces, especially on non-smooth spaces by Bakry and Émery [1]. The curvature-dimension inequality has been extensively studied in the literature. See for example [2] in manifold settings and [3–5] in discrete settings. The following gradient estimate of the semigroup

$$\Gamma(P_t f) \le e^{-2Kt} P_t \Gamma(f) \tag{1}$$

is equivalent to the curvature-dimension inequality $CD(\infty,K)$ for diffusion Laplacians on metric measure spaces (see [6]). On graphs, such a result has been proven by [3,7] in the case of finite graphs and for bounded Laplacians as well as [8,9] for unbounded Laplacians. Furthermore, a version of the strong gradient estimate

$$\sqrt{\Gamma(P_t f)} \le e^{-2Kt} P_t \sqrt{\Gamma(f)} \tag{2}$$

had been proven by the following strong curvature inequality

$$\Gamma(\Gamma(g)) \le 4\Gamma(g)[\Gamma_2(g) - K\Gamma(g)], \tag{3}$$

when the Laplacian generates a diffusion; see [6]. The strong gradient estimate (2) is the key to deriving the Log–Sobolev inequality of the semigroup. Unfortunately, this strong curvature inequality can never be satisfied on a graph. The strong curvature inequality (3) fails, e.g., for $g = \delta_x$.

To prove the discrete version of the Li-Yau inequality, Bauer et al. introduced the modified curvature-dimension inequalities CDE and CDE' and derived the Harnack inequality [10]. The Gaussian heat kernel estimate, volume doubling, and Poincaré inequality were proved in [11] under the assumption of $CDE'(n,0)$. From their papers, $CDE'(n,K)$ is equivalent to $CD(n,K)$ when the Laplacian is diffusion. On a graph, $CDE'(n,K)$ implies $CD(n,K)$, but the reverse is not true (see [10,12]). The typical graphs satisfying the $CDE'(n,0)$ assumption are Abelian Cayley graphs. Unlike the classical curvature-dimension inequality CD, the modified one CDE' is nonlinear and hard to study. The equivalent gradient estimate associated with $CDE'(\infty, K)$ was derived for bounded Laplacian in [3], which states that

$$\Gamma(\sqrt{P_t f}) \leq e^{-2Kt} P_t \Gamma(\sqrt{f}). \tag{4}$$

This form is close to the strong gradient estimate (2), and it is a stronger gradient estimate than the regular one (1) in terms of CD.

It should be pointed out that for the case of unbounded Laplacians, standard techniques for bounded Laplacians could not be used again. Unbounded Laplacians on graphs have been studied in the past decade or so; see for example [13–16]. Hua and the second author proved one of the equivalent semigroup properties, i.e., the gradient estimate (1) of $CD(\infty, K)$ and then the stochastic completeness on a complete graph with a non-degenerate measure [9]. In ref. [8], other equivalent semigroup properties of $CD(\infty, K)$, i.e., the Poincaré inequality and inverse Poincaré inequality, were proved for unbounded Laplacian under the same assumptions. The equivalent semigroup properties of $CD(n, K)$ were also derived. In ref. [17], the Li-Yau inequality and heat kernel estimate have been extended to the case of unbounded Laplacians on graphs satisfying $CDE'(n, 0)$. In this paper, we focus on unbounded Laplacians and explore the equivalent semigroup properties of CDE'.

1.1. Setup and Notation

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. For $x, y \in V$, we call them neighbors if $(x, y) \in E$, i.e., an edge between x and y, which is written as $x \sim y$. We allow loops, i.e., $x \sim x$. G is called locally finite if there are only finite neighbors for any vertex, that is, for any $x \in V$,

$$\#\{y \in V | y \sim x\} < \infty.$$

G is called connected if for any $x, y \in V$, there is a finite sequence $\{x_i\}_{i=0}^n \in V$ satisfying

$$x = x_0 \sim x_1 \sim \cdots \sim x_n = y.$$

In this paper, all the graphs we consider are connected and locally finite.

Given two functions $m : V \to (0, \infty)$ and $\omega : E \to [0, \infty)$ as the measure on V and the weight on E separately, we assume ω is symmetric, i.e., for any $(x, y) \in E$, $\omega_{xy} = \omega_{yx}$. As for $(x, y) \notin E$, we let $\omega_{xy} = 0$. Given a weight function ω and a measure function m, we call $G = (V, E, \omega, m)$ a weighted graph.

We denote by $V^{\mathbb{R}}$ the set of real-value functions on V and by $C_0(V)$ the set of finitely supported functions on V. Let $p \in [1, \infty)$; we denote by ℓ^p_m the spaces of functions on V with respect to the measure m and by ℓ^∞ the spaces of bounded functions.

Given a graph $G = (V, E, \omega, m)$, there is an associated Dirichlet form on $\ell^2(V, m)$

$$Q : D(Q) \times D(Q) \to \mathbb{R}$$
$$Q(f, g) = \frac{1}{2} \sum_{x,y \in V} \omega_{xy}(f(y) - f(x))(g(y) - g(x)),$$

where $D(Q) = \{f \in \ell^2(V, m) | \sum_{x,y \in V} \omega_{xy}(f(y) - f(x))^2 < \infty\}$ and the norm on $D(Q)$ is defined as

$$\|f\|_Q = \sqrt{Q(f) + \|f\|_{\ell^2_m}}.$$

The Laplacian Δ associated with the Dirichlet form on a graph can be written as

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V} \omega_{xy}(f(y) - f(x)), \quad \forall f \in C_0(V).$$

The domain of the Laplacian Δ is defined by $D(\Delta) = \{f \in D(Q) | \Delta f \in \ell^2(V, m)\}$. The associated semigroup of Δ is $P_t = e^{t\Delta}$. It can be seen that the choice of measure m does have a great influence on the Laplacian when the weight function ω is fixed. Usually, the measure m has the following two typical forms:

(1) For any $x \in V$, $m(x) = \sum_{y \sim x} \omega_{xy}$;

(2) For any $x \in V$, $m(x) = 1$.

It is well known that the boundedness of the Laplacian is equivalent to

$$D_\mu := \sup_{x \in V} \frac{\sum_{y \sim x} \omega_{xy}}{m(x)} < \infty,$$

see [14]. Notice that the Laplacian in case (1) is bounded.

In this paper, we focus on the unbounded Laplace operators. We need to assume the measure m is *non-degenerate*, i.e.,

$$\delta := \inf_{x \in V} m(x) > 0.$$

The non-degeneracy of measure is a mild assumption on a graph and plays a significant role in dealing with unbounded Laplacians. Indeed, we have the following useful lemma.

Lemma 1. *Let m be a non-degenerate measure. Then, for any $f \in \ell^p(V, m)$ and $p \in [1, \infty)$,*

$$|f(x)| \le \delta^{-\frac{1}{p}} \|f\|_{\ell^p_m}, \quad \forall x \in V.$$

Moreover, $\ell^p_m \hookrightarrow \ell^q_m$ with $1 \le p < q \le +\infty$.

We further assume that the graph is *complete*, i.e., there exists a non-decreasing sequence $\{\eta_k\}_0^\infty \in C_0(V)$ satisfying

$$\lim_{k \to \infty} \eta_k = \mathbf{1}, \quad \Gamma(\eta_k) \le \frac{1}{k}. \tag{5}$$

Here, $\mathbf{1}$ is the constant function on V, and the limit of (5) is pointwise. This condition is first introduced on the Markov diffusion semigroup in [6] and then on a graph in [9]. The condition of completeness has been proven to be satisfied for a large class of graphs with intrinsic metrics. See Theorem 2.8 in [9]. The following lemma shows that $C_0(V)$ is a dense subset of $D(Q)$ when the graph is complete.

Lemma 2 (Lemma 2.5, [9])**.** *Let $G = (V, E, \omega, m)$ be a complete graph. Then, for any $f \in D(Q)$, we have*

$$\|f\eta_k - f\|_Q \to 0, \quad k \to \infty.$$

Now, we give the definition of the curvature-dimension inequality. First, we introduce the following gradient forms.

Definition 1. *The carré du champ operator Γ and the iterated gradient form Γ_2 are defined by*

$$\Gamma(f, g)(x) = \frac{1}{2}(\Delta(fg)(x) - g\Delta f(x) - f\Delta g(x)),$$

$$\Gamma_2(f, g)(x) = \frac{1}{2}(\Delta\Gamma(f, g)(x) - \Gamma(f, \Delta g)(x) - \Gamma(g, \Delta f)(x)).$$

For convenience, we write $\Gamma(f) = \Gamma(f, f)$ and $\Gamma_2(f) = \Gamma_2(f, f)$.

Next, we introduce the modified curvature-dimension inequality on graphs. As we mentioned before, Bauer et al. [10] modified the curvature-dimension inequalities to prove the Li-Yau inequality. They noticed that the graph Laplacian Δ does not generate a diffusion semigroup except for the square root function $\sqrt{\cdot}$, which motivates the following modification of curvature-dimension inequality. The modification of Γ_2 is defined by

$$\widetilde{\Gamma}_2(f) := \frac{1}{2}\Delta\Gamma(f) - \Gamma\left(f, \frac{\Delta(f^2)}{2f}\right).$$

Definition 2 ($CDE'(n,K)$)**.** *We say $G=(V,E,\omega,m)$ satisfies $CDE'(x,n,K)$ on $x \in V$, if for any positive function f, the following inequality holds true*

$$\widetilde{\Gamma}_2(f)(x) \geq \frac{1}{n} f(x)^2 (\Delta \log f)(x)^2 + K\Gamma(f)(x).$$

G is said to satisfy $CDE'(n,K)$, if for every $x \in V$, $CDE'(x,n,K)$ is true. If $n=\infty$, we say G satisfies $CDE'(\infty,K)$.

1.2. Main Results

Here, we are ready to state our main results.

Theorem 1. *Let $G=(V,E,\omega,m)$ be a complete graph with a non-degenerate measure m. Then, the following statements are equivalent:*

(1) G satisfies $CDE'(\infty,K)$ with $K \in \mathbb{R}$.

(2) For any $0 \leq f \in C_0(V)$, $t \geq 0$,

$$\Gamma(\sqrt{P_t f}) \leq e^{-2Kt} P_t \Gamma(\sqrt{f}).$$

(3) For any $0 \leq f \in D(Q)$, $t \geq 0$,

$$\Gamma(\sqrt{P_t f}) \leq e^{-2Kt} P_t \Gamma(\sqrt{f}).$$

Similarly, we obtain the gradient estimate of $CDE'(n,K)$ with $n \in (0,\infty)$ on a locally finite graph under the same assumptions.

Theorem 2. *Let $G=(V,E,\omega,m)$ be a complete graph with a non-degenerate measure m. Then, the following statements are equivalent:*

(1) G satisfies $CDE'(n,K)$ with $n>0$ and $K \in \mathbb{R}$.

(2) For any $0 \leq f \in C_0(V)$, $t \geq 0$,

$$\Gamma(\sqrt{P_t f}) \leq e^{-2Kt} P_t(\Gamma(\sqrt{f})) - \frac{2}{n}\int_0^t e^{-2Ks} P_s(P_{t-s} f(\Delta \log \sqrt{P_{t-s} f})^2) ds.$$

(3) For any $0 \leq f \in D(Q)$, $t \geq 0$,

$$\Gamma(\sqrt{P_t f}) \leq e^{-2Kt} P_t(\Gamma(\sqrt{f})) - \frac{2}{n}\int_0^t e^{-2Ks} P_s(P_{t-s} f(\Delta \log \sqrt{P_{t-s} f})^2) ds.$$

The proof of the above theorems is based on the semigroup methods, which is a generalization of the result in [3] to unbounded Laplacians. The remaining part of this paper will be organized as follows: In Section 2, we give several preliminary lemmas for our use later. In Section 3, we finish the proof of the main theorems.

2. Preliminaries

We need the following properties of the heat semigroup and Green's formula (see [9,17]).

Lemma 3. *For any $f \in \ell^p(V,m)$ and $p \in [1,\infty]$, we have $P_t f \in \ell^p(V,m)$ and*

$$\|P_t f\|_{\ell^p_m} \le \|f\|_{\ell^p_m}, \quad \forall t \ge 0.$$

Moreover, for any $f \in \ell^2(V,m)$, $P_t f \in D(\Delta)$.

Lemma 4. *Let $G = (V,E,\omega,m)$. For any $f \in D(Q)$ and $g \in D(\Delta)$, we have*

$$\sum_{x\in V} f(x)\Delta g(x)m(x) = -\sum_{x\in V} \Gamma(f,g)m(x).$$

The following lemma is very useful in the proof of main results.

Lemma 5. *For any functions $f,g \in V^{\mathbb{R}}$, if $|f| \le H$ and $|g| \ge h > 0$, then we have*

$$\Gamma\left(\frac{f}{g}\right) \le C_1\Gamma(f) + C_2\Gamma(g),$$

where C_1 and C_2 are constants only depending on H and h.

Proof. By definition of Γ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Gamma\left(\frac{f}{g}\right) &= \frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}\left(\frac{f(y)}{g(y)} - \frac{f(x)}{g(x)}\right)^2 \\
&= \frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}\left[\frac{1}{g(y)}(f(y)-f(x)) + f(x)\left(\frac{1}{g(y)} - \frac{1}{g(x)}\right)\right]^2 \\
&\le \frac{1}{m(x)}\sum_{y\sim x}\omega_{xy}\frac{1}{g^2(y)}(f(y)-f(x))^2 \\
&\quad + \frac{1}{m(x)}\sum_{y\sim x}\omega_{xy}\frac{f^2(x)}{g^2(x)g^2(y)}(g(y)-g(x))^2 \\
&\le \frac{2}{h^2}\Gamma(f) + \frac{2H^2}{h^4}\Gamma(g) = C_1(h)\Gamma(f) + C_2(h,H)\Gamma(g).
\end{aligned}$$

That completes the proof. □

The following Caccioppoli inequality for subsolutions to Poissons equations on graphs was proved in [9]. The authors of [9] derive a uniform upper bound about the solution to the heat equation.

Lemma 6 (Lemma 3.4, [9])**.** *Let $g,h : V \to \mathbb{R}$ satisfy $\Delta g \ge h$. Then, for any $\eta \in C_0(V)$,*

$$\left\|\Gamma(g)\eta^2\right\|_{\ell^1_m} \le C(\left\|\Gamma(\eta)g^2\right\|_{\ell^1_m} + \left\|gh\eta^2\right\|_{\ell^1_m}).$$

Lemma 7 (Lemma 3.6, [9])**.** *Let $G = (V,E,\omega,m)$ be a complete graph. For any $f \in C_0(V)$ and $T > 0$, we have $\max\limits_{[0,T]}\Gamma(P_t f), \max\limits_{[0,T]}|\Gamma(P_t f, \Delta P_t f)| \in \ell^1_m$. Moreover, there exists a positive constant $C(T,f)$ depending on T and f such that*

$$\left\|\max_{[0,T]}\Gamma(P_t f)\right\|_{\ell^1_m}, \left\|\max_{[0,T]}|\Gamma(P_t f, \Delta P_t f)|\right\|_{\ell^1_m} \le C(T,f).$$

Combining Lemmas 5 and 7, we have the following uniform upper bound about the square root of the solution to heat equation.

Lemma 8. *Let $G = (V, E, \omega, m)$ be a complete graph. For any $0 \le f \in C_0(V), \epsilon > 0$, and $T > 0$, we have $\max_{[0,T]} \Gamma(\sqrt{P_t f + \epsilon}) \in \ell^1_m$. Moreover, there exists a positive constant $C(T, f, \epsilon)$ depending on T, f and ϵ such that*

$$\left\| \max_{[0,T]} \Gamma(\sqrt{P_t f + \epsilon}) \right\|_{\ell^1_m} \le C(T, f, \epsilon).$$

Proof. Notice that $P_t f + \epsilon \ge \epsilon$ since $f \ge 0$. Therefore,

$$\begin{aligned}
\left\| \max_{[0,T]} \Gamma(\sqrt{P_t f + \epsilon}) \right\|_{l^1_m} &= \sum_{x \in V} \max_{[0,T]} \left(\frac{1}{2m(x)} \sum_{y \sim x} \omega_{xy} (\sqrt{P_t f + \epsilon}(y) - \sqrt{P_t f + \epsilon}(x))^2 \right) m(x) \\
&= \frac{1}{2} \sum_{x \in V} \max_{[0,T]} \sum_{y \sim x} \omega_{xy} \left(\sqrt{P_t f + \epsilon}(y) - \sqrt{P_t f + \epsilon}(x) \right)^2 \\
&= \frac{1}{2} \sum_{x \in V} \max_{[0,T]} \sum_{y \sim x} \omega_{xy} \left(\frac{P_t f(y) - P_t f(x)}{\sqrt{P_t f + \epsilon}(y) + \sqrt{P_t f + \epsilon}(x)} \right)^2 \\
&\le \frac{1}{4\epsilon} \left(\frac{1}{2} \sum_{x \in V} \max_{[0,T]} \sum_{y \sim x} \omega_{xy} ((P_t f)(y) - (P_t f)(x))^2 \right) \\
&\le C(\epsilon) \left\| \max_{[0,T]} \Gamma(P_t f) \right\|_{l^1_m} \le C(\epsilon) C(T, f) =: C(T, f, \epsilon).
\end{aligned}$$

This proves our case. □

Lemma 9. *Let $G = (V, E, \omega, m)$ be a complete graph with a non-degenerate measure m. If G satisfies $CDE'(\infty, K)$, for any $0 \le f \in C_0(V)$ and $\epsilon > 0$, we have*

$$\Gamma(\sqrt{P_t f + \epsilon}) \in D(Q), \ \ t > 0.$$

Proof. For convenience, let $u = P_t f + \epsilon$. It is easy to see that $\Gamma(\sqrt{u}) \in \ell^2(V, m)$ by Lemma 1 and Lemma 8. If we prove that $Q(\Gamma(\sqrt{u})) < \infty$, the assertion follows.

Let $g = \Gamma(\sqrt{u})$ and $h = 2\Gamma\left(\sqrt{u}, \frac{\Delta u}{2\sqrt{u}}\right) + 2K\Gamma(\sqrt{u})$. Thus, $\Delta g \ge h$ follows from $CDE'(\infty, K)$. According to Caccioppoli inequality (see Lemma 6) and $g \in \ell^2(V, m) \subset \ell^\infty(V)$, we have

$$\begin{aligned}
\left\| \Gamma(g) \eta_k^2 \right\|_{\ell^1_m} &\le C \left(\left\| \Gamma(\eta_k) g^2 \right\|_{\ell^1_m} + \left\| g h \eta_k^2 \right\|_{\ell^1_m} \right) \\
&\le C \left(\frac{1}{k} \|g\|^2_{\ell^2_m} + \|g\|_{\ell^\infty} \left\| \Gamma\left(\sqrt{u}, \frac{\Delta u}{2\sqrt{u}} \right) \right\|_{\ell^1_m} + |2K| \|g\|^2_{\ell^2_m} \right).
\end{aligned}$$

From the Cauchy–Schwarz inequality, we obtain

$$\left\| \Gamma\left(\sqrt{u}, \frac{\Delta u}{2\sqrt{u}} \right) \right\|_{\ell^1_m} \le \frac{1}{2} \|\Gamma(\sqrt{u})\|_{\ell^1_m} + \frac{1}{2} \left\| \Gamma\left(\frac{\Delta u}{2\sqrt{u}} \right) \right\|_{\ell^1_m}.$$

Note that $\epsilon \le u \le \|f\|_\infty + \epsilon$. By Lemma 5, we have

$$\left\| \Gamma\left(\frac{\Delta u}{2\sqrt{u}} \right) \right\|_{\ell^1_m} \le C_1(\epsilon, f) \|\Gamma(\sqrt{u})\|_{\ell^1_m} + C_2(\epsilon, f) \|\Gamma(\Delta u)\|_{\ell^1_m}, \tag{6}$$

where $\|\Gamma(\Delta u)\|_{\ell^1_m} = \|\Gamma(\Delta P_t f)\|_{\ell^1_m} = \|\Gamma(P_t \Delta f)\|_{\ell^1_m} < \infty$ by Lemma 7. Therefore, combining with Lemma 8, it follows that

$$\left\| \Gamma(g) \eta_k^2 \right\|_{\ell^1_m} < \infty.$$

According to the Fatou's lemma, we obtain

$$\left\|\Gamma(\Gamma(\sqrt{P_t f+\epsilon}))\right\|_{\ell^1_m} \leq \liminf_{k\to\infty}\left\|\Gamma(\Gamma(\sqrt{P_t f+\epsilon}))\eta_k^2\right\|_{\ell^1_m} < \infty,$$

which completes the proof. □

3. Proof of Main Results

Proof of Theorem 1. (1) $\Rightarrow$ (2) For any $f,\xi \in C_0(V)$, let

$$G(s) := \sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}\right)(x)P_s\xi(x)m(x), \quad \epsilon > 0.$$

We separate the proof into the following three steps.

Step 1. The derivative of $G(s)$ is as follows.

$$G'(s) = -2\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}, \frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right)(x)P_s\xi(x)m(x) - \sum_{x\in V}\Gamma(\Gamma(\sqrt{P_t f+\epsilon}), P_s\xi)(x)m(x).$$

Indeed, the formal derivative of $G(s)$ is

$$-2\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}, \frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right)(x)P_s\xi(x)m(x) + \sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}\right)(x)\Delta(P_s\xi)(x)m(x) := I_1 + I_2.$$

To prove $G'(s)$ is just the above formula, it is sufficient to show the uniform convergence of the above summations on s. For any $f,\xi \in C_0(V)$, by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}|I_1| &\leq 2||P_s\xi||_{\ell^\infty}\sum_{x\in V}\left|\Gamma\left(\sqrt{P_{t-s}f+\epsilon}, \frac{\Delta(P_{t-s}f+\epsilon)}{2\sqrt{P_{t-s}f+\epsilon}}\right)\right|(x)m(x) \\ &\leq \|\xi\|_{\ell^\infty}\left(\left\|\Gamma(\sqrt{P_{t-s}f+\epsilon})\right\|_{\ell^1_m} + \left\|\Gamma\left(\frac{\Delta(P_{t-s}f+\epsilon)}{2\sqrt{P_{t-s}f+\epsilon}}\right)\right\|_{\ell^1_m}\right),\end{aligned}$$

and

$$|I_2| \leq \sum_{x\in V}\Gamma(\sqrt{P_{t-s}f+\epsilon})(x)|P_s\Delta\xi|(x)m(x) \leq \|\Delta\xi\|_{\ell^\infty}\left\|\Gamma(\sqrt{P_{t-s}f+\epsilon})\right\|_{\ell^1_m}.$$

From Lemma 8 and (6) in Lemma 9, I_1 and I_2 are uniformly convergent on $s \in (\delta, t-\delta)$ for any $0<\delta<t$. Note that $P_s\xi \in D(\Delta)$ for any $\xi \in C_0(V)$, and $\Gamma(\sqrt{P_{t-s}f+\epsilon}) \in D(Q)$ by Lemma 9 from the Green's formula; we finish the proof of Step 1.

Step 2. Under the assumption of $CDE'(\infty, K)$, we have

$$G'(s) \geq 2KG(s). \tag{7}$$

To finish this, we *claim* that for any $h \in D(Q)$,

$$-2\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}, \frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right)(x)h(x)m(x) - \sum_{x\in V}\Gamma\left(\Gamma(\sqrt{P_t f+\epsilon}), h\right)(x)m(x) \geq 2K\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}\right)(x)h(x)m(x). \tag{8}$$

Letting $h = P_s\xi$ in the above inequality, we obtain (7). The rest of Step 2 is to prove the claim.

For any $0 \le h \in C_0(V)$, by the Green's formula, we obtain

$$-\sum_{x\in V}\Gamma\Big(\Gamma(\sqrt{P_t f+\epsilon}),h\Big)(x)m(x) = \sum_{x\in V}\Delta\Gamma\Big(\sqrt{P_{t-s}f+\epsilon}\Big)(x)h(x)m(x),$$

which yields (8) by the $CDE'(\infty,K)$ condition. For any $0 \le h \in D(Q)$, let η_k be defined as shown in (5). Replace h by $h\eta_k \in C_0(V)$ into the above equality. Since

$$\Gamma\left(\sqrt{P_{t-s}f+\epsilon},\frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right),\Gamma\Big(\sqrt{P_{t-s}f+\epsilon}\Big) \in \ell^1(V,m)$$

and $\Gamma\big(\sqrt{P_{t-s}f+\epsilon}\big) \in D(Q)$, it is obvious that

$$\begin{aligned}
&-2\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon},\frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right)(x)h\eta_k(x)m(x)\\
&\to -2\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon},\frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right)(x)h(x)m(x),
\end{aligned}$$

and

$$2K\sum_{x\in V}\Gamma\Big(\sqrt{P_{t-s}f+\epsilon}\Big)(x)h\eta_k(x)m(x) \to 2K\sum_{x\in V}\Gamma\Big(\sqrt{P_{t-s}f+\epsilon}\Big)(x)h(x)m(x),$$

as $k \to \infty$. Moreover, we have

$$\begin{aligned}
&\left|\sum_{x\in V}\Gamma\Big(\Gamma\Big(\sqrt{P_t f+\epsilon}\Big),h\eta_k\Big)(x)m(x) - \sum_{x\in V}\Gamma\Big(\Gamma\Big(\sqrt{P_t f+\epsilon}\Big),h\Big)(x)m(x)\right|\\
&=\left|\sum_{x\in V}\Gamma\Big(\Gamma\Big(\sqrt{P_t f+\epsilon}\Big),h\eta_k - h\Big)(x)m(x)\right|\\
&\le \sum_{x\in V}\sqrt{\Gamma\Big(\Gamma(\sqrt{P_t f+\epsilon}\Big)(x)}\sqrt{\Gamma(h(\eta_k-\mathbf{1}))(x)}m(x)\\
&\le \left(\sum_{x\in V}\Gamma\Big(\Gamma(\sqrt{P_t f+\epsilon}\Big)(x)m(x)\right)^{1/2}\left(\sum_{x\in V}\Gamma(h(\eta_k-\mathbf{1}))(x)m(x)\right)^{1/2} \to 0,
\end{aligned}$$

which finishes the claim.

Step 3. Integrating (7) in Step 2 from 0 to t, we have

$$G(t) \ge e^{2Kt}G(0).$$

For any $y \in V$, let $\xi(x) = \delta_y(x)$ in $G(s)$. From the above inequality and self-adjointness of P_t, it follows that

$$\Gamma(\sqrt{P_t f+\epsilon}) \le e^{-2Kt}P_t(\Gamma(\sqrt{f+\epsilon})). \tag{9}$$

As a result of the local finiteness of the graph, we have

$$\lim_{\epsilon\to 0^+}\Gamma(\sqrt{P_t f+\epsilon}) = \Gamma(\lim_{\epsilon\to 0^+}\sqrt{P_t f+\epsilon}) = \Gamma(\sqrt{P_t f}).$$

Notice that $\Gamma(\sqrt{f+\epsilon}) = \Gamma(\sqrt{f+\epsilon}-\sqrt{\epsilon})$ and $0 \leq \sqrt{f+\epsilon}-\sqrt{\epsilon} \leq \sqrt{f}$ for any $0 \leq f \in C_0(V)$. Then, for every $x \in V$, we have

$$\begin{aligned}\Gamma\left(\sqrt{f+\epsilon}\right)(x) &= \frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}\left[\left(\sqrt{f+\epsilon}-\sqrt{\epsilon}\right)(y)-\left(\sqrt{f+\epsilon}-\sqrt{\epsilon}\right)(x)\right]^2\\ &\leq \frac{1}{m(x)}\sum_{y\sim x}\omega_{xy}\left[\left(\sqrt{f+\epsilon}-\sqrt{\epsilon}\right)^2(y)+\left(\sqrt{f+\epsilon}-\sqrt{\epsilon}\right)^2(x)\right]\\ &\leq \frac{1}{m(x)}\sum_{y\sim x}\omega_{xy}(f(y)+f(x)) = \Delta f(x)+\frac{\deg(x)}{m(x)}f(x).\end{aligned}$$

Note that $P_t(\Delta f + \frac{\deg(\cdot)}{m(\cdot)}f) < \infty$ since $f(x) \in C_0(V)$. By the dominated convergence theorem and the locally finiteness of graph, we obtain

$$\lim_{\epsilon\to 0^+} P_t\Gamma(\sqrt{f+\epsilon}) = P_t(\lim_{\epsilon\to 0^+}\Gamma(\sqrt{f+\epsilon})) = P_t\Gamma(\lim_{\epsilon\to 0^+}\sqrt{f+\epsilon}) = P_t\Gamma(\sqrt{f}). \tag{10}$$

Letting $\epsilon \to 0^+$ in (9), we finish the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) For $0 \leq f \in C_0(V)$, consider

$$F(t) := e^{-2Kt}P_t\Gamma(\sqrt{f}) - \Gamma(\sqrt{P_t f}).$$

Note that $F(0) = 0$ and $F(t) \geq 0$. It follows that $\lim\limits_{t\to 0^+} F'(t) \geq 0$. Then,

$$\begin{aligned}0 \leq \lim_{t\to 0^+} F'(t) &= \lim_{t\to 0^+}\left(-2Ke^{-2Kt}P_t\Gamma(\sqrt{f}) + e^{-2Kt}\Delta P_t\Gamma(\sqrt{f}) - 2\Gamma\left(\sqrt{P_t f}, \frac{\Delta P_t f}{2\sqrt{P_t f}}\right)\right)\\ &= \lim_{t\to 0^+}\left(-2Ke^{-2Kt}P_t\Gamma(\sqrt{f}) + e^{-2Kt}P_t\Delta\Gamma(\sqrt{f}) - 2\Gamma\left(\sqrt{P_t f}, \frac{P_t\Delta f}{2\sqrt{P_t f}}\right)\right)\\ &= -2K\Gamma(\sqrt{f}) + \Delta\Gamma(\sqrt{f}) - 2\Gamma\left(\sqrt{f}, \frac{\Delta f}{2\sqrt{f}}\right) = -2K\Gamma(\sqrt{f}) + 2\Gamma(\sqrt{f}),\end{aligned}$$

which implies $CDE'(\infty, 0)$.

(2) $\Rightarrow$ (3) For any $0 \leq f \in D(Q)$, let η_k be defined as (5). From (2), we have

$$\Gamma\left(\sqrt{P_t(f\eta_k^2)}\right) \leq e^{-2Kt}P_t\Gamma\left(\sqrt{f\eta_k^2}\right).$$

By the local finiteness of the graph and monotone convergence theorem, we obtain

$$\lim_{k\to\infty}\Gamma\left(\sqrt{P_t(f\eta_k^2)}\right) = \Gamma\left(\lim_{k\to\infty}\sqrt{P_t(f\eta_k^2)}\right) = \Gamma\left(\sqrt{P_t f}\right).$$

On the other hand, for any $x \in V$,

$$
\begin{aligned}
\Gamma\left(\sqrt{f\eta_k^2}\right)(x) &= \frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}\left(\sqrt{f(y)}|\eta_k(y)| - \sqrt{f(x)}|\eta_k(x)|\right)^2 \\
&= \frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}\left[\left(\sqrt{(f(y)} - \sqrt{f(x)}\right)|\eta_k(y)| + \sqrt{f(x)}(|\eta_k(y)| - |\eta_k(x)|)\right]^2 \\
&\le \left(1+\frac{1}{\sqrt{k}}\right)\frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}\left(\sqrt{f(y)} - \sqrt{f(x)}\right)^2|\eta_k(y)|^2 \\
&+ \left(1+\sqrt{k}\right)\frac{1}{2m(x)}\sum_{y\sim x}\omega_{xy}f(x)(|\eta_k(y)| - |\eta_k(x)|)^2 \\
&\le \left(1+\frac{1}{\sqrt{k}}\right)\Gamma\left(\sqrt{f}\right)(x) + \frac{1+\sqrt{k}}{k}f(x).
\end{aligned}
$$

In the third step, we use the basic inequality $2ab \le \frac{1}{\sqrt{k}}a^2 + \sqrt{k}b^2$. It follows that

$$
P_t\left(\Gamma\left(\sqrt{f\eta_k^2}\right)\right) \le \left(1+\frac{1}{\sqrt{k}}\right)P_t\Gamma\left(\sqrt{f}\right) + \frac{1+\sqrt{k}}{k}P_tf.
$$

Letting $k \to +\infty$, we obtain

$$
\overline{\lim_{k\to\infty}}\, P_t\left(\Gamma\left(\sqrt{f\eta_k^2}\right)\right) \le \lim_{k\to\infty}\left(1+\frac{1}{\sqrt{k}}\right)P_t\Gamma\left(\sqrt{f}\right) + \lim_{k\to\infty}\frac{1+\sqrt{k}}{k}P_tf = P_t\Gamma\left(\sqrt{f}\right).
$$

Then, for any $0 \le f \in D(Q)$, we have

$$
\Gamma\left(\sqrt{P_tf}\right) = \overline{\lim_{k\to\infty}}\,\Gamma\left(\sqrt{P_tf\eta_k^2}\right) \le e^{-2Kt}\,\overline{\lim_{k\to\infty}}\,P_t\Gamma\left(\sqrt{f\eta_k^2}\right) \le e^{-2Kt}P_t\Gamma\left(\sqrt{f}\right).
$$

(3) $\Rightarrow$ (2) Notice that $C_0(V)$ is a dense subset of $D(Q)$, the proof is obvious. □

Proof of Theorem 2. (1) $\Rightarrow$ (2) For any $f, \xi \in C_0(V)$, consider

$$
H(s) := e^{-2Ks}\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}\right)(x)P_s\xi(x)m(x), \quad \epsilon > 0,
$$

and obtain the formal derivation of H as follows,

$$
\begin{aligned}
&-2e^{-2Ks}\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}, \frac{\Delta(P_{t-s}f+\epsilon)}{\sqrt{P_{t-s}f+\epsilon}}\right)(x)P_s\xi(x)m(x) \\
&+e^{-2Ks}\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}\right)(x)\Delta(P_s\xi)(x)m(x) - 2Ke^{-2Ks}\sum_{x\in V}\Gamma\left(\sqrt{P_{t-s}f+\epsilon}\right)(x)P_s\xi(x)m(x).
\end{aligned}
$$

For any $\xi \in C_0(V)$, we have

$$
\left|2K\sum_{x\in V}\Gamma(\sqrt{P_{t-s}f+\epsilon})P_s\xi(x)m(x)\right| \le 2|K|\|\xi\|_{\ell_m^\infty}\sum_{x\in V}\Gamma(\sqrt{P_{t-s}f+\epsilon})m(x).
$$

Notice that $\left|e^{-2Ks}\right| \le \max\{1, e^{-2Kt}\}$ when $s \in [0,t]$. Combining with Lemma 8 and Step 1 in the proof of Theorem 1, we conclude that the above formal derivation is uniformly convergent to $H'(s)$ on $s \in (\delta, t-\delta)$ for any $0 < \delta < t$. Then, using the $CDE'(n,K)$ condition similar to Step 2 in the proof of Theorem 1, we obtain

$$
H'(s) \ge \frac{2}{n}e^{-2Ks}\sum_{x\in V}(P_{t-s}f+\epsilon)(\Delta\log\sqrt{P_{t-s}f+\epsilon})^2P_s\xi(x)m(x).
$$

Integrating the above inequality from 0 to t, and letting $\xi(x) = \delta_y(x)$, we have

$$\Gamma\left(\sqrt{P_t f+\epsilon}\right) \leq e^{-2Kt} P_t\left(\Gamma\left(\sqrt{f+\epsilon}\right)\right)-\frac{2}{n} \int_0^t e^{-2Ks} P_s\left((P_{t-s} f+\epsilon)\left(\Delta \log \sqrt{P_{t-s} f+\epsilon}\right)^2\right) ds.$$

Let $\{\epsilon_k\}_0^\infty$ be a positive sequence, and $\epsilon_k \to 0^+$ as $k \to \infty$. Replace ϵ with ϵ_k in the above inequality. By the local finiteness of G and Fatou's Lemma, we obtain

$$\begin{aligned}
& -\varliminf_{k \to \infty} \frac{2}{n} \int_0^t e^{-2Ks} P_s\left((P_{t-s} f+\epsilon_k)\left(\Delta \log \sqrt{P_{t-s} f+\epsilon_k}\right)^2\right) ds \\
& \leq -\frac{2}{n} \int_0^t e^{-2Ks} P_s\left(\varliminf_{k \to \infty}(P_{t-s} f+\epsilon_k)\left(\Delta \log \sqrt{P_{t-s} f+\epsilon_k}\right)^2\right) ds \\
& =-\frac{2}{n} \int_0^t e^{-2Ks} P_s\left((P_{t-s} f)\left(\Delta \log \sqrt{P_{t-s} f}\right)^2\right) ds.
\end{aligned}$$

Combining with (10) yields

$$\begin{aligned}
& \Gamma\left(\sqrt{P_t f}\right)=\lim_{k \to \infty} \Gamma\left(\sqrt{P_t f+\epsilon_k}\right) \\
& \leq \varlimsup_{k \to \infty} e^{-2Kt} P_t\left(\Gamma(\sqrt{f+\epsilon_k})\right)-\varliminf_{k \to \infty} \frac{2}{n} \int_0^t e^{-2Ks} P_s\left((P_{t-s} f+\epsilon_k)\left(\Delta \log \sqrt{P_{t-s} f+\epsilon_k}\right)^2\right) ds \\
& \leq e^{-2Kt} P_t\left(\Gamma(\sqrt{f})\right)-\frac{2}{n} \int_0^t e^{-2Ks} P_s\left(P_{t-s} f\left(\Delta \log \sqrt{P_{t-s} f}\right)^2\right) ds.
\end{aligned}$$

(2) $\Rightarrow$ (1) Let

$$L(t)=\Gamma(\sqrt{P_t f})-e^{-2Kt} P_t(\Gamma(\sqrt{f}))+\frac{1}{n} \int_0^t e^{-2Ks} P_s(P_{t-s} f(\Delta \log \sqrt{P_{t-s} f})^2) ds.$$

Notice that $L(0)=0$ and $L(t) \leq 0$ when $t>0$. Then, we have

$$\lim_{t \to 0^+} L'(t) \leq 0,$$

which implies the $CDE'(n,K)$ condition.

(2) $\Leftrightarrow$ (3) This follows from a density argument. □

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