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The Gravity Force Generated by a Non-Rotating Level Ellipsoid of Revolution with Low Eccentricity as a Series of Spherical Harmonics

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Abstract: The gravity force of a gravity field generated by a non-rotating level ellipsoid of revolution enclosing mass M is given as a solution of a partial differential equation along with a boundary condition of Dirichlet type. The partial differential equation is formulated herein on the basis of the behavior of spherical gravity fields. A classical solution to this equation is represented on the basis of spherical harmonics. The series representation of the solution is exploited in order to conduct a rigorous asymptotic analysis with respect to eccentricity. Finally, the Dirichlet boundary problem is solved for the case of an ellipsoid of revolution (spheroid) with low eccentricity. This has been accomplished on the basis of asymptotic analysis, which resulted in the determination of the coefficients participating in the spherical harmonics expansion. The limiting case of this series expresses the gravity force of a non-rotating sphere.

Keywords: gravity field; gravity force; equipotential surface; ellipsoid; spherical harmonics

MSC: 35A09; 35C10; 35J25; 33C05; 33C75

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1. Introduction

The gravity field which is under consideration is generated by a non-rotating ellipsoid of revolution with semiaxes (a, b), (with $a > b$).In our case, it has an equipotential surface (level ellipsoid).According to [1], this gravity field was first given by Pizzetti in 1894 and was further elaborated upon by C. Somigliana in 1929.Descriptions of this field can be found in several books, for example, in [2–4].

According to Xiong Li [3], although the Earth is not an exact ellipsoid, the equipotential ellipsoid furnishes a simple, consistent, and uniform reference system for all purposes of geodesy, as well as geophysics.

The gravity field of an ellipsoid is of fundamental practical importance because it is easy to handle mathematically, and the deviations of the actual gravity field from the ellipsoidal "theoretical" or "normal" field are small. This splitting of the Earth's gravity field into a "normal" and a remaining small "disturbing" or "anomalous" field considerably simplifies many problems, including the determination of the geoid (for geodesists) and the use of gravity anomalies to understand the Earth's interior (for geophysicists). In addition [5], this kind of gravity field plays a significant role in planetary geodesy since the shape of large bodies in the solar system can be well approximated by ellipsoids of revolutions (case of planets) or [6,7] by triaxial ellipsoids (case of many natural satellites).

Various relations which were formulated for the description of this field gravity force are given for the surface of the ellipsoid in several coordinate systems (see, for example, [8]) and for the outer space of the ellipsoid. Although these formulae are widely used, they are not solutions to some partial differential equation (for example, in contrast to the gravity potential, which is a solution of the Laplace equation).

In this work, we prove the fact that the gravity force constitutes the solution toa suitable boundary value problem consisting of a partial differential equation and a boundary condition of the Dirichlet type. This is exploitable in the sense that the spectral analysis of the boundary value problem offers the possibility to express the solution in terms of adequate eigenfunctions involving the well-known Legendre functions.

Having introduced the appropriate basis inspired by the boundary value problem, the gravity force acquires a very efficient representation. Via this representation, the unknown character of the problem has been transferred from the gravity force to the coefficients of expansion in terms of the structural elements related to the aforementioned partial differential equation.

The eccentricity of the ellipsoid, which is a small parameter, plays an important role in defining the deviation from the spherical case. Its particular involvement in the problem via the implication of spherical functions is convenient for the application of asymptotic techniques, establishing, in a rigorous manner, a robust determination of the aforementioned expansion coefficients.

2. Formulation of a Partial Differential Equation Related to Spherical Gravity Force

Let S be a sphere of radius R and mass M. Let W be the gravity potential of this gravity field. The equipotential surfaces of this field are spheres which have the same center as the sphere S.

Let (X, Y, Z) be a Cartesian system, and the equation of the sphere is

$$X^2 + Y^2 + Z^2 = R^2 \tag{1}$$

The gravity field outside of the sphere in Cartesian coordinates is a function:

$$\xi : V \to \Re^3 : (X,Y,Z) \to \xi(X,Y,Z) = (W_X(X,Y,Z), W_Y(X,Y,Z), W_Z(X,Y,Z)) \tag{2}$$

where W_X, W_Y, and W_Z are the first-order partial derivatives of the potential W, and V is the part of the three dimensional space which is outside of the sphere. The family of the equipotential surfaces of this gravity field is spheres and can be represented as

$$W(X,Y,Z) = w \quad , \quad 0 < w \leq w_0 \tag{3}$$

A point P with coordinates (X_P,Y_P,Z_P) is on an arbitrary equipotential surface with gravity potential W_P. This equipotential surface is also referred to as $W = W_P$.

In addition, let (x, y, z) be a Cartesian system with its center at point P. The x-axis is tangent to the meridian of the equipotential surface (pointing to the North Pole), the z-axis is vertical to the tangent plane of the equipotential surface at point P (pointing outwards), and y-axis makes the system right-handed. The rotation matrix between the axles X, Y, Z and x, y, z is as follows [9]:

$$\begin{bmatrix} X - X_P \\ Y - Y_P \\ Z - Z_P \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\lambda & -\sin\lambda & \sin\theta\cos\lambda \\ \cos\theta\sin\lambda & \cos\lambda & \sin\theta\sin\lambda \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}_P \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$

The angles θ and λ are spherical angles, i.e.,

$$\begin{aligned} X &= r\sin\theta\cos\lambda \\ Y &= r\sin\theta\sin\lambda \\ Z &= r\cos\theta \end{aligned} \tag{5}$$

In this system, $W_z(P)$ is not zero, and from the implicit function theorem, we knowthat the vector equation of the equipotential surface $W = W_P$ around point P is

$$\bar{s} : (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1) \to \Re^3 : (x,y) \to \bar{s}(x,y) = (x, y, z(x,y)) \tag{6a}$$

The first-order partial derivatives of the above vector equation are

$$\bar{s}_x(x,y) = \left(1, 0, -\frac{W_x}{W_z}\right) \tag{6b}$$

$$\bar{s}_y(x,y) = \left(0, 1, -\frac{W_y}{W_z}\right) \tag{6c}$$

The fundamental elements of the first kind [10] of the equipotential surface are defined to be

$$E(x,y) \equiv E = \langle \bar{s}_x, \bar{s}_x \rangle = \langle \left(1, 0, -\frac{W_x}{W_z}\right), \left(1, 0, -\frac{W_x}{W_z}\right) \rangle = \frac{W_x^2 + W_z^2}{W_z^2} \tag{7a}$$

$$F(x,y) \equiv F = \langle \bar{s}_x, \bar{s}_y \rangle = \langle \left(1, 0, -\frac{W_x}{W_z}\right), \left(1, 0, -\frac{W_y}{W_z}\right) \rangle = \frac{W_x W_y}{W_z^2} \tag{7b}$$

$$G(x,y) \equiv G = \langle \bar{s}_y, \bar{s}_y \rangle = \langle \left(1, 0, -\frac{W_y}{W_z}\right), \left(1, 0, -\frac{W_y}{W_z}\right) \rangle = \frac{W_y^2 + W_z^2}{W_z^2} \tag{7c}$$

The normal vector and the unit normal vector are

$$\overline{N}_C(x,y) = \frac{\partial \bar{s}}{\partial x} \times \frac{\partial \bar{s}}{\partial y} = \left(\frac{W_x}{W_z}, \frac{W_y}{W_z}, 1\right) \tag{8}$$

$$\overline{N}(x,y) = |W_z| \frac{1}{\sqrt{W_x^2 + W_y^2 + W_z^2}} \overline{N}_C(x,y) = |W_z| \frac{1}{g(x,y)} \overline{N}_C(x,y) \tag{9}$$

The gravity force g in the above relation is equal to

$$g(x,y) = \sqrt{W_x^2 + W_y^2 + W_z^2} \tag{10a}$$

The fundamental quantities of the second kind L, M, and N of the equipotential surface describe the projections of the vectors

$$\bar{s}_{xx} \quad , \quad \bar{s}_{xy} \quad , \quad \bar{s}_{yy} \tag{10b}$$

on the axis of the unit normal vector of the equipotential surface. The above vectors are the second-order partial derivatives of the vector equation of the equipotential surface. The fundamental quantities L, M, and N [10] are defined to be

$$L(x,y) = \langle \overline{N}, \bar{s}_{xx} \rangle = |W_z| \frac{-W_{xx} W_z^2 + 2W_{xz} W_x W_z - W_{zz} W_x^2}{W_z^3 g} \tag{11}$$

$$M(x,y) = \langle \overline{N}, \bar{s}_{xy} \rangle = |W_z| \frac{W_y W_z W_{xz} + W_x W_z W_{yz} - W_{xy} W_z^2 - W_{zz} W_x W_y}{W_z^3 g} \tag{12}$$

$$N(x,y) = \langle \overline{N}, \bar{s}_{yy} \rangle = |W_z| \frac{-W_{yy} W_z^2 + 2W_{yz} W_y W_z - W_{zz} W_y^2}{W_z^3 g} \tag{13}$$

The expressions of the fundamental elements and gravity force at point P in (x, y, z) coordinates are

$$E(0,0) \equiv E(P) = 1 \quad , \quad F(P) = 0 \quad , \quad G(P) = 1 \tag{14}$$

$$L(P) = -\left.\frac{|W_z|}{W_z}\right|_P \left.\frac{W_{xx}}{g}\right|_P \tag{15}$$

$$M(P) = -\frac{|W_z|}{W_z}\bigg|_P \frac{W_{xy}}{g}\bigg|_P \tag{16}$$

$$N(P) = -\frac{|W_z|}{W_z}\bigg|_P \frac{W_{yy}}{g}\bigg|_P \tag{17}$$

$$g(P) = |W_z(P)| \tag{18}$$

The gravity vector at point P has the opposite direction of the z-axis; therefore, $W_z(P) < 0$. Hence, the above three formulae become

$$L(P) = \frac{W_{xx}}{g}\bigg|_P \tag{19}$$

$$M(P) = \frac{W_{xy}}{g}\bigg|_P \tag{20}$$

$$N(P) = \frac{W_{yy}}{g}\bigg|_P \tag{21}$$

Since $F(P) = 0$, the parametric lines $x = 0$ and $y = 0$ are vertical to one another. The equipotential surface is a spherical surface; therefore, all directions are principal directions, i.e.,

$$M(P) = 0 \Rightarrow W_{xy}(P) = 0 \tag{22}$$

The curvatures along the x-direction and y-direction, respectively, are

$$k_1(P) = \frac{W_{xx}}{g}\bigg|_P \tag{23}$$

$$k_2(P) = \frac{W_{yy}}{g}\bigg|_P \tag{24}$$

The first-order partial derivatives of the gravity force g (see Equation (10a)) at point P in the (x, y, z) system are

$$g_x(P) = \frac{1}{g(P)}(W_x W_{xx} + W_y W_{xy} + W_z W_{xz})_P \tag{25}$$

$$g_y(P) = \frac{1}{g(P)}(W_x W_{xy} + W_y W_{yy} + W_z W_{yz})_P \tag{26}$$

$$g_z(P) = \frac{1}{g(P)}(W_x W_{xz} + W_y W_{yz} + W_z W_{zz})_P \tag{27}$$

The second-order partial derivatives of the gravity force g are defined as follows, if Equations (10) and (18) are taken into account:

$$g_{xx}(P) = \left\{ [W_{xx}^2 + W_{xz}^2 + W_z W_{xxz}]\frac{1}{g} - \frac{W_{xz}^2}{g} \right\}_P \tag{28}$$

$$g_{yy}(P) = \left\{ [W_{yy}^2 + W_{yz}^2 + W_z W_{yyz}]\frac{1}{g} - \frac{W_{yz}^2}{g} \right\}_P \tag{29}$$

$$g_{zz}(P) = \left\{ [W_{xz}^2 + W_{yz}^2 + W_{zz}^2 + W_z W_{zzz}]\frac{1}{g} - \frac{W_{zz}^2}{g} \right\}_P \tag{30}$$

Adding the three above relations and taking into consideration that W is an harmonic function,

$$W_{xx} + W_{yy} + W_{zz} = 0 \tag{31}$$

$$W_{xxz} + W_{yyz} + W_{zzz} = 0 \tag{32}$$

we find that

$$\begin{gathered} g_{xx}(P) + g_{yy}(P) + g_{zz}(P) = \Big\{ [W_{xx}^2 + W_{xz}^2 + W_z W_{xxz}]\tfrac{1}{g} - \tfrac{W_{xz}^2}{g} + \\ + [W_{yy}^2 + W_{yz}^2 + W_z W_{yyz}]\tfrac{1}{g} - \tfrac{W_{yz}^2}{g} + [W_{xz}^2 + W_{yz}^2 + W_{zz}^2 + W_z W_{zzz}]\tfrac{1}{g} - \tfrac{W_{zz}^2}{g} \Big\}_P \end{gathered} \tag{33}$$

or

$$g_{xx}(P) + g_{yy}(P) + g_{zz}(P) = \left(\frac{W_{xx}^2}{g} + \frac{W_{yy}^2}{g} + \frac{W_{xz}^2}{g} + \frac{W_{yz}^2}{g} \right)_P \tag{34}$$

Along the plumbline which passes at point P, it holds that

$$\left. \frac{dx}{dt} \right|_P = W_x(P) = 0 \tag{35}$$

$$\left. \frac{dy}{dt} \right|_P = W_y(P) = 0 \tag{36}$$

$$\left. \frac{dz}{dt} \right|_P = W_z(P) \tag{37}$$

$$\left. \frac{d^2x}{dt^2} \right|_P = (W_{xx}W_x + W_{xy}W_y + W_{xz}W_z)_P = W_{xz}(P)W_z(P) \tag{38}$$

$$\left. \frac{d^2y}{dt^2} \right|_P = (W_{xy}W_x + W_{yy}W_y + W_{yz}W_z)_P = W_{yz}(P)W_z(P) \tag{39}$$

$$\left. \frac{d^2z}{dt^2} \right|_P = (W_{xz}W_x + W_{yz}W_y + W_{zz}W_z)_P = W_{zz}(P)W_z(P) \tag{40}$$

The curvature of the plumbline at point P is equal to

$$k_{pl}(P) = \frac{\| \frac{d\bar{r}}{dt}\big|_P \times \frac{d^2\bar{r}}{dt^2}\big|_P \|}{\| \frac{d\bar{r}}{dt}\big|_P \|^3} = \left. \frac{|W_{yz}|}{W_z} \right|_P \tag{41}$$

Since the plumblines of the gravity field ξ are straight lines, then $k_{pl}(P) = 0$, and, consequently, $W_{xz}(P) = W_{yz}(P) = 0$. The relation (34) now becomes

$$(g_{xx} + g_{yy} + g_{zz})_P = (k_1^2 + k_2^2 + k_{pl}^2)_P g(P) \tag{42}$$

or

$$(\nabla^2 g)(P) - (k_1^2 + k_2^2)_P g(P) = 0 \tag{43}$$

However, the value of the curvature along the x-direction and y-direction of the spherical equipotential surface is equal to $1/R$. Hence, Equation (43) becomes

$$(\nabla^2 g)(P) - \frac{2}{R^2} g(P) = 0 \tag{44}$$

The above relation holds for point P. This relation can be considered as an operator, i.e.,

$$\nabla^2 - \frac{2}{r^2} \tag{45}$$

The above operator is invariant under rotations; therefore, the following partial differential equation holds:

$$\nabla^2 g - \frac{2}{r^2} g = 0 \tag{46}$$

where g is the gravity force.

3. Solution to the Partial Differential Equation

Expressing the above partial differential equation in spherical coordinates, (r, θ', λ) becomes

$$g_{rr} + \frac{2}{r} g_r + \frac{1}{r^2} g_{\theta\theta} + \frac{\tan\theta'}{r^2} g_\theta + \frac{1}{r^2 \cos^2\theta'} g_{\lambda\lambda} - \frac{2}{r^2} g = 0 \tag{47}$$

$$\begin{aligned} x &= r\cos\theta' \cos\lambda \\ y &= r\cos\theta' \sin\lambda \\ z &= r\sin\theta' \end{aligned} \tag{48}$$

A classical solution to the above partial differential equation can be found with the method of separating variables [11], hence:

$$g(r, \theta, \lambda) = F_1(r) F_2(\theta') F_3(\lambda) \tag{49}$$

and substituting to the equation, we have the following system:

$$F_{3,\lambda\lambda} + m^2 F_3 = 0 \tag{50}$$

$$\cos^2\theta' F_{2,\theta\theta} - 2\sin\theta' F_{2,\theta} + \left(n(n+1) - \frac{m^2}{\cos^2\theta'} \right) F_2 = 0 \tag{51}$$

$$F_{1,rr} + \frac{2}{r} F_{1,r} - \left[\frac{2}{r^2} + \frac{n(n+1)}{r^2} \right] F_1 = 0 \tag{52}$$

The first two differential equations have the following solutions:

$$F_3(\lambda) = c_{31} \cos m\lambda + c_{32} \sin m\lambda \tag{53}$$

$$F_2(\theta) = c_{21} P_{nm}(\sin\theta') + c_{22} Q_{nm}(\sin\theta') \tag{54}$$

The third differential equation is an Euler equation, and in order to find the general solution, we set $F_1 = r^a$ and the equation becomes

$$a(a-1) r^{a-2} + 2a r^{a-2} - 2\left[\frac{n(n+1)}{2} + 1 \right] r^{a-2} = 0 \tag{55}$$

Since r is not zero, the unknown a will be determined from the second-order algebraic equation

$$a^2 + a - 2\left[\frac{n(n+1)}{2} + 1 \right] = 0 \tag{56}$$

$$a_1 = \frac{-1 + \sqrt{9 + 4n(n+1)}}{2} \tag{57}$$

$$a_2 = \frac{-1 - \sqrt{9 + 4n(n+1)}}{2} \tag{58}$$

Hence, the solution to this Euler equation is

$$F_1(r) = c_1 r^{\frac{-1+\sqrt{9+4n(n+1)}}{2}} + c_2 r^{\frac{-1-\sqrt{9+4n(n+1)}}{2}} \quad (59)$$

We combine Equations (53), (54) and (59) to construct the separate eigensolutions $F_1(r)$, $F_2(\theta')$,and $F_3(\lambda)$. The general solution to Equation (47) is expanded upon in terms of these eigensolutions as follows:

$$g(r,\theta',\lambda) = \sum_{n=0}^{+\infty} r^{\frac{-1+\sqrt{9+4n(n+1)}}{2}} \sum_{m=0}^{n} [a_{nm} P_{nm}(\sin\theta')\cos m\lambda + b_{nm} P_{nm}(\sin\theta')\sin m\lambda] \quad (60)$$

$$g(r,\theta',\lambda) = \sum_{n=0}^{+\infty} r^{\frac{-1-\sqrt{9+4n(n+1)}}{2}} \sum_{m=0}^{n} [a_{nm} P_{nm}(\sin\theta')\cos m\lambda + b_{nm} P_{nm}(\sin\theta')\sin m\lambda] \quad (61a)$$

The coefficients α_{nm} and b_{nm} are real numbers. Making all coefficients equal to zero except the term a_{00}, we have two cases: In the first case, a_{00} = GM/a^3, and Equation (60) becomes

$$g_{\text{int}}(r,\theta',\lambda) = \frac{GM}{a^3} r \quad (61b)$$

which describes the gravity force inside of a sphere of radius a and mass M. In the second case, a_{00} = GM, and Equation (61) becomes

$$g_{ext}(r,\theta',\lambda) = \frac{GM}{r^2} \quad (61c)$$

which describes the gravity force outside of a sphere of radius a and mass M. On the surface of the sphere, Equations (61b) and (61c) give the same result, and, therefore, are compatible:

$$g_{ext}(a,\theta',\lambda) = g_{\text{int}}(a,\theta',\lambda) = \frac{GM}{a^2} \quad (61d)$$

4. The Gravity Force Generated by a Non-Rotating Ellipsoid of Revolution as a Solution of a Dirichlet Problem of the Suggested Partial Differential Equation

The level ellipsoid has constant gravity potential W_0 on its surface. This surface is described in geodetic coordinates [12] as

$$(x(\phi,\lambda), y(\phi,\lambda), z(\phi,\lambda)) = \frac{1}{\sqrt{1-e^2\sin^2\phi}}(a\cos\phi\cos\lambda, a\cos\phi\sin\lambda, a(1-e^2)\sin\phi) \quad (62)$$

where e is the ellipsoid's first eccentricity. The gravity force on the surface of the ellipsoid is equal to [2]:

$$g(\phi) = \frac{ag_a\cos^2\phi + bg_b\sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \quad (63)$$

The symbols "g_α" and "g_b" stand for the value of gravity force on the equator and at the poles, respectively. Their values are equal to

$$g_a = \frac{GM}{ab} \quad (64)$$

$$g_b = \frac{GM}{a^2} \quad (65)$$

From Equation (62), we know that

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \sqrt{\cos^2 \phi + (1 - e^2)^2 \sin^2 \phi} \tag{66}$$

From Equations (62) and (66), we derive the relationship between spherical latitude θ' and geodetic latitude φ:

$$\cos^2 \phi = \frac{(1 - e^2)^2 \cos^2 \theta'}{\sin^2 \theta' + (1 - e^2)^2 \cos^2 \theta'} \tag{67}$$

$$\sin^2 \phi = \frac{\sin^2 \theta'}{\sin^2 \theta' + (1 - e^2)^2 \cos^2 \theta'} \tag{68}$$

Hence, the gravity force on the surface S of the ellipsoid becomes

$$g(\theta) = \frac{a(1 - e^2)^2 g_a \cos^2 \theta' + b g_b \sin^2 \theta'}{\sqrt{\sin^2 \theta' + (1 - e^2)^2 \cos^2 \theta'} \sqrt{a^2 (1 - e^2)^2 \cos^2 \theta' + b^2 \sin^2 \theta'}} \tag{69}$$

The Dirichlet problem for the determination of the gravity force g is

$$\nabla^2 g - \frac{2}{r^2} g = 0 \tag{70}$$

$$g|_S(\theta) = g(\theta) = \frac{a(1 - e^2)^2 g_a \cos^2 \theta' + b g_b \sin^2 \theta'}{\sqrt{\sin^2 \theta' + (1 - e^2)^2 \cos^2 \theta'} \sqrt{a^2 (1 - e^2)^2 \cos^2 \theta' + b^2 \sin^2 \theta'}} \tag{71}$$

$$\begin{aligned} x &= r \cos \theta' \\ z &= r \sin \theta' \end{aligned} \tag{72}$$

Replacing Equations (67) and (68) in Equation (66),we obtain the following relation:

$$r = \frac{a\sqrt{1 - e^2}}{\sqrt{1 - e^2 \cos^2 \theta'}} \tag{73}$$

Let D = (x_0, z_0) be an arbitrary point on the ellipse (see Figure 1). We consider point Z with coordinates $(x_0 e^2, 0)$, whose position is illustrated in Figure 1 (e is the first eccentricity of the ellipse).

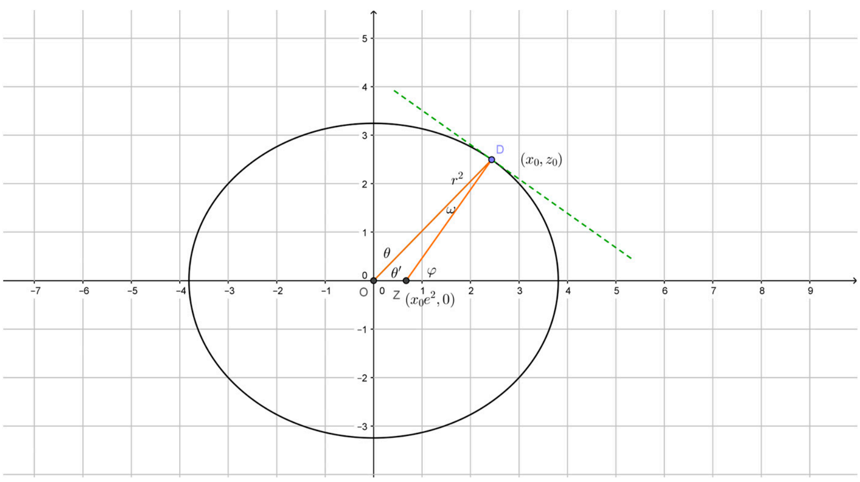

Figure 1. Let γ be an ellipse with semimajor axis a and semiminor axis b.

From the triangle ZOD, it holds that $\varphi = \theta' + \omega$, and

$$\sin\phi = \sin(\theta' + \omega) = \sin\theta' \cos\omega + \cos\theta' \sin\omega \tag{74}$$

Line segment OD is equal to r. Line segment DZ is equal to

$$DZ = \sqrt{x_0^2 (1-e^2)^2 + z_0^2} = r\sqrt{(1-e^2)^2 \cos^2\theta' + \sin^2\theta'} \tag{75}$$

In addition,

$$OZ^2 = OD^2 + DZ^2 - 2OD \cdot DZ \cos\omega \Rightarrow \cos\omega = \frac{OD^2 + DZ^2 - OZ^2}{2OD \cdot DZ} \tag{76}$$

$$\begin{aligned}\cos\omega &= \frac{r^2 + r^2\left[(1-e^2)^2 \cos^2\theta' + \sin^2\theta\prime\right] - e^4 r^2 \cos^2\theta'}{2r^2\sqrt{(1-e^2)^2 \cos^2\theta' + \sin^2\theta'}} = \\ &= \frac{1 + \cos^2\theta' + e^4\cos^2\theta' - 2e^2\cos^2\theta' + \sin^2\theta' - e^4\cos^2\theta'}{2\sqrt{(1-e^2)^2 \cos^2\theta' + \sin^2\theta'}} = \frac{(1 - e^2\cos^2\theta')}{\sqrt{(1-e^2)^2 \cos^2\theta' + \sin^2\theta'}}\end{aligned} \tag{77}$$

Hence

$$\cos\omega = \frac{1 - e^2\cos^2\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}} \tag{78}$$

and

$$\sin\omega = \frac{e^2\sin\theta'\cos\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}} \tag{79}$$

Therefore,

$$\begin{aligned}&P_{2n}(\sin\phi) = P_{2n}(\sin(\theta' + \omega)) = P_{2n}(\sin\theta'\cos\omega + \cos\theta'\sin\omega) \\ &= P_{2n}\left(\frac{1 - e^2\cos^2\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}}\sin\theta' + \frac{e^2\sin\theta'\cos\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}}\cos\theta'\right) \\ &= P_{2n}\left(\frac{\sin\theta' - e^2\cos^2\theta'\sin\theta' + e^2\sin\theta'\cos^2\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}}\right) = P_{2n}\left(\frac{\sin\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}}\right)\end{aligned} \tag{80}$$

Equation (73) can also be written as

$$r = \frac{b}{\sqrt{1 - e^2\cos^2\theta'}} \tag{81}$$

$$\begin{aligned}&\frac{\sin\theta'}{\sqrt{1 - e^2(2-e^2)\cos^2\theta'}} = \sin\theta'\left[1 - e^2(2-e^2)\cos^2\theta'\right]^{-\frac{1}{2}} \\ &= \sin\theta'\left[1 - \frac{e^2(2-e^2)}{2}\cos^2\theta' + \frac{3e^4(2-e^2)^2}{8}\cos^4\theta' - \frac{15e^6(2-e^2)^4}{36}\cos^6\theta' + \ldots\right]\end{aligned} \tag{82}$$

$$\begin{aligned}&\frac{b}{\sqrt{1 - e^2\cos^2\theta'}} = b\left(1 - e^2\cos^2\theta'\right)^{-\frac{1}{2}} \\ &= b\left(1 + \frac{e^2}{2}\cos^2\theta' - \frac{3e^4}{8}\cos^4\theta' + \frac{15e^6}{36}\cos^6\theta' - \frac{105e^8}{192}\cos^8\theta' - \ldots\right)\end{aligned} \tag{83}$$

We intend to find a solution to this Dirichlet problem, making a first-level approximation of gravity force g while keeping only terms of e^2. The gravity field of the level ellipsoid has rotational symmetry and equatorial plane symmetry. Therefore, the solution contains only Legendre polynomials of even degrees. For small eccentricities, we knowthat (we set $\varepsilon^2 = e^2(2 - e^2)$)

$$P_{2n}(\sin\phi) = P_{2n}\left(\sin\theta' - \frac{\varepsilon^2}{2}\sin\theta'\cos^2\theta'\right) \tag{84}$$

and

$$P_{2n}\left(\sin\theta' - \frac{\varepsilon^2}{2}\sin\theta'\cos^2\theta'\right) = P_{2n}(\sin\theta') + P_{2n}{}'(\sin\theta')\frac{\varepsilon^2}{2}\sin\theta'\cos^2\theta' \tag{85}$$

The prime on Legendre polynomials means derivation with respect to the argument θ'. Equation (83) becomes

$$\frac{b}{\sqrt{1-e^2\cos^2\theta'}} = b\left(1+\frac{e^2}{2}\cos^2\theta' - \frac{3e^4}{8}\cos^4\theta'\right) \tag{86}$$

The crucial Equation (69) must be subjected to similar approximation with respect to eccentricity. We remark that

$$[\sin^2\theta' + (1-e^2)^2\cos^2\theta']^{-\frac{1}{2}} = (\sin^2\theta' + \cos^2\theta' - \varepsilon^2\cos^2\theta')^{-\frac{1}{2}} = 1 - \frac{\varepsilon^2}{2}\cos^2\theta' \tag{87}$$

$$\begin{aligned}&[a^2(1-e^2)^2\cos^2\theta' + b^2\sin^2\theta']^{-\frac{1}{2}} = b^{-1}[(1-e^2)\cos^2\theta' + \sin^2\theta']^{-\frac{1}{2}}\\ &= b^{-1}(1-e^2\cos^2\theta')^{-\frac{1}{2}} = \tfrac{1}{b}\left(1+\tfrac{e^2}{2}\cos^2\theta' - \tfrac{3e^4}{8}\cos^4\theta'\right)\end{aligned} \tag{88}$$

$$a(1-e^2)^2 g_a\cos^2\theta' + bg_b\sin^2\theta' = ag_a\cos^2\theta' + bg_b\sin^2\theta' - \varepsilon^2 ag_a\cos^2\theta' \tag{89}$$

The Dirichlet condition now becomes

$$\begin{aligned}g(\theta') &= \left(1-\tfrac{\varepsilon^2}{2}\cos^2\theta'\right)\left(1+\tfrac{e^2}{2}\cos^2\theta' - \tfrac{3e^4}{8}\cos^4\theta'\right)\\ &\quad\cdot[(1-\varepsilon^2)\tfrac{1}{\sqrt{1-e^2}}g_a\cos^2\theta' + g_b\sin^2\theta]\\ &= \left(1+\tfrac{e^2}{2}\cos^2\theta' - \tfrac{3e^4}{8}\cos^4\theta' - \tfrac{\varepsilon^2}{2}\cos^2\theta'\right)\left[(1-\varepsilon^2)\left(1+\tfrac{e^2}{2}-\tfrac{3e^4}{8}\right)g_a\cos^2\theta' + g_b\sin^2\theta\right]\end{aligned} \tag{90}$$

Replacing $\varepsilon^2 = -e^4 + 2e^2$, we have that

$$\begin{aligned}&\left(1+\tfrac{e^2}{2}\cos^2\theta' - \tfrac{3e^4}{8}\cos^4\theta' - \tfrac{\varepsilon^2}{2}\cos^2\theta'\right)\left[(1-\varepsilon^2)\left(1+\tfrac{e^2}{2}-\tfrac{3e^4}{8}\right)g_a\cos^2\theta' + g_b\sin^2\theta\right]\\ &= \left(1+\tfrac{e^2}{2}\cos^2\theta' - \tfrac{3e^4}{8}\cos^4\theta' - \tfrac{-e^4+2e^2}{2}\cos^2\theta'\right)\cdot\\ &\left[(1+e^4-2e^2)\left(1+\tfrac{e^2}{2}-\tfrac{3e^4}{8}\right)g_a\cos^2\theta' + g_b\sin^2\theta\right] =\\ &= \left(1-\tfrac{e^2}{2}\cos^2\theta' - \tfrac{3e^4}{8}\cos^4\theta' + \tfrac{e^4}{2}\cos^2\theta'\right)\cdot\\ &[(1+e^4-2e^2)g_a\cos^2\theta' - e^2(e^2+2)g_a\cos^2\theta' + g_b\sin^2\theta]\end{aligned} \tag{91}$$

Therefore,

$$\begin{aligned}&g(\theta') = (1+e^4-2e^2)g_a\cos^2\theta' - e^2(e^2+2)g_a\cos^2\theta' + g_b\sin^2\theta\\ &-\tfrac{e^2}{2}(1-2e^2)g_a\cos^4\theta' + e^4g_a\cos^4\theta' - \tfrac{e^2}{2}g_b\cos^2\theta'\sin^2\theta' - \tfrac{3e^4}{8}g_a\cos^6\theta'\\ &-\tfrac{3e^4}{8}g_b\cos^4\theta'\sin^2\theta' + \tfrac{e^4}{2}g_a\cos^4\theta' + \tfrac{e^4}{2}g_b\cos^2\theta'\sin^2\theta'\end{aligned} \tag{92}$$

Keeping only the e^2 terms, we transform the above relation as follows:

$$\begin{aligned}&g(\theta') = (1-2e^2)g_a\cos^2\theta' - 2e^2g_a\cos^2\theta' + g_b\sin^2\theta - \tfrac{e^2}{2}g_a\cos^4\theta'\\ &-\tfrac{e^2}{2}g_b\cos^2\theta'\sin^2\theta' =\\ &g_a\cos^2\theta' + g_b\sin^2\theta - 4e^2g_a\cos^2\theta' - \tfrac{e^2}{2}g_a\cos^4\theta' - \tfrac{e^2}{2}g_b\cos^2\theta'\sin^2\theta'\end{aligned} \tag{93}$$

The solution is a series expansion of the form

$$\begin{aligned}&g(r,\theta') = \sum_{n=0}^{+\infty} a_{2n}r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}P_{2n}\left(\sin\theta' - \tfrac{\varepsilon^2}{2}\sin\theta'\cos^2\theta'\right)\\ &= \sum_{n=0}^{+\infty} a_{2n}r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}P_{2n}(\sin\theta') + \tfrac{\varepsilon^2}{2}\sum_{n=0}^{+\infty} a_{2n}r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}P_{2n}{}'(\sin\theta')\sin\theta'\cos^2\theta'\end{aligned} \tag{94}$$

On the surface of the ellipsoid, it holds that (again, we keep only the e^2 terms and replace $\varepsilon^2 = -e^4 + 2e^2$ on the right-hand side of the equation)

$$\begin{aligned}&g_a \cos^2\theta' + g_b \sin^2\theta - 4e^2 g_a \cos^2\theta' - \tfrac{e^2}{2} g_a \cos^4\theta' - \tfrac{e^2}{2} g_b \cos^2\theta' \sin^2\theta' \\ &= \sum_{n=0}^{+\infty} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} P_{2n}(\sin\theta') + e^2 \sum_{n=0}^{+\infty} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} P_{2n}{\prime}(\sin\theta') \sin\theta' \cos^2\theta\end{aligned} \tag{95}$$

We intend to determine the coefficients a_{2n}. Integrating both parts and substituting the radial distance from the e^2 approximation of Equation (86), the above relation becomes

$$\begin{aligned}&\int_{-1}^{1} [g_a \cos^2\theta' + g_b \sin^2\theta - 4e^2 g_a \cos^2\theta' - \tfrac{e^2}{2} g_a \cos^4\theta' - \tfrac{e^2}{2} g_b \cos^2\theta' \sin^2\theta'] \\ &\cdot P_{2m}(\sin\theta') d(\sin\theta') = \int_{-1}^{1} \sum_{n=0}^{+\infty} a_{2n} \left[b\left(1 - e^2\cos^2\theta'\right)^{-\frac{1}{2}} \right]^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} \\ &\cdot [P_{2n}(\sin\theta') + e^2 P_{2n}{}'(\sin\theta') \sin\theta' \cos^2\theta] P_{2m}(\sin\theta') d(\sin\theta')\end{aligned} \tag{96}$$

We first determine [13] the integrals involving the terms $g_a\cos^2\theta'$ and $g_b\sin^2\theta'$on the left-hand side of Equation (96). This material is shown in Appendix A.

For a chosen truncation value $n = n_0$, Equation (96) can be written in a matrix form:

$$(A + k^2 B) X = C \tag{97}$$

The involved terms are those for which it holds that $n = m - 1$, $n = m$, and $n = m + 1$. Establishing

$$\hat{a}_{2n} = \frac{a_{2n}}{b^{\frac{1+\sqrt{9+4n(n+1)}}{2}}} \tag{98}$$

the matrix A is a diagonal matrix of the form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & A_{22} & 0 & 0 & \ldots & 0 \\ 0 & 0 & A_{33} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & A_{n_0 n_0} \end{bmatrix} \tag{99}$$

Along a column m = const. and along a line n = const., the A_{ii} terms (i = 1,2,3, ... , n_0) are the fundamental terms of the expansions and constitute independent terms of the eccentricity (see Equations (A20), (A25) and (A28)). Matrix B is a matrix of the form

$$B = k^2 \begin{bmatrix} B_{11} & B_{12} & 0 & 0 & \ldots & 0 \\ B_{21} & B_{22} & B_{23} & 0 & \ldots & 0 \\ 0 & B_{32} & B_{33} & B_{34} & \ldots & 0 \\ 0 & 0 & B_{43} & B_{44} & B_{45} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & B_{n_0 n_0 - 1} \\ 0 & 0 & \ldots & 0 & B_{n_0 - 1 n_0} & B_{n_0 n_0} \end{bmatrix} \tag{100}$$

Matrix B, in its original form, has a very interesting structure. The principal diagonal is built again with terms which are eccentricity insensitive, while the adjacent diagonals have elements of order $O(k^{2j})$ (see Equations (A24)–(A33)), where j refers to the diagonal numbering (enumeration away from the principal one).

Omitting all terms which offer $O(k^4)$ contribution, the final form of matrix B is a diagonal matrix. The matrix X is of the form

$$\mathrm{X} = \begin{bmatrix} \hat{a}_0 & \hat{a}_2 & \hat{a}_4 & \hat{a}_6 & \ldots & \hat{a}_{2n_0} \end{bmatrix}^T \tag{101}$$

The matrix C is of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & \dots & c_{n_0} \end{bmatrix}^T \tag{102}$$

The term c_0 involves the already-determined integral of Equation (96) for m = 0, i.e., for the integral

$$c_0 = \int_{-1}^{1} [g_a \cos^2\theta' + g_b \sin^2\theta - 4e^2 g_a \cos^2\theta' - \tfrac{e^2}{2} g_a \cos^4\theta' - \tfrac{e^2}{2} g_b \cos^2\theta' \sin^2\theta'] \, d(\sin\theta') \tag{103}$$

Term l_1 contains another family of calculated integrals:

$$\begin{aligned} c_1 = \int_{-1}^{1} & [g_a \cos^2\theta' + g_b \sin^2\theta - 4e^2 g_a \cos^2\theta' - \tfrac{e^2}{2} g_a \cos^4\theta' - \tfrac{e^2}{2} g_b \cos^2\theta' \sin^2\theta'] \\ & \cdot P_2(\sin\theta') + e^2(g_a \cos^2\theta' + g_b \sin^2\theta) P_2{}'(\sin\theta') \sin\theta' \cos^2\theta' d(\sin\theta') \end{aligned} \tag{104}$$

which appear on the left-hand side of Equation(96) for m = 1, and so on for the rest of the coefficients c_i. In conclusion, we have the following system:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & A_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & A_{33} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A_{n_0 n_0} \end{bmatrix} + \right.$$
$$\left. +k^2 \begin{bmatrix} B_{11} & B_{12} & 0 & 0 & \dots & 0 \\ B_{12} & B_{22} & B_{23} & 0 & \dots & 0 \\ 0 & B_{23} & B_{33} & B_{34} & \dots & 0 \\ 0 & 0 & B_{43} & B_{44} & B_{45} & \dots \\ \dots & \dots & \dots & \dots & \dots & B_{n_0-1\,,\,n_0} \\ 0 & 0 & \dots & 0 & B_{n_0\,,n_0-1} & B_{n_0\,,\,n_0} \end{bmatrix} \right) \begin{bmatrix} \hat{a}_0 \\ \hat{a}_2 \\ \hat{a}_4 \\ \hat{a}_6 \\ \dots \\ \hat{a}_{2n_0} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \dots \\ c_{n_0} \end{bmatrix} \tag{105}$$

We suppose that b > 1, k^2 << 1 (since e^2 << 1); therefore, the above system can be solved. It holds that

$$\begin{aligned} & (A + k^2 B)X = C \Rightarrow [A(I + k^2 A^{-1} B)]X = C \Rightarrow X = [A(I + k^2 A^{-1} B)]^{-1} C \\ & \Rightarrow X = (I - k^2 A^{-1} B) A^{-1} C \Rightarrow X = (A^{-1} - k^2 A^{-1} B A^{-1}) C \end{aligned} \tag{106}$$

The above relation gives the coefficients in Equation (101) up to n = n_0. It is worth mentioning that in the limiting case for which e^2 = 0 (b = a), Equation (96) becomes

$$\int_{-1}^{1} g_a P_{2m}(\sin\theta') d(\sin\theta') = \sum_{n=0}^{+\infty} a^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} a_{2n} \int_{-1}^{1} P_{2n}(\sin\theta') P_{2m}(\sin\theta') d(\sin\theta') \tag{107}$$

For n = 0 and m = 0,

$$\int_{-1}^{1} g_a P_0(\sin\theta') d(\sin\theta') = 2g_a = \frac{2GM}{a^2} \tag{108}$$

$$\sum_{n=0}^{+\infty} a^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} a_{2n} \int_{-1}^{1} P_{2n}(\sin\theta') P_{2m}(\sin\theta') d(\sin\theta') = \frac{1}{a^2} a_0 \int_{-1}^{1} d(\sin\theta') = \frac{2a_0}{a^2} \tag{109}$$

Therefore,

$$\frac{2GM}{a^2} = \frac{2a_0}{a^2} \Rightarrow a_0 = GM \tag{110}$$

Making all other coefficients a_{2n} (n = 1, 2, 3, ...) equal to zero, we end up with the gravity force of a sphere with mass M and radius a.

$$g(r, \theta') = \frac{GM}{a^2} \tag{111}$$

From system (105), we have the degenerate case

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \frac{a_0}{a^2} \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{GM}{a^2} \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \tag{112}$$

Since the coefficients a_{2n} (n = 1, 2, 3, ... , n_0) are known, the desired expression for the gravity force of the non-rotating level ellipsoid of revolution is (see Equations (61a), (86) and (94))

$$\begin{aligned} g(r, \theta') = &\sum_{n=0}^{n_0} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} P_{2n}(\sin\theta') \\ &+e^2 \sum_{n=0}^{n_0} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} P_{2n}{}'(\sin\theta') \sin\theta' \cos^2\theta' \end{aligned} \tag{113}$$

or [14]

$$\begin{aligned} g(r, \theta') = &\sum_{n=0}^{n_0} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} P_{2n}(\sin\theta') \\ &+e^2 \sum_{n=0}^{n_0} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} (2n+1)\sin\theta'[P_{2n}(\sin\theta')\sin\theta' - P_{2n+1}(\sin\theta')] \end{aligned} \tag{114}$$

or

$$\begin{aligned} g(r, \theta') = &\sum_{n=0}^{n_0} a_{2n} r^{-\frac{1+\sqrt{9+4n(n+1)}}{2}} \\ &\cdot\left\{P_{2n}(\sin\theta') + e^2(2n+1)\sin\theta'[P_{2n}(\sin\theta')\sin\theta' - P_{2n+1}(\sin\theta')]\right\} \end{aligned} \tag{115}$$

with (see Equation(81))

$$r \geq \frac{b}{\sqrt{1 - e^2\cos^2\theta'}} \tag{116}$$

5. Conclusions

In this work, we expressed the gravity force of a non-rotating level ellipsoid of revolution, containing mass M, as a series of spherical harmonics. In the second section ("Formulation of a partial differential equation which is related to the spherical gravity force"), we demonstrated that it is possible to formulate a partial differential equation which can be used in order to solve our problem.

In the third section, a classical solution to this partial differential equation was expressed as a series of spherical harmonics. A significant difference of this series (compared to other similar series) was that the powers of the radial distance were irrational numbers (except the first term).

In the fourth section, a Dirichlet problem was formulated in order to find an expression for the gravity force, which was generated by a non-rotating level ellipsoid of revolution. To solve this problem, we imposed a restriction, i.e., the eccentricity of the ellipsoid should be very small. This allowed us to make several approximations, which were for the radial distance of the ellipsoid's points, the Dirichlet condition, and the Legendre polynomials.

The purpose of future works should be to better define the level of accuracy of the results by incorporating higher powers of eccentricity.

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Appendix A

The relation under consideration is

$$\begin{aligned}&\int_{-1}^{1}[g_a\cos^2\theta' + g_b\sin^2\theta - 4e^2 g_a\cos^2\theta' - \tfrac{e^2}{2}g_a\cos^4\theta' - \tfrac{e^2}{2}g_b\cos^2\theta'\sin^2\theta']\\ &\cdot P_{2m}(\sin\theta')d(\sin\theta') = \int_{-1}^{1}\sum_{n=0}^{+\infty}a_{2n}\left[b(1-e^2\cos^2\theta')^{-\frac{1}{2}}\right]^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\ &\cdot[P_{2n}(\sin\theta') + e^2 P_{2n}{}'(\sin\theta')\sin\theta'\cos^2\theta]P_{2m}(\sin\theta')d(\sin\theta')\end{aligned} \tag{A1}$$

It holds that [13]

$$\begin{aligned}&\int_{-1}^{1}P_{2m}(\sin\theta')g_a\cos^2\theta' d(\sin\theta') = g_a\int_{-1}^{1}P_{2m}(\sin\theta')(1-\sin^2\theta')d(\sin\theta')\\ &= g_a\int_{-1}^{1}P_{2m}(\sin\theta')d(\sin\theta') - g_a\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta')\\ &= -g_a\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta') = \begin{cases} -g_a\frac{2^{2m+1}}{3}\dfrac{\begin{pmatrix} m+1\\ 2m\end{pmatrix}}{\begin{pmatrix} 2m+3\\ 2\end{pmatrix}} = 0 & , \quad m=0,1\\ 0 \quad , \quad m\geq 2\end{cases}\end{aligned} \tag{A2}$$

$$\begin{aligned}&\int_{-1}^{1}P_{2m}(\sin\theta')g_b\sin^2\theta' d(\sin\theta') = g_b\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta')\\ &= \begin{cases} g_b\frac{2^{2m+1}}{3}\dfrac{\begin{pmatrix} m+1\\ 2m\end{pmatrix}}{\begin{pmatrix} 2m+3\\ 2\end{pmatrix}} = 0 & , \quad m=0,1\\ 0 \quad , \quad m\geq 2\end{cases}\end{aligned} \tag{A3}$$

The remaining terms on the left-hand side of Equation (A1) can be written as

$$-e^2\int_{-1}^{1}(1-\sin^2\theta')\left[4g_a - \frac{1}{2}g_a(1-\sin^2\theta') - \frac{1}{2}g_b\sin^2\theta'\right]P_{2m}(\sin\theta')d(\sin\theta') \tag{A4}$$

The above relation is split into two parts:

$$-e^2\int_{-1}^{1}\left[4g_a - \frac{1}{2}g_a(1-\sin^2\theta') - \frac{1}{2}g_b{}'\sin^2\theta'\right]P_{2m}(\sin\theta')d(\sin\theta') \tag{A5}$$

and

$$
\begin{aligned}
&-e^2\int_{-1}^{1}(-\sin^2\theta')\left[4g_a-\tfrac{1}{2}g_a(1-\sin^2\theta')-\tfrac{1}{2}g_b{}'\sin^2\theta'\right]P_{2m}(\sin\theta')d(\sin\theta')\\
&=-4e^2g_a\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta')-\tfrac{e^2}{2}g_a\int_{-1}^{1}\sin^2\theta' P_{2m}(\sin\theta')d(\sin\theta')\\
&+\tfrac{e^2}{2}g_a\int_{-1}^{1}\sin^4\theta' P_{2m}(\sin\theta')d(\sin\theta')-\tfrac{e^2}{2}g_b\int_{-1}^{1}\sin^4\theta' P_{2m}(\sin\theta')d(\sin\theta')
\end{aligned}
\tag{A6}
$$

For Equation (A5), we know that

$$
\int_{-1}^{1}\left(-4g_ae^2+\frac{e^2}{2}g_a\right)P_{2m}(\sin\theta')d(\sin\theta')=0\quad,\quad m\neq 0 \tag{A7}
$$

$$
\int_{-1}^{1}\left(-4g_ae^2+\frac{e^2}{2}g_a\right)P_{2m}(\sin\theta')d(\sin\theta')=-7e^2g_a\quad,\quad m=0 \tag{A8}
$$

$$
\begin{aligned}
&-\tfrac{e^2}{2}g_a\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta')\\
&=\begin{cases}-\tfrac{e^2}{2}g_a\ \dfrac{2^{2m+1}}{3}\dfrac{\begin{pmatrix}m+1\\2m\end{pmatrix}}{\begin{pmatrix}2m+3\\2m\end{pmatrix}}=0 & ,\quad m=0,1\\[2ex] 0\quad,\quad m\geq 2\end{cases}
\end{aligned}
\tag{A9}
$$

$$
\frac{e^2}{2}g_b\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta')=\begin{cases}\tfrac{e^2}{2}g_b\dfrac{2^{2m+1}}{3}\dfrac{\begin{pmatrix}m+1\\2m\end{pmatrix}}{\begin{pmatrix}2m+3\\2\end{pmatrix}}=0 & ,\quad m=0,1\\[2ex] 0\quad,\quad m\geq 2\end{cases}
\tag{A10}
$$

From Equation (A6), we know that

$$
\begin{aligned}
&-4g_ae^2\int_{-1}^{1}P_{2m}(\sin\theta')\sin^2\theta' d(\sin\theta')\\
&=\begin{cases}-4e^2\ g_a\dfrac{2^{2m+1}}{3}\dfrac{\begin{pmatrix}m+1\\2m\end{pmatrix}}{\begin{pmatrix}2m+3\\2m\end{pmatrix}}=0 & ,\quad m=0,1\\[2ex] 0\quad,\quad m\geq 2\end{cases}
\end{aligned}
\tag{A11}
$$

$$
-\frac{e^2}{2}g_a\int_{-1}^{1}\sin^2\theta' P_{2m}(\sin\theta')d(\sin\theta')=\begin{cases}-\tfrac{e^2}{2}g_a\dfrac{2^{2m+1}}{3}\dfrac{\begin{pmatrix}m+1\\2m\end{pmatrix}}{\begin{pmatrix}2m+3\\2\end{pmatrix}}=0 & ,\quad m=0,1\\[2ex] 0\quad,\quad m\geq 2\end{cases}
\tag{A12}
$$

$$\begin{aligned}&\frac{e^2}{2}g_a\int_{-1}^{1}\sin^4\theta' P_{2m}(\sin\theta')d(\sin\theta')\\&=\begin{cases}\frac{e^2}{2}g_a\frac{2^{2m+1}}{5}\frac{\begin{pmatrix}m+2\\2m\end{pmatrix}}{\begin{pmatrix}2m+5\\4\end{pmatrix}} & , \quad m=0,1,2\\ 0 \quad , \quad m\geq 3\end{cases}\end{aligned} \tag{A13}$$

$$-\frac{e^2}{2}g_b\int_{-1}^{1}\sin^4\theta' P_{2m}(\sin\theta')d(\sin\theta')=\begin{cases}-\frac{e^2}{2}g_b\frac{2^{2m+1}}{5}\frac{\binom{m+2}{2m}}{\binom{2m+5}{4}} & , \quad m=0,1,2\\ 0 \quad , \quad m\geq 3\end{cases} \tag{A14}$$

The following integral

$$\begin{aligned}&\int_{-1}^{1}\sum_{n=0}^{+\infty}a_{2n}\left[b(1-e^2\cos^2\theta')^{-\frac{1}{2}}\right]^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\\&\cdot[P_{2n}(\sin\theta')+e^2P_{2n}{}'(\sin\theta')\sin\theta'\cos^2\theta]P_{2m}(\sin\theta')d(\sin\theta')\end{aligned} \tag{A15}$$

was determined with the program Mathematica. Making the transformation

$$\begin{aligned}&(1-e^2\cos^2\theta')=(1-e^2+e^2\sin^2\theta')=(1-e^2)\left(1+\frac{e^2}{1-e^2}\sin^2\theta'\right)\\&=(1-e^2)(1+k^2\sin^2\theta')\end{aligned} \tag{A16}$$

the integral in Equation (A15) is equal to

$$\begin{aligned}&\sum_{n=0}^{+\infty}a_{2n}(1-e^2)^{\frac{1+\sqrt{9+4n(n+1)}}{4}}b^{-\frac{1+\sqrt{9+4n(n+1)}}{2}}\int_{-1}^{1}(1+k^2\sin^2\theta')^{\frac{1+\sqrt{9+4n(n+1)}}{4}}\\&\cdot[P_{2n}(\sin\theta')+e^2P_{2n}{}'(\sin\theta')\sin\theta'\cos^2\theta]P_{2m}(\sin\theta')d(\sin\theta')\end{aligned} \tag{A17}$$

In order to proceed to the integration procedure, we set

$$\begin{aligned}&I\left(\sin\theta',k,\frac{1+\sqrt{9+4n(n+1)}}{4},n,m\right)\\&=\int_{-1}^{1}(1+k^2\sin^2\theta')^{\frac{1+\sqrt{9+4n(n+1)}}{4}}[P_{2n}(\sin\theta')+e^2P_{2n}{}'(\sin\theta')\sin\theta'\cos^2\theta]\\&\quad\cdot P_{2m}(\sin\theta')d(\sin\theta')\end{aligned} \tag{A18}$$

The determination of the above integral involved the hypergeometric function [15]

$${}_2F_1(a,b,c,d)=1+\sum_{l=1}^{+\infty}\frac{a(a+1)\dots(a+l-1)b(b+1)\dots(b+l-1)}{l!c(c+1)\dots(c+l-1)}d^l \tag{A19}$$

Integral (A18) can be determined for chosen indices m and n. We give some examples: For $n = m = 0$,

$$I(\sin\theta',k,1,0,0)=2+\frac{2k^2}{3} \tag{A20}$$

For $n = 2$ and $m = 0$,

$$I\left(\sin\theta', k, \tfrac{1+\sqrt{33}}{4}, 2, 0\right) = \tfrac{1}{8(1+k^2)}\Big[6(1+k^2)\,{}_2F_1\left(\tfrac{1}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{3}{2}, -k^2\right) \\ -20(1+3k^2)\,{}_2F_1\left(\tfrac{3}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{5}{2}, -k^2\right) + (14+94k^2)\,{}_2F_1\left(\tfrac{5}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{7}{2}, -k^2\right) \\ +40k^2\,{}_2F_1\left(\tfrac{7}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{9}{2}, -k^2\right)\Big] \tag{A21}$$

For $n = m = 1$,

$$I\left(\sin\theta', k, \tfrac{1+\sqrt{17}}{4}, 1, 1\right) = -\tfrac{1}{4(1+k^2)}\Big[-2(1+k^2)\,{}_2F_1\left(\tfrac{1}{2}, -\tfrac{1+\sqrt{17}}{4}, \tfrac{3}{2}, -k^2\right) \\ +2(1+2k^2)\,{}_2F_1\left(\tfrac{3}{2}, -\tfrac{1+\sqrt{17}}{4}, \tfrac{5}{2}, -k^2\right) + (2+4k^2)\,{}_2F_1\left(\tfrac{3}{2}, -\tfrac{1+\sqrt{17}}{4}, \tfrac{5}{2}, -k^2\right) \\ +\tfrac{42}{35}(3+11k^2)\,{}_2F_1\left(\tfrac{5}{2}, -\tfrac{1+\sqrt{17}}{4}, \tfrac{7}{2}, -k^2\right) - \tfrac{180}{35}k^2\,{}_2F_1\left(\tfrac{7}{2}, -\tfrac{1+\sqrt{17}}{4}, \tfrac{9}{2}, -k^2\right)\Big] \tag{A22}$$

For $n = m = 2$,

$$I\left(\sin\theta', k, \tfrac{1+\sqrt{33}}{4}, 2, 2\right) = -\tfrac{1}{64(1+k^2)}\Big[-18(1+k^2)\,{}_2F_1\left(\tfrac{1}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{3}{2}, -k^2\right) \\ +120(1+2k^2)\,{}_2F_1\left(\tfrac{3}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{5}{2}, -k^2\right) - (444+1404k^2)\,{}_2F_1\left(\tfrac{5}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{7}{2}, -k^2\right) \\ +\left(600 + \tfrac{21240}{7}k^2\right)\,{}_2F_1\left(\tfrac{7}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{9}{2}, -k^2\right) \\ -\left(\tfrac{2450+24850k^2}{9}\right)\,{}_2F_1\left(\tfrac{9}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{11}{2}, -k^2\right) + \tfrac{9800}{11}k^2\,{}_2F_1\left(\tfrac{11}{2}, -\tfrac{1+\sqrt{33}}{4}, \tfrac{13}{2}, -k^2\right)\Big] \tag{A23}$$

The integral I for $n = m$ include a constant term; for $m = n + 1$, only $O(k^4)$ terms; for $m = n + 2$, only $O(k^6)$ terms; and so on. Since we do not have a general formula for the integral I, we provide some examples which were solved by Mathematica. These examples involve Taylor series (keeping the principal terms) of integral I for specific values of n and m.

For $n = 1$ and $m = 0$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{17}}{4}, 1, 0\right)\right] = \tfrac{-454+110\sqrt{17}}{-375+90\sqrt{17}}k^2 - \tfrac{-146+34\sqrt{17}}{-875+90\sqrt{17}}k^4 \\ + \tfrac{-2682+626\sqrt{17}}{-7875+1890\sqrt{17}}k^6 + O(k^8) \tag{A24}$$

For $n = 1$ and $m = 1$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{17}}{4}, 1, 1\right)\right] = \tfrac{2}{5} + \left(\tfrac{1}{6} + \tfrac{11\sqrt{17}}{210}\right)k^2 + \left(-\tfrac{3}{280} + \tfrac{\sqrt{17}}{56}\right)k^4 \\ + \left(\tfrac{1349}{18480} - \tfrac{1289\sqrt{17}}{55440}\right)k^6 + O(k^8) \tag{A25}$$

For $n = 1$ and $m = 2$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{17}}{4}, 1, 2\right)\right] = \tfrac{2}{105}(-7+\sqrt{17})k^2 + \left(\tfrac{599}{3465} - \tfrac{13\sqrt{17}}{693}\right)k^4 \\ + \left(-\tfrac{53}{308} + \tfrac{3691\sqrt{17}}{180180}\right)k^6 + O(k^8) \tag{A26}$$

For $n = 1$ and $m = 3$,

$$T\left[I\left(\sin\theta', k, \frac{1+\sqrt{17}}{4}, 1, 3\right)\right] = -\frac{1+9\sqrt{17}}{1001}k^4 + \left(-\frac{29}{2310} + \frac{1037\sqrt{17}}{90090}\right)k^6 + O(k^8) \tag{A27}$$

For $n = 2$ and $m = 2$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{33}}{4}, 2, 2\right)\right] = \tfrac{2}{9} + \left(\tfrac{17}{198} + \tfrac{14}{13\sqrt{33}}\right)k^2 + \tfrac{12215+2151\sqrt{33}}{360360}k^4 \\ + \left(\tfrac{4567}{144144} - \tfrac{61}{336\sqrt{33}}\right)k^6 + O(k^8) \tag{A28}$$

For $n = 2$ and $m = 1$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{33}}{4}, 2, 1\right)\right] = \tfrac{2}{105}(21+\sqrt{33})k^2 + \left(-\tfrac{25}{99} + \tfrac{173}{105\sqrt{33}}\right)k^4 + \left(\tfrac{107}{252} - \tfrac{2027}{1092\sqrt{33}}\right)k^6 + O(k^8) \tag{A29}$$

For $n = 2$ and $m = 0$,

$$T\left[I\left(\sin\theta', k, \frac{1+\sqrt{33}}{4}, 2, 0\right)\right] = \frac{2}{315}(35+19\sqrt{33})k^4 + \left(\frac{13}{126} - \frac{97}{42\sqrt{33}}\right)k^6 + O(k^8) \tag{A30}$$

For $n = 2$ and $m = 3$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{33}}{4}, 2, 3\right)\right] = \tfrac{5}{429}(-15+\sqrt{33})k^2 + \left(\tfrac{541}{2574} - \tfrac{421}{546\sqrt{33}}\right)k^4 + \left(-\tfrac{292517}{1225224} + \tfrac{34603}{37128\sqrt{33}}\right)k^6 + O(k^8) \tag{A31}$$

For $n = 2$ and $m = 4$,

$$T\left[I\left(\sin\theta', k, \tfrac{1+\sqrt{33}}{4}, 2, 4\right)\right] = -\left(\tfrac{14}{21879} + \tfrac{14}{39\sqrt{33}}\right)k^4 + \left(-\tfrac{1397}{37791} + \tfrac{461}{969\sqrt{33}}\right)k^6 + O(k^8) \tag{A32}$$

For $n = 2$ and $m = 5$,

$$T\left[I\left(\sin\theta', k, \frac{1+\sqrt{33}}{4}, 2, 5\right)\right] = \left(-\frac{10}{969} + \frac{350}{12597\sqrt{33}}\right)k^6 + O(k^8) \tag{A33}$$

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