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# Superposition Formulas and Evolution Behaviors of Multi-Solutions to the (3+1)-Dimensional Generalized Shallow Water Wave-like Equation

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**Abstract:** The superposition formulas of multi-solutions to the (3+1)-dimensional generalized shallow water wave-like Equation (GSWWLE) are proposed. There are arbitrary test functions in the superposition formulas of the mixed solutions and the interaction solutions, and we generalized to the sum of any $N$ terms. By freely selecting the test functions and the positive integer $N$, we have obtained abundant solutions for the GSWWLE. First, we introduced new mixed solutions between two arbitrary functions and the multi-kink solitons, and the abundant mixed solutions were obtained through symbolic computation. Next, we constructed the multi-localized wave solutions which are the superposition of N-even power functions. Finally, the novel interaction solutions between the multi-localized wave solutions and the multi-arbitrary function solutions for the GSWWLE were obtained. The evolution behaviors of the obtained solutions are shown through 3D, contour and density plots. The received results have immensely enriched the exact solutions of the GSWWLE in the available literature.

**Keywords:** mixed solution; multi-localized wave solution; interaction solution; generalized bilinear equation; (3+1)-dimensional generalized shallow water wave-like equation

**MSC:** 37K10; 35C08; 37K40

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## 1. Introduction

One of the important subjects to research is nonlinear localized waves in nonlinear mathematical physics; its theory has extensive applications in nonlinear fields, for example, nonlinear optics, fluid mechanics, bio-physics, and so on. Soliton [1–3], rogue wave [4–6], breathers [7,8] and lump solution [9–11] are nonlinear localized waves according to dynamic characteristics and physical properties. With the progress of technologies and the improvement of computing power, researchers are interested in numerical [12,13] and symbolic computation [1–11] in recent years. By using symbolic computation, many useful and interesting localized solutions are studied [14,15]. In addition to this, optical solitons [16–19], traveling wave solutions [20,21] and interaction solutions [22–25] have been studied as nonlinear waves. Recently, N-soliton solutions of integrable equations have been studied systematically by introducing a detailed algorithm for checking the Hirota conditions for N-soliton solutions [26]. On the other hand, N-soliton solutions have been carefully studied in the case of nonlocal integrable equations as well [27].

Linear superposition principle does not hold well in nonlinear systems because of the nonlinear terms. Nonetheless, it was found to apply for some specific cases [28,29]. Recently, based on bilinear equation and superposition formula, the researchers studied various exact solutions to nonlinear evolution equations (NLEEs), such as breathers [7,8], lump solution [9–11], interaction solutions [22–25], lump-type solution [30], high-order lump-type solution [31–35], localized wave solutions [36,37] and breather lump-kink solitons [38]. Particularly, the researchers studied the superposition formula to the sum of any $N$ terms, such as resonant multi-soliton solutions [29], lump-multi-solitions [39,40], lump-multi-stripe solutions [39–41], interaction solutions between a multiple solitary wave and a triangular periodic wave [41], interaction solutions between the multi-localized wave solutions and the multi-kink soliton solutions [42,43], etc. In order to construct a general form of the solution to NLEEs, we will give the superposition formulas of multi-solutions which can include the above solutions [7–11,22–25,29–43]. There are arbitrary test functions in these superposition formulas, and we can generalize to the sum of any $N$ terms. By freely selecting the test functions and the positive integer $N$, we can obtain the abundant solutions of NLEEs, and we can analyze the interaction behaviors among $N$ function solutions.

It is known that shallow water waves have an important influence on marine ecology, atmospheric science, ocean engineering, and so on [44]. The shallow water equations investigate the motion forms of water. The (l+l)-dimensional shallow water wave equation has been discussed in detail [45]. Some have begun to study the higher-dimensional shallow water wave because it contains rich dynamic behaviors [46]. As a generalization of the (l+l)-dimensional shallow water wave equation, Jimbo and Miwa introduced a (3+1)-dimensional generalized shallow water wave equation (GSWWE) [47]

$$u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + u_{yt} - u_{xz} = 0. \tag{1}$$

Equation (1) is the second equation in the Kadomtsev–Petviashvili hierarchy. It has been widely applied in tidal waves, ocean engineering weather simulations, tsunami predication and so on. Equation (1) has been studied by using the different methods such as the soliton-type solutions [48–50], the traveling wave solutions [51–54], the non-traveling wave solutions [52,53], the periodic solitary wave solutions [55,56], the rational solutions [57,58], the multiple-soliton solutions [58,59], the lump solutions and the interaction solutions [60] for Equation (1), etc.

Under a scale transformation $x \to -x$, Equation (1) can be written as

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xz} = 0. \tag{2}$$

So, we can study the solutions of equivalent Equation (2). Researchers have studied Grammian and Pfaffian solutions [61], the lump-type solutions and their interaction solutions [62–64], the breather wave solutions [63], the periodic wave solutions [65], the high-order breather solutions, the high-order lump solutions and the hybrid solutions [66], the solitary wave solutions, the periodic wave solutions and the interactional solutions [67] of Equation (2). Through the dependent variable transformation

$$u = 2(\ln f)_x, \tag{3}$$

the generalized bilinear equation (GBE) of Equation (2) is derived as follows

$$\begin{aligned} \mathrm{GBE}_{\mathrm{GSW}}(f) &:= (D_{3,x}^3 D_{3,y} - D_{3,y}D_{3,t} - D_{3,x}D_{3,z})f \cdot f \\ &= 2(3f_{xx}f_{xy} + f_y f_t - f_{yt}f + f_x f_z - f_{xz}f) = 0, \end{aligned} \tag{4}$$

where $D$ is the generalized bilinear differential operator [68]. Based on the GBE (4), we can derive the following (3+1)-dimensional generalized shallow water wave-like Equation (GSWWLE) under the transformation $f = e^{\int (u/2)dx}$,

$$\begin{aligned} &\frac{(D_{3,x}^3 D_{3,y} - D_{3,y}D_{3,t} - D_{3,x}D_{3,z})f \cdot f}{f^2} \\ &= \frac{3}{2}u_x u_y + \frac{3}{4}uu_x\partial_x^{-1}u_y + \frac{3}{4}u^2 u_y + \frac{3}{8}u^3\partial_x^{-1}u_y - \partial_x^{-1}u_{yt} - u_z = 0, \end{aligned} \tag{5}$$

which possesses the same bilinear type as GSWWE (2). When $z = x$, the rational solutions and the lump solutions of the GSWWLE (5) are studied [69]. The breather solutions, the three-wave solutions, the high-order lump-type solutions and the interaction solutions of the GSWWLE (5) are obtained in [70]. In the following, we will study new mixed solutions, the multi-localized wave solutions and the interaction solutions of the GSWWLE (5). If $f$ solves the GBE (4), then $u = 2(\ln f)_x$ will present a solution to the GSWWLE (5) according to the generalized bilinear method [68] and Bell polynomial theories of integrable equations [71].

In the present paper, we will give the superposition formulas of multi-solutions to the GSWWLE. In Section 2, we will introduce new mixed solutions between two arbitrary functions and multi-kink solitons. In Section 3, we will obtain the multi-localized wave solutions by using the superposition of N-even power functions. Furthermore, we will study the novel interaction solutions between the multi-localized wave solutions and the multi-arbitrary function solutions. The dynamical features of these waves will be shown via the various figures. Section 4 will conclude this paper.

## 2. Mixed Solutions between Two Arbitrary Functions and the Multi-Kink Solitons of the GSWWLE

In this section, we propose new mixed solutions between two arbitrary functions and the multi-kink solitons as a general solution of the GSWWLE. We suppose that the solution $f$ of the GBE (4) is in the form of

$$f = \alpha_0 + F(\xi_1) + G(\xi_2) + \sum_{j=1}^{M} m_j e^{\eta_j}, \tag{6}$$

where $\xi_i = \alpha_{i1}x + \alpha_{i2}y + \alpha_{i3}z + \alpha_{i4}t + \alpha_{i5}, \eta_j = \beta_{j1}x + \beta_{j2}y + \beta_{j3}z + \beta_{j4}t + \beta_{j5}$ and $\alpha_0, \alpha_{ik}, m_j, \beta_{jk}$ $(i = 1, 2; j = 1, \ldots, M; k = 1, \ldots, 5)$ are arbitrary real constants and $M$ is an arbitrary positive integer. $F(\xi_1)$ and $G(\xi_2)$ are arbitrary functions. The constants $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$ and $\beta_{j1}, \beta_{j2}, \beta_{j3}$ indicate the wave velocity in the $x, y, z$ direction, respectively. $\alpha_{i4}, \beta_{j4}$ means the frequency of the wave and $\alpha_{i5}, \beta_{j5}$ represent the invariance of variables.

By substituting the test function (6) into the GBE (4), and collecting all terms with the same order of $F(\xi_1), G(\xi_2), F'(\xi_1), G'(\xi_2), \ldots, e^{\eta_j}$ together, a complicated equation can be obtained. Equating each coefficient of these different power terms to zero yields the following system of nonlinear algebraic equations.

$$\begin{aligned}
&\alpha_{i1}^3\alpha_{i2} = 0, \beta_{j1}^3\beta_{j2} = 0, \alpha_{i1}\alpha_{i3} + \alpha_{i2}\alpha_{i4} = 0, \alpha_{11}\alpha_{21}(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}) = 0,\\
&\alpha_{11}\alpha_{23} + \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{21} + \alpha_{14}\alpha_{22} = 0, \beta_{j1}\beta_{j3} + \beta_{j2}\beta_{j4} = 0,\\
&\alpha_{i1}\beta_{j3} + \alpha_{i2}\beta_{j4} + \alpha_{i3}\beta_{j1} + \alpha_{i4}\beta_{j2} = 0, \alpha_{i1}\beta_{j1}(\alpha_{i1}\beta_{j2} + \alpha_{i2}\beta_{j1}) = 0,\\
&3\beta_{k1}\beta_{j1}(\beta_{k1}\beta_{j2} + \beta_{k2}\beta_{j1}) - (\beta_{k1}\beta_{j3} + \beta_{k2}\beta_{j4} + \beta_{k3}\beta_{j1} + \beta_{k4}\beta_{j2}) = 0,\ 1 \le k < j \le M
\end{aligned} \tag{7}$$

where $i = 1, 2; j = 1, 2, \ldots, M$. Solving the algebraic Equation (7), we obtain the following relations of the parameters $\alpha_{ik}, \beta_{jk}$ in Cases 1.1–1.6.

**Case 1.1**: $\alpha_{i1} = \alpha_{i4} = 0, \beta_{j2} = \beta_{j3} = 0, \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} = 0, \alpha_{i2}\beta_{j4} + \alpha_{i3}\beta_{j1} = 0,$

**Case 1.2**: $\alpha_{11} = \alpha_{14} = 0, \alpha_{22} = \alpha_{23} = 0, \beta_{j1} = \beta_{j4} = 0, \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{21} = 0,$ $\alpha_{21}\beta_{j3} + \alpha_{24}\beta_{j2} = 0,$

**Case 1.3**: $\alpha_{11} = \alpha_{14} = 0, \alpha_{22} = \alpha_{23} = 0, \beta_{j2} = \beta_{j3} = 0, \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{21} = 0,$ $\alpha_{12}\beta_{j4} + \alpha_{13}\beta_{j1} = 0,$

**Case 1.4**: $\alpha_{i2} = \alpha_{i3} = 0, \beta_{j1} = \beta_{j4} = 0, \alpha_{11}\alpha_{24} - \alpha_{14}\alpha_{21} = 0, \alpha_{i1}\beta_{j3} + \alpha_{i4}\beta_{j2} = 0,$

**Case 1.5**: $\alpha_{12} = \alpha_{13} = 0, \alpha_{21} = \alpha_{24} = 0, \beta_{j1} = \beta_{j4} = 0, \alpha_{11}\alpha_{23} + \alpha_{14}\alpha_{22} = 0,$ $\alpha_{11}\beta_{j3} + \alpha_{14}\beta_{j2} = 0,$

**Case 1.6**: $\alpha_{12} = \alpha_{13} = 0, \alpha_{21} = \alpha_{24} = 0, \beta_{j2} = \beta_{j3} = 0, \alpha_{11}\alpha_{23} + \alpha_{14}\alpha_{22} = 0,$ $\alpha_{22}\beta_{j4} + \alpha_{23}\beta_{j1} = 0,$

where $i = 1, 2; j = 1, 2, \ldots, M$.

By freely selecting the test functions $F(\xi_1), G(\xi_2)$, we can obtain the abundant solutions of the GSWWLE (5) with the transformation (3). For example, the breather lump-kink solitons ($M = 1, 2, 3, 4$) [38], one lump-multi-stripe solutions [39,41], the interaction solution between the multiple solitary wave and the triangular periodic wave [41], etc.

As an example, we choose $F(\xi_1) = \cos(\xi_1), G(\xi_2) = \cosh(\xi_2)$ as follows

$$f_1(x, y, z, t) = \alpha_0 + \cos(\xi_1) + \cosh(\xi_2) + \sum_{j=1}^{M} m_j e^{\eta_j}. \tag{8}$$

Substituting the Cases 1.1–1.6 of parameters into (8), we can obtain the breather lump-kink solitons of the GSWWLE (5) through the transformation (3). To show the evolution behaviors of the breather lump-kink solitons with the change of kink-soliton number $M$, we choose the appropriate values to the parameters of Case 1.1 as follows

$$\begin{aligned}&\alpha_0 = 1, \alpha_{12} = 3, \alpha_{15} = 2, \alpha_{22} = 1.1, \alpha_{23} = 1, \alpha_{25} = 5.96, \beta_{11} = 13.1, \beta_{15} = 1.5,\\ &\beta_{21} = -5, \beta_{25} = -6, \beta_{31} = -3, \beta_{35} = 6, m_1 = -30, m_2 = 1, m_3 = 60, t = 1, z = x.\end{aligned} \tag{9}$$

Figure 1 shows the 3D plots of the breather lump-kink solitons when $M = 1, 2, 3$. The interaction behaviors of the breather lump-kink solitons are studied on the basis of the increase in the number $M$, and we take the breather lump-M-soliton as the main research object when $M = 1, 2, 3$. In fact, the evolution behaviors of the breather lump-M-soliton solutions are similar when $M \to +\infty$.

As can be seen in Figure 1, the mixed solution consists of the M-kink wave and the breather wave. Fistly, we can see the interactions among the kink solitons. It shows a single kink wave when $M = 1$. A single kink wave is divided into two kink waves when $M = 2$, and a single kink wave is divided into three kink waves when $M = 3$. This is a non-elastic collisions fission phenomenon. With the change of time, the breather wave interacts with the kink wave, and the two waves begin to change in velocities, shapes and amplitudes.

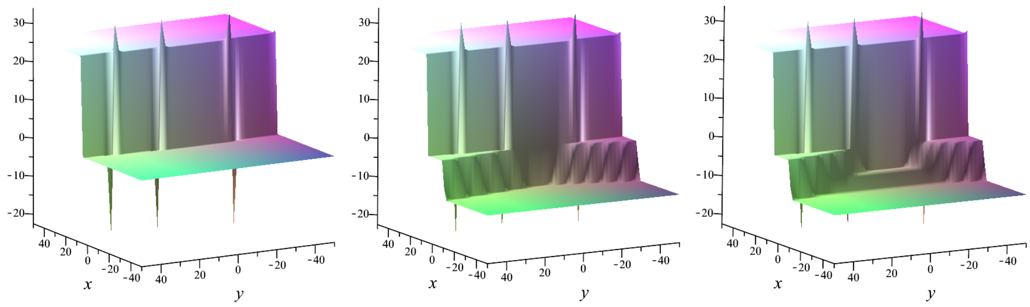


**Figure 1.** 3D plots ($M = 1, 2, 3$) corresponding to Case 1.1 at time $t = 1$.

## 3. Multi-Localized Wave Solutions and Interaction Solutions of the GSWWLE

### *3.1. Multi-Localized Wave Solutions*

In this section, we construct new multi-localized wave solutions of the GSWWLE (5) by utilizing the superposition of N-even power functions. To generate multi-localized wave solutions, we take an ansatz to the GBE (4)

$$f = \alpha_0 + \sum_{i=1}^{N} \xi_i^{2n_i}, \tag{10}$$

where $\xi_i = \alpha_{i1}x + \alpha_{i2}y + \alpha_{i3}z + \alpha_{i4}t + \alpha_{i5}$, and $\alpha_0, \alpha_{ik}$ $(i = 1, \ldots, N; k = 1, \ldots, 5)$ are real unknowns that will be determined subsequently. $N, n_i$ are arbitrary positive integers. By freely choosing the values of $N, n_i$ in (10), we can obtain various kinds of exact analytical solutions to NLEEs, such as the lump solutions [9–11], the lump-type solutions [30], the high-order lump-type solutions [31–35], the localized wave solutions [36,37], etc.

When we choose $n_i = 1$ in (10), the test function (10) is written as

$$f = \alpha_0 + \sum_{i=1}^{N} \xi_i^2. \tag{11}$$

The solution (11) of the GBE (4) is the superposition of N-quadratic functions. We substitute (11) into the GBE (4) and gather the coefficients of the resulting polynomial in $x, y, z, t$, to obtain a nonlinear algebraic system for $\alpha_0, \alpha_{ik}$. By solving the above equations with the aid of Maple, we obtain the following results.

**Case 2.1**: $\alpha_{i3} = -\frac{\alpha_{14}\alpha_{i2}}{\alpha_{11}}, \alpha_{i4} = \frac{\alpha_{14}\alpha_{i1}}{\alpha_{11}}$, and $\alpha_{i1}\alpha_{i2} = -\alpha_{i-1,1}\alpha_{i-1,2}$($i$ is even),

where $i = 1, 2, \ldots, N(N \geq 2)$. If $N$ is odd, $\alpha_{N1}\alpha_{N2} = 0$.

For example, we give the $N = 5$ corresponding solutions $f_2(x, y, z, t)$ and $f_3(x, y, z, t)$ of the GBE (4) to Case 2.1 as follows.

$$\begin{aligned} f_2(x,y,z,t) &= (\alpha_{11}x + \alpha_{12}y - \frac{\alpha_{12}\alpha_{14}}{\alpha_{11}}z + \alpha_{14}t + \alpha_{15})^2 + (\alpha_{21}x - \frac{\alpha_{11}\alpha_{12}}{\alpha_{21}}y + \frac{\alpha_{12}\alpha_{14}}{\alpha_{21}}z \\ &+ \frac{\alpha_{21}\alpha_{14}}{\alpha_{11}}t + \alpha_{25})^2 + (\alpha_{31}x + \alpha_{32}y - \frac{\alpha_{32}\alpha_{14}}{\alpha_{11}}z + \frac{\alpha_{31}\alpha_{14}}{\alpha_{11}}t + \alpha_{35})^2 \\ &+ (\alpha_{41}x - \frac{\alpha_{31}\alpha_{32}}{\alpha_{41}}y + \frac{\alpha_{14}\alpha_{31}\alpha_{32}}{\alpha_{11}\alpha_{41}}z + \frac{\alpha_{14}\alpha_{41}}{\alpha_{11}}t + \alpha_{45})^2 + (\alpha_{52}y - \frac{\alpha_{14}\alpha_{52}}{\alpha_{11}}z + \alpha_{55})^2 + \alpha_0. \end{aligned} \tag{12}$$

$$\begin{aligned} f_3(x,y,z,t) &= (\alpha_{11}x + \alpha_{12}y - \frac{\alpha_{12}\alpha_{14}}{\alpha_{11}}z + \alpha_{14}t + \alpha_{15})^2 + (\alpha_{21}x - \frac{\alpha_{11}\alpha_{12}}{\alpha_{21}}y + \frac{\alpha_{12}\alpha_{14}}{\alpha_{21}}z \\ &+ \frac{\alpha_{21}\alpha_{14}}{\alpha_{11}}t + \alpha_{25})^2 + (\alpha_{31}x + \alpha_{32}y - \frac{\alpha_{32}\alpha_{14}}{\alpha_{11}}z + \frac{\alpha_{31}\alpha_{14}}{\alpha_{11}}t + \alpha_{35})^2 \\ &+ (\alpha_{41}x - \frac{\alpha_{31}\alpha_{32}}{\alpha_{41}}y + \frac{\alpha_{14}\alpha_{31}\alpha_{32}}{\alpha_{11}\alpha_{41}}z + \frac{\alpha_{14}\alpha_{41}}{\alpha_{11}}t + \alpha_{45})^2 + (\alpha_{51}x + \frac{\alpha_{14}\alpha_{51}}{\alpha_{11}}t + \alpha_{55})^2 + \alpha_0. \end{aligned} \tag{13}$$

When $N = 6$, we obtain the solution $f_4(x, y, z, t)$ of the GBE (4) as follows.

$$\begin{aligned} f_4(x,y,z,t) &= (\alpha_{11}x + \alpha_{12}y - \frac{\alpha_{12}\alpha_{14}}{\alpha_{11}}z + \alpha_{14}t + \alpha_{15})^2 + (\alpha_{21}x - \frac{\alpha_{11}\alpha_{12}}{\alpha_{21}}y + \frac{\alpha_{12}\alpha_{14}}{\alpha_{21}}z \\ &+ \frac{\alpha_{21}\alpha_{14}}{\alpha_{11}}t + \alpha_{25})^2 + (\alpha_{31}x + \alpha_{32}y - \frac{\alpha_{32}\alpha_{14}}{\alpha_{11}}z + \frac{\alpha_{31}\alpha_{14}}{\alpha_{11}}t + \alpha_{35})^2 \\ &+ (\alpha_{41}x - \frac{\alpha_{31}\alpha_{32}}{\alpha_{41}}y + \frac{\alpha_{14}\alpha_{31}\alpha_{32}}{\alpha_{11}\alpha_{41}}z + \frac{\alpha_{14}\alpha_{41}}{\alpha_{11}}t + \alpha_{45})^2 + (\alpha_{51}x + \alpha_{52}y - \frac{\alpha_{52}\alpha_{14}}{\alpha_{11}}z \\ &+ \frac{\alpha_{51}\alpha_{14}}{\alpha_{11}}t + \alpha_{55})^2 + (\alpha_{61}x - \frac{\alpha_{51}\alpha_{52}}{\alpha_{61}}y + \frac{\alpha_{14}\alpha_{51}\alpha_{52}}{\alpha_{11}\alpha_{61}}z + \frac{\alpha_{14}\alpha_{61}}{\alpha_{11}}t + \alpha_{65})^2 + \alpha_0. \end{aligned} \tag{14}$$

The following results are interesting. By choosing the values of $n_i$, we can obtain various high-order multi-localized wave solutions.

(1) When $N$ is odd, we set $n_i = 1 (i = 1, \ldots, N-1)$ and $n_N$ is the arbitrary positive integer, namely

$$f = \alpha_0 + \sum_{i=1}^{N-1} \xi_i^2 + \xi_N^{2n_N}. \tag{15}$$

The coefficients of the solution (15) still satisfy Case 2.1.

(2) When $n_i$ is an arbitrary positive integer and $N$ is even, we obtain the following results

**Case 2.2**: $\alpha_{i2} = \alpha_{i3} = 0, \alpha_{i4} = \frac{\alpha_{14}\alpha_{i1}}{\alpha_{11}}, \alpha_{N1} = \alpha_{N4} = 0, \alpha_{N3} = -\frac{\alpha_{14}\alpha_{N2}}{\alpha_{11}},$

**Case 2.3**: $\alpha_{i1} = \alpha_{i4} = 0, \alpha_{i3} = \frac{\alpha_{13}\alpha_{i2}}{\alpha_{12}}, \alpha_{N2} = \alpha_{N3} = 0, \alpha_{N4} = -\frac{\alpha_{13}\alpha_{N1}}{\alpha_{12}},$

where $i = 1, 2, \ldots, N-1$.

Substituting the three cases 2.1–2.3 of parameters into the test functions $f$ in (11) and (15), we can obtain the multi-localized wave solutions for the GSWWLE (5) under the transformation (3)

$$u = \frac{2f_x}{f}, \tag{16}$$

where $f$ is the positive function solution of the GBE (4). We can easily verify that the multi-localized wave solutions (16) are also the multi-localized wave solutions corresponding to Case 2.1 of the GSWWE (2).

From (16), we can see that at any fixed time $t$, the localized wave solution $u \to 0$ if and only if $\xi_1^2 + \cdots + \xi_N^2 \to \infty$, namely

$$\lim_{x^2+y^2+z^2 \to +\infty} u(x,y,z,t) = 0.$$

The localized wave solution corresponding to (16) is rationally localized in all directions in the space.

To exhibit the localized characteristics of the localized wave solution $u(x,y,z,t)$ corresponding to $f_4(x,y,z,t)$ of the GSWWLE (5) clearly, a 3D plot, a contour plot and a density plot with particular choices of the involved parameters are shown in Figure 2. The involved parameters adopted are

$$\begin{aligned}
&\alpha_0 = 1, \alpha_{11} = 1.5, \alpha_{12} = 1, \alpha_{14} = 0.5, \alpha_{15} = 0, \alpha_{21} = 2, \alpha_{25} = 0, \alpha_{31} = 1, \alpha_{32} = 2, \\
&\alpha_{35} = 0, \alpha_{41} = 1, \alpha_{45} = 0, \alpha_{51} = 1, \alpha_{52} = 0.3, \alpha_{55} = 0, \alpha_{61} = 1, \alpha_{65} = 0, z = x.
\end{aligned} \tag{17}$$

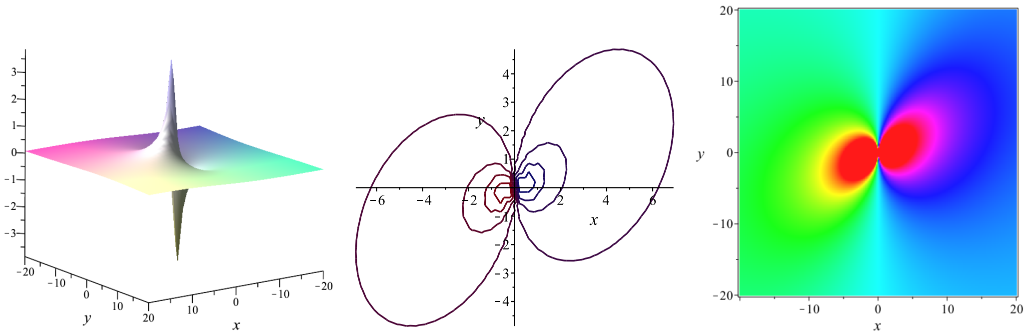


**Figure 2.** 3D plot, contour plot and density plot at time $t = 0$.

If the conditions $\alpha_{11}\alpha_{21}\alpha_{41}\alpha_{61} \neq 0$ and $\alpha_0 > 0$ are satisfied, the function $f_4(x,y,z,t)$ in (14) is positive. In Figure 2, we choose the parameters $\alpha_{i5} = 0$, which show the localized wave centered at the origin $(0,0)$ when $t = 0$. The crest and trough of the localized wave are symmetric about the origin $(0,0)$, as shown in Figure 2, and so it can be thought to be the bright-dark wave because the height of the crest and the depth of the trough are equal.

When $N = 2$, we can give the general formula of the original coordinates of lump

$$\left( \frac{[(\alpha_{23}z + \alpha_{24}t + \alpha_{25})\alpha_{12} - (\alpha_{13}z + \alpha_{14}t + \alpha_{15})\alpha_{22}]\sqrt{\alpha_{11}^2 + \alpha_{21}^2} \pm \sqrt{\alpha_0}|\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}|}{(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\sqrt{\alpha_{11}^2 + \alpha_{21}^2}}, \right.$$
$$\left. \frac{(\alpha_{13}z + \alpha_{14}t + \alpha_{15})\alpha_{21} - (\alpha_{23}z + \alpha_{24}t + \alpha_{25})\alpha_{11}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \right) \tag{18}$$

where $\alpha_0, \alpha_{ik}$ are given in Case 2.1. Then the simplified form in $x, y$ plane can be obtained by substituting $\alpha_{15} = \alpha_{25} = 0, z = 0$ into Formula (18)

$$\left( \frac{(\alpha_{12}\alpha_{24} - \alpha_{14}\alpha_{22})\sqrt{\alpha_{11}^2 + \alpha_{21}^2}\,t \pm \sqrt{\alpha_0}|\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}|}{(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\sqrt{\alpha_{11}^2 + \alpha_{21}^2}}, \frac{\alpha_{14}\alpha_{21} - \alpha_{11}\alpha_{24}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}t \right). \tag{19}$$

From Formula (19), we know that the initial velocities in $x$ direction and $y$ direction of the lump are $v_x = \frac{\alpha_{12}\alpha_{24} - \alpha_{14}\alpha_{22}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}$, $v_y = \frac{\alpha_{14}\alpha_{21} - \alpha_{11}\alpha_{24}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}$.

### *3.2. Interaction Solutions between the Multi-Localized Wave Solutions and the Multi-Arbitrary Function Solutions of the GSWWLE*

In this section, we will pay attention to the interaction solutions between the multi-localized wave solutions and the multi-arbitrary function solutions for the GSWWLE (5). In order to derive the interaction solutions, we assume that the GBE (4) has the novel superposition formula of exact solution

$$f = \alpha_0 + \sum_{i=1}^{N} \xi_i^{2n_i} + \sum_{j=1}^{M} F_j(\eta_j), \tag{20}$$

where $\xi_i = \alpha_{i1}x + \alpha_{i2}y + \alpha_{i3}z + \alpha_{i4}t + \alpha_{i5}, \eta_j = \beta_{j1}x + \beta_{j2}y + \beta_{j3}z + \beta_{j4}t + \beta_{j5}$, and $\alpha_0, \alpha_{ik}, \beta_{jk}$ $(i = 1, \ldots, N; j = 1, \ldots, M; k = 1, \ldots, 5)$ are real parameters. $N, n_i, M$ are arbitrary positive integers, and $F_j(\eta_j)$ are arbitrary functions. By freely choosing the values of $N, n_i, M$ and the test functions $F_j(\eta_j)$ in (20), we can obtain various kinds of interaction solutions, such as the interaction solutions [22,23], the lump-multi-strip solutions and the lump-multi-solitions [39,41] and the interaction solutions of lump-M-cosh solitons [42] to NLEEs.

When we choose $n_i = 1$, the exact solution of the GBE (4) is written

$$f = \alpha_0 + \sum_{i=1}^{N} \xi_i^2 + \sum_{j=1}^{M} F_j(\eta_j). \tag{21}$$

By substituting (21) into the GBE (4), and collecting the coefficients, we obtain a nonlinear algebraic system for $\alpha_0, \alpha_{ik}, \beta_{jk}$. By solving the above equations with the aid of Maple, we obtain the solutions for the GBE (4). In fact, the Cases 2.1–2.3 correspond to the following two cases, respectively:

$$(\mathrm{I})\beta_{j1} = \beta_{j4} = 0, \beta_{j3} = -\frac{\alpha_{14}\beta_{j2}}{\alpha_{11}}; \;\; (\mathrm{II})\beta_{j2} = \beta_{j3} = 0, \beta_{j4} = \frac{\alpha_{14}\beta_{j1}}{\alpha_{11}}; \tag{22}$$

where $j = 1, \ldots, M$ and $\alpha_{11} \neq 0$.

By substituting the parameters $\alpha_0, \alpha_{ik}, \beta_{jk}$ in Case 2.1 and (I), (II) into the expressions (21) and using (16), we obtain the interaction solutions between multi-localized wave solutions and multi-arbitrary function solutions for the GSWWLE (5). We can easily verify that the interaction solutions (16) corresponding to (21) and Case 2.1, (I), (II) are also the interaction solutions of the GSWWE (2).

By freely choosing the values of the functions $F_j(\eta_j)$ in the expression (21) and using (16), we can obtain various kinds of interaction solutions for the GSWWLE (5) and GSWWE (2). For example, when $F_j(\eta_j) = e^{\eta_j}$, we obtain the interaction solutions between the multi-localized wave solutions and the multi-kink soliton solutions [42,43], etc.

As the example, we choose $N = 6, M = 1, 2$ and $F_j(\eta_j) = e^{\eta_j}$ corresponding to the solution of the GBE (4) to Case 2.1 and (II) as follows

$$f_5(x, y, z, t) = f_4(x, y, z, t) + e^{\beta_{11}x + \frac{\alpha_{14}\beta_{11}}{\alpha_{11}}t + \beta_{15}} + \alpha_0. \tag{23}$$

$$f_6(x, y, z, t) = f_4(x, y, z, t) + e^{\beta_{11}x + \frac{\alpha_{14}\beta_{11}}{\alpha_{11}}t + \beta_{15}} + e^{\beta_{21}x + \frac{\alpha_{14}\beta_{21}}{\alpha_{11}}t + \beta_{25}} + \alpha_0. \tag{24}$$

By substituting (23) and (24) into the transformation (16), we obtain the interaction solutions $u(x, y, z, t)$ between the localized wave solution and multi-kink soliton solution of the GSWWLE (5). In the following, we will study the evolution behaviors of the interaction solutions with the change of soliton number $M$. We choose the appropriate values for the parameters as follows

$$\begin{aligned}&\alpha_0 = 1, \alpha_{11} = 1.5, \alpha_{12} = 1, \alpha_{14} = 0.5, \alpha_{15} = 0, \alpha_{21} = 2, \alpha_{25} = 0, \alpha_{31} = 1, \alpha_{32} = 2,\\&\alpha_{35} = 0, \alpha_{41} = 1, \alpha_{45} = 0, \alpha_{51} = 1, \alpha_{52} = 0.3, \alpha_{55} = 0, \alpha_{61} = 1, \alpha_{65} = 0, \beta_{11} = 0.5,\\&\beta_{15} = 1, \beta_{21} = -0.5, \beta_{25} = -1, z = x.\end{aligned} \tag{25}$$

Figures 3 and 4 show the dynamic processes of the interaction solution corresponding to (23) and (24). As can be seen in Figures 3 and 4, the interaction solution consists of the M-kink wave and the localized wave. Fistly, we can see the interactions among the kink solitons. It displays a single kink wave only for $M = 1$ in Figure 3 and single kink wave split into two kink waves when $M = 2$ in Figure 4. From figures, the interaction behaviors of the localized wave and multi-kink soliton are seen on the basis of the increase in the number $M$, and we take the localized wave-M-kink soliton as the main research object when $M = 1, 2$. In fact, the evolution behaviors of the localized wave-M-kink soliton are similar when $M \to +\infty$. The figures show the 3D plots, contour plots and density plots in the $(x, y)$-plane when $t = 0$. The localized structures and the energy distribution of the interaction solution are shown on the 3D plots and the density plots, respectively. On Figures 3 and 4, the localized waves interact with the kink soliton solutions and move forward in the $y$ direction.

With the change of time, the localized wave interacts with the kink, and the two waves begin to change in velocities, shapes and amplitudes. Comparing Figure 3 with Figure 2, it is observed that after interaction of localized wave and kink waves, the height and intensity of localized wave reduces compared with the single localized wave.

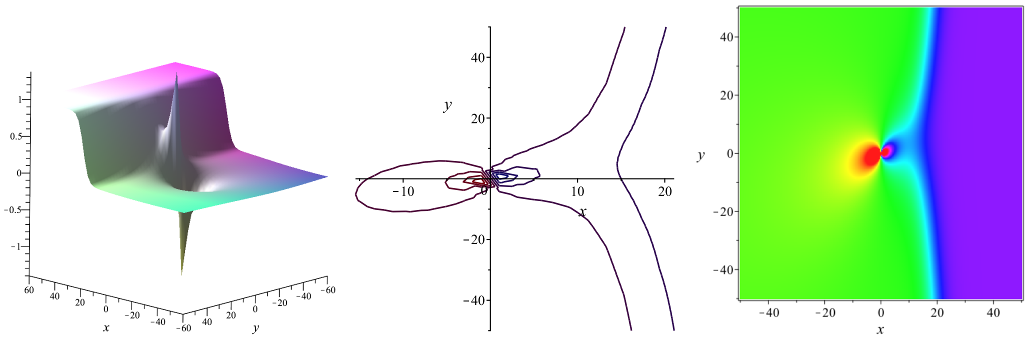

**Figure 3.** 3D plot, contour plot and density plot corresponding to $f_5(x, y, z, t)$ at times $t = 0$.

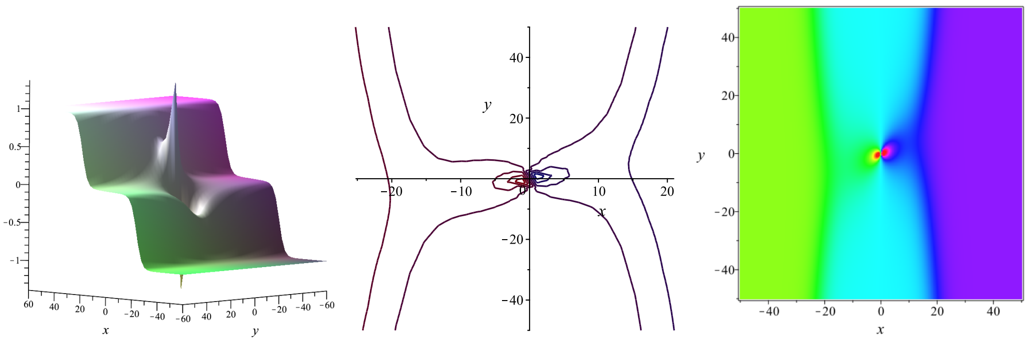

**Figure 4.** 3D plot, contour plot and density plot corresponding to $f_6(x, y, z, t)$ at times $t = 0$.

## 4. Conclusions

In this paper, we proposed the superposition formulas of multi-solutions to the GSWWLE. Based on the superposition formulas of multi-solutions, we have investigated novel mixed solutions, the multi-localized wave solutions and the interaction solutions of the GSWWLE by using symbolic computation. There are arbitrary test functions in the superposition formulas of the mixed solutions (6) and the interaction solutions (20), and we generalized to the sum of any $N$ terms. By freely selecting the test functions and the positive integer $N$, we obtained abundant solutions of the GSWWLE. At first, we gave the mixed solutions between two arbitrary functions and the multi-kink solitons of the GSWWLE. Next, we successfully structured the abundant multi-localized wave solutions which are the superposition of N-even power functions of the GSWWLE. Finally, the interaction solutions between the multi-localized wave solutions and the multi-arbitrary function solutions of the GSWWLE are obtained. Through 3D plots, contour plots and density plots, we illustrated the dynamical features of the mixed solutions, the localized wave solutions and the interaction solutions. It is worth mentioning that the received multi-localized wave solutions and the interaction solutions are also the exact solutions of the GSWWE (2). In addition, the received results have enriched the exact solutions of the GSWWLE [69,70] and the GSWWE [47–67] in the available literature.

The exact solutions of NLEEs play a crucial role in the study of nonlinear physical or natural phenomena. It is always a research goal to construct new exact solutions of NLEEs. Moreover, our method may provide an effective and direct tool to apply the high-order nonlinear wave and the collision phenomena to many other NLEEs in mathematical physics.

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