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A Note on the Geometry of Closed Loops

Nir Shvalb [1], Mark Frenkel [2], Shraga Shoval [3] and Edward Bormashenko [2,*]

[1] Department of Mechanical Engineering & Mechatronics, Faculty of Engineering, Ariel University, P.O. Box 3, Ariel 407000, Israel
[2] Chemical Engineering Department, Engineering Faculty, Ariel University, P.O. Box 3, Ariel 407000, Israel
[3] Department of Industrial Engineering and Management, Faculty of Engineering, Ariel University, P.O. Box 3, Ariel 407000, Israel
* Correspondence: edward@ariel.ac.il

Abstract: In this paper, we utilize the Ramsey theory to investigate the geometrical characteristics of closed contours. We begin by examining a set of six points arranged on a closed contour and connected as a complete graph. We assign the downward-pointing edges a red color, while coloring the remaining edges green. Our analysis establishes that the curve must contain at least one monochromatic triangle. This finding has practical applications in the study of dynamical billiards. Our second result is derived from the Jordan curve theorem and the Ramsey theorem. Finally, we discuss Ramsey constructions arising from differential geometry. Applications of the Ramsey theory are discussed.

Keywords: Ramsey theory; closed contour; Jordan theorem; complete graph

MSC: 05D10; 05C15; 37C83

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1. Introduction

Ramsey theory is a branch of graph theory that focuses on the appearance of interconnected substructures within a structure/graph of a known size [1–9]. Ramsey theory states that any structure will necessarily contain an interconnected substructure [1–6]. Ramsey's theorem, in one of its graph-theoretic forms, states that one will find monochromatic cliques in any edge labelling (e.g., with colors) of a sufficiently large complete graph [6]. Ramsey theory is a branch of graph theory that focuses on the appearance of inter-connected substructures within a structure/graph of a known size [1–9]. A graph is defined as a pair (V, E), where V is the set of vertices, and the set of edges (also called links) $E \subseteq V \times V$ is an anti-reflective and symmetric binary relation of V (this definition is true for non-directed/undirected graphs). If $X \subseteq V$ is such that $(x, x') \in E$, for all distinct $x, x' \in X$, we define X as a clique. In the case when $(x, x') \notin E$ for all distinct $x, x' \in X$, we define X as an anticlique. The infinite Ramsey theorem (also called Ramsey's theorem for pairs) states that if (V, E) is an infinite graph, then (V, E) either contains an infinite clique or an infinite anticlique [7].

The Ramsey theorem may be re-shaped in the language of colorings. Given a set X and $m \in \mathbb{N}$, we label by $X^{[m]}$ the set of m-element subsets of X. If $X \subseteq \mathbb{N}$, it is convenient to identify $X^{[m]}$ with the set of pairs $\{(x_1, x_2 \ldots x_m \in X^m : x_1 < x_2 < \ldots < x_m)\}$. Given $\in \mathbb{N}$, a k-coloring of $X^{[m]}$ is a function of c: $X^{[m]} \rightarrow [1, \ldots, k]$. Now, we refer to the elements of $[1, \ldots k]$ as colors. A subset $Y \subseteq X$ is monochromatic for the coloring c if the restriction of c to $Y^{[m]}$ is constant. The infinite Ramsey theorem is re-formulated as follows in terms of colors: for any $m \in \mathbb{N}$, any infinite set V and any k-coloring c of $V^{[m]}$, there is an infinite subset of V that is monochromatic for the coloring c.

Now, let us formulate the finite Ramsey theorem. Let us introduce the following notation: given $k, l, m, n \in \mathbb{N}$, we denote $l \rightarrow (n)_k^m$ if every coloring of $[l]^m$ with k colors has a homogeneous set of size n. Now, we are ready to formulate the finite Ramsey theorem, which sounds as follows: for every $k, m, n \in \mathbb{N}$, there is $l \in \mathbb{N}$, such that $l \to (n)_k^m$ [7].

Let us re-formulate the Ramsey theorem in view of its numerous physics, chemistry and engineering applications. Simply speaking, the Ramsey theorem states that any structure will necessarily contain an interconnected substructure, or when speak in the graph-theoretic form language, it states that one will find monochromatic cliques in any edge labelling (e.g., with colors) of a sufficiently large complete graph [1–6].

One more example of Ramsey-like thinking is delivered by the van der Waerden's theorem: colorings of the integers by finitely many colors must have long monochromatic arithmetic progressions [6]. The rigorous formulation of the van der Waerden theorem sounds as follows: For any integers $k, m \geq 1$, there exists an integer $N = N_{vdW}(k, m) \geq 1$, such that every coloring c: $\{1, \ldots N\} \to \{1, \ldots, m\}$ of $\{1, \ldots N\}$ into m colors contains at least one monochromatic arithmetic progression of length k (i.e., a progression in $\{1, \ldots N\}$ of the cardinality k on which c is constant [8]). One of the most successful application of the Ramsey-theory approach is the ergodic-theoretic proof of the Szemerédi theorem (stating that a subset of natural numbers possessing a positive upper density contains infinitely many arithmetic progressions of length l for all positive integers l), carried out by Furstenberg [9]. Applications of the Ramsey theory are discussed in [10,11].

An accessible introduction to the Ramsey theory is found in [1–4]. A rigorous approach is laid out in [5–7]. Problems in Ramsey theory typically ask a question of the form: "How big must some structure be to guarantee that a particular property holds?" We apply Ramsey theory to the analysis of some geometrical properties of closed contours.

2. Results

2.1. Ramsey Theory and Geometry of Closed Curves

Consider the closed curve $\mathcal{L}$ depicted in Figure 1 as a solid black line. We connect each pair of points in pairs with the straight colored segments, as shown in Figure 1B. The equations of these straight lines are given by:

$$y_{i,k}(x) = \alpha_{i,k}x + \beta_{i,k}(i, k \in \mathcal{L}), \tag{1}$$

this being the equation of the line connecting the points numbered i and k, respectively . It is always possible to choose six points laying in a closed 2D curve in such a way that $\alpha_{i,k} \neq 0$ holds. This is the case since Jordan's theorem holds for any closed simple curves in the plane, including exotic ones such as the Warsaw circle. Nevertheless, for the sake of simplicity, let us restrict ourselves to well-behaved piecewise-polynomial curves F over an algebraically closed field. Since the curve {y = 0} and F do not share a common factor, F ∩ {y = 0} is a finite set of points (a generalized case is provided in [12]). Thus, for any piecewise-polynomial curve as Figure 1A, excluding a finite number of points will suffice for the above statement to hold.

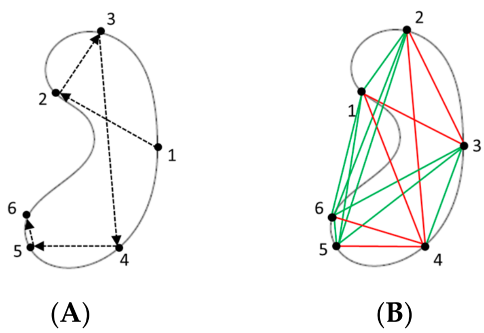

Figure 1. Dynamic billiard as the Ramsey system. (**A**) Trajectory of the point within the pool (billiard) is shown with violet arrows. (**B**) Complete graph arising from six points, depicting reflections from the boundary.

Restricting our focus solely to slopes that are positive or negative, we proceed to color the edges that connect the points with red or green hues, as illustrated in Figure 1B. This coloring scheme gives rise to a complete bi-color graph, and by virtue of the Ramsey theorem, we can infer that the graph must contain at least one monochromatic triangle of either red or green color. Specifically, the Ramsey number $R(3,3)$ is equal to 6, which implies the appearance of at least one such triangle.

2.2. Dynamical Billiards

This simple observation enables the re-shaping of the problem of dynamical billiards in the spirit of the Ramsey theory. Consider the dynamical billiard depicted in Figure 2.

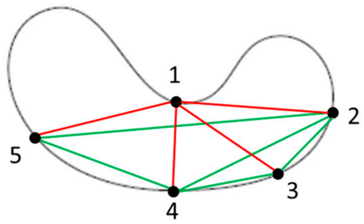

Figure 2. A closed curve possessing segments with differently signed curvatures is depicted. The curvature is positive at the black segment of the curvature and negative at the violet segment of the curve. Five points are placed on the curve. Green edges connect the points placed on the segments where the curvatures are of the same sign; red edges connect the points placed on segments where curvatures are differently signed.

In a dynamical billiard, a particle moves along a straight line and is reflected from the boundaries. Billiards are Hamiltonian idealizations of the known billiard game, in which the boundaries have a general geometric shape (rather than a rectangular shape). In Figure 1A, a particle alternates between free motion (presupposed to be a straight line, depicted with violet arrows) and specular reflections from a boundary. This class of systems is called a dynamical billiard [13]. Dynamical billiards are described by mathematical models that appear in a diversity of physical phenomena. The dynamical properties of such models are determined by the shape of the walls of the container, and they may vary from completely regular (integrable) to fully chaotic [12–14]. Thus, consider the simplest system in which the particles move in a 2D container and collide with its walls/boundary, as shown in Figure 1A. The reflection points are marked as circles. Consider the set of first six reflections from the boundary shown with green circles in Figure 1A and numbered $1 \leq i \leq 6$. We connect the points in pairs with straight line segments, as shown in Figure 1B. The equations of these straight lines are again provided by Equation (1). Following the Ramsey procedure and approach discussed in Section 2.1, we color the edges connecting the reflection points numbered i an k, for which $\alpha_{i,k} > 0$ holds, in red, and the links for which $\alpha_{i,k} < 0$ in green, as shown in Figure 1B. Again, $\alpha_{i,k} \neq 0$ is presumed. Thus, the complete bi-color graph emerges, and according to the Ramsey theorem, at least one (red or green) monochromatic triangle should necessarily appear within the graph (the Ramsey number $R(3,3) = 6$). Indeed, triangle "165" appearing in Figure 1B is built of green edges only. This result will be true for any closed boundary, for any starting point of the body moving within the billiard. Moreover, the reflections may not be exactly specular. Note that we also do not specify the Hamiltonian, representing the energy of the particle. Any set of six reflection points located on the closed boundary will generate the complete graph, which fulfils the Ramsey theorem, and at least one monochromatic triangle will necessarily appear in the graph.

Dynamic billiards are of much interest in view of deterministic chaos, and it was demonstrated that this chaos will never be complete; substructures built of the segments of the trajectory of the particle will necessarily appear, as illustrated in Figure 1B.

2.3. Transitive Ramsey Numbers and Differential Geometry of Closed Curves

Consider now the closed curve, shown in Figure 2. The curvature of this curve is positive at the black segment of the curve and negative at the violet segment of the curve. Recall that the signed curvature κ is given by Equation (2):

$$\kappa = \frac{y''}{\left(1+y'^2\right)^{3/2}} \tag{2}$$

Consider a set of n points placed on the curve. Some of the points are placed on the segments with the positive curvature (they are denoted with the symbol "+" in Figure 2), and some are placed on the segments with the negative curvature (they are denoted with the symbol "−" in Figure 2). According to the Ramsey approach, we connect the points with the same sign with the green edge, and we connect differently signed points with the red edge, as shown in Figure 2. This procedure gives rise to the complete graph. Let us answer the following question: what is the minimal number of points that necessarily provides the formation of the monochromatic triangle in the complete graph? At first glance, it also seems that the answer is supplied by the Ramsey theorem and that $R(3,3) = 6$ is kept. However, this answer is wrong, due to the fact that the presented relations between the vertices of the graph are transitive/intransitive in their nature [15].

Consider first the equally signed vertices. When the points numbered "n", "k" and "l" are of the same sign, i.e., if the edges $\{n,k\}$ and $\{k,l\}$ are of the same color, then the third edge $\{n,l\}$ will be as well. Thus, the vertices in the addressed case are connected via a *transitive relation* (see Figure 3A).

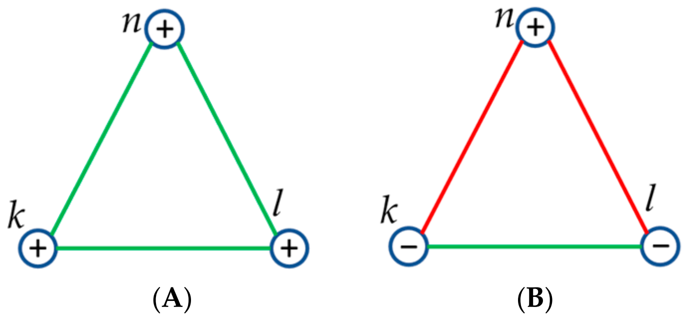

Figure 3. Logical interrelations between the points representing positive and negative curvatures are shown. Green edges connect the points placed on the segments where the curvatures are of the same sign; red edges connect the points placed on segments where the curvatures are differently signed. (**A**) Points in which the curvature is positive are placed in the vertices of the graph. Only green edges appear in the graph. (**B**) Points in which the curvature is positive and negative are placed in the vertices of the graph. The monochromatic triangle is impossible in this case.

Now, consider differently signed vertices. When the pair of vertices $\{n, k\}$ and the pair of vertices $\{k, l\}$ are differently signed, the pair $\{n, l\}$ is necessarily of the same sign (see Figure 3B), which implies an *intransitivity relation*. This is exactly the situation that is inherent to the relation between three vertices of different signs, as shown in Figure 3B. It should be emphasized that no monochromatic triangle will appear when points of various signs of the curvature are located in its vertices; however, the clique built of two monochromatic edges will be necessarily present, as shown in Figure 3B. Using notions from the Ramsey theory, we conclude that $R_{tr,int}(2,3) = 3$ is true for the vertices of the graph connected by transitive/intransitive relations. This proves the following:

Proposition1. *$R_{tr,int}(r,k) < R(r,k)$ for all positive r and k.*

2.4. Ramsey Approach and the Jordan Curves

Consider one more property of closed curves emerging from the Jordan curve theorem and the Ramsey approach. According to the Jordan theorem, every Jordan curve (a plane simple closed curve) divides the plane into an "interior" region bounded by the curve and an "exterior" region containing all the near and distant exterior points. In what follows, we prove that:

Theorem 1. *For any closed convex curve Γ in the plane and any $n = R_{tr,int}(r,k)$ points $\{P_1, P_2, ..n\}$, there is always a k-complete graph that is either completely within $Int\ \Gamma$ or completely within $Ext\ \Gamma$.*

Proof. Let us choose six points, some of which belong to the interior region and some of which belong to the exterior region, as shown in Figure 4 (the closed curve is shown by the black solid line).

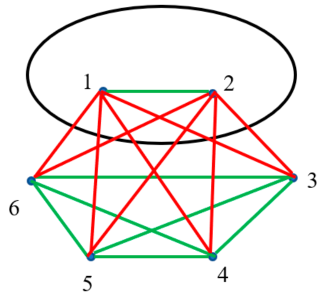

Figure 4. Black solid line depicts the Jordan curve dividing the plane into the interior and exterior regions (separated by the solid black line). Points labeled "1" and "2" are located within the interior region, and points "3", "4", "5" and "6" are located in the exterior region. Red edges connect points located in different regions; green edges connect points located in the same region. Triangles "456", "345", "356" and "346" are monochromatic.

In the case illustrated in Figure 4, the points numbered "1" and "2" are chosen within the interior region, i.e., within the contour, whereas the points numbered "2", "4", "5" and "6" are located in the exterior region of the plane, i.e., outside the contour. Let us connect the points located within the same region (whether interior or exterior) with green edges and the points placed within the different regions with the red edges, as shown in Figure 4. No monochromatic, red triangle is possible in this case. Indeed, if the vertices "n" and "k" and "k" and "l" are located in different regions, the vertices "n" and "l" are necessarily located in the same region. No monochromatic, red triangle is possible in this case, as can be seen in Figure 4, and we return to intransitive logical relations for the points located in different regions, which was already discussed in detail in the previous section. In turn, the logical relation between the points located in the same region are transitive [15]. Thus, a monochromatic green triangle will necessarily appear when the three points are placed in the same region of the plane, as shown in Figure 4. Using the notions of the Ramsey theory, we derive $R_{trans,intrans}(2,3) = 3$ for the graph, as shown in Figure 4.

Now, consider three Jordan curves (numbered "I", "II" and "III" in Figure 5) dividing the plane in four regions, as depicted in Figure 5. Six points labeled "1" … "6" are placed on the plane, as shown in Figure 5.

Let us connect the points located within the same region (whichever it is) with green edges and the points placed within the different regions with red edges, as shown in Figure 5. Fewer than three points are located within the same region (this renders transitive/intransitive relations between vertices impossible in the situation depicted in Figure 6). Thus, according to the Ramsey theorem, a monochromatic triangle will necessarily appear in the complete graph, as depicted in Figure 5. Indeed, the triangles "136", "236", "346", "356", "234" and "156" are monochromatic (red). No monochromatic green triangles are recognized in the graph, as shown in Figure 5.

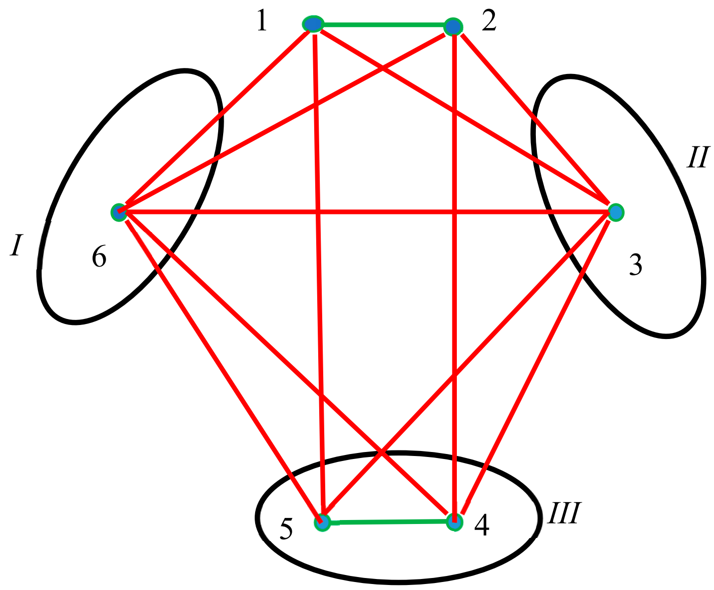

Figure 5. Black solid line depicts three closed Jordan curves dividing the plane into the interior and exterior regions Red edges connect points located in different regions; green edges connect points located in the same region. Triangles "136", "236", "346", "356", "234" and "156" are monochromatic.

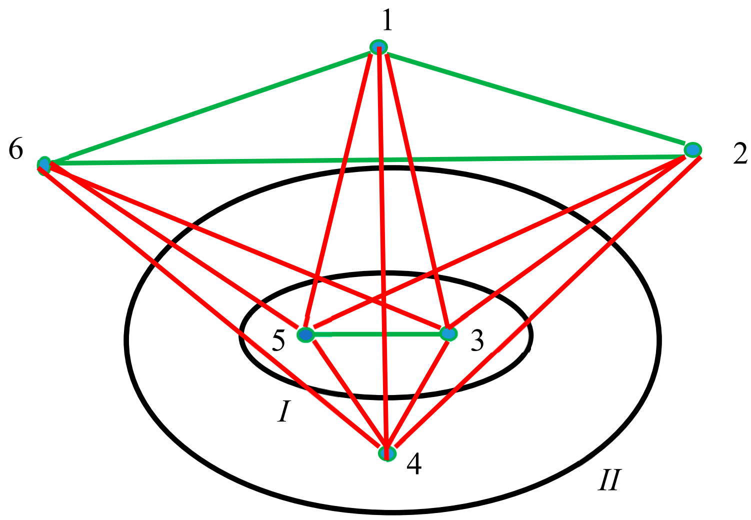

Figure 6. The Jordan curve "*I*" is located within the Jordan curve "*II*", thus dividing the plane into three regions. Red edges connect points located in different regions; green edges connect points located in the same region. The triangle "126" is green; triangles "456" and "234" are red.

Consider one more possible configuration when the Jordan curve "*I*" is located within the Jordan curve "*II*", as shown in Figure 6. In this case, two Jordan curves divide the plane into three distinguishable regions. Consider the set of six points labeled "1" ... "6", which are located as shown in Figure 6. Again, we connect the points located within the same region (whichever it is) with green edges and the points placed within the different regions with red edges; thus, the complete graph emerges, as shown in Figure 6.

Three monochromatic triangles, namely green "126", red "456" and "234" ones, are recognized in the complete graph, as shown in Figure 6. The green monochromatic triangle "126" should necessarily appear due to the transitivity of logical relations between points "1", "2" and "6", located in the same region, whereas the red triangles emerge from the Ramsey construction depicted in Figure 6. □

3. Conclusions

The Ramsey theory/approach shows potential for a diversity of mathematical and physical applications, including graph theory [7], ergodic theory [7–9], unimaginable numbers, Goodstein sequences, Knuth powers, Planar geometry [16], labeled graphs [17], theory of dynamic billiards [12–14,18], statistical physics [19], axiomatic thermodynamics [20] and analysis of many-body interactions [21]. We demonstrate that the Ramsey theory enables the instructive analysis of a discrete sets of points located on closed curves. Monochromatic triangles necessarily appear within the curves when the points are linked with straight lines. The color of the link is assigned by the slope of the straight line. For a set comprising six points located on the closed curve, at least one monochromatic triangle should necessarily appear (Ramsey number $R(3,3) = 6$). This result immediately provides a non-trivial prediction related to the reflective motion of the point within the dynamical billiard. The nearest application of this theorem is related to the design of optical cavities, in which the motion of rays resembles that of elastic balls encountering the curved boundaries of 2D billiards [22]. The second theorem emerges from the combination of the Jordan closed curve and Ramsey theorem. We propose to connect the points located within the same region with green edges and the points placed within different regions with red edges. In this case, the transitivity/intransitivity of the relations between the points should be considered. Ramsey constructions arising from the differential geometry of closed contours are discussed. In this case, transitive/intransitive Ramsey graphs also emerge. The properties of these graphs are discussed.

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