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Article

Extension of Almost Primary Ideals to Noncommutative Rings and the Generalization of Nilary Ideals

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Abstract: In this paper, we introduce the concepts of almost right primary ideals and almost nilary ideals and study their related results. We compare almost right primary ideals with other types of ideals, such as right primary ideals and weakly right primary ideals, and investigate their forms in decomposable rings. Moreover, we study the prime radical of an ideal of the product rings. Finally, we provide a definition of fully almost right primary rings and demonstrate that the homomorphic image of a fully almost right primary ring is again a fully almost right primary ring. We also investigate the quotient structure of fully almost right primary rings.

Keywords: almost right primary ideals; almost nilary ideals; noncommutative rings

MSC: 16N60; 16N99; 16W99

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1. Introduction

In commutative rings, almost prime ideals were introduced by Bhatwadekar and Sharma [1]. Weakly primary ideals were introduced by Atani and Farzalipour [2], while almost primary ideals have been studied in [3]. Recall that a proper ideal P of a commutative ring R is called almost primary if $a, b \in R$ with $ab \in P - P^2$, either $a \in P$ or $b^n \in P$ [3].

The generalization of mathematical topics into noncommutative rings has been a topic of interest for many researchers. We mention, for example, research related to the paper [4,5], which extends the concept of primary ideals (weakly primary ideals) to noncommutative rings. The concept of nilary ideals was introduced in [6], where it was called semi-primary, while (principally) right primary ideals were first introduced in [7] as generalized right primary ideals. Birkenmeier et al. introduced the concept of principally nilary ideals and rings in [4]. A proper ideal P of an arbitrary ring R is called a (principally) nilary ideal if, whenever A and B are (principal) ideals of R with $AB \subseteq P$, then either $A^m \subseteq P$ or $B^n \subseteq P$ for some positive integers m, n. The ideal P of R is called the right primary (right weakly primary) ideal if, whenever A and B are ideals of R with $AB \subseteq P$ ($0 \neq AB \subseteq P$), then either $A \subseteq P$ or $B^n \subseteq P$ for some positive integer n [4,5]. Moreover, P is called almost prime ideal if $AB \subseteq P$ and $AB \nsubseteq P^2$, then either $A \subseteq P$ or $B \subseteq P$ [8].

In this paper, we generalize the concept of almost primary ideals to noncommutative rings by defining principally almost primary ideals and principally almost nilary ideals in Definition 1. We explore the relationships between almost right primary ideals, right primary ideals, and weakly right primary ideals. We determine the forms of almost right primary ideals of decomposable rings and show that if the prime radical of an ideal of the product rings is a principally almost right primary ideal, then it is an idempotent ideal. As in [8–10], there have been studies on the structure of rings, not necessarily commutative, in which all ideals are weakly prime, prime, or almost prime. We also define the concept of a fully almost right primary ring, which is a ring in which every ideal is an almost primary ideal. We prove that the homomorphic image of a fully almost right primary ring is again a fully almost right primary ring and show that if R is a fully almost right primary ring, then so is R/I, where I is an ideal of R.

Throughout this paper, all rings are associative, noncommutative, and without identity unless stated otherwise; by ideal, we mean two-sided ideal. For ideals I and J of a ring R, we adopt the following notation:

(1) $(I : J)^* = \{x \in R \mid xJ \subseteq I\}$ and $I : J = \{x \in R \mid Jx \subseteq I\}$.

(2) The pseudo-radical of an ideal I, $\sqrt{I}$ is the sum of all ideals W of R such that $W^n \subseteq I$ for some $n \in \mathbb{Z}^+$.

(3) $Rad(I)$ is the prime radical of I, i.e., the intersection of all prime ideals of R containing I. $\sqrt{I} \subseteq Rad(I)$.

Lemma 1 (Lemma 1.2. of [4])**.** *Let A, B, and I be ideals of a ring R. Then we have:*

(1) $A \subseteq B$ implies $\sqrt{A} \subseteq \sqrt{B}$.

(2) Assume that $A \subseteq \sqrt{I}$. If A is finitely generated or $(\sqrt{I})^m \subseteq I$ for some positive integer m, then $A^n \subseteq I$ for some positive integer n. In particular, if $\sqrt{I}$ is finitely generated, then $(\sqrt{I})^n \subseteq I$ for some positive integer n.

(3) If $(\sqrt{I})^m \subseteq I$ for some positive integer m, then $\sqrt{I} = Rad(I) = \sqrt{\sqrt{I}}$.

2. Almost Right Primary Ideals

Definition 1. *Let R be a ring.*

(1) An ideal P of R is called a principally almost right primary ideal if, whenever A and B are (principal) ideals of R with $AB \subseteq P$ and $AB \nsubseteq P^2$, then either $A \subseteq P$ or $B^n \subseteq P$, for some $n \in \mathbb{Z}^+$. In the case that $n = 1$, then an almost right primary ideal is referred to as an almost prime ideal.

(2) An ideal P of R is called a principally almost nilary ideal if, whenever A and B are (principal) ideals of R with $AB \subseteq P$ and $AB \nsubseteq P^2$, then either $A^m \subseteq P$ or $B^n \subseteq P$, for some $m, n \in \mathbb{Z}^+$. In the case that $m = 1$, then a principally almost nilary ideal is referred to as a principally almost right primary ideal.

(3) A right ideal P of R is called a principally almost right primary right ideal if, whenever A and B are (principal) right ideals of R with $AB \subseteq P$ and $AB \nsubseteq P^2$, then either $A \subseteq P$ or $B^n \subseteq P$, for some $n \in \mathbb{Z}^+$.

Similar to (1) in Definition 1, a principally almost left primary can be defined, a prime ideal is the right or left almost primary ideal, and every almost right or left primary ideal is an almost nilary ideal. In addition, an almost nilary ideal is an almost right or left primary ideal.

Example 1. *(i) It is clear from Definition 1 that 0 is always an almost right primary (almost nilary) ideal; however, it is not a nilary or right primary ideal in general.*

(ii) One can see that every prime, almost prime, right primary, and weakly right primary ideal is an almost right primary ideal. Hence, the concept of almost right primary ideals is a generalization of almost prime and weakly prime ideals, and, therefore, of prime ideals.

(iii) Every almost right primary ideal is an almost nilary ideal.

(iv) Let $R = F[x, y]$, where F is a field and $P = \langle x^2, xy\rangle$. P is not a primary ideal since $xy \in P$ but $x \notin P$ and $y^m \notin P$ for all $m \in \mathbb{Z}^+$. However, P is an almost nilary ideal since it is a nilary ideal via [5].

(v) Example 2 provides an almost right primary ideal that is neither a nilary nor right primary ideal.

Proposition 1. *Every (principally) weakly right primary ideal P of a ring R is a principally almost right primary ideal.*

Proof. Suppose that $AB \subseteq P$ and $AB \nsubseteq P^2$ for any (principal) ideals A and B of R, then $AB \neq 0$. Thus, either $A \subseteq P$ or $B^n \subseteq P$, for some $n \in \mathbb{Z}^+$. □

Recall that an ideal P of a ring R is called a semiprime ideal, whenever $A^2 \subseteq P$ implies $A \subseteq P$ for any ideal A of R.

Proposition 2. *Let P be a semiprime ideal of a ring R, then P is an almost prime ideal if and only if P is an almost right primary (an almost nilary) ideal.*

Proof. If P is an almost prime ideal, then clearly it is an almost right primary (an almost nilary) ideal. Conversely suppose that $AB \subseteq P$, and $AB \nsubseteq P^2$, for any ideals A and B of R. If $A \nsubseteq P$, then there exists $n \in \mathbb{Z}^+$ such that $B^n \subseteq P$ and, hence, $B^m \subseteq P$ for every $m \geq n$. Thus, $[B^{2^{n-1}}]^2 = B^{2^n} \subseteq P$ since P is a semiprime ideal. Then, $B^{2^{n-1}} \subseteq P$; by repeating the process, we can obtain $B^2 \subseteq P$ and, hence, $B \subseteq P$. $\square$

Remark 1. *Let R be a commutative ring with identity. An ideal P of R satisfying the condition in Definition 1 (1) is an almost primary ideal; this is because for any a and b of R, with $ab \in P - P^2$, we have $\langle a\rangle\langle b\rangle = \langle ab\rangle \subseteq P$, $\langle a\rangle\langle b\rangle \nsubseteq P^2$; thus, either $a \in \langle a\rangle \subseteq P$ or $b^n \in (\langle b\rangle)^n \subseteq P$. However, the converse does not hold in general. Our definition of an almost right primary ideal of a ring with identity is equivalent to the following condition (condition 4 in Theorem 1). For any $a, b \in R$. If $aRb \subseteq P$ and $aRb \nsubseteq P^2$, then either $(a) \subseteq P$ or $((b))^n \subseteq P$, for some $n \in \mathbb{Z}^+$. This is clearly different from the definition of an almost primary ideal of a commutative ring mentioned in the introduction.*

Proposition 3. *Let R be a ring with identity, and P be an ideal of R. Then P is an almost right primary right ideal if and only if P is an almost right primary ideal.*

Proof. Let P be an almost right primary right ideal. Then, clearly, P is an almost right primary ideal. Conversely, suppose that $AB \subseteq P$, and $AB \nsubseteq P^2$, for right ideals A and B of R. then $AR = A$, and $(RA)(RB) = RAB \subseteq RP = P$, for ideals RA and RB. Assume that $(RA)(RB) \subseteq P^2$, then $AB \subseteq RAB = (RA)(RB) \subseteq P^2$, which is a contradiction. Thus, $(RA)(RB) \nsubseteq P^2$, and by (2) we have either $A \subseteq RA \subseteq P$ or $B^m \subseteq (RB)^m \subseteq P$, for some $m \in \mathbb{Z}^+$. $\square$

Proposition 4. *Let R be a ring with identity, and P be an ideal of R. Then P is an almost right primary left ideal if and only if P is an almost right primary ideal.*

Proof. Similar to the proof of Proposition 3. $\square$

Theorem 1. *Let R be a ring with identity, and P be an ideal of R. Then the following statements are equivalent.*

(1) *P is a principally almost nilary ideal.*
(2) *For any $a, b \in R$. If $\langle a\rangle\langle b\rangle \subseteq P$ and $\langle a\rangle\langle b\rangle \nsubseteq P^2$, then either $(\langle a\rangle)^m \subseteq P$ or $(\langle b\rangle)^n \subseteq P$, for some $m, n \in \mathbb{Z}^+$.*
(3) *For any $a, b \in R$. If $(a)\langle b\rangle \subseteq P$ and $(a)\langle b\rangle \nsubseteq P^2$, then either $((a))^m \subseteq P$ or $(\langle b\rangle)^n \subseteq P$, for some $m, n \in \mathbb{Z}^+$.*
(4) *For any $a, b \in R$. If $aRb \subseteq P$ and $aRb \nsubseteq P^2$, then either $((a))^m \subseteq P$ or $((b))^n \subseteq P$, for some $m, n \in \mathbb{Z}^+$.*

Recall that when $m = 1$, then the proof also applies to an almost right primary ideal.

Proof. (1) $\Rightarrow$ (2) Suppose that $\langle a\rangle\langle b\rangle \subseteq P$ and $\langle a\rangle\langle b\rangle \nsubseteq P^2$ then $\langle a\rangle\langle b\rangle \subseteq P$, $\langle a\rangle\langle b\rangle \nsubseteq P^2$; thus, either $(\langle a\rangle)^m \subseteq P$ or $(\langle b\rangle)^n \subseteq P$ and, hence, either $(\langle a))^m \subseteq P$ or $(\langle b))^n \subseteq P$ for some $m, n \in \mathbb{Z}^+$.

(2) $\Rightarrow$ (3) Let $a, b \in R$, such that $(a\rangle(b\rangle \subseteq P$ and $(a\rangle(b\rangle \nsubseteq P^2$, then $\langle a)(b\rangle \subseteq P$ and $\langle a)\langle b) \nsubseteq P^2$, thus by (2) either $(\langle a))^m \subseteq P$ or $(\langle b))^n \subseteq P$ for some $m, n \in \mathbb{Z}^+$ and, hence, $((a\rangle)^m \subseteq P$ or $((b\rangle)^n \subseteq P$.

(3) $\Rightarrow$ (4) Let $a, b \in R$, such that $aRb \subseteq P$, $aRb \nsubseteq P^2$. Then, $(a)\langle b\rangle \subseteq P$ and $(a)\langle b\rangle \nsubseteq P^2$, and by (3) we are done.

(4) $\Rightarrow$ (1) Let $a, b \in R$, such that $\langle a\rangle\langle b\rangle \subseteq P$ and $\langle a\rangle\langle b\rangle \nsubseteq P^2$, then $aRb \subseteq P$ and $aRb \nsubseteq P^2$, hence, by (4), either $((a))^m \subseteq P$ or $((b))^n \subseteq P$, for some $m, n \in \mathbb{Z}^+$. Thus, either $(a) \subseteq \sqrt{P}$ or $(b) \subseteq \sqrt{P}$. If $(a) \subseteq \sqrt{P}$ then $\langle a\rangle \subseteq \sqrt{P}$, and by Lemma 1 we obtain $(\langle a\rangle)^m \subseteq P$ for some $m \in \mathbb{Z}^+$. If $(b) \subseteq \sqrt{P}$, then $(\langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{Z}^+$. $\square$

Proposition 5. *Let R be a ring with identity, and let P be an ideal of R, such that P^2 is the right primary ideal. Then the ideal P is a principally almost nilary ideal if and only if, for any ideals, A and B of R, with $AB \subseteq P$ and $AB \nsubseteq P^2$, it is the case that either $A \subseteq \sqrt{P}$ or $B \subseteq \sqrt{P}$.*

Proof. Suppose that P is a principally almost nilary ideal. For any ideals, A or B of R, suppose that $AB \subseteq P$ and $AB \nsubseteq P^2$. If $A \nsubseteq \sqrt{P}$ then there exists $a \in A$, such that $\langle a\rangle \nsubseteq \sqrt{P}$ and, thus, $(\langle a\rangle)^n \nsubseteq P$ for all $n \in \mathbb{Z}^+$. For any $b \in B$, $\langle a\rangle\langle b\rangle \subseteq AB \subseteq P$. If $\langle a\rangle\langle b\rangle \subseteq P^2$ then either $\langle a\rangle \subseteq P \subseteq \sqrt{P}$, which is a contradiction, or $(\langle b\rangle)^m \subseteq P$ for some $m \in \mathbb{Z}^+$. Thus, $\langle b\rangle \subseteq \sqrt{P}$ for all $b \in B$ and, hence, $B \subseteq \sqrt{P}$. On the other hand, if $\langle a\rangle\langle b\rangle \nsubseteq P^2$, then by assumption we have $B \subseteq \sqrt{P}$.

Conversely, suppose that $\langle a\rangle\langle b\rangle \subseteq P$ and $\langle a\rangle\langle b\rangle \nsubseteq P^2$, then by assumption, either $\langle a\rangle \subseteq \sqrt{P}$ or $\langle b\rangle \subseteq \sqrt{P}$, thus by Lemma 1, either $(\langle a\rangle)^m \subseteq P$ or $(\langle b\rangle)^n \subseteq P$ for some $m, n \in \mathbb{Z}^+$. $\square$

Remark 2. *Observe that in Proposition 5, the mentioned condition always implies that the ideal P is principally almost nilary ideal.*

Proposition 6. *Let P be an ideal of a ring R.*

(1) *If $\sqrt{P}$ is a principally right primary (principally nilary) ideal, then P is a principally right primary (principally nilary) ideal and, hence, a principally almost right primary (principally almost nilary) ideal.*

(2) *If $(\sqrt{P})^2 \subseteq P^2$, i.e., $(\sqrt{P})^2 = P^2$, and $\sqrt{P}$ is an almost right primary (almost nilary) ideal, then so is P.*

Proof. (1) For any principal ideals, A and B of R, suppose that $AB \subseteq P$, then $AB \subseteq \sqrt{P}$, thus either $A \subseteq \sqrt{P}$ ($A^m \subseteq \sqrt{P}$ for some $m \in \mathbb{Z}^+$) or $B^n \subseteq \sqrt{P}$ for some $n \in \mathbb{Z}^+$. Thus, by Lemma 1, either $A^{k_1} \subseteq P$ ($A^{mk_1} \subseteq P$) for some $k_1 \in \mathbb{Z}^+$ or $B^{nk_2} \subseteq P$ for some $k_2 \in \mathbb{Z}^+$. Hence, P is a principally right primary (principally nilary) ideal.

(2) For any ideals, A and B of R, suppose that $AB \subseteq P$ and $AB \nsubseteq P^2$, then $AB \subseteq \sqrt{P}$ and $AB \nsubseteq (\sqrt{P})^2$ and, thus, either $A \subseteq \sqrt{P}$ ($A^m \subseteq \sqrt{P}$) or $B^n \subseteq \sqrt{P}$ for some $m, n \in \mathbb{Z}^+$, and since $(\sqrt{P})^2 \subseteq P^2 \subseteq P$, then by Lemma 1, either $A^{k_1} \subseteq P$ ($A^{mk_1} \subseteq P$) or $B^{nk_2} \subseteq P$ for some $k_1, k_2 \in \mathbb{Z}^+$. $\square$

Theorem 2. *Let R be the ring with identity, and P be an ideal of R. Then the following statements are equivalent.*

(1) *P is a principally almost nilary ideal.*

(2) *For all $a \in R\backslash\sqrt{P}$. Either $(P : \langle a\rangle) = (P^2 : \langle a\rangle)$ or $(P : \langle a\rangle) \subseteq \sqrt{P}$, and either $(P : \langle a\rangle)^* = (P^2 : \langle a\rangle)^*$ or $(P : \langle a\rangle)^* \subseteq \sqrt{P}$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R\backslash\sqrt{P}$. For any $b \in P : \langle a\rangle$, we have $\langle a\rangle b \subseteq P$, thus $\langle a\rangle\langle b\rangle \subseteq PR \subseteq P$.

If $\langle a\rangle\langle b\rangle \subseteq P^2$, then $\langle a\rangle b \subseteq P^2$ and, hence, $b \in P^2 : \langle a\rangle$, thus $P : \langle a\rangle \subseteq P^2 : \langle a\rangle$. Since $P^2 : \langle a\rangle \subseteq P : \langle a\rangle$, we have $P : \langle a\rangle = P^2 : \langle a\rangle$.

If $\langle a\rangle\langle b\rangle \nsubseteq P^2$, then by (1) either $(\langle a\rangle)^m \subseteq P$ for some $m \in \mathbb{Z}^+$, which implies that $\langle a\rangle \subseteq \sqrt{P}$ (a contradiction). Or $(\langle b\rangle)^n \subseteq P$, which means $\langle b\rangle \subseteq \sqrt{P}$, hence $P : \langle a\rangle \subseteq \sqrt{P}$.

Similar to the previous proof, the validity of the other relationship can be proven.

(2) $\Rightarrow$ (1) Suppose that $AB \subseteq P$, such that $A^m \nsubseteq P$ and $B^n \nsubseteq P$ for all $m, n \in \mathbb{Z}^+$, for some principal ideals, A and B of R, then we prove that $AB \subseteq P^2$. Let $a \in A^m \backslash P$. Then, we have $\langle a \rangle B \subseteq AB \subseteq P$, which implies $B \subseteq P : \langle a \rangle$. By (2) either $B \subseteq P^2 : \langle a \rangle$ or $B \subseteq \sqrt{P}$. If $B \subseteq \sqrt{P}$, then by Lemma 1 $B^k \subseteq P$ for some $k \in \mathbb{Z}^+$ (a contradiction). Therefore, $aB \subseteq \langle a \rangle B \subseteq P^2$. Consequently, $(A \backslash P)B \subseteq P^2$.

Let $b \in B^n \backslash P$. Then, $A\langle b \rangle \subseteq AB \subseteq P$, and so $A \subseteq (P : \langle b \rangle)^*$. By (2), we obtain $A \subseteq (P^2 : \langle b \rangle)^* = (P : \langle b \rangle)^*$, because $A \nsubseteq \sqrt{P}$. Thus, $Ab \subseteq A\langle b \rangle \subseteq P^2$, which implies that $A(B \backslash P) \subseteq P^2$. Finally, we have

$$\begin{aligned} AB &= (A \backslash P)B + (A \cap P)(B \backslash P) + (A \cap P)(B \cap P) \\ &\subseteq (A \backslash P)B + A(B \backslash P) + (A \cap P)(B \cap P) \subseteq P^2, \end{aligned}$$

which completes the proof. □

Theorem 3. *Let R be a ring with identity, and P be an almost right primary ideal of R. Then for all $a \in R \backslash P$, the following holds.*

(1) Either $(P : \langle a \rangle) = (P^2 : \langle a \rangle)$ or $(P : \langle a \rangle) \subseteq \sqrt{P}$.
(2) Either $(P : \langle a \rangle)^* = (P^2 : \langle a \rangle)^*$ or $(P : \langle a \rangle)^* \subseteq \sqrt{P}$.

Proof. Similar to the proof of the above theorem. □

Theorem 4. *Let R be a ring, I be an ideal of R. Let P be an ideal of R, such that $I \subseteq P$. If P is an almost right primary ideal of R then P/I is an almost right primary ideal of R/I.*

Proof. Suppose that $\bar{A}\bar{B} \subseteq \bar{P} = P/I$ and $\bar{A}\bar{B} \nsubseteq \bar{P}^2$ for ideals $\bar{A}, \bar{B}$ in R/I. Assume that $\bar{A} = A/I$ and $\bar{B} = B/I$ for some ideals $A \supseteq I$ and $B \supseteq I$. Then, $(AB + I)/I \subseteq P/I$ and $(AB + I)/I \nsubseteq (P^2 + I)/I$, which implies that $AB \subseteq P$ and $AB \nsubseteq P^2$. So, either $A \subseteq P$ or $B^m \subseteq P$, for some $m \in \mathbb{Z}^+$. If $B^m \subseteq P$, then $B^m + I \subseteq P + I$ and, hence, $(\bar{B})^m = (B/I)^m \subseteq \bar{P}$. Or $\bar{A} \subseteq \bar{P}$. □

Theorem 5. *Let R be a ring, and P be an ideal of R. Then P is an almost right primary ideal of R if and only if P/P^2 is a weakly right primary ideal of R/P^2.*

Proof. Suppose that P is an almost right primary ideal. Let $\bar{I}, \bar{J}$ be ideals of R/P^2, such that $\bar{0} \neq \bar{I}\bar{J} \subseteq \bar{P} = P/P^2$. Thus, there exist ideals $I \supseteq P^2$ and $J \supseteq P^2$ of R, such that $\bar{I} = I/P^2$ and $\bar{J}$=J/P^2. Therefore, $\bar{0} \neq (IJ + P^2)/P^2 \subseteq P/P^2$, thus $P^2 \neq IJ \subseteq P$. By assumption, we have that either $I \subseteq P$ or $J^m \subseteq P$, for some $m \in \mathbb{Z}^+$ since $IJ \nsubseteq P^2$. This implies that either $\bar{I} \subseteq \bar{P}$ or $(\bar{J})^m \subseteq \bar{P}$.

For the converse, suppose that I, J are ideals of R, such that $IJ \subseteq P$ and $IJ \nsubseteq P^2$. Then, $\bar{I}$=$(I + P^2)/P^2$, $\bar{J} = (J + P^2)/P^2$ are ideals of R/P^2. Note that,

$$\bar{I}\bar{J} = (IJ + IP^2 + P^2J + P^4 + P^2)/P^2 \subseteq P/P^2 = \bar{P},$$

and $\bar{I}\bar{J} \nsubseteq \bar{P}^2$. Thus, $\bar{0} \neq \bar{I}\bar{J} \subseteq \bar{P}$ and by assumption, either $\bar{I} \subseteq \bar{P}$ or $(\bar{J})^m \subseteq \bar{P}$, for some $m \in \mathbb{Z}^+$. Consequently, $I \subseteq P$ or $J^m \subseteq P$. □

Theorem 6. *Let I be an almost right primary ideal of a ring R. If $\bar{P}$ is a weakly right primary ideal of R/I, then there exists an almost right primary ideal P of R with $I \subseteq P$, such that $\bar{P} = P/I$.*

Proof. It is clear that $\bar{P} = P/I$ where P is an ideal of R with $I \subseteq P$. Suppose that $AB \subseteq P$, $AB \nsubseteq P^2$, for any ideals, A and B of R. Obviously, $AB \nsubseteq I^2$.

If $AB \subseteq I$, then either $A \subseteq I \subseteq P$ or $B^n \subseteq I \subseteq P$, for some $n \in \mathbb{Z}^+$.

If $AB \nsubseteq I$, then $I/I \neq (AB + I)/I \subseteq P/I$, and since P/I is a weakly right primary ideal, then either $(A + I)/I \subseteq P/I$ or $[(B + I)/I]^m \subseteq P/I$ for some $n \in \mathbb{Z}^+$; thus, either $A \subseteq P$ or $B^m \subseteq P$. Hence, P is an almost right primary ideal. □

Theorem 7. *Let f: $R \to S$ be a ring epimorphism, and P be an almost right primary ideal of R, such that* $\ker f \subseteq P$*. Then $f(P)$ is an almost right primary ideal of S.*

Proof. Suppose that $A_2B_2 \subseteq f(P)$ and $A_2B_2 \nsubseteq (f(P))^2$ for any ideals A_2, B_2 of S. Then, the inverse images A_1 and B_1 of A_2 and B_2, respectively, are ideals of R containing the kernel of f. Since f is an epimorphism, then $f(A_1) = A_2$ and $f(B_1) = B_2$. Then, we have: $f(A_1B_1) = A_2B_2 \subseteq f(P)$ and $f(A_1B_1) \nsubseteq (f(P))^2 = f(P^2)$. Thus, $A_1B_1 \subseteq f^{-1}(f(A_1B_1)) \subseteq f^{-1}(f(P)) = P$ and $A_1B_1 \nsubseteq P^2$.

By assumption, either $A_1 \subseteq P$ or $B_1^m \subseteq P$, for some $m \in \mathbb{Z}^+$. If $A_1 \subseteq P$, then $A_2 \subseteq f(P)$, and if $B_1^m \subseteq P$, then $B_2^m = [f(B_1)]^m = f(B_1^m) \subseteq f(P)$. □

Corollary 1. *Let $f : R \to S$ be a ring epimorphism, and B be an ideal of S, such that $f^{-1}(B)$ is an almost right primary ideal of R. Then, B is an almost right primary ideal of S.*

Proof. Since the inverse image of any ideal of S is an ideal of R containing $\ker f$, the proof follows from Theorem 7. □

Theorem 8. *Let $f : R \to S$ be a ring epimorphism, and P be an ideal of R, such that* $\ker f \subseteq P^2$*. If $f(P)$ is an almost right primary ideal of S, then P is an almost right primary ideal of R.*

Proof. Suppose that $A_1B_1 \subseteq P$ and $A_1B_1 \nsubseteq P^2$ for any ideals A_1, B_1 of R. Then, $f(A_1)f(B_1) = f(A_1B_1) \subseteq f(P)$. Assume that $f(A_1B_1) \subseteq f(P^2)$, then $A_1B_1 \subseteq f^{-1}(f(A_1B_1)) \subseteq f^{-1}(f(P^2)) = P^2$, which is a contradiction. Hence, $f(A_1)f(B_1) = f(A_1B_1) \nsubseteq (f(P))^2$. Since $f(P)$ is an almost right primary ideal of S, then either $f(A_1) \subseteq f(P)$ or $f(B_1^m) = [f(B_1)]^m \subseteq f(P)$, for some $m \in \mathbb{Z}^+$. Thus, either $A_1 \subseteq f^{-1}(f(A_1)) \subseteq f^{-1}(f(P)) = P$ or $B_1^m \subseteq P$. □

Corollary 2. *Let $f : R \to S$ be a ring epimorphism and B be an almost right primary ideal of S, such that* $\ker f \subseteq (f^{-1}(B))^2$*. Then, $f^{-1}(B)$ is an almost right primary ideal of R.*

Proof. Let $P = f^{-1}(B)$. Then, P is an almost right primary ideal of R by Theorem 8, since $\ker f \subseteq P^2$ and $f(P) = f(f^{-1}(B)) = B$ is an almost right primary ideal of S. □

As in Theorems 4, 7, and 8, the almost nilary version can be proven analogously, and we obtain the following theorems.

Theorem 9. *Let R be a ring, I be an ideal of R. Let P be an ideal of R, such that $I \subseteq P$. If P is an almost nilary ideal of R, then P/I is an almost nilary ideal of R/I.*

Theorem 10. *Let f: $R \to S$ be a ring epimorphism and P be an almost nilary ideal of R, such that* $\ker f \subseteq P$*. Then $f(P)$ is an almost nilary ideal of S.*

Theorem 11. *Let $f : R \to S$ be a ring epimorphism, and P be an ideal of R, such that* $\ker f \subseteq P^2$*. If $f(P)$ is an almost nilary ideal of S, then P is an almost nilary ideal of R.*

In the next Theorem, we show the analogy between the right primary and almost right primary.

Theorem 12. *Let R be the ring with identity, and P be an ideal of R, such that $(P^2 : \langle c \rangle) \subseteq P$ for any $c \in P$. Then P is the right primary ideal if and only if P is a principally almost right primary ideal.*

Proof. Suppose that P is the right primary ideal. Then clearly P is a principally almost right primary ideal.

For the converse implication, assume that the principally almost right primary ideal P is not the right primary ideal. Then, there exist principal ideals, A or B of R, such that $AB \subseteq P$ with $A \nsubseteq P$ and $B^m \nsubseteq P$, for every $m \in \mathbb{Z}^+$, so by Lemma 1, $B \nsubseteq \sqrt{P}$. Hence, by assumption, we obtain $AB \subseteq P^2$. Let $a \in A \backslash P$ and $b \in B \backslash \sqrt{P}$ Then, for any $x \in P$, we have:

$$(\langle a \rangle + \langle x \rangle)\langle b \rangle = \langle a \rangle . \langle b \rangle + \langle x \rangle \langle b \rangle \subseteq AB + P\langle b \rangle \subseteq P.$$

If $(\langle a \rangle + \langle x \rangle)\langle b \rangle \subseteq P^2$, then $\langle x \rangle \langle b \rangle \subseteq P^2$. This implies that $\langle x \rangle b \subseteq P^2$ and, thus, $b \in (P^2 : \langle x \rangle) \subseteq P \subseteq \sqrt{P}$. This contradicts with $b \in B \backslash \sqrt{P}$. If $(\langle a \rangle + \langle x \rangle)\langle b \rangle \nsubseteq P^2$, then since P is a principally almost right primary ideal, we either have $\langle a \rangle + \langle x \rangle \subseteq P$ or $[\langle b \rangle\,]^n \subseteq P$, for some $n \in \mathbb{Z}^+$, which implies either $a \in P$ or $b \in \langle b \rangle \subseteq \sqrt{P}$, respectively (a contradiction). □

Corollary 3. *Let R be the ring and P be an ideal of R, such that $P^2 = 0$. Then P is a weakly right primary ideal if and only if P is an almost right primary ideal.*

Proof. If P is a weakly right primary, then P is an almost right primary ideal.

Now suppose that P is an almost right primary ideal. Let A and B be any ideals of R, such that $0 \neq AB \subseteq P$. Then $AB \nsubseteq P^2 = 0$. Thus, we are done. □

The next result is a consequence of Corollary 3.

Corollary 4. *Let R be the ring, such that $R^2 = 0$, and let P be an ideal of R. Then P is a weakly right primary ideal if and only if P is an almost right primary ideal.*

Proposition 7. *Let (R, M) be a local ring, and let P be an ideal of R, such that $P^2 = M^2$. Then P is an almost right primary ideal.*

Proof. Let A and B be ideals of R. Then, $A \subseteq M$ and $B \subseteq M$. Thus, $AB \subseteq M^2 = P^2$, which yields that P is an almost right primary ideal. □

3. Decomposable Rings

Definition 2. *A ring R is called decomposable if $R = R_1 \times R_2$ for some nontrivial rings R_1 and R_2.*

It is well known that if the rings R_1, R_2 are with identities, then any ideal of $R_1 \times R_2$ has the form $I \times J$, where I and J are ideals of R_1 and R_2, respectively. If I is an ideal of R_1, then $\sqrt{I \times R_2} = \sqrt{I} \times R_2$, and $\sqrt{R_1 \times J} = R_1 \times \sqrt{J}$, for any ideal J of R_2. Theorem 6 in [11] states that an ideal $\mathcal{P}$ of the direct product of commutative rings R, S is prime if and only if $\mathcal{P}$ has the form $P \times S$ where P is a prime ideal of R or $R \times Q$ where Q is a prime ideal of S. In the following, we show that this theorem is just a special property from a more general case.

Lemma 2. *Let R and S be any rings with identities. An ideal $\mathcal{P}$ of $R \times S$ is prime if and only if $\mathcal{P}$ has the form $P \times S$ where P is a prime ideal of R or $R \times Q$ where Q is a prime ideal of S.*

Proof. $(\Rightarrow)$ Suppose that $\mathcal{P} = P \times Q$ is a prime ideal of $R \times S$ where P, Q are ideals of R, S, respectively. Then $(P \times S)(R \times Q) \subseteq \mathcal{P}$, then either $(P \times S) \subseteq P \times Q$, which implies $S = Q$ or $(R \times Q) \subseteq P \times Q$, which implies $R = P$. The rest of the proof (that the ideal P (Q) of $P \times S$ $(R \times Q)$ is prime) is straightforward.

$(\Leftarrow)$ Can be easily verified. □

Lemma 3. *Let R and S be any rings with identities. Let $I \times J$ be an ideal of $R \times S$. Then $Rad(I \times J) = Rad(I) \times Rad(J)$.*

Proof. $Rad(I) \times Rad(J) = \underset{i \in L}{\cap} A_i \times \underset{j \in N}{\cap} B_j$ where for every $i \in L$ $(j \in N)$, A_i (B_j) is a prime ideal of R (S) containing I (J). Thus

$$Rad(I) \times Rad(J) = (A_1 \times S) \cap \ldots \cap (A_l \times S) \cap (R \times B_1) \cap \ldots \cap (R \times B_n)$$

Thus, $Rad(I) \times Rad(J) = \underset{i \in K}{\cap} \mathcal{P}_i$, where each $\mathcal{P}_i$ is a prime ideal of $R \times S$ containing $I \times J$ by Lemma 2. Hence, $Rad(I) \times Rad(J) = Rad(I \times J)$ □

Now we are ready to characterize the forms of almost right primary ideals of decomposable rings. Recall that Lemma 8 (2) of [11] states that if Q is a primary ideal of $R \times S$ with $\sqrt{Q} \neq R \times S$ where R, S are commutative rings, then either $Q = Q_1 \times S$ where Q_1 is a primary ideal of R or $Q = R \times Q_2$, where Q_2 is a primary ideal of S. However the same characteristic is not quite true of the almost right primary ideals, as we shall see below.

Theorem 13. *Let R_1 and R_2 be any rings, and P be an ideal of R_1. Then the following statements are equivalent.*

(1) *P is a principally almost right primary ideal of R_1.*
(2) *$P \times R_2$ is a principally almost right primary ideal of $R_1 \times R_2$.*

Proof. $(1) \Rightarrow (2)$ Let $(A_1 \times B_1)(A_2 \times B_2) \subseteq (P \times R_2)$, and

$$(A_1 \times B_1)(A_2 \times B_2) \nsubseteq (P \times R_2)^2,$$

where A_1, A_2 are principal ideals of R_1, and B_1, B_2 are principal ideals of R_2. Then,

$$(A_1 A_2) \times (B_1 B_2) \subseteq P \times R_2,$$

and $(A_1 A_2) \times (B_1 B_2) \nsubseteq (P^2 \times R_2^2)$. Thus, $A_1 A_2 \subseteq P$ and $A_1 A_2 \nsubseteq P^2$. Hence, by (1) either $A_1 \subseteq P$ or $A_2 \subseteq \sqrt{P}$. This implies that either $A_1 \times B_1 \subseteq P \times R_2$ or $A_2 \times B_2 \subseteq \sqrt{P} \times R_2 = \sqrt{P \times R_2}$. Hence, either $A_1 \times B_1 \subseteq P \times R_2$ or $(A_2 \times B_2)^m \subseteq P \times R_2$ for some $m \in \mathbb{Z}^+$, by Lemma 1.

$(2) \Rightarrow (1)$ Let I, J be principal ideals of R_1, such that $IJ \subseteq P$ and $IJ \nsubseteq P^2$. Then $(I \times R_2)(J \times R_2) \subseteq (P \times R_2)$ and $(I \times R_2)(J \times R_2) \nsubseteq (P \times R_2)^2$. Thus, by (2) either $I \times R_2 \subseteq P \times R_2$ or $(J \times R_2)^m \subseteq P \times R_2$, which implies that either $I \subseteq P$ or $J^m \subseteq P$, for some $m \in \mathbb{Z}^+$. □

Remark 3. *As a modification of Theorem 13, one can easily show that if P is an ideal of R_2, then P is a principally almost right primary ideal of R_2 if and only if $R_1 \times P$ is a principally almost right primary ideal of $R_1 \times R_2$.*

Corollary 5. *Let $R = \prod_{i=1}^{n} R_i$ for the rings $R_1, \ldots, R_n$. If for some $j \in \{1, 2, ..., n\}$, P_j is a principally almost primary ideal of R_j, then $R_1 \times R_2 \times \ldots \times P_j \times \ldots \times R_n$ is a principally almost right primary ideal of R.*

Theorem 14. *Let P be an ideal of $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities. If $P = I \times J$ is a principally almost right primary ideal of R, where I, J are proper ideals of R_1 and R_2, respectively. Then P is the idempotent ideal.*

Proof. If both I and J are idempotent then P is idempotent. Thus, assume that either ideal I or J is not idempotent, without loss of generality, assume that I is not an idempotent ideal of R_1. Then, there exists an element $x \in I \backslash I^2$. Thus, $\langle x \rangle \subseteq I$ and $\langle x \rangle \nsubseteq I^2$. Therefore,

$$\langle x \rangle \times 0 \subseteq I \times J \text{ and } \langle x \rangle \times 0 \nsubseteq I^2 \times J^2.$$

Now assume that $(\langle x \rangle \times R_2)(R_1 \times 0) \subseteq I^2 \times J^2$.

Thus, $\langle x\rangle \times 0 = \langle x\rangle R_1 \times R_2 \cdot 0 = (\langle x\rangle \times R_2)(R_1 \times 0) \subseteq I^2 \times J^2$, which is a contradiction. Therefore, $(\langle x\rangle \times R_2)(R_1 \times 0) \nsubseteq I^2 \times J^2$. On the other hand, $(\langle x\rangle \times R_2)(R_1 \times 0) = \langle x\rangle \times 0 \subseteq I \times J$. Because the ideal $I \times J$ is a principally almost right primary ideal, and $\langle x\rangle \times R_2$, $R_1 \times 0$ are principal ideals generated by $(x, 1_{R_2})$ and $(1_{R_1}, 0)$ of R_1, R_2, respectively, we have that either $\langle x\rangle \times R_2 \subseteq I \times J$ or $(R_1 \times 0)^m \subseteq I \times J$. Hence, either $R_2 = J$ or $R_1 = I$, which yields a contradiction. Thus, I must be idempotent; hence, P is an idempotent ideal. □

Theorem 15. *Let P be an ideal of $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities. Then P is a principally almost right primary ideal of R if and only if it has one of the following forms.*

(1) *$I \times R_2$, where I is a principally almost right primary ideal of R_1.*
(2) *$R_1 \times J$, where J is a principally almost right primary ideal of R_2.*
(3) *$I \times J$, where I and J are idempotent ideals of R_1 and R_2, respectively.*

Proof. We come up with the proof by Theorem 13, Remark 3, Theorem 14, and the fact that every idempotent ideal is an almost right primary ideal. □

Example 2. *Let $R = \mathbb{Z}_{12} \times S$, where S is any ring with identity. Let I and J be any nonzero idempotent ideals of S, such that $IJ = 0$. The ideal $P = \langle 4\rangle \times 0$ is an almost right primary ideal as a consequence of the fact that P is idempotent. However, by Lemma 2, the ideal $P = \langle 4\rangle \times 0$ is not prime ideal. Moreover, since $(\langle 2\rangle \times I)(\langle 6\rangle \times J) \subseteq \langle 4\rangle \times 0$, $(\langle 2\rangle \times I)^m \nsubseteq \langle 4\rangle \times 0$ and $(\langle 6\rangle \times I)^n \nsubseteq \langle 4\rangle \times 0$, for all $m, n \in \mathbb{Z}^+$, then P is neither a nilary nor the right primary ideal.*

Theorem 16. *Let R and S be any rings with identities. Let $I \times J$ be an ideal of $R \times S$. If $Rad(I \times J)$ is a principally almost right primary ideal of $R \times S$, then it is an idempotent ideal.*

Proof. Suppose that $P = Rad(I \times J)$ is a principally almost right primary ideal for any ideals I, J, of R, S, respectively. Then by Theorem 15, we have three cases.

(1) If $P = A \times S$ where A is a principally almost right primary ideal of R then by Lemma 3 we obtain $Rad(J) = S$, which is a contradiction since $Rad(J)$ is proper of the ring S.
(2) If $P = R \times B$, where B is a principally almost right primary ideal of S then by Lemma 3, we obtain $Rad(I) = R$ (a contradiction).
(3) If $P = A \times B$, where A, B are idempotent ideals of R and S, respectively, then by Lemma 3, we obtain $Rad(I) = A$ and $Rad(J) = B$, which are idempotent ideals and, hence, so is P.

□

Proposition 8. *Let R_1 and R_2 be rings with identities. If every ideal of R_1 and R_2 is a product of a principally almost right primary ideal, then every ideal of $R_1 \times R_2$ is a product of a principally almost right primary ideal.*

Proof. Let I and J be ideals of R_1 and R_2, respectively, and $I = A_1 \cdots A_n$ and $J = B_1 \cdots B_m$ for principally almost right primary ideals A_i and B_j, and let P be an ideal of $R_1 \times R_2$. Then P must have one of the following three forms by Theorem 15.

(1) If $P = I \times R_2$, then

$$P = (A_1 \cdots A_n) \times R_2 = (A_1 \times R_2) \cdots (A_n \times R_2).$$

(2) If $P = R_1 \times J$, then

$$P = R_1 \times (B_1 \cdots B_m) = (R_1 \times B_1) \cdots (R_1 \times B_m).$$

(3) Finally, if $P = I \times J$, then

$$P = (A_1 \cdots A_n) \times (B_1 \cdots B_m) = (A_1 \times R_2) \cdots (A_n \times R_2)(R_1 \times B_1) \cdots (R_1 \times B_m).$$

In all cases, we obtain a product of principally almost right primary ideals of R due to Theorem 15. □

4. Fully almost Right Primary Rings

Definition 3. *A ring in which every ideal is a principally almost right primary ideal is called a fully (principally) almost right primary ring.*

Note that every fully prime ring (fully weakly prime ring, fully idempotent ring, fully almost prime, fully weakly right primary) is a fully almost right primary ring.

The next result is a consequence of Corollary 3.

Corollary 6. *Let R be a ring, such that $P^2 = 0$ for every ideal P of R. Then, R is a fully almost right primary ring if and only if R is a fully weakly right primary ring.*

Remark 4. *Corollary 4 suggests that the assumption of Corollary 6 can be replaced by $R^2 = 0$.*

Corollary 7. *Every local ring (R, M) with $M^2 = 0$ is a fully almost right primary ring.*

Proof. For any ideal P of R, we have that $P^2 = M^2 = 0$. Thus, P is an almost right primary ideal by Proposition 7. □

Theorem 17. *Let R be a ring, and I be an ideal of R. If R is a fully almost right primary ring, so is R/I.*

Proof. Suppose $\bar{P}$ is an ideal of R/I. Then, there exists an ideal $P \supseteq I$ of R, such that $\bar{P} = P/I$. Clearly, P is an almost right primary ideal of R. Hence, by Theorem 4, $\bar{P}$ is an almost right primary ideal of R/I. □

Theorem 18. *Let $f : R \to S$ be a ring epimorphism. If R is a fully almost right primary ring, so is S.*

Proof. Let P be an ideal of S. Then $f^{-1}(P) \supseteq \ker f$ is an almost right primary ideal of R. Then, by Theorem 7, $f(f^{-1}(P)) = P$ is an almost right primary ideal of S. □

Theorem 19. *Let $f : R \to S$ be a ring epimorphism, such that $\ker f \subseteq I^2$, for any ideal I of R. If S is a fully almost right primary ring, so is R.*

Proof. Let P be an ideal of R. Then, $f(P)$ is an almost right primary ideal of the fully almost right primary ring S. Hence, by Theorem 8, P is an almost right primary ideal of R. □

Remark 5. *The nilary version of Theorems 17, 18, and 19, can be obtained by using the Theorems 9, 10, and 11 respectively.*

Note that every fully idempotent ring is a fully almost right primary ring. However, the converse does not hold in general. In the following theorem, we show that the equivalence holds in direct product rings.

Theorem 20. *Let $R = R_1 \times R_2$, where R_1, R_2 are rings with identities. Then the following statements are equivalent.*

(1) R is a fully idempotent ring.
(2) R is a fully principally almost right primary ring.

Proof. $(1) \Rightarrow (2)$ Clear.

$(2) \Rightarrow (1)$ Assume that P is an ideal of R, which is not idempotent. Then by Theorem 15, we have three cases for the form of P.

Case 1. $P = I \times R_2$, where I is a principally almost right primary ideal of R_1. Then I is not an idempotent ideal of R_1. So there exists an element $x \in I \backslash I^2$, which yields that $\langle x \rangle \subseteq I$ and $\langle x \rangle \nsubseteq I^2$. Hence,

$$\langle x \rangle \times 0 \subseteq I \times 0 \ \text{ and } \ \langle x \rangle \times 0 \nsubseteq I^2 \times 0.$$

Thus $(\langle x \rangle \times R_2)(R_1 \times 0) \subseteq I \times 0$ and

$$(\langle x \rangle \times R_2)(R_1 \times 0) \nsubseteq I^2 \times 0^2 = (I \times 0)^2.$$

Since $I \times 0$ is an almost principally right primary ideal, then either $\langle x \rangle \times R_2 \subseteq I \times 0$ or $(R_1 \times 0)^n \subseteq I \times 0$, for some $n \in \mathbb{Z}^+$. This implies that either $R_2 = 0$ or $I = R_1$, which is a contradiction. So I must be idempotent.

Case 2. $P = R_1 \times J$ where J is a principally almost right primary ideal of R_2. Similar to the proof of case (1), one can see that J must be idempotent.

Case 3. $P = I \times J$, where I and J are idempotent ideals of R_1 and R_2, respectively. Clearly, P is an idempotent ideal of R. □

Corollary 8. *Let R_1 and R_2 be rings with identities. Then the following statements are equivalent.*

(1) $R_1 \times R_2$ *is a fully principally almost right primary ring.*
(2) R_1 *and* R_2 *are fully idempotent rings.*

Theorem 21. *Let $R = \prod_{i=1}^{n} R_i$ for the rings $R_1, \ldots, R_n$. If R is a fully principally almost right primary ring, then so is R_i for every $i = 1, \ldots, n$.*

Proof. Let $\pi_i : R \to R_i$ be the projective epimorphism, where

$$\pi_i(a_1, a_2, \ldots, a_i, \ldots, a_n) = a_i,$$

for all $i = 1, \ldots, n$. Then, R_i is a fully principally almost right primary ring for all $i = 1, \ldots, n$, by Theorem 18. □

Remark 6. *Theorem 15 shows that the converse of Theorem 21 is not true in general, in other words, the direct product of a fully almost right primary ring does not need to be a fully principally almost right primary ring. The following corollary gives a special case, such that the direct product of a fully principally almost right primary ring is a fully principally almost right primary ring.*

Corollary 9. *If a ring R is a fully principally almost right primary ring, then so are the rings R/I_1 R/I_2 $(R/I_1) \times (R/I_2)$, where I_1 and I_2 are any comaximal ideals of R.*

Proof. R/I_1 and R/I_2 are fully principally almost right primary rings by Theorem 17. Now, by the epimorphism $\phi : R \to (R/I_1) \times (R/I_2)$, defined as $\phi(r) = (r + I_1, r + I_2)$, the proof is complete by Theorem 18. □

5. Conclusions

In this paper, we introduce the concepts of almost right primary ideals and almost nilary ideals and study their related results. We compare almost right primary ideals with other types of ideals, such as right primary ideals and weakly right primary ideals, and investigate their forms in decomposable rings. Moreover, we study the prime radical of an ideal of the product rings. Finally, we provide a definition of fully almost right primary rings and demonstrate that the homomorphic image of a fully almost right primary ring is again a fully almost right primary ring. We also investigate the quotient structures of fully almost right primary rings.

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