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# A Brief Survey and an Analytic Generalization of the Catalan Numbers and Their Integral Representations †

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**Abstract:** In the paper, the authors briefly survey several generalizations of the Catalan numbers in combinatorial number theory, analytically generalize the Catalan numbers, establish an integral representation of the analytic generalization of the Catalan numbers by virtue of Cauchy's integral formula in the theory of complex functions, and point out potential directions for further study.

**Keywords:** Catalan number; generalized Catalan function; generalized Catalan number; Cauchy's integral formula; generalization; generating function; integral representation

**MSC:** 05A15; 11B75; 11B83; 26A09; 30E20; 41A58

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## 1. A Brief Survey of the Catalan Numbers and Their Generalizations

The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)} \tag{1}$$

form a sequence of integers [1–3], have combinatorial interpretations [2,4], have a long history [1,5], and can be generated [4,6] by

$$G(x) = \frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n, \tag{2}$$

where

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \ne 0, -1, -2, \dots$$

is the classical Euler gamma function [7].

A generalization of the Catalan numbers $C_n$ was defined in References [8–10] by

$$_p d_n = \frac{1}{n}\binom{pn}{n-1} = \frac{1}{(p-1)n+1}\binom{pn}{n} \tag{3}$$

for $n, p \ge 1$. It is obvious that $C_n = {}_2d_n$. In Reference [2] (pp. 375–376), the generalization $_{p+1}d_n$ is denoted by $C(n,p)$ for $p \ge 0$ and is termed the generalized Catalan numbers. In Reference [2] (pp. 377–378), the Fuss numbers

$$F(m,n)=\frac{1}{mn+1}\binom{mn+1}{n} \tag{4}$$

were given and discussed. It is apparent that $F(2,n)=C_n$.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p,r)$ are defined [11] as numbers of the form

$$A_n(p,r)=\frac{r}{np+r}\binom{np+r}{n}=\frac{r\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}. \tag{5}$$

It is easy to see that

$$A_n(p,1)=F(p,n), \quad A_n(2,1)=C_n, \quad n\ge 0$$

and

$$A_{n-1}(p,p)={}_pd_n=C(n,p-1), \quad n,p\ge 1.$$

There has been some discussion in the literature, such as in References [2,6,12], on the investigation of the Fuss–Catalan numbers $A_n(p,r)$.

In the paper [13], starting from the second expression in Equation (1) in terms of gamma functions, the Catalan numbers $C_n$ were analytically generalized to

$$C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a),\Re(b)>0, \quad \Re(z)\ge 0 \tag{6}$$

with

$$C\left(\frac{1}{2},2;n\right)=C_n, \quad n\ge 0. \tag{7}$$

It is not difficult to verify that

$$C(n+1,2;(m-1)n)=\left(\frac{2}{n+1}\right)^{(m-1)n}{}_md_n=\left(\frac{2}{n+1}\right)^{(m-1)n}C(n,m-1)$$

for $m,n\ge 1$. Thereafter, the Catalan–Qi function $C(a,b;z)$ and its analytic generalizations were thoroughly investigated in References [5,14–23] and closely related sources therein.

In Reference [13], it was determined that

$$\begin{aligned}C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z&\frac{(z+a)^z}{(z+b)^{z+b-a}}\\ &\times\exp\left[b-a+\int_0^\infty\frac{1}{t}\left(\frac{1}{1-\mathrm{e}^{-t}}-\frac{1}{t}-a\right)\left(\mathrm{e}^{-at}-\mathrm{e}^{-bt}\right)\mathrm{e}^{-zt}\,\mathrm{d}\,t\right]\end{aligned}$$

for $\Re(a),\Re(b)>0$ and $\Re(z)\ge 0$. In Theorem 1.1 of Reference [16], we discovered several relations between the Fuss–Catalan numbers $A_n(p,r)$ and the Catalan–Qi numbers $C(a,b;n)$, one of which reads that

$$A_n(p,r)=r^n\frac{\prod_{k=0}^{p-1}C\left(\frac{k+r}{p},1;n\right)}{\prod_{k=0}^{p-2}C\left(\frac{k+r+1}{p-1},1;n\right)} \tag{8}$$

for integers $n\ge 0$, $p>1$, and $r>0$. In the series of papers [13,15–17,19,24], among other things, some properties, including the general expression and a generalization of the asymptotic expansion

$$\frac{4^x\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(x+2)}\sim\frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}}-\frac{9}{8}\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\cdots\right),$$

the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities of the Catalan numbers $C_n$, the Catalan–Qi numbers $C(a,b;n)$, the Catalan–Qi function $C(a,b;z)$, and the Fuss–Catalan numbers $A_n(p,r)$ were established.

In Reference [18], the notion

$$Q(a,b;p,q;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z}[\Gamma(z+1)]^{q-p}\frac{\Gamma(pz+a)}{\Gamma(qz+b)} \tag{9}$$

was introduced, where $\Re(a),\Re(b)>0$, $\Re(p),\Re(q)>0$, and $\Re(z)\ge 0$. We call the quantity $Q(a,b;p,q;z)$ the Fuss–Catalan–Qi function and, when taking $z=n\ge 0$, call $Q(a,b;p,q;n)$ the Fuss–Catalan–Qi numbers. It is easy to see that

$$Q\left(\frac{1}{2},2;1,1;n\right)=Q(1,2;2,1;n)=C_n,$$
$$Q(r,r+1;p,p-1;n)=A_n(p,r),$$
$$Q(p,p+1;p,p-1;n-1)={}_pd_n=C(n,p-1),$$
$$Q(a,b;1,1;z)=C(a,b;z).$$

For $\Re(a),\Re(b)>0$ and $\Re(z)\ge 0$, when $p,q\in\mathbb{N}$, we generalized Equation (8) as

$$Q(a,b;p,q;z)=\left[\left(\frac{b}{a}\right)^{q-p+1}\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(q+b)}\right]^z\frac{\prod_{k=0}^{p-1}C\left(\frac{k+a}{p},1;z\right)}{\prod_{k=0}^{q-1}C\left(\frac{k+b}{q},1;z\right)}.$$

The Catalan numbers $C_n$ for $n\ge 0$ have several integral representations which have been surveyed in Section 2 of Reference [5]. The integral representation

$$C_n=\frac{1}{2\pi}\int_0^4\sqrt{\frac{4-x}{x}}\,x^n\,\mathrm{d}\,x,\quad n\ge 0 \tag{10}$$

was highlighted in Reference [25] and applied in Reference [21]. An alternative integral representation

$$C_n=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)^{n+2}}\,\mathrm{d}\,t \tag{11}$$

was derived from the integral representation

$$\frac{1}{1+\sqrt{1-4x}}=\frac{1}{2\pi}\int_0^\infty\frac{\sqrt{t}}{1/4+t}\frac{1}{1/4+t-x}\,\mathrm{d}\,t,\quad x\in\left(-\infty,\frac{1}{4}\right] \tag{12}$$

of the generating function $G(x)$ as given in Equation (2), which was established in Theorem 1.3 of Reference [19] by virtue of Cauchy's integral formula in the theory of complex functions.

The generalized Catalan function $C(a,b;z)$ defined by Equation (6) also has several integral representations which have been surveyed in Section 2 of Reference [5]. For example, corresponding to integral representations in Equations (10) and (11), integral representations

$$C(a,b;x)=\left(\frac{a}{b}\right)^{b-1}\frac{1}{B(a,b-a)}\int_0^{b/a}\left(\frac{b}{a}-t\right)^{b-a-1}t^{x+a-1}\,\mathrm{d}\,t \tag{13}$$

and

$$C(a,b;x)=\left(\frac{a}{b}\right)^{a}\frac{1}{B(a,b-a)}\int_0^\infty\frac{t^{b-a-1}}{(t+a/b)^{x+b}}\,\mathrm{d}\,t. \tag{14}$$

for $b > a > 0$ and $x \ge 0$ were established in Theorem 4 of Reference [17], where the classical beta function $B(z,w)$ can be defined or expressed [26] by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}\,t = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}\,\mathrm{d}\,t = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

for $\Re(z), \Re(w) > 0$. We note that, when letting $a = \frac{1}{2}$ and $b = 2$, the integral representations in Equations (13) and (14) become those in Equations (10) and (11), respectively.

The generating function $G(x)$ in Equation (2) can be regarded as a special case $a = \frac{1}{2}$, $b = \frac{1}{4}$, and $c = 1$ of the function

$$G_{a,b,c}(x) = \frac{1}{a+\sqrt{b-cx}}, \quad a \ge 0, b, c > 0.$$

Essentially, it is better to regard the function

$$G_{a,b}(x) = \frac{1}{a+\sqrt{b-x}}, \quad a \ge 0, b > 0 \tag{15}$$

as a generalization of the generating function $G(x)$, because

$$G_{1/2,1/4}(x) = G(x), \quad G_{a,b}(x) = G_{a,b,1}(x), \quad G_{a,b,c}(x) = \frac{G_{a/\sqrt{c},b/c}(x)}{\sqrt{c}},$$

but we can not express $G_{a,b}(x)$ in terms of $G(x)$.

Now we would like to pose the following three questions.

(1) Can one establish an explicit formula for the sequence $\mathcal{C}(a,b;n)$ generated by

$$G_{a,b}(x) = \frac{1}{a+\sqrt{b-x}} = \sum_{n=0}^{\infty} \mathcal{C}(a,b;n)x^n \tag{16}$$

for $a \ge 0$ and $b > 0$?

(2) Can one find an integral representation for the sequence $\mathcal{C}(a,b;n)$ by finding an integral representation of the generating function $G_{a,b}(x)$ in Equation (15)?

(3) Can one combinatorially interpret the sequence $\mathcal{C}(a,b;n)$ or some special case of $\mathcal{C}(a,b;n)$ except the case $a = \frac{1}{2}$ and $b = \frac{1}{4}$?

It is easy to see that

$$\lim_{a\to 0^+} \mathcal{C}(a,b;n) = \frac{(-1)^n}{n!}\left\langle -\frac{1}{2}\right\rangle_n \frac{1}{b^{(2n+1)/2}} \tag{17}$$

and

$$\mathcal{C}\left(\frac{1}{2},\frac{1}{4};n\right) = C_n \tag{18}$$

for $n \ge 0$, where the notation

$$\langle \alpha \rangle_n = \prod_{k=0}^{n-1}(\alpha - k) = \begin{cases} \alpha(\alpha-1)\cdots(\alpha-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

for $\alpha \ne 0$ is called the falling factorial [27,28]. Comparing Equation (18) with Equation (7) reveals that $C(a,b;n) \not\equiv \mathcal{C}(a,b;n)$, although it is possible that

$$\{C(a,b;n) : n \ge 0, a \ge 0, b > 0\} = \{\mathcal{C}(a,b;n) : n \ge 0, a \ge 0, b > 0\}$$

or that there exist two 2-tuples $(a_n, b_n) \in (0,\infty) \times (0,\infty)$ and $(\alpha_n, \beta_n) \in (0,\infty) \times (0,\infty)$ such that $C(a_n, b_n; n) = \mathcal{C}(\alpha_n, \beta_n; n)$ for all $n \ge 0$.

For our own convenience and referencing to the convention in the mathematical community, while calling $C(a,b;n)$ for $n \ge 0$, $a \ge 0$, and $b > 0$ generalized Catalan numbers of the first kind, and we call $\mathcal{C}(a,b;n)$ for $n \ge 0$, $a \ge 0$, and $b > 0$ generalized Catalan numbers of the second kind.

In order to make this article more readable, we provide the following overview of all of the definitions used:

(1) $C_n$: Catalan numbers,
(2) $C(a,b;z)$: generalized Catalan function of the first kind,
(3) $C(a,b;n)$: generalized Catalan numbers of the first kind,
(4) $\mathcal{C}(a,b;n)$: generalized Catalan numbers of the second kind,
(5) $\mathcal{C}(a,b;z)$: generalized Catalan function of the second kind.

In this paper, we will give solutions to the first two problems above—establishing an explicit formula for generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$ and finding an integral representation for generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$ by finding an integral representation of the generating function $G_{a,b}(x)$ in Equation (15)—while leaving the third problem above to interested combinatorists.

## 2. An Explicit Formula for Generalized Catalan Numbers of the Second Kind

In this section, we will establish an explicit formula for generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$, which gives a solution to the first problem posed in Equation (16).

**Theorem 1.** *The generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$ for $n \ge 0$, $a \ge 0$, and $b > 0$ can be explicitly computed by*

$$\mathcal{C}(a,b;n) = \frac{1}{(2n)!!b^{n+1/2}} \sum_{k=0}^{n} \binom{2n-k-1}{2(n-k)} \frac{k![2(n-k)-1]!!}{\left(1+a/\sqrt{b}\right)^{k+1}}, \tag{19}$$

*where the double factorial of negative odd integers $-(2\ell+1)$ is defined by*

$$(-2\ell-1)!! = \frac{(-1)^{\ell}}{(2\ell-1)!!} = (-1)^{\ell}\frac{(2\ell)!!}{(2\ell)!}, \quad \ell \ge 0.$$

**Proof.** The Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ are defined in Reference [29] (p. 134, Theorem A) by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i\ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The famous Faà di Bruno formula can be described [29] (p. 139, Theorem C) in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\, x^n} f \circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x))\, \mathrm{B}_{n,k}\left(h'(x), h''(x), \dots, h^{(n-k+1)}(x)\right), \tag{20}$$

where $f \circ h$ denotes the composite of the $n$-time differentiable functions $f$ and $h$.

Let $h = h(x) = \sqrt{b-x}$. Then

$$h^{(k)}(x) = (-1)^k \left\langle \frac{1}{2} \right\rangle_k (b-x)^{1/2-k} \to (-1)^k \left\langle \frac{1}{2} \right\rangle_k b^{1/2-k}, \quad x \to 0$$

for $k \geq 0$ and, in light of Formula (20) for $f(u) = \frac{1}{a+u}$,

$$
\begin{aligned}
\frac{\mathrm{d}^n G_{a,b}(x)}{\mathrm{d}\, x^n} &= \sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}\, h^k}\left(\frac{1}{a+h}\right) \mathrm{B}_{n,k}\left(h'(x), h''(x), \dots, h^{(n-k+1)}(x)\right) \\
&= \sum_{k=0}^{n} (-1)^k \frac{k!}{[a+h(x)]^{k+1}} \mathrm{B}_{n,k}\left(h'(x), h''(x), \dots, h^{(n-k+1)}(x)\right) \\
&\to \sum_{k=0}^{n} (-1)^k \frac{k!}{[a+h(0)]^{k+1}} \mathrm{B}_{n,k}\left(-\left\langle \frac{1}{2} \right\rangle_1 b^{-1/2}, \left\langle \frac{1}{2} \right\rangle_2 b^{-3/2}, \right. \\
&\qquad \left. \dots, (-1)^{n-k+1} \left\langle \frac{1}{2} \right\rangle_{n-k+1} b^{1/2-(n-k+1)}\right), \quad x \to 0 \\
&= \sum_{k=0}^{n} (-1)^k \frac{k!}{\left(a+\sqrt{b}\right)^{k+1}} (-1)^n b^{k/2-n} \mathrm{B}_{n,k}\left(\left\langle \frac{1}{2} \right\rangle_1, \left\langle \frac{1}{2} \right\rangle_2, \dots, \left\langle \frac{1}{2} \right\rangle_{n-k+1}\right) \\
&= \frac{1}{2^n b^{n+1/2}} \sum_{k=0}^{n} k![2(n-k)-1]!! \binom{2n-k-1}{2(n-k)} \left(\frac{\sqrt{b}}{a+\sqrt{b}}\right)^{k+1},
\end{aligned}
$$

where we used the formulas

$$
\mathrm{B}_{n,k}\left(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\right) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})
$$

and

$$
\mathrm{B}_{n,k}\left(\left\langle \frac{1}{2} \right\rangle_1, \left\langle \frac{1}{2} \right\rangle_2, \dots, \left\langle \frac{1}{2} \right\rangle_{n-k+1}\right) = (-1)^{n+k}[2(n-k)-1]!! \left(\frac{1}{2}\right)^n \binom{2n-k-1}{2(n-k)} \tag{21}
$$

for $n \geq k \geq 0$, see p. 135 of Reference [29] and Formula (3.6) in the first two lines on p. 168 of [27], respectively. By the way, Formula (21) is connected with Remark 1 of Reference [30], Section 1.3 of Reference [28], Theorem 4 of Reference [24], and closely related sources therein.

The Equation (16) means that

$$
n!\mathcal{C}(a,b;n) = \lim_{x \to 0} \frac{\mathrm{d}^n G_{a,b}(x)}{\mathrm{d}\, x^n}.
$$

Consequently, we obtain the explicit formula

$$
\mathcal{C}(a,b;n) = \frac{1}{(2n)!!b^{n+1/2}} \sum_{k=0}^{n} k![2(n-k)-1]!! \binom{2n-k-1}{2(n-k)} \left(\frac{\sqrt{b}}{a+\sqrt{b}}\right)^{k+1},
$$

which can be rearranged as Equation (19). The proof of Theorem 1 is complete. □

**Corollary 1** ([24], Theorem 1.3)**.** *The Catalan number $C_n$ for $n \geq 0$ can be explicitly computed by*

$$
C_n = \frac{1}{n!} \sum_{\ell=0}^{n} \binom{n+\ell-1}{2\ell} 2^\ell (n-\ell)!(2\ell-1)!!. \tag{22}
$$

**Proof.** This follows from utilizing Relation (18) and applying $a = \frac{1}{2}$ and $b = \frac{1}{4}$ in (19). The proof of Corollary 1 is complete. □

## 3. An Integral Representation for Generalized Catalan Numbers of the Second Kind

In this section, we will discover an integral representation for generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$ by finding an integral representation of the generating function $G_{a,b}(x)$ in Equation (15), which gives a solution to the second problem posed in Equation (16).

**Theorem 2.** *The principal branch of the generating function $G_{a,b}(z)$ for $a \ge 0$ and $b > 0$ can be represented by*

$$G_{a,b}(z) = \frac{1}{a+\sqrt{b-z}} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{1}{b+t-z}\,\mathrm{d}\,t, \quad z \in \mathbb{C}\setminus[b,\infty). \tag{23}$$

*Consequently, generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$ for $a \ge 0$ and $b > 0$ can be represented by*

$$\mathcal{C}(a,b;n) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{n+1}}\,\mathrm{d}\,t, \quad n \ge 0. \tag{24}$$

**Proof.** Let

$$F(z) = \frac{1}{a+\exp\frac{\ln(-z)}{2}}, \quad z \in \mathbb{C}\setminus[0,\infty), \quad \arg z \in (0,2\pi),$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and $\arg z$ stands for the principal value of the argument of $z$. By virtue of Cauchy's integral formula (p. 113 of Reference [31]) in the theory of complex functions, for any fixed point $z_0 = x_0 + \mathrm{i}\,y_0 \in \mathbb{C}\setminus[0,\infty)$, we have

$$F(z_0) = \frac{1}{2\pi\,\mathrm{i}}\int_L \frac{F(\xi)}{\xi - z_0}\,\mathrm{d}\,\xi,$$

where $L$ is a positively oriented contour $L(r,R)$ in $\mathbb{C}\setminus[0,\infty)$, as shown in Figure 1, satisfying

(1) $0 < r < |z_0| < R$;
(2) $L(r,R)$ consists of the half circle $z = r\,\mathrm{e}^{\mathrm{i}\,\theta}$ for $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$;
(3) $L(r,R)$ consists of the line segments $z = x \pm \mathrm{i}\,r$ for $x \in (0, R(r)]$, where $R(r) = \sqrt{R^2 - r^2}$;
(4) $L(r,R)$ consists of the circular arc $z = R\,\mathrm{e}^{\mathrm{i}\,\theta}$ for

$$\theta \in \left(\arctan\frac{r}{R(r)}, 2\pi - \arctan\frac{r}{R(r)}\right);$$

(5) the line segments $z = x \pm \mathrm{i}\,r$ for $x \in (0, R(r)]$ cut the circle $|z| = R$ at the points $R(r) \pm \mathrm{i}\,r$ and $R(r) \to R$ as $r \to 0^+$.

The integral on the circular arc $z = R\,\mathrm{e}^{\mathrm{i}\,\theta}$ equals

$$\begin{aligned}
&\frac{1}{2\pi\,\mathrm{i}}\int_{\arcsin[r/R(r)]}^{2\pi-\arcsin[r/R(r)]} \frac{R\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\theta}}{\left(R\,\mathrm{e}^{\mathrm{i}\,\theta} - z_0\right)\left[a+\exp\frac{\ln(-R\,\mathrm{e}^{\mathrm{i}\,\theta})}{2}\right]}\,\mathrm{d}\,\theta \\
&= \frac{1}{2\pi}\int_{\arcsin[r/R(r)]}^{2\pi-\arcsin[r/R(r)]} \frac{1}{\left(1-\frac{z_0}{R\,\mathrm{e}^{\mathrm{i}\,\theta}}\right)\left[a+\exp\frac{\ln(-R\,\mathrm{e}^{\mathrm{i}\,\theta})}{2}\right]}\,\mathrm{d}\,\theta \\
&= \frac{1}{2\pi}\int_{\arcsin[r/R(r)]}^{2\pi-\arcsin[r/R(r)]} \frac{1}{\left(1-\frac{z_0}{R\,\mathrm{e}^{\mathrm{i}\,\theta}}\right)\left[a+\exp\frac{\ln R + \mathrm{i}\arg(-R\,\mathrm{e}^{\mathrm{i}\,\theta})}{2}\right]}\,\mathrm{d}\,\theta \\
&\to 0
\end{aligned}$$

uniformly as $R \to \infty$.

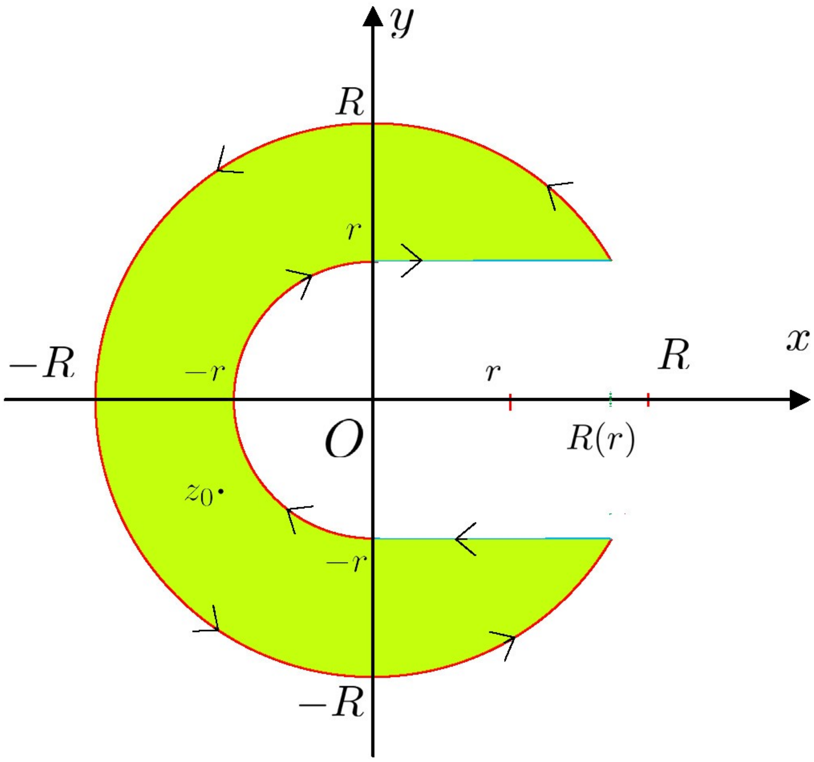


**Figure 1.** The positively oriented contour $L(r,R)$ in $\mathbb{C}\setminus[0,\infty)$.

The integral on the half circle $z = r\,e^{i\theta}$ for $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is

$$\begin{aligned}
&-\frac{1}{2\pi\,i}\int_{\pi/2}^{3\pi/2}\frac{r\,i\,e^{i\theta}}{\left(r\,e^{i\theta}-z_0\right)\left[a+\exp\frac{\ln(-r\,e^{i\theta})}{2}\right]}\,d\theta\\
&=-\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{r\,e^{i\theta}}{r\,e^{i\theta}-z_0}\frac{1}{a+\exp\frac{\ln(-r\,e^{i\theta})}{2}}\,d\theta\\
&=-\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{r\,e^{i\theta}}{r\,e^{i\theta}-z_0}\frac{1}{a+\exp\frac{\ln r+i\arg(-r\,e^{i\theta})}{2}}\,d\theta\\
&=-\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{r\,e^{i\theta}}{r\,e^{i\theta}-z_0}\frac{1}{a+\sqrt{r}\,\exp\frac{i\arg(-r\,e^{i\theta})}{2}}\,d\theta\\
&\to 0
\end{aligned}$$

uniformly as $r \to 0^+$.

Since

$$\begin{aligned}
F(x+i\,r)&=\frac{1}{a+\exp\frac{\ln(-x-r\,i)}{2}}\\
&=\frac{1}{a+\exp\frac{\ln\sqrt{x^2+r^2}+i[\arctan(r/x)-\pi]}{2}}\\
&=\frac{1}{a+\sqrt[4]{x^2+r^2}\left[\cos\frac{\arctan(r/x)-\pi}{2}+i\sin\frac{\arctan(r/x)-\pi}{2}\right]}\\
&=\frac{1}{a+\sqrt[4]{x^2+r^2}\left[\sin\frac{\arctan(r/x)}{2}-i\cos\frac{\arctan(r/x)}{2}\right]}\\
&=\frac{1}{a+\sqrt[4]{x^2+r^2}\,\sin\frac{\arctan(r/x)}{2}-i\sqrt[4]{x^2+r^2}\,\cos\frac{\arctan(r/x)}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a + \sqrt[4]{x^2+r^2}\,\sin\frac{\arctan(r/x)}{2} + \mathrm{i}\,\sqrt[4]{x^2+r^2}\,\cos\frac{\arctan(r/x)}{2}}{\left[a + \sqrt[4]{x^2+r^2}\,\sin\frac{\arctan(r/x)}{2}\right]^2 + \left[\sqrt[4]{x^2+r^2}\,\cos\frac{\arctan(r/x)}{2}\right]^2} \\
&\to \frac{a + \mathrm{i}\sqrt{x}}{a^2 + x}
\end{aligned}
$$

as $r \to 0^+$ and $\overline{F(z)} = F(\bar{z})$, the integral on the line segments $z = x \pm \mathrm{i}\,r$ for $x \in (0, R(r)]$ is equal to

$$
\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\left[\int_0^{R(r)} \frac{F(x+\mathrm{i}\,r)}{x+\mathrm{i}\,r - z_0}\,\mathrm{d}\,x + \int_{R(r)}^0 \frac{F(x-\mathrm{i}\,r)}{x-\mathrm{i}\,r-z_0}\,\mathrm{d}\,x\right] \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{R(r)} \frac{(x-\mathrm{i}\,r-z_0)F(x+\mathrm{i}\,r) - (x+\mathrm{i}\,r-z_0)F(x-\mathrm{i}\,r)}{(x+\mathrm{i}\,r-z_0)(x-\mathrm{i}\,r-z_0)}\,\mathrm{d}\,x \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{R(r)} \frac{(x-z_0)[F(x+\mathrm{i}\,r) - F(x-\mathrm{i}\,r)] - \mathrm{i}\,r[F(x+\mathrm{i}\,r) + F(x-\mathrm{i}\,r)]}{(x+\mathrm{i}\,r-z_0)(x-\mathrm{i}\,r-z_0)}\,\mathrm{d}\,x \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{R(r)} \frac{(x-z_0)\left[F(x+\mathrm{i}\,r) - F\left(\overline{x+\mathrm{i}\,r}\right)\right] - \mathrm{i}\,r\left[F(x+\mathrm{i}\,r) + F\left(\overline{x+\mathrm{i}\,r}\right)\right]}{(x+\mathrm{i}\,r-z_0)(x-\mathrm{i}\,r-z_0)}\,\mathrm{d}\,x \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{R(r)} \frac{(x-z_0)\left[F(x+\mathrm{i}\,r) - \overline{F(x+\mathrm{i}\,r)}\right] - \mathrm{i}\,r\left[F(x+\mathrm{i}\,r) + \overline{F(x+\mathrm{i}\,r)}\right]}{(x+\mathrm{i}\,r-z_0)(x-\mathrm{i}\,r-z_0)}\,\mathrm{d}\,x \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{R(r)} \frac{(x-z_0)[2i\Im(F(x+\mathrm{i}\,r))] - \mathrm{i}\,r[2\Re(F(x+\mathrm{i}\,r))]}{(x+\mathrm{i}\,r-z_0)(x-\mathrm{i}\,r-z_0)}\,\mathrm{d}\,x \\
&\to \frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{2i}{x-z_0}\frac{\sqrt{x}}{a^2+x}\,\mathrm{d}\,x, \quad r \to 0^+, \quad R \to \infty \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{x}}{(a^2+x)(x-z_0)}\,\mathrm{d}\,x.
\end{aligned}
$$

Consequently, it follows that

$$
\frac{1}{a + \exp\frac{\ln(-z_0)}{2}} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{x}}{(a^2+x)(x-z_0)}\,\mathrm{d}\,x \tag{25}
$$

for any $z_0 \in \mathbb{C} \setminus [0, \infty)$ and $\arg z_0 \in (0, 2\pi)$. Due to the point $z_0$ in Equation (25) being arbitrary, the integral Formula (25) can be rearranged as

$$
F(z) = \frac{1}{a + \exp\frac{\ln(-z)}{2}} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{x}}{(a^2+x)(x-z)}\,\mathrm{d}\,x \tag{26}
$$

for $z \in \mathbb{C} \setminus [0, \infty)$ and $\arg z \in (0, 2\pi)$.

Let

$$
f(z) = \frac{1}{a + \exp\frac{\ln(b-z)}{2}}, \quad z \in \mathbb{C} \setminus [b, \infty), \quad \arg(z-b) \in (0, 2\pi).
$$

Then $f(z) = F(z-b)$. Therefore, from Equation (26), it follows that

$$
f(z) = \frac{1}{a + \exp\frac{\ln(b-z)}{2}} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{x}}{(a^2+x)(b+x-z)}\,\mathrm{d}\,x
$$

for $z \in \mathbb{C} \setminus [b, \infty)$ and $\arg(z-b) \in (0, 2\pi)$. The integral representation in Equation (23) is thus proved.

Differentiating $n \ge 0$ times with respect to $z$ on both sides of Equation (23) and taking the limit $z \to 0$ yields

$$\begin{aligned} G_{a,b}^{(n)}(z) &= \frac{\mathrm{d}^n}{\mathrm{d}\, z^n}\left(\frac{1}{a+\sqrt{b-z}}\right) \\ &= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{\mathrm{d}^n}{\mathrm{d}\, z^n}\left(\frac{1}{b+t-z}\right)\mathrm{d}\, t \\ &= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{n!}{(b+t-z)^{n+1}}\,\mathrm{d}\, t \\ &\to \frac{n!}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{n+1}}\,\mathrm{d}\, t, \quad z \to 0. \end{aligned}$$

As a result, by virtue of Equation (16), we have

$$\mathcal{C}(a,b;n) = \frac{G_{a,b}^{(n)}(0)}{n!} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{n+1}}\,\mathrm{d}\, t.$$

The integral representation in Equation (24) for generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$ is thus proved. The proof of Theorem 2 is complete. □

## 4. Potential Directions to Further Study

In this section, we will try to point out two potential directions for further study.

### *4.1. Generalized Catalan Function of the Second Kind*

Motivated by the integral representation in Equation (24) for generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$, we can consider the function

$$\mathcal{C}(a,b;z) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{z+1}}\,\mathrm{d}\, t, \quad a \ge 0, \quad b > 0, \quad \Re(z) \ge 0 \tag{27}$$

and call it a generalized Catalan function of the second kind, while calling $C(a,b;z)$ in (6) generalized Catalan function of the first kind.

We can study the generalized Catalan function of the second kind $\mathcal{C}(a,b;z)$ as a function of three variables: $a, b$, and $z$. It is easy to see that

$$\begin{aligned} \frac{\mathrm{d}^n\, \mathcal{C}(a,b;z)}{\mathrm{d}\, b^n} &= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{\mathrm{d}^n}{\mathrm{d}\, b^n}\left[\frac{1}{(b+t)^{z+1}}\right]\mathrm{d}\, t \\ &= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{\langle -(z+1)\rangle_n}{(b+t)^{z+n+1}}\,\mathrm{d}\, t \\ &= (-1)^n\frac{(z+1)_n}{\pi}\int_0^\infty \frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{z+n+1}}\,\mathrm{d}\, t, \end{aligned}$$

where the rising factorial $(z)_n$ is defined [27] by

$$(z)_n = \prod_{\ell=0}^{n-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

which is also known as the Pochhammer symbol or shifted factorial in the theory of special functions [7,26]. This means that the generalized Catalan function of the second kind $\mathcal{C}(a,b;z)$ is a completely monotonic function [32–34] with respect to $b \in (0,\infty)$. Utilizing complete monotonicity [33–35], we can derive many new analytic properties of the generalized Catalan function of the second kind $\mathcal{C}(a,b;z)$.

In summation, by employing the integral representation in Equation (27), we believe that we can discover some new properties of the generalized Catalan function of the second kind $\mathcal{C}(a,b;z)$, of the generalized Catalan numbers of the second kind $\mathcal{C}(a,b;n)$, and of the Catalan numbers $C_n$. For the sake of the length limit of this paper, we would not like to study this in further detail here.

*4.2. Central Binomial Coefficients*

It is known that central binomial coefficients $\binom{2n}{n}$ can be generated by

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n=1+2x+6x^2+20x^3+70x^4+252x^5+\cdots, \tag{28}$$

It has been an attracting point for mathematicians to study identities involving central binomial coefficients $\binom{2n}{n}$. For example, we can rewrite (Lemma 3 of Reference [36]) as

$$\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{(k+1)4^k}=2\left[1-\frac{1}{4^n}\binom{2n}{n}\right]$$

and

$$\sum_{k=0}^{n-1}\binom{2k}{k}\frac{4^{n-k}}{n-k}=2\binom{2n}{n}\sum_{k=1}^{n}\frac{1}{2k-1}.$$

For more information on results at this point, please refer to References [5,36–46] and closely related sources therein.

Combining Equation (28) with Equations (16) and (24) yields

$$\begin{aligned}\binom{2n}{n}&=\frac{1}{n!}\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(\frac{1}{\sqrt{1-4x}}\right)\\&=\frac{1}{n!}\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(\frac{1}{2}\lim_{\substack{a\to0^+\\b\to1/4}}\frac{1}{a+\sqrt{b-x}}\right)\\&=\frac{1}{n!}\frac{1}{2}\lim_{\substack{a\to0^+\\b\to1/4}}\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(\frac{1}{a+\sqrt{b-x}}\right)\\&=\frac{1}{2}\lim_{\substack{a\to0^+\\b\to1/4}}\mathcal{C}(a,b;n)\\&=\frac{1}{2}\lim_{\substack{a\to0^+\\b\to1/4}}\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{n+1}}\,\mathrm{d}\,t\\&=\frac{1}{2\pi}\int_0^\infty\frac{1}{\sqrt{t}}\frac{1}{(1/4+t)^{n+1}}\,\mathrm{d}\,t\\&=\frac{2^{2n+1}}{\pi}\int_0^\infty\frac{1}{(1+t^2)^{n+1}}\,\mathrm{d}\,t.\end{aligned}\tag{29}$$

The last three integral representations should provide effective tools for further studying central binomial coefficients $\binom{2n}{n}$. These integral representations of central binomial coefficients $\binom{2n}{n}$ were extended and investigated in Reference [47].

## 5. Remarks

In this section, we give several remarks on our main results and related things.

**Remark 1.** *Taking $a \to 0^+$ on both sides of Formula (19) and employing Equation (17) results in*

$$\sum_{k=0}^{n} \binom{n+\ell-1}{2\ell} (n-\ell)!(2\ell-1)!! = (2n-1)!!.$$

*Combining Equation (22) with the first equality in Equation (1) gives*

$$\sum_{\ell=0}^{n} \binom{n+\ell-1}{2\ell} (n-\ell)!(2\ell-1)!!2^{\ell} = \frac{n!}{n+1}\binom{2n}{n}.$$

*Stimulated by these two identities and Formula (19) in Theorem 1, we would like to ask a question: can one use a simple quantity to express the sum*

$$\sum_{\ell=0}^{n} \binom{n+\ell-1}{2\ell} (n-\ell)!(2\ell-1)!!t^{\ell}$$

*for $t \in \mathbb{R} \setminus \{0,1,2\}$?*

*One of anonymous reviewers pointed out that the famous software Wolfram Mathematica 12 gives*

$$\sum_{\ell=0}^{n} \binom{n+\ell-1}{2\ell} (n-\ell)!(2\ell-1)!!t^{\ell} = \frac{1-t}{(1-t/2)^{n+1}}\Gamma(n+1) + \frac{\Gamma(2n+1)}{\Gamma(n+2)}\,{}_2F_1\left(2,1+2n;2+n;\frac{t}{2}\right),$$

*where the notation ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function [26,48]. This formula does not fit for $t = 2$. For $t = 2$, the summation can be directly computed.*

**Remark 2.** *The double factorials are given by $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$ and $(2n)!! = 2^n n!$. Application and simplification gives*

$$\mathcal{C}(a,b;n) = \frac{1}{4^n b^{n+1/2}\left(1+a/\sqrt{b}\right)^n} \frac{\Gamma(2n-1)}{\Gamma(n)\Gamma(n+1)}\,{}_2F_1\left(2,1-n;2-2n;1+\frac{a}{\sqrt{b}}\right).$$

*This can be converted into the formula found in Reference [49].*

**Remark 3.** *When taking $z = x \in (-\infty, b)$, the integral formula (23) becomes*

$$\frac{1}{a+\sqrt{b-x}} = \frac{1}{\pi}\int_0^{\infty} \frac{\sqrt{t}}{a^2+t}\frac{1}{b+t-x}\,\mathrm{d}\,t, \quad a \ge 0, b > 0. \tag{30}$$

*When taking $x \to b^-$, the integral in Equation (30) converges. Consequently, the integral representation in Equation (30) is valid on $(-\infty, b]$.*

**Remark 4.** *When taking $a = \frac{1}{2}$ and $b = \frac{1}{4}$, the integral representations in Equations (23) and (24) reduce to Equations (12) and (11), respectively.*

**Remark 5.** *Combining the explicit formula (19) with the integral representation in Equation (24) and simplifying leads to*

$$\int_0^{\infty} \frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{n+1}}\,\mathrm{d}\,t = \frac{\pi}{(2n)!!b^{n+1/2}} \sum_{k=0}^{n} \binom{2n-k-1}{2(n-k)} \frac{k![2(n-k)-1]!!}{\left(1+a/\sqrt{b}\right)^{k+1}}$$

*for $a \ge 0$, $b > 0$, and $n \ge 0$.*

**Remark 6.** *An anonymous reviewer commented on this paper as follows.*

*This paper constitutes a further extension of a vast body of work which is aimed at finding integral representations of various generalizations of Catalan numbers. Let me stress the following distinction: all the integral representations can be subdivided into two classes:*

*(a)* *representations as Hausdorff-type power moments, and*
*(b)* *other representations.*

*For example, Equation (10) for the Catalan numbers belongs to class (a), whereas the three last equations in (29) belong to the class (b).*

*It turns out that the integral representations of generalized Catalan numbers as moments [class (a)] are unique via the Hausdorff theorem, whereas those of class (b) are not unique. Therefore it seems that the search for unique integral representations is somewhat more relevant than that for the non-unique ones.*

*The paper is quite complete and is written in an understandable way.*

*The papers [50–54] treat the moment representations of several extensions of the Catalan numbers.*

**Remark 7.** *An anonymous reviewer commented on this paper as follows.*

*Let $P_n(x)$ be a family of polynomials defined by $P_0(x) = 1$, $P_n(x) = x - u_0$, and*

$$xP_n(x) = P_{n+1}(x) + u_n P_n(x) + a_{n-1} P_{n-1}(x)$$

*for $n \geq 2$, with coefficients $a_n > 0$ and $b_n \in \mathbb{R}$ for $n \geq 0$. By the Favard theorem, there exists a probability distribution $\mu$ on $\mathbb{R}$ such that $P_n$ are orthogonal with respect to $\mu$. In general, this leads to the theory of orthogonal polynomials. A particular class, when $a_1 = a_2 = \dots$ and $u_1 = u_2 = \dots$, depending on four parameters $a = a_0$, $b = a_1 = a_2 = \dots$, $u = u_0$, and $v = u_1 = u_2 = \dots$ (by affine transformation of the distribution one can assume that $a_0 = 1$ and $u_0 = 0$), has been studied in References [55–58], particularly in the context of free probability, and the corresponding probability distributions are now called "Meixner". The special subclass corresponding to that in this paper is studied in detail (up to a proper normalization) in Reference [59] (see Corollary 2.2 and Section 3) from an alternative angle and the coefficients indeed have combinatorial interpretation as ballot numbers (see [12]); the integral representation is given in Theorem 3.1.*

*These comments tell us that the main results in this paper are intrinsic, significant, and of deep backgrounds of combinatorics.*

**Remark 8.** *One of anonymous reviewers gave the following critique.*

*Suppose there is a function $\rho(a,b;u)$ such that*

$$C(a,b;n) = \int_I \rho(a,b;u) u^n \,\mathrm{d}\, u$$

*for some interval $I$. Then for $z \notin I$*

$$\frac{1}{z}\sum_{p=0}^{\infty} \frac{C(a,b;n)}{z^p} = \frac{1}{z} G_{a,b}\left(\frac{1}{z}\right) = \int_I \frac{\rho(a,b;u)}{z-u} \,\mathrm{d}\, u.$$

*On the other hand, knowledge of the functional form of $\frac{1}{z}G_{a,b}\left(\frac{1}{z}\right)$ allows $\rho(a,b;u)$ to be computed by the inverse Stieltjes transform formula*

$$\rho(a,b;u) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi\,\mathrm{i}}\left[\frac{1}{z}G_{a,b}\left(\frac{1}{z}\right)\bigg|_{z=u-\epsilon\,\mathrm{i}} - G_{a,b}\left(\frac{1}{z}\right)\bigg|_{z=u+\epsilon\,\mathrm{i}}\right] = \frac{1}{\pi x}\frac{\sqrt{1/x-b}}{a^2-b+1/x}$$

*supported on $x \in \left(0, \frac{1}{b}\right)$. This gives a more straightforward viewpoint of the results of Theorem 2 in this paper.*

**Remark 9.** *This paper is a revised version of the electronic preprints [60] and a companion of the paper [61].*

## 6. Conclusions

In this paper, the authors briefly surveyed several generalizations, such as Equations (3)–(6) and (9), of the Catalan numbers $C_n$, analytically generalized the Catalan numbers $C_n$ as $\mathcal{C}(a,b;n)$ by Equations (16) and (27), established an integral representation (24) of the analytic generalization $\mathcal{C}(a,b;n)$ of the Catalan numbers $C_n$ by virtue of Cauchy's integral formula in the theory of complex functions.

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