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Synergistic Mechanism of Designing Information Granules with the Use of the Principle of Justifiable Granularity

Dan Wang, Yukang Liu and Zhenhua Yu *

Institute of System Security and Control, School of Computer Science and Technology, Xi'an University of Science and Technology, Xi'an 710054, China
* Correspondence: zhenhuayu@xust.edu.cn

Abstract: The construction of information granules is a significant and interesting topic of Granular Computing (GrC) in which information granules play a vital role in representing and describing data, and it has become one of the most effective frameworks for solving complex problems. In this study, we are interested in the collaborative impacts of several different characteristics on constructing information granules, and a novel synergistic mechanism of the principle of justifiable granularity is utilized in developing information granules. The synergistic mechanism is finalized with a two-phase process—to start with, the principle of justifiable granularity and Fuzzy C-Means Clustering method are combined to develop a collection of information granules. First, the available experimental data is transformed (normalized) into fuzzy sets following the standard Fuzzy C-Means Clustering method. Then, information granules are developed based on the elements located in different clusters with the use of the principle of justifiable granularity. In the sequel, the positions of information granules are updated by considering the collaborative impacts of the other information granules with the parameters of specifying the level of influence. Experimental studies are conducted to illustrate the nature and feasibility of the proposed framework based on the synthetic data as well as a series of publicly available datasets coming from KEEL machine learning repositories.

Keywords: information granules; the principle of justifiable granularity; synergistic mechanism; collaborative construction

MSC: 68T37

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1. Introduction

As a new interdisciplinary research field, Granular Computing (GrC) [1] plays a vital role in data mining by generalizing different forms of information granules. The concept of granular computing achieves rapid development as a human-centric data processing pattern that mainly concentrates on information granules and the processing of information granules. Information granules [2,3] are a collection of general and abstract entities that converge based on their indistinguishability, compactness, functionality, or similarity. As is commonly recognized, information granules can be expressed in several different forms, such as sets and intervals, fuzzy sets [1], rough sets [4], shadowed sets [5], probabilistic sets [6], and so on. The development of information granules has occupied a key position in describing data in the framework of granular computing.

The construction of information granules in different formalisms has attracted much attention in describing and representing data [7–11]. Since the theory of fuzzy sets proposed by L. A. Zadeh [12], it has become a generalized form to represent information and knowledge. To be more specific, fuzzy sets show a strong capacity for dealing with information with fuzziness and uncertainty. For example, Zhang et al. [13] present an original design of interval type-2 information granules based on a collection of type-1 fuzzy sets by engaging the principle of justifiable granularity, where an information granule is generated by maximizing the product of two generic characteristics of coverage and specificity.

Based on the correlation among features, an improved version of a top-down granulation model is offered by incorporating the principal component analysis [14]. Ouyang et al. [15] design a model for rule formation based on the information sub-granules, which are constructed by combining the Fuzzy C-Means and Density-Based Spatial Clustering of Applications with Noise (DBSCAN). At the same time, many scholars have also proposed targeted methods to build information granules in both the input and output spaces. For instance, Lu et al. [16,17] realize the formation of input hyper-box information granules by implementing the hyper-box iteration granulation algorithm governed by information granularity on the input space. Shan et al. [18] present the concept of interval granular fuzzy models. Jing et al. [19] associate information granules and propose an approach to construct granular models directly based on information granules expressed both in input and output spaces. In addition, considering the distribution of the data sets, information granules of higher order or information granules of higher type are developed [20,21], which achieve better performance to some extent.

Many scholars focus their attention on developing ways of constructing information granules. The principle of justifiable granularity is regarded as a general method to design a single information granule based on the available experimental evidence [22–24]. While the typical clustering algorithms, for instance, Fuzzy C-Means Clustering (FCM), are utilized to construct a series of information granules with respect to different features of the entire data space [25]. To achieve satisfactory results for the construction process, the principle of justifiable granularity and the FCM algorithm are commonly utilized in a collaborative manner—FCM is applied to cluster the data, and the principle of justifiable granularity is used to construct an information granule based on the elements in different clusters. Several related works focus on developing augmented frameworks for granular models, such as fuzzy classifiers [26,27] and fuzzy rule-based models [28–30]. Various variant fuzzy rule-based models have also been studied and implemented in various fields. For example, the model based on Mamdani and Larsen fuzzy inference is used to select the color constancy algorithm for dark image enhancement [31], and the Zadeh dominated fuzzy rule composition algorithm is used to solve the positioning problem in iris recognition [32]. The combination of the Mamdani model and hierarchical clustering is utilized to predict fault severity in industrial manufacturing [33]. At the same time, the granular models have been widely used in various fields, such as quantifying the quality of numerical models [34], data fusion [35], transferring knowledge [36], facial semantic description [37], time series prediction [38], long-term prediction [39], controlling complex nonlinear systems [40], anomaly detection [41], implementing missing data interpolation [42], and conformance checking techniques [43], attacks detection [44–46], image classification [47], and some other practical complex problems [48,49], and achieving quite good results.

As one of the most important ways of developing information granules, the principle of justifiable granularity processes with the optimization of the upper and lower bounds by achieving a compromise of two conflicting criteria, i.e., coverage and specificity, which consider both the completeness as well as the accuracy in expressing the experimental evidence. However, there is an emerging challenge: with the data scale and the diversity of the feature space increasing, the complexity of data processing becomes more difficult. A more special situation arises when the interaction among the features in different data spaces or different features in the same data space will directly influence the locations of information granules. In other words, the information granules of the same data space or different data spaces will be influenced by each other.

The purpose of this paper is to focus on the collaborative development of information granules with the use of the principle of justifiable granularity, which fully takes into consideration that the information granules in the data space will impact the position of a selected information granule. Compared to the traditional ways of developing information granules, the proposed method, named synergistic mechanism of the principle of justifiable granularity, takes all the features as a whole, and the construction process of a selected information granule is realized by considering the impacts of other features with the parameters of specifying the levels of influence. First, the feature space is clustered following the basic Fuzzy C-Means Clustering, and information granules are decided based on the elements positioned in different clusters, respectively. Second, the locations of the information granules are updated by including the impacts of other information granules in determining the coverage criterion. Finally, the optimal positions of the information granules in the feature space are decided using the principle of justifiable granularity, and the overall performance of the synergistic mechanism is evaluated with the *AUC* values.

The study is structured as follows: Section 2 introduces the basic ideas of the theory and development of fuzzy sets with the use of the principle of justifiable granularity. In Section 3, the synergistic mechanism of the principle of justifiable granularity in building different information granules is illustrated in detail. Experimental studies are carried out in Section 4 to verify the feasibility of the proposed mechanism, in which synthetic data and a series of publicly available machine learning data sets are applied in the experiments. Section 5 presents the conclusions and future directions.

2. Construction of Fuzzy Sets with the Use of the Principle of Justifiable Granularity

In real-world systems, we usually encounter objects whose belongingness to a given category is neither full belongingness nor full exclusion. For instance, considering the height of an adult male person, we qualify a person as being tall if they have a height of 1.9 m, whereas we consider a person to be short if they have a height of 1 m. Usually, we cannot distinguish a difference of 0.1 m in height, so we may qualify a person with a height of 1.89 m as being tall. However, questions come with that: how do we qualify a person of 1.7 m as being tall or short? What are the height range values to qualify a person as being tall? It is obvious that no single number can be given to qualify a person as tall or short. Fuzzy sets provide an idea to deal with cases where an object of the universe is compatible with a class by membership values. Conceptually, a fuzzy set A [12] is described by a membership function, which maps the elements of a universe X to the unit interval [0, 1].

$$A : X \to [0, 1], \tag{1}$$

where [0, 1] means real numbers between 0 and 1 (including 0 and 1).

The α-cut of a fuzzy set A, denoted by A_α, is a set which consists of the elements of the universe whose membership values are equal to or larger than a certain value α, where $\alpha \in [0, 1]$ [44]. It is expressed by the following equality:

$$A_\alpha = \{x \in X | A(x) \geq \alpha\}. \tag{2}$$

An illustration of the concept of the α-cut is presented in Figure 1b.

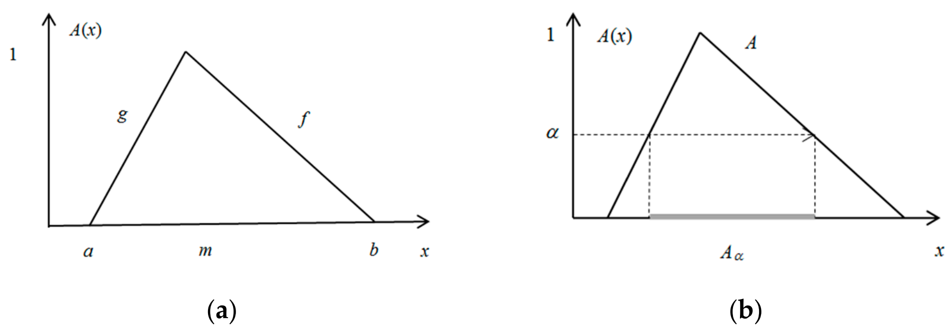


Figure 1. Membership function of a fuzzy set (**a**) and α-cut of a fuzzy set (**b**).

Considering the formation of information granules in the form of fuzzy sets with a finite support [a, b]. Any fuzzy set can be described by a certain unimodal membership function of some parametric form (g, f) with $g(a) = 0$, $g(m) = f(m) = 1$, and $f(b) = 0$, where g is a non-decreasing function and f is a non-increasing function, as shown in Figure 1a.

The principle of justifiable granularity provides a general idea of how to construct a single information granule by achieving a sound compromise between two essential requirements: coverage and specificity. It is intuitively apparent that the two requirements are in conflict, as follows:

- Coverage (*cov*) implies the ability of the information granule to reflect the experimental data. In other words, it is anticipated that the information granule will "cover" more experimental data. For instance, if an information granule is an interval, then the more data included in the bounds of the interval, the better. In the case of fuzzy sets, we expect that the sum value of the membership grades of the data included in the bounds of the information granule will be as high as possible. However, it is required that the information granule be specific enough.
- The specificity (*sp*) criterion concerns the semantic meaning of the information granules. This requires the information granules to be highly detailed (more specific), so we expect a smaller information granule.

The principle of justifiable granularity aims at the optimization of the fuzzy sets by independently adjusting the lower and upper bounds a and b, respectively. The modal value of m is taken as a numeric representative of x_k, say their mean or median. Let us consider a one-dimensional data set $\{x_k\}$, $k = 1, 2, \ldots, N$. The optimization criterion Q involves two conflicting requirements: coverage and specificity, denoted by *cov* and *sp*, respectively. Assuming the following form of Q:

$$Q = cov * sp. \tag{3}$$

Considering the elements positioned on the right side of m, i.e., $[m, b]$, the coverage and specificity are defined in the usual way:

$$cov = \mathrm{card}\{x_k | x_k \in [m, b]\}, \tag{4}$$

$$sp = 1 - \frac{|b - m|}{|x_{\max} - m|}. \tag{5}$$

As for the elements positioned in $[a, m]$, we have the two criteria and optimize the lower bound a in a similar manner:

$$cov = \mathrm{card}\{x_k | x_k \in [a, m]\}, \tag{6}$$

$$sp = 1 - \frac{|m - \mathrm{a}|}{|\mathrm{m} - x_{min}|}. \tag{7}$$

The generalized version of the problem arises when the data are weighted, i.e., we have (x_k, w_k), $k = 1, 2, \ldots, N$. Then the coverage is modified in the following way:

$$cov = \sum_{k: x_k \in [m,b]} \min(A(x_k), w_k), \tag{8}$$

$$cov = \sum_{k: x_k \in [a,m]} \min(A(x_k), w_k). \tag{9}$$

The specificity remains the same as the formulas presented in the previous case.

The maximization of Q leads to the optimal value of the upper bound of the fuzzy set A, say as follows:

$$b_{opt} = \arg \mathrm{Max}_b Q(b). \tag{10}$$

The optimal value of the lower bound of the fuzzy set A is obtained in an analogous way:

$$a_{opt} = \arg \mathrm{Max}_a Q(a). \tag{11}$$

3. Synergistic Mechanism of the Principle of Justifiable Granularity

We are interested in forming information granules for individual variables in the case of multivariable data, such as pairs of data (x_k, y_k), $k = 1, 2, \ldots, N$, by invoking a certain mechanism of synergy in their formation. Considering a collection of one-dimensional weighted data (x_k, w_k) and (y_k, v_k), the same construction as above is realized for both the two data sets. As a result, we produce fuzzy sets A with the bounds a and b and modal value m for $\{x_k\}$, as well as fuzzy sets B with the bounds c and d and modal value n for $\{y_k\}$. The coverage and specificity expressions are analogous to the ones presented above.

As x_k and y_k are related, it is anticipated that A and B are related, and they could be constructed together in some synergistic way. To accomplish that, we consider that x_k and y_k are "weighted" by the membership grades of the other fuzzy set. This means that the determination of A is influenced by the membership grades of B and vice versa. Considering the way in which we are proceeding with the formation of A and B, we are referring to their formation as being synergistic.

In a nutshell, in the construction of A, we are using the weight of x_k resulting from B, say $B(y_k)$. More generally, let us consider a certain function F for $B(y_k)$, say $F(B(y_k))$. Likewise, in the construction of B, we weight the data y_k by the membership grades of A, say $F(A(x_k))$.

The first step is the formation of fuzzy sets A for x_k and B for y_k individually. It can be processed using the principle of justifiable granularity. In what follows, we can optimize A by taking into account the membership grades of B as well as optimize B by taking into account the membership grades of A. We have the weighted average value to serve as the numeric representative of x_k, and it is presented as follows:

$$\mathrm{m} = \frac{\sum_{k=1}^{N} x_k B(y_k)}{\sum_{k=1}^{N} B(y_k)}, \tag{12}$$

For y_k:

$$\mathrm{n} = \frac{\sum_{k=1}^{N} y_k A(x_k)}{\sum_{k=1}^{N} A(x_k)}. \tag{13}$$

In terms of the optimized criterion Q used for the construction of fuzzy set A, we have the expression as follows:

$$\mathrm{Q} = cov * sp. \tag{14}$$

For the upper bound b, the coverage is expressed as follows:

$$cov = \sum_{k: x_k \in [m,b]} A(x_k) B^{\alpha_1}(y_k), \tag{15}$$

where $\alpha_1 \in [0, 1]$ indicates the parameter for specifying the impact of fuzzy set B on the determination of fuzzy set A. It should be noticed that when $\alpha_1 = 0$, the impact of fuzzy set B should be ignored, and the synergistic mechanism is the same as the traditional principle of justifiable granularity.

Additionally, the specificity is expressed as the same as in Equation (5).

As for the construction of fuzzy set B, we have analogous expressions.

Similarly, for the upper bound d, the coverage is expressed as follows:

$$cov = \sum_{k: y_k \in [n,d]} B(y_k) A^{\beta_1}(x_k), \tag{16}$$

where $\beta_1 \in [0, 1]$ implies the parameter of specifying the impact of fuzzy set A on the determination of fuzzy set B. Moreover, when $\beta_1 = 0$, there is no impact of fuzzy set A

on the construction of fuzzy set B. The synergistic mechanism performs as well as the traditional principle of justifiable granularity.

In Algorithm 1, we present the synergistic mechanism of constructing information granules with the use of the principle of justifiable granularity.

Algorithm 1: Synergistic mechanism of the principle of justifiable granularity

Input: Two-dimensional data set in pairs (x_k, y_k), k = 1, 2, ... , N; number of clusters c; fuzzy coefficient m = 2.0; impact factor α_1, β_1, iteration of the optimization process *epoch*;
Output: Optimized information granules A and B
1: Initialize the position of information granules A and B following standard FCM
2: Select the prototypes v_i and w_i, i = 1, 2, ... , c, as the numeric representative of the principle of justifiable granularity
3: **while** *iter* = 1, 2, ... , *epoch* **do**
4: update the position of information granules A and B
5: $cov = \sum\limits_{k:x_k \in [m,b]} A(x_k) B^{\alpha_1}(y_k)$
6: $sp = 1 - \frac{|b-m|}{|x_{\max}-m|}$
7: $Q = cov * sp$
8: **end while**
9: $b_{opt} = \arg \mathrm{Max}_b Q(b)$

As an extension, let us consider a more complex situation when another data space $\{z_k\}$, k = 1, 2, ... , N is included and it results in a fuzzy set C that further influences the construction of fuzzy sets A and B. To determine the optimal bounds of C, the numeric representative w is taken as the weighted mean value of $\{z_k\}$, and the upper and lower bounds are f and e, respectively, which are decided following the pattern of the principle of justifiable granularity.

In this case, the coverage for constructing fuzzy set A is expressed as follows:

$$cov = \sum_{k:x_k \in [m,b]} A(x_k) B^{\alpha_1}(y_k)\, C^{\alpha_2}(z_k), \tag{17}$$

where α_1, $\alpha_2 \in [0, 1]$ imply the impacts of fuzzy sets B and C on the design of fuzzy set A, respectively. When α_1, α_2 = 0, there is no impact from fuzzy sets B and C, and it performs the same as the principle of justifiable granularity.

To generalize a comprehensive construction process, the development of fuzzy sets B and C can be expressed as follows:

For the upper bound, the coverage for B is expressed as follows:

$$cov = \sum_{k:y_k \in [n,d]} B(y_k) A^{\beta_1}(x_k) C^{\beta_2}(z_k), \tag{18}$$

where β_1, $\beta_2 \in [0, 1]$ means the influence of A and C on the design of fuzzy set B. To be specific, the impacts of fuzzy sets A and C are not considered in the construction process.

The coverage for C is described as follows:

$$cov = \sum_{k:z_k \in [w,f]} C(z_k) A^{\gamma_1}(x_k) B^{\gamma_2}(y_k), \tag{19}$$

where γ_1, $\gamma_2 \in [0, 1]$ means the influence of A and B on the design of fuzzy set C.

The general idea about the collaborative development of fuzzy sets A, B, and C is illustrated in Figure 2.

As shown in Figure 2a, two information granules (A and B) are developed in the proposed synergistic manner. To be more specific, the construction of A is influenced by the information granule B with a factor of α_1, whereas the construction of B is influenced by the information granule A with a factor of β_1. Considering the collaborative construction for three information granules A, B, and C, as presented in Figure 2b, the construction of A is influenced by the information granules B and C with factors α_1 and α_2, respectively, while B are developed with the influence of A and C (with factors β_1 and β_2) and C are developed with the influence of A and B (with factors γ_1 and γ_2). It should be noticed that when a certain factor equals 0, for instance, when we have $\alpha_1 = 0$ in Figure 2a, then fuzzy set B has no influence on the construction of A. In Figure 2b, when $\alpha_1 = 0$, it indicates that there is no influence of B on the construction of A; moreover, when $\alpha_1 = \alpha_2 = 0$, there is no impact of B and C on A.

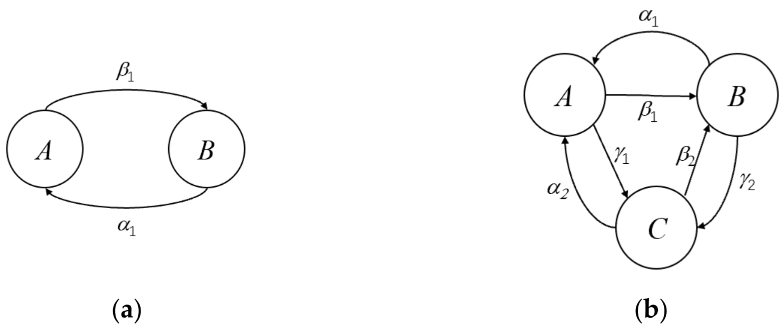


Figure 2. Collaborative development of information granules with different impact factors. (**a**) Design of information granules A and B (**b**) Design of information granules A, B, and C.

4. Experimental Studies

In this section, a series of experiments are presented to qualify the performance of the synergistic mechanism of the principle of justifiable granularity. The experiments are conducted on a computer with an AMD 6-core r5-3600 4.2 GHZ CPU. NVIDIA GeForce RTX 3060Ti 8 GB, and DDR4 16 GB memory. Python, MATLAB, and several data analysis tools, including NumPy, Pandas, Matplotlib, etc., are used to complete the verification experiments.

4.1. Synthetic Data Set

To start with the examples based on the synthetic data, let us consider a group of three two-dimensional Gaussian distributed data sets $\{x_k, y_k\}$, $k = 1, 2, \ldots, N$, where $N = 300$. The mean vectors m and the covariance matrix δ are summarized as follows:

$$Group\ 1:\ m = [2,\ 3],\ \delta = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}, N_1 = 100,$$

$$Group2: m = [-2, -3],\ \delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N_2 = 100,$$

$$Group3: m = [1, -1],\ \delta = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, N_3 = 100.$$

The plot of the data set is presented in Figure 3.

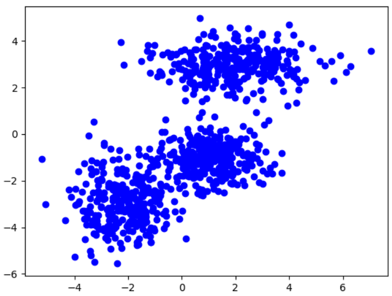


Figure 3. Plot of the experimental data set.

As the most commonly used ways of developing information granules, the principle of justifiable granularity and FCM usually perform in a cooperative manner. First, FCM is utilized to generate several information granules by conducting the typical clustering process; then, the principle of justifiable granularity is applied to design a single information granule based on the available experimental evidence located in different clusters. To be specific, to develop a collection of information granules for the synergistic data, the data set is normalized following the basic Fuzzy C-Means Clustering method, the prototypes p_j, $j = 1, 2, \ldots, c$, and the membership grades u_{ij} are updated as follows iteratively:

$$p_j = \frac{\sum_{i=1}^{N} u_{ij}^m * x_i}{\sum_{i=1}^{N} u_{ij}^m}, \tag{20}$$

$$u_{ij} = \frac{1}{\sum_{k=1}^{c} \left(\frac{\|x_i - c_j\|}{\|x_i - c_k\|} \right)^{\frac{2}{m-1}}}, \tag{21}$$

where m means the fuzziness coefficient, and usually $m = 2.0$.

As shown in Figure 4, the data set is clustered into three clusters following the FCM method with an iteration of 20.

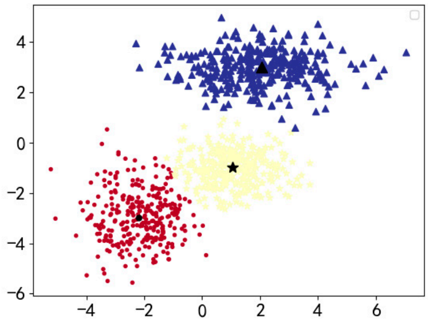


Figure 4. Results of fuzzy clustering.

In the sequel, the membership grades are used for the collaborative design of information granules with the proposed synergistic mechanism of the principle of justifiable granularity in the two-dimensional data space. Let us consider different values of the parameters α_1 and α_2; the positions of the information granules are illustrated as shown in Figure 5. Here α_1 and α_2 are both unified influence parameters, that is, when one of the fuzzy sets is constructed, the other two fuzzy sets have an influence on the constructed fuzzy set. For example, when building fuzzy set 0, the influence parameters of fuzzy set 1 and fuzzy set 2 are α_1 and α_2. When building fuzzy set 1, α_1 is the influence of fuzzy set 0 on fuzzy set 1, and α_2 is the influence of fuzzy set 2 on fuzzy set 0. Similarly, when building fuzzy set 2, α_1 is the influence of fuzzy set 0 on fuzzy set 2, and α_2 is the influence of fuzzy set 1 on fuzzy set 2. With different values of α_1 and α_2, the corresponding fuzzy sets have different degrees of influence on the constructed fuzzy sets.

As shown in Figure 5, it is evident that with the increase of the impact parameters α_1, and α_2 of the synergistic mechanism, the location of the constructed information granule becomes closer to the corresponding information granules. Take the 0th cluster (marked as the ▲ colored blue) as an example; it can be seen that with the increasing values of α_1 (which specifies the impact of the 1th cluster, marked as the • colored red), the location of information granule 0 will become closer to the 1th cluster. In a similar manner, with the increasing values of α_2 (which specifies the impact of the 2th cluster, marked as the ★ colored yellow), the location of information granule 0 will become closer to the 2th cluster. This situation is also applicable to the construction of information granules 1 and

2. It should be noted that when $\alpha_1 = \alpha_2 = 0$, it indicates that the information granules are constructed without the impact of the other information granules; in other words, when $\alpha_1 = \alpha_2 = 0$, it is a special case that the synergistic mechanism performs the same as the traditional pattern of the principle of justifiable granularity.

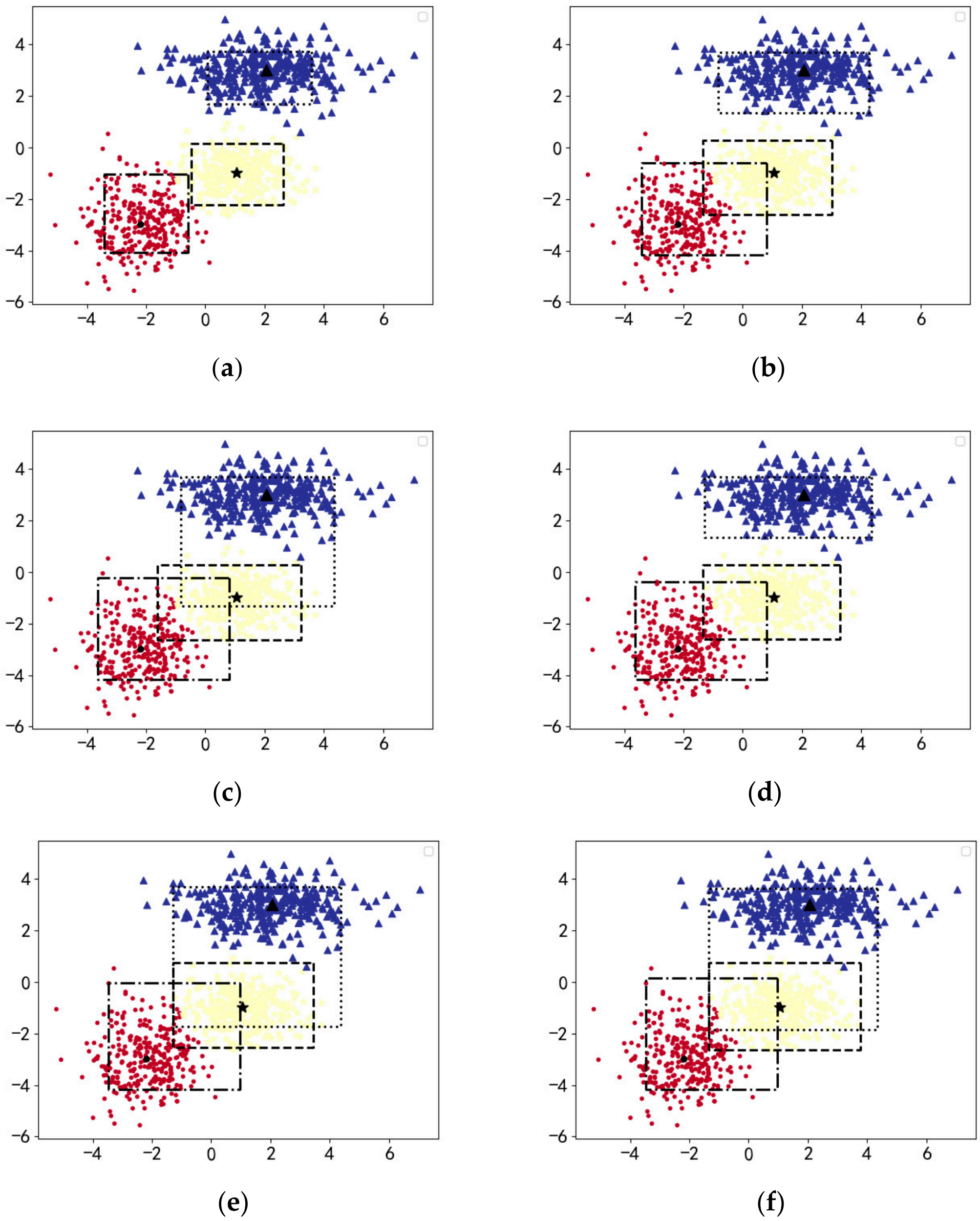


Figure 5. Information granules constructed in a collaborative pattern with different values of parameters α_1 and α_2 (**a**) $\alpha_1 = \alpha_2 = 0$ (**b**) $\alpha_1 = 0.3$ $\alpha_2 = 0.5$ (**c**) $\alpha_1 = 0.3$ $\alpha_2 = 0.8$ (**d**) $\alpha_1 = 0.5$ $\alpha_2 = 0.5$ (**e**) $\alpha_1 = 0.8$ $\alpha_2 = 0.5$ (**f**) $\alpha_1 = \alpha_2 = 1$.

To evaluate the performance of the information granules, we present the curves by plotting the coverage-specificity (*cov-sp*) values for obtaining the upper bounds with a pair of selected parameters when $\alpha_1 = \alpha_2 = 0.5$ (which is correlated to the results shown in Figure 5d). As shown in Figure 6.

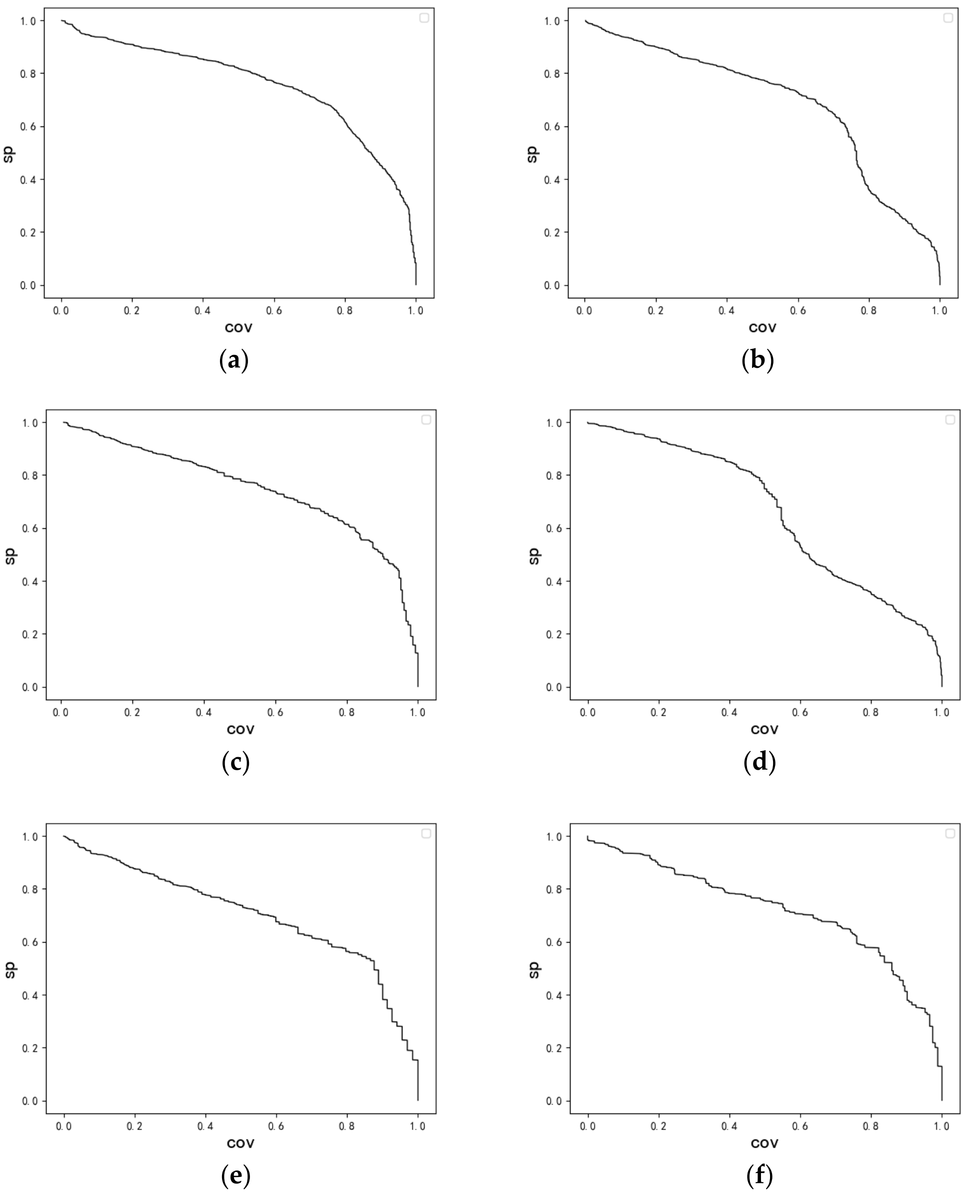


Figure 6. The curve of *cov-sp* values for the upper bound. (**a**) Cluster 0, 1st dimension (**b**) Cluster 0, 2nd dimension (**c**) Cluster 1, 1st dimension (**d**) Cluster 1, 2nd dimension (**e**) Cluster 2, 1st dimension (**f**) Cluster 2, 2nd dimension.

As shown in Figure 6, regardless of the dimensionality for each cluster of the data set, it is obvious that with the increase of the *cov*, the values of *sp* keep decreasing, which shows the same performance of the construction of information granules with the use of the principle of justifiable granularity.

Moreover, the positions of different information granules as well as the *AUC* values, which specify the overall performance of the proposed synergistic mechanism by calculating the areas of *cov-sp* curves, *x*- and *y*-ordinates, are summarized in Table 1.

From the results presented in Table 1, the *AUC* values obtained for both dimensions of the upper and lower bounds show good performance with a collection of reasonable values that are greater than 0.6. Furthermore, the ranges determined with the constructed information granules can cover most of the elements in the data set.

When changing the values of α_1 and α_2 the upper and lower bounds of the information granules also change. Here we take cluster 0 in the 1st dimension as an example, and the results are recorded in Table 2.

Table 1. The summary of the *AUC* values and bounds of different information granules.

Dimensions	Upper/Lower Bounds	0th Cluster (Marked as ▲ Colored Blue)		1st Cluster (Marked as ● Colored Red)		2nd Cluster (Marked as ★ Colored Yellow)	
		AUC	Values of *a*/*b*	AUC	Values of *a*/*b*	AUC	Values of *a*/*b*
1st dimension	upper bound *b*	0.703	4.342	0.755	0.802	0.742	3.292
	lower bound *a*	0.695	−1.300	0.704	−3.632	0.732	−1.349
2nd dimension	upper bound *b*	0.720	3.691	0.682	−0.366	0.654	0.272
	lower bound *a*	0.703	1.352	0.627	−4.162	0.762	−2.610

Table 2. Summary of the bounds of information granules and *AUC* values for different values of α_1 and α_2.

Values of Parameters α_1 And α_2	Bounds of Information Granule $[a, b]$	Values of AUC	
		Upper Bound	Lower Bound
$\alpha_1 = 0, \alpha_2 = 0$	[0.072, 3.597]	0.790	0.823
$\alpha_1 = 0.1, \alpha_2 = 0.1$	[−0.133, 4.037]	0.765	0.795
$\alpha_1 = 0.1, \alpha_2 = 0.5$	[−0.476, 4.272]	0.733	0.767
$\alpha_1 = 0.1, \alpha_2 = 0.8$	[−0.833, 4.272]	0.726	0.756
$\alpha_1 = 0.5, \alpha_2 = 0.1$	[−1.300, 4.272]	0.720	0.704
$\alpha_1 = 0.5, \alpha_2 = 0.5$	[−1.300, 4.342]	0.703	0.695
$\alpha_1 = 0.5, \alpha_2 = 0.8$	[−1.300, 4.342]	0.707	0.695
$\alpha_1 = 0.8, \alpha_2 = 0.1$	[−1.300, 4.342]	0.692	0.627
$\alpha_1 = 0.8, \alpha_2 = 0.5$	[−1.300, 4.382]	0.686	0.634
$\alpha_1 = 0.8, \alpha_2 = 0.8$	[−1.300, 4.342]	0.696	0.645
$\alpha_1 = 1.0, \alpha_2 = 1.0$	[−1.351, 4.342]	0.704	0.622

In Table 2, the results show that with the use of the proposed synergistic mechanism, the positions of the 1st dimension (x-coordinate) of the 0th information granule are changing with different values of α_1 and α_2. When $\alpha_1 = 0$, $\alpha_2 = 0$, it indicates that the determination of information granule 0 is not influenced by information granules 1 and 2. In this case, the information granules are developed using the principle of justifiable granularity, ignoring their synergistic nature. Compared with the information granule constructed based on the principle of justifiable granularity, the parameters of α_1 and α_2 are included for considering the influence of other information granules. To be more specific, for a given parameter α_1, with an increasing value of α_2, it is obvious that the range of the bounds of the information granules for each dimension becomes wider. Similarly, for a given parameter α_2, with an increasing value of α_1, the range of the bounds of the information granules for each dimension becomes wider as well. It implies that with the increasing values of the parameters of the synergistic mechanism, the coverage of information granules is also increasing.

At the same time, the *AUC* values are obtained for the upper and lower bounds to evaluate the performance of the construction process. As the *AUC* values presented in Table 1 show, their values are greater than 0.6. This also means that, compared with the information granules constructed based on the principle of justifiable granularity, the information granules constructed under the synergistic mechanism not only have higher coverage of the data set but also consider the specificity of the data.

To further verify the overall performance of the proposed model, the constructed multidimensional information granule is regarded as a whole to compute the coverage and specificity, referred to here as *COV* and *SP*. The two criteria are expressed as follows:

COV implies that the data is covered in the area of the information granule:

$$COV_i = \frac{1}{N_i}\text{card}\{x_k | x_k \in M_i\}, \tag{22}$$

where N_i is the number of data that belong to the ith cluster. M_i means the area of the ith information granule.

SP is taken as the product of the specificities of each dimension, and it can be expressed as follows:

$$SP_i = \left(1 - \frac{|a_{i1}^{+} - a_{i1}^{-}|}{|x_{i1,\max} - x_{i1,\min}|}\right) * \left(1 - \frac{|a_{i2}^{+} - a_{i2}^{-}|}{|x_{i2,\max} - x_{i2,\min}|}\right) * \ldots\ldots * \left(1 - \frac{|a_{in}^{+} - a_{in}^{-}|}{|x_{in,\max} - x_{in,\min}|}\right). \tag{23}$$

Considering a two-dimensional information granule, M_i can be expressed as $M_i = [a_{i1}{}^{-}, a_{i1}{}^{+}] \times [a_{i2}{}^{-}, a_{i2}{}^{+}]$, then *COV* can be rewritten as follows:

$$COV_i = \frac{1}{N_i}\text{card}\left\{x_k \middle| x_{k1} \in [a_{i1}{}^{-}, a_{i1}{}^{+}] \cup x_{k2} \in [a_{i2}{}^{-}, a_{i2}{}^{+}]\right\}, \tag{24}$$

where x_{k1} and x_{k2} are the 1st dimension and the 2nd dimension of the data x_k, respectively.

SP can be expressed as follows:

$$SP_i = \left(1 - \frac{|a_{i1}^{+} - a_{i1}^{-}|}{|x_{i1,\max} - x_{i1,\min}|}\right) * \left(1 - \frac{|a_{i2}^{+} - a_{i2}^{-}|}{|x_{i2,\max} - x_{i2,\min}|}\right). \tag{25}$$

We plot the curve of coverage and specificity with respect to different values of α_1 and α_2, the results are plotted as shown in Figure 7.

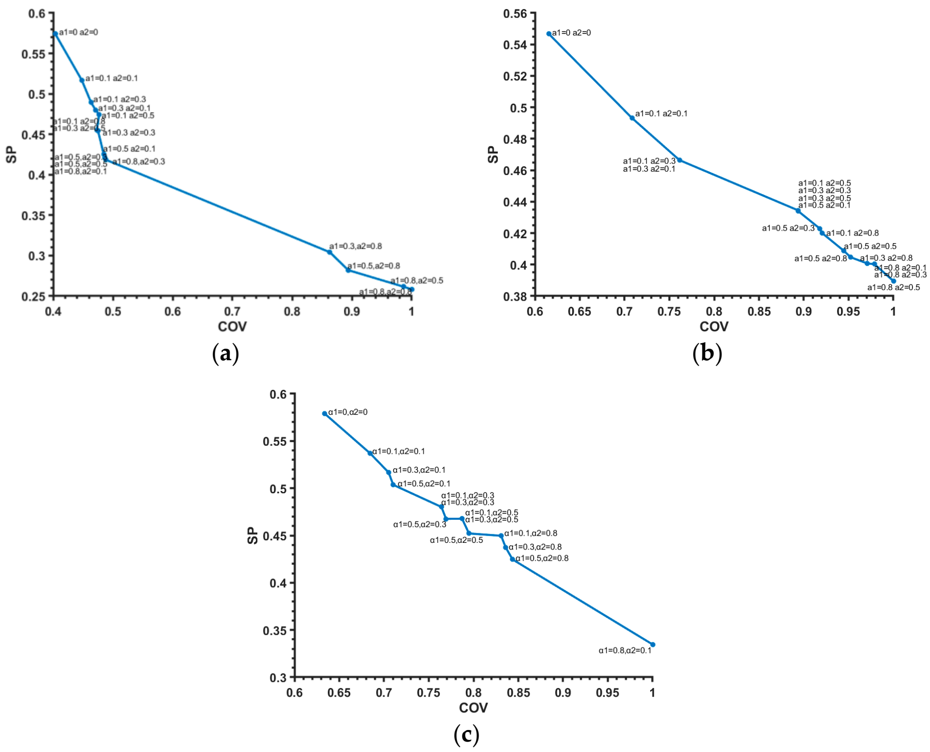


Figure 7. The curves of *COV*-*SP* with different values of α_1 and α_2 (**a**) Cluster 0 (**b**) Cluster 1 (**c**) Cluster 2.

As shown in Figure 7, a similar conclusion can be obtained as above: with increasing values of *COV*, the values of *SP* keep decreasing.

In addition, we consider the overall performance of the global coverage and specificity by calculating the entire data sets regardless of the different clusters.

The global coverage, Cov_g, is decided as follows:

$$Cov_g = \sum_{i=1}^{c} COV_i, \tag{26}$$

where c is the number of clusters. It should be noted here that if an element is covered by more than one cluster at the same time, it is counted only once.

We also define the global specificity, $Spec_g$, as follows:

$$Spec_g = \frac{1}{c}\sum_{i=1}^{c} SP_i. \tag{27}$$

Finally, the optimization criterion V_g can be expressed as follows:

$$V_g = Cov_g * Spec_g. \tag{28}$$

We plot the curves of Cov_g, $Spec_g$, and V_g with respect to different numbers of clusters, say c = 3, 4, 5, 6, 7, 8, 9, and 10. The results are as shown in Figure 8.

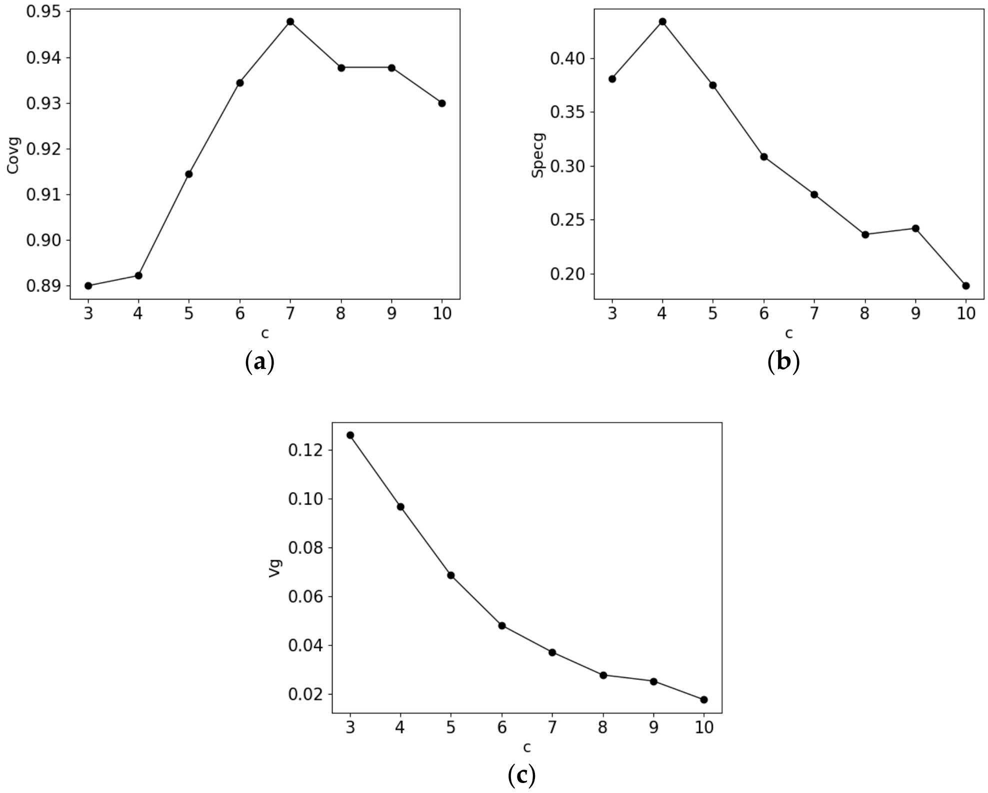


Figure 8. The curves of Cov_g, $Spec_g$, and V_g under different c values. (**a**) $\mathrm{Cov_g}$ curve (**b**) $\mathrm{Spec_g}$ curve (**c**) $\mathrm{V_g}$ curve.

As shown in Figure 8, with the increasing number of clusters c (which means more information granules are to be constructed and the construction process will be realized in a collaborative way), the overall performance index V_g decreases accordingly. The global specificity of $Spec_g$ shows an increase when the number of clusters increases from 3 to 4 and a decrease from 4 to 8; moreover, there is a slight increase when c = 9 and a decrease when c = 10. The global specificity of $Spec_g$ achieves its optimal value when c = 4. At the same time, the global coverage of Cov_g increases when the value of c increases from 3 to 7, decreases with the value of c from 7 to 10, and reaches its highest when c = 7.

4.2. KEEL Machine Learning Data Sets

In this part, several publicly available machine learning data sets coming from the KEEL machine learning repository [45] are considered in the experiments.

4.2.1. Iris Data Set

The Iris data set consists of 50 instances and four attributes, which contain three classes of features, and each class refers to a type of iris plant. One class is linearly separable from the other two; the latter are NOT linearly separable from each other. The four attributes are referred to as sepal length, sepal width, petal length, and petal width, respectively. The *AUC* values and the bounds of information granules are summarized in Table 3.

Table 3. The summary of the *AUC* values and optimized bounds of information granules ($\alpha_1 = \alpha_2 = 0.5$).

Attributes	Upper/Lower Bounds	0th Cluster		1st Cluster		2nd Cluster	
		AUC	Values of *a*/*b*	AUC	Values of *a*/*b*	AUC	Values of *a*/*b*
Sepal Length	upper bound *b*	0.525	7.200	0.717	6.701	0.678	6.003
	lower bound *a*	0.687	5.800	0.689	5.500	0.683	4.899
Sepal Width	upper bound *b*	0.779	3.400	0.814	3.200	0.550	3.901
	lower bound *a*	0.687	2.700	0.643	2.499	0.554	2.800
Petal Length	upper bound *b*	0.529	6.100	0.754	5.102	0.587	4.003
	lower bound *a*	0.768	4.397	0.731	3.298	0.628	1.300
Petal Width	upper bound *b*	0.526	2.300	0.701	1.801	0.568	1.502
	lower bound *a*	0.727	1.399	0.737	0.999	0.660	0.200

From the results presented in Table 3, we can find that for each information granule constructed based on different cluster numbers from 0 to 2, the *AUC* values obtained for both dimensions for obtaining the upper bound and lower bound show good performance with a collection of reasonable values that are greater than 0.5, and this indicates that the areas determined with the constructed information granules can cover most of the elements in the data set. The maximal *AUC* value is obtained when optimizing the upper bound of the 1st cluster for attribute sepal width, while the minimal value is obtained when optimizing the upper bound of the 0th cluster for attribute sepal length. Coordinately, the upper bound and lower bound of information granules for different clusters are optimized as shown in the table.

The *cov-sp* curves of the information granules obtained based on different clusters are plotted in Figure 9.

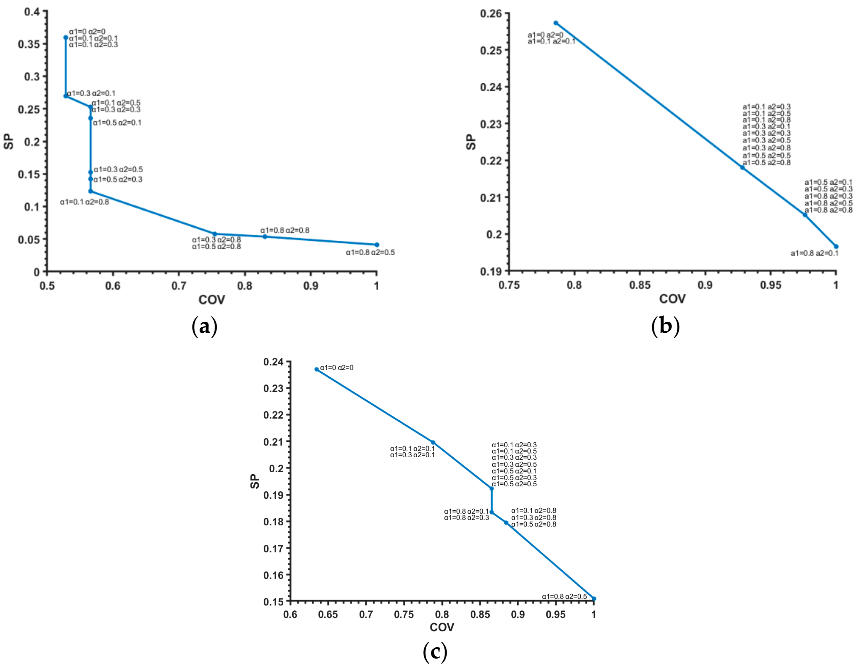


Figure 9. The curves of *cov-sp* with several pairs of α_1, α_2 values. (**a**) Cluster 0 (**b**) Cluster 1 (**c**) Cluster 2.

As shown in Figure 9, the curves of *cov-sp* are plotted with respect to different pairs of α_1, and α_2 values for the three clusters, respectively. In general, despite the values of α_1, and α_2, it can be found that a greater value of coverage corresponds to a smaller value of specificity, which implies the same tendency for the changing of coverage and specificity in the traditional principle of justifiable granularity.

4.2.2. Banana Data Set

The Banana data set is an artificial data set where the instances belong to several clusters with a banana shape. There are two attributes, At1 and At2, which correspond to the x axis and y axis, respectively. There are two types of 5300 data in total.

The *AUC* values and the bounds of information granules are shown in Table 4.

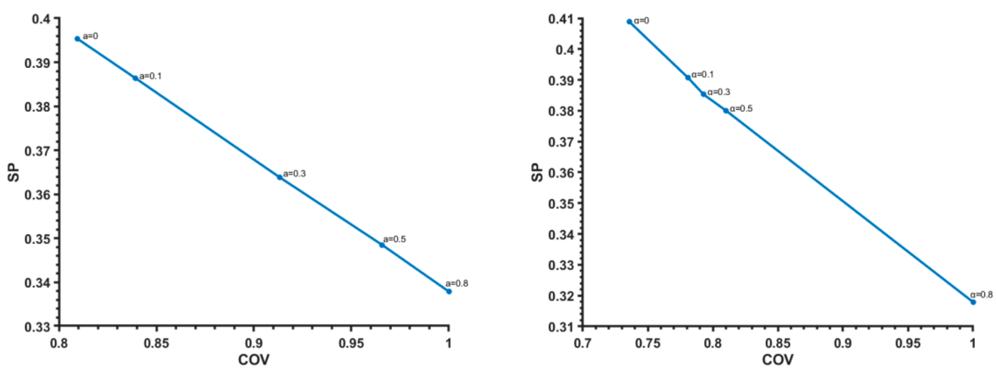
Table 4. The summary of the *AUC* values and bounds of information granules (α = 0.5).

Attributes	Upper/Lower Bounds	0th Cluster		1st Cluster	
		AUC	Values of *a*/*b*	AUC	Values of *a*/*b*
At1	upper bound *b*	0.698	0.624	0.708	1.471
	lower bound *a*	0.735	−1.640	0.714	−1.042
At2	upper bound *b*	0.724	0.732	0.758	1.721
	lower bound *a*	0.716	−1.401	0.671	−0.472

As the results in Table 4 present, we can find that for each information granule, the *AUC* values of both dimensions of the upper bounds and lower bounds show good performance with a maximal value of 0.758 (when optimizing the upper bound of the 1st cluster for attribute At2) and a minimal value of 0.671 (when optimizing the lower bound of the 1st cluster for attribute At2). This indicates that the areas determined with the constructed information granules (expressed with the upper and lower bounds) can cover most of the elements in the data set.

In Figure 9, we plot the *cov-sp* curves for different information granules designed based on the clusters.

The curves of *cov-sp* values under different α values for the Banana Data Set are as shown in Figure 10. In this case, only one granularity parameter α is included in the synergistic mechanism; the value of *cov* keeps increasing with an increasing value of α, while the value of *sp* decreases.



(**a**) (**b**)

Figure 10. The curve of *cov-sp* values under different α values. (**a**) The 0th cluster (**b**) The 1st cluster.

4.2.3. Appendicitis Data Set

The Appendicitis data set records patients with appendicitis. The data represents seven medical measures taken over 106 patients. The seven medical measures are recorded as At1, At2, At3, At4, At5, At6, and At7.

The *AUC* values and the bounds of information granules are shown in Table 5.

Table 5. The summary of the *AUC* values and bounds of information granules (α = 0.5).

Attributes	Upper/Lower Bounds	0th Cluster		1st Cluster	
		AUC	Values of *a*/*b*	AUC	Values of *a*/*b*
At1	upper bound *b*	0.732	0.520	0.740	0.628
	lower bound *a*	0.624	0.187	0.687	0.351
At2	upper bound *b*	0.719	0.627	0.726	0.884
	lower bound *a*	0.559	0.187	0.513	0.360
At3	upper bound *b*	0.740	0.472	0.740	0.684
	lower bound *a*	0.574	0.089	0.662	0.360
At4	upper bound *b*	0.737	0.471	0.682	0.520
	lower bound *a*	0.584	0.089	0.538	0.098
At5	upper bound *b*	0.703	0.471	0.675	0.521
	lower bound *a*	0.673	0.058	0.485	0.058
At6	upper bound *b*	0.731	0.628	0.813	0.796
	lower bound *a*	0.563	0.187	0.521	0.360
At7	upper bound *b*	0.734	0.471	0.756	0.627
	lower bound *a*	0.523	0.089	0.683	0.351

From the results presented in Table 5, we can find that for each information granule, the *AUC* values obtained for both dimensions of the upper bounds and lower bounds show good performance, among which the maximal value is 0.813 (when optimizing the upper bound of the 1st cluster for attribute At6). The minimal *AUC* value is 0.485 when optimizing the lower bound of the 1st cluster for attribute At5. Corresponding to the obtained *AUC* values, the upper and lower bounds that determine the updated optimal position of the information granules developed with the synergistic mechanism of the principle of justifiable granularity are presented.

As shown in Figure 11, we present the curves of *cov*-*sp* with different pairs of α_1, and α_2 values. Similar to the changing of coverage and specificity in the traditional principle of justifiable granularity, the results of the proposed mechanism indicate that a greater value of coverage corresponds to a smaller value of specificity. Considering the parameter of granularity α, the value of *cov* keeps increasing with an increasing value of α, while the value of *sp* decreases.

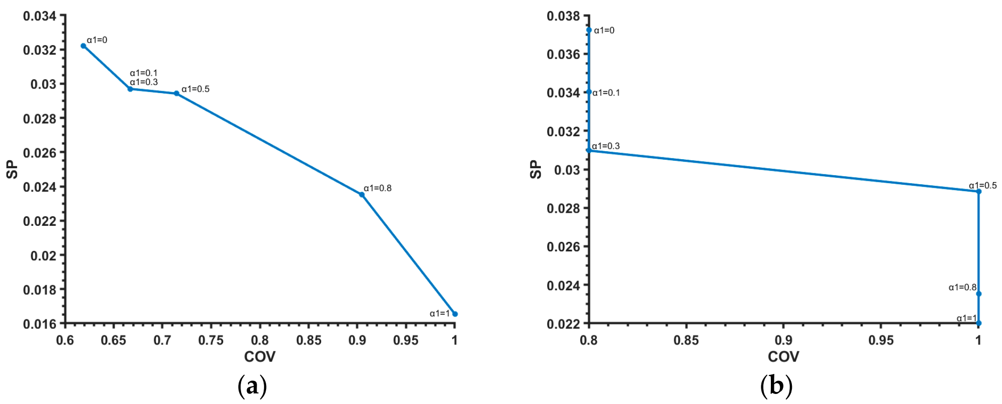


Figure 11. The curve of *cov*-*sp* values under different values of α. (**a**) The 0th cluster (**b**) The 1st cluster.

5. Conclusions

Information granules play an important role in describing and expressing data in different forms, including sets and intervals, fuzzy sets, rough sets, and so on. Fuzzy sets show good performance in dealing with information with fuzziness and uncertainty. In practical complex environments, with the rapid development of data size, the data with multimodality become significantly more challenging to process and manage. There is a commonly encountered situation where data is collected with different characteristics, and those with different features will be influenced by each other. In this study, a synergistic mechanism of the principle of justifiable granularity in developing information granules is proposed, in which information granules are constructed by considering the influence of other information granules in the same feature space or different feature spaces. In other words, a collection of information granules is designed in a collaborative manner rather than based on some experimental evidence alone. First of all, the Fuzzy C-Means Clustering method is utilized to cluster the data space into different clusters, and then information granules are developed with the use of the principle of justifiable granularity based on the data located in the different clusters. Then, the position of each information granule is updated while considering the impact of the other information granules in the data space, and the level of impact is specified according to the parameters of information granularity α. The main significance of this study lies in the fact that the synergistic mechanism of the principle of justifiable granularity takes full consideration of the collaborative impacts of different information granules. The performance of the proposed mechanism is evaluated with the *AUC* values that are determined based on the coverage and specificity of the principle of justifiable granularity. Finally, a series of experimental studies are conducted to verify the feasibility of the proposed mechanism. From the results of the experiments, we can find that (i) with increasing values of the impact parameters, the location of the information granule becomes closer to the other ones that influence its development; (ii) the *AUC* values can achieve a reasonable value of greater than 0.7, and it can be found that the information granules constructed under the synergistic mechanism show good performance.

In future studies, it is worth considering information granules developed based on heterogeneous or separately non-linear data. Moreover, an interesting idea lies in the application of the synergistic mechanism of the principle of justifiable granularity to deal with real industry problems such as mine pressure prediction, image edge detection, and so on.

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