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Subclasses of p-Valent Functions Associated with Linear q-Differential Borel Operator

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Abstract: The aim of the present paper is to introduce and study some new subclasses of p-valent functions by making use of a linear q-differential Borel operator. We also deduce some properties, such as inclusion relationships of the newly introduced classes and the integral operator $\mathcal{J}_{\mu,p}$.

Keywords: p-valent; analytic function; starlike functions; convex functions; close-to-convex functions; fractional derivative; linear q-differential Borel operator

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$\mathcal{F}(\varsigma) = \varsigma^p + \sum_{j=p+1}^{\infty} a_j \varsigma^j \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic in the open unit disc $\Delta = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$.

Let $\mathcal{P}_{p,k}(\alpha)$ be the class of functions $h(\varsigma)$ analytic in Δ satisfying the properties $h(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\Re\{h(\varsigma)\} - \alpha}{p - \alpha} \right| d\theta \leq k\pi, \tag{2}$$

where $\varsigma = re^{i\theta}$, $k \geq 2$ and $0 \leq \alpha < p$. This class was introduced by (Aouf [1] with $\lambda = 0$).

We note that

(i) $\mathcal{P}_{1,k}(\alpha) = \mathcal{P}_k(\alpha)$ $(k \geq 2,\ 0 \leq \alpha < 1)$ (see Padmanabhan and Parvatham [2]);
(ii) $\mathcal{P}_{1,k}(0) = \mathcal{P}_k$ $(k \geq 2)$ (see Pinchuk [3] and Robertson [4]);
(iii) $\mathcal{P}_{p,2}(\alpha) = \mathcal{P}(p,\alpha)$ $(0 \leq \alpha < p,\ p \in \mathbb{N})$, where $\mathcal{P}(p,\alpha)$ is the class of functions with a positive real part greater than α (see [1]);
(iv) $\mathcal{P}_{p,2}(0) = \mathcal{P}(p)$ $(p \in \mathbb{N})$, where $\mathcal{P}(p)$ is the class of functions with a positive real part (see [1]).

From (2), we have $h(\varsigma) \in \mathcal{P}_{p,k}(\alpha)$ if and only if there exists $h_1, h_2 \in \mathcal{P}_p(\alpha)$ such that

$$\mathcal{G}(\varsigma) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(\varsigma) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(\varsigma) \ \ (\varsigma \in \Delta). \tag{3}$$

For two functions $\mathcal{F}(\varsigma)$ given by (1) and $\mathcal{H}(\varsigma)$ given by

$$\mathcal{H}(\varsigma) = \varsigma^p + \sum_{j=p+1}^{\infty} b_j \varsigma^j$$

the Hadamard product (or convolution) is defined by

$$(\mathcal{F} * \mathcal{H})(\varsigma) = \varsigma^p + \sum_{j=p+1}^{\infty} a_j b_j \varsigma^j = (\mathcal{H} * \mathcal{F})(\varsigma). \tag{4}$$

Define here a Borel distribution with parameter λ, which is a discrete random variable denoted by χ. This variable takes the values $1, 2, 3, \ldots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \ldots$, respectively.

Wanas and Khuttar [5] recently introduced the Borel distribution (BD) whose probability mass function is (see [6,7])

$$P(\chi = \rho) = \frac{(\rho\lambda)^{\rho-1} e^{-\lambda\rho}}{\rho!}, \quad \rho = 1, 2, 3, \ldots$$

Wanas and Khuttar studied a series $\mathcal{M}(\lambda; \varsigma)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\begin{aligned} \mathcal{M}_p(\lambda; \varsigma) &= \varsigma^p + \sum_{j=p+1}^{\infty} \frac{[\lambda(j-p)]^{j-p-1} e^{-\lambda(j-p)}}{(j-p)!} \varsigma^j, \ (0 < \lambda \leq 1), \\ &= \varsigma^p + \sum_{j=p+1}^{\infty} \phi_{j,p}(\lambda) \varsigma^k, \ (0 < \lambda \leq 1), \end{aligned} \tag{5}$$

where

$$\phi_{j,p}(\lambda) = \frac{[\lambda(j-p)]^{j-p-1} e^{-\lambda(j-p)}}{(j-p)!} \tag{6}$$

We propose a linear operator $\mathcal{D}(p, \lambda; \varsigma)\mathcal{F} : \mathcal{A}_p \to \mathcal{A}_p$ as follows

$$\begin{aligned} \mathcal{D}(p, \lambda; \varsigma)\mathcal{F}(\varsigma) &= \mathcal{M}_p(\lambda; \varsigma) * \mathcal{F}(\varsigma) \\ &= \varsigma^p + \sum_{j=p+1}^{\infty} \frac{[\lambda(j-p)]^{j-p-1} e^{-\lambda(j-p)}}{(j-p)!} a_j \, \varsigma^j, \ (0 < \lambda \leq 1). \end{aligned}$$

In a recent paper, Srivastava [8] studied various types of operators regarding q-calculus. We recall further some important definitions and notations. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\mu; q)_j = \begin{cases} 1 & j = 0, \\ (1-\mu)(1-\mu q)\ldots(1-\mu q^{j-1}) & j \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(\varsigma)$, we get

$$(q^\mu; q)_j = \frac{(1-q)^j \, \Gamma_q(\mu + j)}{\Gamma_q(\mu)}, \quad (j \in \mathbb{N}_0),$$

where (see [9])

$$\Gamma_q(\varsigma) = (1-q)^{1-\varsigma} \frac{(q; q)_\infty}{(q^\varsigma; q)_\infty}, \quad (|q| < 1).$$

Furthermore, we note that

$$(\mu;q)_\infty = \prod_{j=0}^{\infty}\left(1-\mu q^j\right), \quad (|q|<1),$$

and, the q-gamma function $\Gamma_q(\varsigma)$ is known

$$\Gamma_q(\varsigma+1) = [j]_q\,\Gamma_q(\varsigma),$$

where $[j]_q$ denotes the basic q-number defined as follows

$$[j]_q := \begin{cases} \frac{1-q^j}{1-q}, & j\in\mathbb{C}, \\ 1+\sum\limits_{i=1}^{j-1} q^i, & j\in\mathbb{N}. \end{cases} \tag{7}$$

Using the definition from (7), we have the next two products:

(i) For a non negative integer j, the *q-shifted factorial* is defined by

$$[j]_q! := \begin{cases} 1, & \text{if } \ j=0, \\ \prod\limits_{n=1}^{j}[n]_q, & \text{if } \ j\in\mathbb{N}. \end{cases}$$

(ii) For a positive number r, the *q-generalized Pochhammer symbol* is given by

$$[r]_{q,j} := \begin{cases} 1, & \text{if } \ j=0, \\ \prod\limits_{n=r}^{r+j-1}[n]_q, & \text{if } \ j\in\mathbb{N}. \end{cases}$$

In terms of the classical (Euler's) gamma function $\Gamma(\varsigma)$, we have

$$\Gamma_q(\varsigma)\to\Gamma(\varsigma) \qquad \text{as } q\to 1^-.$$

Furthermore, we notice that

$$\lim_{q\to 1^-}\left\{\frac{(q^\mu;q)_j}{(1-q)^j}\right\} = (\mu)_j.$$

2. Preliminaries

In order to establish our new results, we have to recall the construct of a q-derivative operator. Considering $0<q<1$, the *q-derivative operator* [10] (see also other specific and generalized results [11–15]) for $\mathcal{D}(p,\lambda;\varsigma)\mathcal{F}$ is defined by

$$\begin{aligned} \mathcal{D}_q(\mathcal{D}(p,\lambda;\varsigma)\mathcal{F}(\varsigma)) \quad : &= \frac{\mathcal{D}(p,\lambda;\varsigma)\mathcal{F}(\varsigma)-\mathcal{D}(p,\lambda;\varsigma)\mathcal{F}(q\varsigma)}{\varsigma(1-q)} \\ &= [p]_q\varsigma^{p-1}+\sum_{j=p+1}^{\infty}[j]_q\frac{[\lambda(j-p)]^{j-p-1}e^{-\lambda(j-p)}}{(j-p)!}a_j\,\varsigma^{j-1}, \end{aligned}$$

where $[j]_q$ is defined in (7)

For $\alpha>-1$ and $0<q<1$, we obtain the linear operator $\mathcal{D}^{\mu,q}_{p,\lambda}\mathcal{F}:\mathcal{A}_p\to\mathcal{A}_p$ by

$$\mathcal{D}^{\mu,q}_{p,\lambda}\mathcal{F}(\varsigma)*\mathcal{N}^{q}_{p,\mu+1}(\varsigma) = \frac{\varsigma}{[p]_q}\,\mathcal{D}_q(\mathcal{D}(p,\lambda;\varsigma)\mathcal{F}(\varsigma)),\ \varsigma\in\Delta,$$

where the function $\mathcal{N}^{q}_{p,\alpha+1}$ is given by

$$\mathcal{N}^{q}_{p,\mu+1}(\varsigma) := \varsigma^{p} + \sum_{j=p+1}^{\infty} \frac{[\mu+1]_{q,j-p}}{[j-1]_{q}!}\varsigma^{j}, \ \varsigma \in \Delta.$$

A simple computation shows that

$$\begin{aligned} \mathcal{D}^{\mu,q}_{p,\lambda}\mathcal{F}(\varsigma) \quad : \quad &= \varsigma^{p} + \sum_{j=p+1}^{\infty} \frac{[j]_{q}![\lambda(j-p)]^{j-p-1}e^{-\lambda(j-p)}}{[p]_{q}[\mu+1]_{q,j-p}(j-p)!} a_{j}\,\varsigma^{j} \\ &= z^{p} + \sum_{j=p+1}^{\infty} \phi_{j} a_{j}\,\varsigma^{j} \quad (0<\lambda\leq 1,\ \mu>p,\ 0<q<1,\ \varsigma\in\Delta). \end{aligned} \tag{8}$$

where

$$\phi_{j} = \frac{[j]_{q}![\lambda(j-p)]^{j-p-1}e^{-\lambda(j-p)}}{[p]_{q}[\mu+1]_{q,j-p}(j-p)!}. \tag{9}$$

For $\delta \geq 0$, with the aid of the operator $\mathcal{D}^{\mu,q}_{p,\lambda}$ one can defined the linear q-differential Borel operator $\mathcal{A}_{p} \to \mathcal{A}_{p}$ as follows:

$$\begin{aligned} \mathcal{G}^{\mu,0}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \quad : \quad &= \mathcal{D}^{\mu,q}_{p,\lambda}\mathcal{F}(\varsigma) \\ \mathcal{G}^{\mu,1}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \quad : \quad &= (1-\delta)\mathcal{G}^{\mu,0}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) + \delta\frac{\varsigma}{p}\left(\mathcal{G}^{\mu,0}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)' \\ &= \varsigma^{p} + \sum_{j=p+1}^{\infty} \frac{[j]_{q}![\lambda(j-p)]^{j-p-1}e^{-\lambda(j-p)}}{[p]_{q}[\mu+1]_{q,j-p}(j-p)!}\left[1+\delta\left(\frac{j}{p}-1\right)\right] a_{j}\,\varsigma^{j} \\ \mathcal{G}^{\mu,2}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \quad : \quad &= (1-\delta)\mathcal{G}^{\mu,1}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) + \delta\frac{\varsigma}{p}\left(\mathcal{G}^{\mu,1}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)' \\ &= \varsigma^{p} + \sum_{j=p+1}^{\infty} \frac{[j]_{q}![\lambda(j-p)]^{j-p-1}e^{-\lambda(j-p)}}{[p]_{q}[\mu+1]_{q,j-p}(j-p)!}\left[1+\delta\left(\frac{j}{p}-1\right)\right]^{2} a_{j}\,\varsigma^{j} \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \quad : \quad &= \varsigma^{p} + \sum_{j=p+1}^{\infty} \frac{[j]_{q}![\lambda(j-p)]^{j-p-1}e^{-\lambda(j-p)}}{[p]_{q}[\mu+1]_{q,j-p}(j-p)!}\left[1+\delta\left(\frac{j}{p}-1\right)\right]^{m} a_{j}\,\varsigma^{j}, \\ &(m\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\},\ \delta\geq 0,\ 0<\lambda\leq 1,\ \mu>p,\ 0<q<1). \end{aligned} \tag{10}$$

From the relation (10), we can easily deduce that the next relations held for all $\mathcal{F} \in \mathcal{A}_{p}$:

$$\text{(i)} \quad \varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)' = \mu\,\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) - (\mu-p)\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma), \tag{11}$$

and

$$\text{(ii)} \ \delta\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)' = p\,\mathcal{G}^{\mu,m+1}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) - p\,(1-\delta)\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \tag{12}$$

Remark 1. *By particularizing the parameters p and m, we derive the following operators based on Borel distribution:*

(1) *Letting* $p = 1$, *we obtain that* $\mathcal{G}^{\mu,m}_{1,q,\lambda,\delta} =: \mathcal{I}^{\mu,m}_{q,\lambda,\delta}$, *where the operator* $\mathcal{I}^{\mu,m}_{q,\lambda,\delta}$ *is defined as follows:*

$$\mathcal{I}^{\mu,m}_{q,\lambda,\delta}\mathcal{F}(\varsigma) := \varsigma + \sum_{j=2}^{\infty} \frac{[j]_q![\lambda(j-1)]^{j-2}e^{-\lambda(j-1)}}{[\mu+1]_{q,j-1}(j-1)!}[1+\delta(j-1)]^m a_j\, \varsigma^j;$$

(2) *Letting* $p = 1$ *and* $m = 0$, *we deduce that* $\mathcal{G}^{\mu,0}_{1,q,\lambda,\delta} =: \mathcal{B}^{\mu,q}_{\lambda}$, *where the operator* $\mathcal{B}^{\mu,q}_{\lambda}$, *introduced by El-Deeb and Murugusundaramoorthy [16];*

(3) *Letting* $q \to 1^-$ *and* $p = 1$, *we deduce that* $\lim\limits_{q\to 1^-} \mathcal{G}^{\mu,m}_{1,q,\lambda,\delta} := \mathcal{R}^{\mu,m}_{\lambda,\delta}$, *where the operator* $\mathcal{R}^{\mu,m}_{\lambda,\delta}$ *is defined as follows*

$$\mathcal{R}^{\mu,m}_{\lambda,\delta}\mathcal{F}(\varsigma) := \varsigma + \sum_{j=2}^{\infty} \tfrac{j[\lambda(j-1)]^{j-2}e^{-\lambda(j-1)}}{(\mu+1)_{j-1}}[1+\delta(j-1)]^m a_j \varsigma^j;$$

(4) *Putting* $q \to 1^-$, $p = 1$ *and* $m = 0$, *we obtain that* $\lim\limits_{q\to 1^-} \mathcal{G}^{\mu,0}_{1,q,\lambda,\delta} := \mathcal{M}^{\mu}_{\lambda}$, *where the operator* $\mathcal{M}^{\mu}_{\lambda}$, *studied by El-Deeb and Murugusundaramoorthy [16].*

Now we introduce the following classes $\mathcal{S}^k_p(\alpha)$, $\mathcal{C}^k_p(\alpha)$ and $\mathcal{K}^k_p(\beta,\alpha)$ of the class $\mathcal{A}_p$ for $0 \le \alpha, \beta < p,\ p \in \mathbb{N}$ and $k \ge 2$ as follows:

$$\mathcal{S}^k_p(\alpha) = \left\{ \mathcal{F} :\ \mathcal{F} \in \mathcal{A}_p \text{ and } \frac{\varsigma\mathcal{F}'(\varsigma)}{\mathcal{F}(\varsigma)} \in \mathcal{P}_{p,k}(\alpha),\ \varsigma \in \Delta \right\},$$

$$\mathcal{C}^k_p(\alpha) = \left\{ \mathcal{F} :\ \mathcal{F} \in \mathcal{A}_p \text{ and } 1 + \frac{\varsigma\mathcal{F}''(\varsigma)}{\mathcal{F}'(\varsigma)} \in \mathcal{P}_{p,k}(\alpha),\ \varsigma \in \Delta \right\},$$

and

$$\mathcal{K}^k_p(\beta,\alpha) = \left\{ \mathcal{F} :\ \mathcal{F} \in \mathcal{A}_p,\ g \in \mathcal{S}^2_p(\alpha) \text{ and } \frac{\varsigma\mathcal{F}'(\varsigma)}{g(\varsigma)} \in \mathcal{P}_{p,k}(\beta),\ \varsigma \in \Delta \right\}.$$

Obviously, we know that

$$\mathcal{F}(\varsigma) \in \mathcal{C}^k_p(\alpha) \Leftrightarrow \frac{\varsigma\mathcal{F}'(\varsigma)}{p} \in \mathcal{S}^k_p(\alpha). \tag{13}$$

Remark 2. *By particularizing the parameter* k, *we obtain the following classes:*

(i) $\mathcal{S}^2_p(\alpha) = \mathcal{S}^*_p(\alpha)$ $(0 \le \alpha < p,\ p \in \mathbb{N})$, *where* $\mathcal{S}^*_p(\alpha)$ *is the well-known class of* $p-$*valently starlike functions of order* α *and was studied by Patil and Thakare [17];*

(ii) $\mathcal{C}^2_p(\alpha) = \mathcal{C}_p(\alpha)$ $(0 \le \alpha < p,\ p \in \mathbb{N})$, *where* $\mathcal{C}_p(\alpha)$ *is the well-known class of* $p-$*valently convex functions of order* α *and was studied by Owa [18];*

(iii) $\mathcal{K}^2_p(\beta,\alpha) = \mathcal{K}_p(\beta,\alpha)$ $(0 \le \alpha < p,\ p \in \mathbb{N})$, *where* $\mathcal{K}_p(\beta,\alpha)$ *is the class of all* $p-$*valently close-to-convex functions of order* β *and type* α *and was introduced by Aouf [19].*

Next, by making use of the operator defined by (10), we obtain the following subclasses $\mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$, $\mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$ and $\mathcal{K}^{\mu,m,k}_{p,q,\lambda,\delta}(\beta,\alpha)$ of the class $\mathcal{A}_p$ as follows:

$$\mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) = \left\{ \mathcal{F} :\ \mathcal{F} \in \mathcal{A}_p \text{ and } \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \in \mathcal{S}^k_p(\alpha),\ \varsigma \in \Delta \right\}, \tag{14}$$

$$\mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) = \left\{ \mathcal{F} :\ \mathcal{F} \in \mathcal{A}_p \text{ and } \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \in \mathcal{C}^k_p(\alpha),\ \varsigma \in \Delta \right\}, \tag{15}$$

and

$$\mathcal{K}^{\mu,m,k}_{p,q,\lambda,\delta}(\beta,\alpha) = \left\{ \mathcal{F} :\ \mathcal{F} \in \mathcal{A}_p \text{ and } \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \in \mathcal{K}^k_p(\beta,\alpha),\ \varsigma \in \Delta \right\}. \tag{16}$$

We can easily see that

$$\mathcal{F}(\varsigma) \in \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) \Leftrightarrow \frac{\varsigma \mathcal{F}'(\varsigma)}{p} \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) \tag{17}$$

In order to establish our main results, we will require the following lemmas.

Lemma 1 ([20,21])**.** *Let $\Phi(r,s)$ be complex valued function, $\Phi : \mathcal{D} \to \mathbb{C}$, $\mathcal{D} \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $r = r_1 + ir_2$, $s = s_1 + is_2$. Suppose that $\Phi(r,s)$ satisfies the following conditions:*

(i) $\Phi(r,s)$ *is continuous in a domain* $\mathcal{D}$;

(ii) $(1,0) \in \mathcal{D}$ *and* $\Re\{\Phi(1,0)\} > 0$;

(iii) $\Re\{\Phi(ir_2,s_1)\} \le 0$ *for all* $(ir_2,s_1) \in \mathcal{D}$ *and such that* $s_1 \le -\frac{1}{2}(1+r_2^2)$.

Let $h(\varsigma) = 1 + \sum_{m=1}^{\infty} c_m \varsigma^m$, *be regular in* Δ *such that* $(h(\varsigma), \varsigma h'(\varsigma)) \in \mathcal{D}$ *for all* $\varsigma \in \Delta$. *If*

$$\Re\left\{\Phi(h(\varsigma), \varsigma h'(\varsigma))\right\} > 0 \quad (\varsigma \in \Delta),$$

then

$$\Re\{h(\varsigma)\} > 0 \quad (\varsigma \in \Delta).$$

Lemma 2 ([22])**.** *Let Φ be convex and $\mathcal{F}$ be starlike in Δ. Then, for Υ analytic in Δ with $\Upsilon(0) = 1$, $\frac{\Phi * \Upsilon\mathcal{F}}{\Phi * \mathcal{F}}$ is contained in the convex hull of $\Upsilon(\Delta)$.*

3. Inclusion Properties Involving the Operator $\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}$

Further, we assume throughout this paper that $k \ge 2$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\delta \ge 0$, $0 < \lambda \le 1$, $0 < q < 1$, $\varsigma \in \Delta$ and the power are the principal values.

Theorem 1. *For $0 \le \zeta \le \alpha < p$ and $\mu > p$, then*

$$\mathcal{S}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\alpha) \subset \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\zeta),$$

where ζ is given by

$$\zeta = \frac{2[p - 2\alpha(p-\mu)]}{\sqrt{(2\mu - 2p - 2\alpha + 1)^2 + 8(p - 2\alpha(p-\mu))} + (2\mu - 2p - 2\alpha + 1)}. \tag{18}$$

Proof. Assume that $\mathcal{F} \in \mathcal{S}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\alpha)$ and let

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)} = M(\varsigma) = (p-\zeta)h(\varsigma) + \zeta. \tag{19}$$

where

$$h(\varsigma) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(\varsigma) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(\varsigma) \tag{20}$$

and $h_i(z)$ $(i = 1, 2)$ are analytic in Δ with $h_i(0) = 1,\ i = 1, 2$. Using (11) and (19), we have

$$\mu \frac{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)} = (p-\zeta)h(\varsigma) + \zeta - \mu + p. \tag{21}$$

By computing the logarithmical derivative of (21) with respect to ς, we have

$$\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)} - \alpha = \zeta - \alpha + (p-\zeta)h(\varsigma) + \frac{(p-\zeta)\varsigma h'(\varsigma)}{(p-\zeta)h(\varsigma) + \zeta - \mu + p}. \tag{22}$$

Now we show that $M(\varsigma) \in \mathcal{P}_{p,k}(\alpha)$ or $h_i(\varsigma) \in \mathcal{P},\ i = 1, 2$. From (20) and (22), we have

$$\begin{aligned}\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)} - \alpha = {}& \left(\frac{k}{4} + \frac{1}{2}\right)\left\{\zeta - \alpha + (p-\zeta)h_1(\varsigma) + \frac{(p-\zeta)\varsigma h_1'(\varsigma)}{(p-\zeta)h_1(\varsigma) + \zeta - \mu + p}\right\} \\ & - \left(\frac{k}{4} - \frac{1}{2}\right)\left\{\zeta - \alpha + (p-\zeta)h_2(\varsigma) + \frac{(p-\zeta)\varsigma h_2'(\varsigma)}{(p-\zeta)h_2(\varsigma) + \zeta - \mu + p}\right\}\end{aligned}$$

and this implies that

$$\Re\left\{\zeta - \alpha + (p-\zeta)h_i(\varsigma) + \frac{(p-\zeta)\varsigma h_i'(\varsigma)}{(p-\zeta)h_i(\varsigma) + \zeta - \mu + p}\right\} > 0\ (\varsigma \in \Delta;\ i = 1, 2).$$

We form the function $\Phi(r, s)$ by choosing $r = h_i(\varsigma)$ and $s = \varsigma h_i'(\varsigma)$. Thus

$$\Phi(r, s) = \zeta - \alpha + (p-\zeta)r + \frac{(p-\zeta)s}{(p-\zeta)r + \zeta - \mu + p}.$$

Then, we have

(i) $\Phi(r, s)$ is continuous function in $\mathcal{D} = \left(\mathbb{C}\backslash\frac{\zeta-\mu+p}{\zeta-p}\right) \times \mathbb{C}$;

(ii) $(1, 0) \in \mathcal{D}$ and $\Re\{\Phi(1, 0)\} = p - \alpha > 0$;

(iii)

$$\begin{aligned}\Re\{\Phi(ir_2, s_1)\} &= \Re\left\{\zeta - \alpha + (p-\zeta)ir_2 + \frac{(p-\zeta)s_1}{(p-\zeta)ir_2 + \zeta - \mu + p}\right\} \\ &= \zeta - \alpha + \frac{(p-\zeta)(\zeta-\mu+p)s_1}{(p-\zeta)^2 r_2^2 + (\zeta-\mu+p)^2} \\ &\leq \zeta - \alpha - \frac{(p-\zeta)(\zeta-\mu+p)(1+r_2^2)}{2\left[(p-\zeta)^2 r_2^2 + (\zeta-\mu+p)^2\right]} \\ &= \frac{R + Er_2^2}{2C},\end{aligned}$$

for all $(ir_2, s_1) \in \mathcal{D}$ such that $s_1 \leq -\frac{1}{2}(1 + r_2^2)$,
where

$$R = 2(\zeta-\alpha)(\zeta-\mu+p)^2 - (p-\zeta)(\zeta-\mu+p),$$
$$E = 2(\zeta-\alpha)(p-\zeta)^2 - (p-\zeta)(\zeta-\mu+p),$$
$$C = (p-\zeta)^2 r_2^2 + (\zeta-\mu+p)^2.$$

We note that $\Re\{\Phi(ir_2, s_1)\} < 0$, if and only if $R \le 0,\ E < 0$ and $C > 0$. From $R \le 0$, we obtain ζ as given by (18), and from $0 \le \zeta < \alpha < p$, we have $E < 0$. By applying Lemma 1, $h_i(\varsigma) \in \mathcal{P}\ (i = 1, 2)$ and consequently $M(\varsigma) \in \mathcal{P}_{p,k}(\gamma)$ for $\varsigma \in \Delta$. This completes the proof of Theorem 1. □

Theorem 2. *For $0 \le \zeta \le \alpha < p$ and $\mu > p$, then*

$$\mathcal{C}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\alpha) \subset \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\zeta),$$

where ζ is given by (18).

Proof. Let

$$\begin{aligned}
\mathcal{F} \in \mathcal{C}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\alpha) &\Rightarrow \mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \in \mathcal{C}^{k}_{p}(\alpha)\\
&\Rightarrow \frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{p} \in \mathcal{S}^{k}_{p}(\alpha)\\
&\Rightarrow \mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\left(\frac{\varsigma\mathcal{F}'(\varsigma)}{p}\right) \in \mathcal{S}^{k}_{p}(\alpha)\\
&\Rightarrow \frac{\varsigma\mathcal{F}'(\varsigma)}{p} \in \mathcal{S}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\alpha) \subset \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\zeta)\\
&\Rightarrow \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\left(\frac{\varsigma\mathcal{F}'(\varsigma)}{p}\right) \in \mathcal{S}^{k}_{p}(\zeta)\\
&\Rightarrow \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) \in \mathcal{C}^{k}_{p}(\zeta)\\
&\Rightarrow \mathcal{F} \in \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\zeta).
\end{aligned}$$

This completes the proof of Theorem 2. □

Theorem 3. *For $0 \le \beta \le \alpha < p$ and $\mu > p$, then*

$$\mathcal{K}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\beta, \alpha) \subset \mathcal{K}^{\mu,m,k}_{p,q,\lambda,\delta}(\beta, \alpha).$$

Proof. Let $\mathcal{F} \in \mathcal{K}^{\mu-1,m,k}_{p,q,\lambda,\delta}(\beta, \alpha)$. Then, there exists $G(\varsigma) \in \mathcal{S}^{2}_{p}(\alpha) \equiv S^{*}_{p}(\alpha)$ such that

$$\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{G(\varsigma)} \in \mathcal{P}_{p,k}(\beta). \tag{23}$$

Then

$$G(\varsigma) = \mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\,g(\varsigma) \in \mathcal{S}^{\mu-1,m,2}_{p,q,\lambda,\delta}(\alpha).$$

We set

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\,g(\varsigma)} = \mathcal{R}(\varsigma) = (p - \beta)h(\varsigma) + \beta, \tag{24}$$

where $h(\varsigma)$ is given by (20). By using (11) in (23), we get

$$\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\,g(\varsigma)} = \frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\left(\varsigma\mathcal{F}'(\varsigma)\right)\right)' + (\mu - p)\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\left(\varsigma\mathcal{F}'(\varsigma)\right)}{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\,g(\varsigma)\right)' + (\mu - p)\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\,g(\varsigma)}. \tag{25}$$

Furthermore, $G(\varsigma) \in \mathcal{S}^{\mu-1,m,2}_{p,q,\lambda,\delta}(\alpha)$ and by using Theorem 1, with $k = 2$, we have $G(\varsigma) \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(\alpha)$. Therefore, we can write

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}g(\varsigma)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}g(\varsigma)} = \mathcal{R}_0(\varsigma) = (p-\alpha)q(\varsigma) + \alpha \quad (q \in \mathcal{P}_k), \tag{26}$$

where $q(\varsigma) = 1 + c_1\varsigma + c_2\varsigma^2 + \ldots$ is analytic and $q(0) = 1$ in Δ. By differentiating (24) with respect to ς, we have

$$\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\left(\varsigma\mathcal{F}'(\varsigma)\right)\right)' = \varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}g(\varsigma)\right)'\mathcal{R}(\varsigma) + \varsigma\mathcal{R}'(\varsigma)\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}g(\varsigma)$$

then

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\left(\varsigma f'(\varsigma)\right)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}g(\varsigma)} = \varsigma\mathcal{R}'(\varsigma) + \mathcal{R}_0(\varsigma)\mathcal{R}(\varsigma). \tag{27}$$

From (25) and (27), we obtain

$$\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}g(\varsigma)} = \frac{\varsigma\mathcal{R}'(\varsigma) + \mathcal{R}_0(\varsigma)\mathcal{R}(\varsigma) + (\mu-p)\mathcal{R}(\varsigma)}{\mathcal{R}_0(\varsigma) + (\mu-p)}$$

so that

$$\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}f(\varsigma)\right)'}{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}g(\varsigma)} = \mathcal{R}(\varsigma) + \frac{\varsigma\mathcal{R}'(\varsigma)}{\mathcal{R}_0(\varsigma) + (\mu-p)}. \tag{28}$$

Let

$$\mathcal{R}(\varsigma) = \left(\tfrac{k}{4} + \tfrac{1}{2}\right)\{(p-\beta)h_1(\varsigma) + \beta\} - \left(\tfrac{k}{4} - \tfrac{1}{2}\right)\{(p-\beta)h_2(\varsigma) + \beta\}$$

and

$$\mathcal{R}_0(\varsigma) + (\mu-p) = (p-\alpha)q(\varsigma) + (\alpha+\mu-p).$$

We intend to show that $\mathcal{R} \in \mathcal{P}_{p,k}(\beta)$ or $h_i \in \mathcal{P}$ for $i = 1,2$. Then, we can say that $\Re\{\mathcal{R}_0(\varsigma) + (\mu-p)\} > 0$. From (24) and (28), we have

$$\begin{aligned}\frac{\varsigma\left(\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu-1,m}_{p,q,\lambda,\delta}g(\varsigma)} - \beta &= \left(\tfrac{k}{4} + \tfrac{1}{2}\right)\left\{(p-\beta)h_1(\varsigma) + \frac{(p-\beta)\varsigma h_1'(\varsigma)}{(p-\alpha)q(\varsigma) + (\alpha+\mu-p)}\right\} \\ &\quad - \left(\tfrac{k}{4} - \tfrac{1}{2}\right)\left\{(p-\beta)h_2(\varsigma) + \frac{(p-\beta)\varsigma h_2'(\varsigma)}{(p-\alpha)q(\varsigma) + (\alpha+\mu-p)}\right\}\end{aligned}$$

and this implies that

$$\Re\left\{(p-\beta)h_i(\varsigma) + \frac{(p-\beta)\varsigma h_i'(\varsigma)}{(p-\alpha)q(\varsigma) + (\alpha+\mu-p)}\right\} > 0 \; (\varsigma \in \Delta,\ i = 1,2).$$

We form the function $\Phi(r,s)$ by choosing $r = h_i(\varsigma)$ and $s = \varsigma h_i'(\varsigma)$. Thus,

$$\Phi(r,s) = (p-\beta)r + \frac{(p-\beta)s}{(p-\alpha)q(\varsigma) + (\alpha+\mu-p)}. \tag{29}$$

Then

(i) $\Phi(r,s)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$;
(ii) $(1,0) \in D$ and $\Re\{\Phi(1,0)\} = p - \beta > 0$;

(iii)

$$\begin{aligned}
\Re\{\Phi(ir_2, s_1)\} &= \Re\left\{(p-\beta)iu_2 + \frac{(p-\beta)v_1}{(p-\alpha)(q_1+iq_2)+(\alpha+\mu-p)}\right\} \\
&= \frac{(p-\beta)[(p-\alpha)q_1+\alpha+\mu-p]s_1}{[(p-\alpha)q_1+\alpha+\mu-p]^2+(p-\alpha)^2q_2^2} \\
&\leq -\frac{(p-\beta)[(p-\alpha)q_1+\alpha+\mu-p](1+r_2^2)}{2\left\{[(p-\alpha)q_1+\alpha+\mu-p]^2+(p-\alpha)^2q_2^2\right\}} < 0,
\end{aligned}$$

for all $(ir_2, s_1) \in D$ such that $s_1 \leq -\frac{1}{2}(1+r_2^2)$.

By applying Lemma 1, we have $\Re\{h_i(\varsigma)\} > 0$ for $(i = 1,2)$ and consequently $\mathcal{R}(\varsigma) \in \mathcal{P}_{p,k}(\beta)$ for $\varsigma \in \Delta$. This completes the proof of Theorem 3. □

4. Inclusion Properties Involving the Integral Operator $\mathcal{J}_{\delta,p}$

The generalized Bernardi operator is defined by (see [23])

$$\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma) = \frac{\delta+p}{\varsigma^\delta}\int_0^\varsigma t^{\delta-1}\mathcal{F}(t)dt \quad (\delta > -p), \tag{30}$$

which satisfies the following relationship:

$$\varsigma\left(\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma)\right)' = (\delta+p)\mathcal{F}(\varsigma) - \delta\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma). \tag{31}$$

Theorem 4. *If $0 \leq \alpha < p$, $k \geq 2$ and $\mathcal{F} \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$, then $\mathcal{J}_{\delta,p}(\mathcal{F}) \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$ $(\delta \geq 0)$.*

Proof. Let

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma)} = \mathcal{R}(\varsigma) = (p-\alpha)h(\varsigma) + \alpha, \tag{32}$$

where $h(\varsigma)$, given by (20). Using (31), we have

$$\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma)\right)' = (\delta+p)\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma) - \delta\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma). \tag{33}$$

From (32) and (33), we have

$$(\delta+p)\frac{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma)} = (p-\alpha)h(\varsigma) + \alpha + \delta. \tag{34}$$

By computing the logarithmical derivative of (34) with respect to ς and multiplying by ς, we have

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)} - \alpha = (p-\alpha)h(\varsigma) + \frac{(p-\alpha)\varsigma h'(\varsigma)}{(p-\alpha)h(\varsigma)+\alpha+\delta}. \tag{35}$$

Now, we show that $\mathcal{R}(\varsigma) \in \mathcal{P}_{p,k}(\alpha)$ or $h_i \in \mathcal{P}$ for $i = 1, 2$. From (20) and (35), we have

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)\right)'}{\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}(\varsigma)} - \alpha = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\alpha)h_1(\varsigma) + \frac{(p-\alpha)\varsigma h_1'(\varsigma)}{(p-\alpha)h_1(\varsigma)+\alpha+\delta}\right\}$$
$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\alpha)h_2(\varsigma) + \frac{(p-\alpha)\varsigma h_2'(\varsigma)}{(p-\alpha)h_2(\varsigma)+\alpha+\delta}\right\}$$

and this implies that

$$\Re\left\{(p-\alpha)h_i(\varsigma) + \frac{(p-\alpha)\varsigma h_i'(\varsigma)}{(p-\alpha)h_i(\varsigma)+\alpha+\delta}\right\} > 0 \; (\varsigma \in \Delta;\ i = 1, 2).$$

We form the function $\Phi(r, s)$ by choosing $r = h_i(\varsigma)$ and $s = \varsigma h_i'(\varsigma)$. Thus

$$\Phi(r, s) = (p-\alpha)r + \frac{(p-\alpha)s}{(p-\alpha)r+\alpha+\delta}. \tag{36}$$

Clearly, conditions (i), (ii) and (iii) of Lemma 1 are satisfied. By applying Lemma 1, we have $\Re\{h_i(\varsigma)\} > 0$ for $(i = 1, 2)$ and consequently $\mathcal{J}_{\delta,p}(\mathcal{F}) \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$ for $\varsigma \in \Delta$. This completes the proof of Theorem 1. □

Theorem 5. *If $0 \le \alpha < p$, $k \ge 2$ and $\mathcal{F} \in \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$, then $\mathcal{J}_{\delta,p}(\mathcal{F}) \in \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha)$ $(\delta \ge 0)$.*

Proof. Let

$$\mathcal{F} \in \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) \Leftrightarrow \frac{\varsigma\mathcal{F}'(\varsigma)}{p} \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha).$$

By applying Theorem 4, we have

$$\mathcal{J}_{\delta,p}\left(\frac{\varsigma\mathcal{F}'(\varsigma)}{p}\right) \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) \Leftrightarrow \frac{\varsigma\left(\mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma)\right)'}{p} \in \mathcal{S}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha) \Leftrightarrow \mathcal{J}_{\delta,p}(\mathcal{F})(\varsigma) \in \mathcal{C}^{\mu,m,k}_{p,q,\lambda,\delta}(\alpha),$$

which evidently proves Theorem 5. □

5. Inclusion Properties by Convolution

Theorem 6. *Let Φ be a convex function and $\mathcal{F} \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$, then $G \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$, where $G = \mathcal{F} * \Phi$ and $0 \le \gamma < 1$.*

Proof. To show that $G = \mathcal{F} * \Phi \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$ $(0 \le \gamma < 1)$, it sufficient to show that $\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G}$ contained in the convex hull of $\Upsilon(\Delta)$. Now

$$\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G} = \frac{\Phi * \Upsilon\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}}{\Phi * \mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}}, \tag{37}$$

where $\Upsilon = \frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}\mathcal{F}}$ is analytic in Δ and $\Upsilon(0) = 1$. From Lemma 2, we can see that $\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G}$ is contained in the convex hull of $\Upsilon(\Delta)$, since $\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta}G}$ is analytic in Δ and

$$\Upsilon(\Delta) \subseteq \Omega = \left\{ w : \frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta} w(\varsigma)\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta} w(\varsigma)} \in \mathcal{P}(\gamma) \right\}, \tag{38}$$

then $\frac{\varsigma\left(\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta} G\right)'}{p\mathcal{G}^{\mu,m}_{p,q,\lambda,\delta} G}$ lies in Ω, this implies that $G = \mathcal{F} * \Phi \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$. □

Theorem 7. *Let Φ be a convex function and $\mathcal{F} \in \mathcal{C}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$, then $G \in \mathcal{C}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$, where $G = \mathcal{F} * \Phi$ and $0 \leq \gamma < 1$.*

Proof. Let $\mathcal{F} \in \mathcal{C}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$, then, by using (13), we have

$$\frac{\varsigma\mathcal{F}'(\varsigma)}{p} \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$$

and hence by using Theorem 6, we get

$$\begin{aligned} \frac{\varsigma\mathcal{F}'(\varsigma)}{p} * \Phi(\varsigma) \quad &\in \quad \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma) \\ &\Rightarrow \quad \frac{\varsigma(\mathcal{F} * \Phi)'(\varsigma)}{p} \in \mathcal{S}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma). \end{aligned}$$

Now applying (13) again, we obtain $G = \mathcal{F} * \Phi \in \mathcal{C}^{\mu,m,2}_{p,q,\lambda,\delta}(p\gamma)$, which evidently proves Theorem 7. □

Remark 3. *Particularizing the parameters q and m in the results of this paper, we derive various results for different operators.*

6. Conclusions

In the present survey, we propose new subclasses of p-valent functions by making use of the linear q-differential Borel operator. The applications of this interesting operator are discussed. Inclusion properties and certain integral preserving relations were aimed to be our main concern.

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