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Threshold Analysis of a Stochastic SIRS Epidemic Model with Logistic Birth and Nonlinear Incidence

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Abstract: The paper mainly investigates a stochastic SIRS epidemic model with Logistic birth and nonlinear incidence. We obtain a new threshold value (R_0^m) through the Stratonovich stochastic differential equation, different from the usual basic reproduction number. If $R_0^m < 1$, the disease-free equilibrium of the illness is globally asymptotically stable in probability one. If $R_0^m > 1$, the disease is permanent in the mean with probability one and has an endemic stationary distribution. Numerical simulations are given to illustrate the theoretical results. Interestingly, we discovered that random fluctuations can suppress outbreaks and control the disease.

Keywords: stochastic SIRS epidemic model; Logistic birth; nonlinear incidence; global stability; stationary distribution

MSC: 60H05; 60H35

1. Introduction

In the past few decades, researchers have provided various SIR models to study the spread of epidemics [1–5]. For example, Beretta and Takeuchi [1] proposed a deterministic SIR model with varying population sizes, and Lu et al. [2] analyzed a SIR epidemic model with horizontal and vertical transmission. In practice, the persistence of infectious diseases and disease-related deaths may lead to changes in birth rates as the population size increases toward its carrying capacity. In other words, these SIR models should consider density-dependent limited growth. Suppose that the total population $N(t)$ can be divided into three compartments at time t: susceptible $S(t)$, infectious $I(t)$, and removed $R(t)$. Zhang, Li, and Ma [3] introduced a SIR epidemic model with Logistic birth as follows

$$\begin{cases} \mathrm{d}S(t) = \left[\left(b - r\dfrac{N(t)}{K}\right)N(t) - \mu S(t) - \eta S(t)I(t)\right]\mathrm{d}t, \\ \mathrm{d}I(t) = [\eta S(t)I(t) - (\mu+\phi+\kappa)I(t)]\mathrm{d}t, \\ \mathrm{d}R(t) = [\phi I(t) - \mu R(t)]\mathrm{d}t, \end{cases} \quad (1)$$

where K is the environmental capacity, b is the birth rate, η is the exposure rate, μ is the natural mortality rate, $r(= b - \mu)$ is the intrinsic rate, κ is the mortality rate due to disease, and ϕ is the recovery rate of the infected.

For some diseases, the recovered individuals will lose immunity after a certain period and become susceptible again. The SIRS model can be used to describe this phenomenon [6–9]. However, most SIRS models often ignore the density-dependent demographics and heterogeneous populations. Inspired by the above models, we propose

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a deterministic SIRS model with Logistic birth and a nonlinear incidence rate, which is formed by

$$\begin{cases} dS(t) = \left[\left(b - r\frac{N(t)}{K}\right)N(t) - \mu S(t) - \eta f(S(t), I(t), R(t)) + \psi R(t)\right]dt, \\ dI(t) = [\eta f(S(t), I(t), R(t)) - (\mu + \phi + \kappa)I(t)]dt, \\ dR(t) = [\phi I(t) - (\mu + \psi)R(t)]dt, \end{cases} \tag{2}$$

where f is a positive function, and ψ denotes the immunity loss rate of the recovered. The model (1) is a special case of model (2) with $\psi = 0$ and $f(S(t), I(t), R(t)) = S(t)I(t)$.

In the real world, all dynamic systems are affected by environmental noise [10–13]. It is necessary to study a stochastic SIRS epidemic model with Logistic birth and a nonlinear incidence rate is necessary. Considering the effect of random perturbations, we assume that the perturbations in the environment are expressed as a parameter η change to the random variable $\eta \to \eta + \sigma dB(t)$ and obtain the following stochastic SIRS model

$$\begin{cases} dS(t) = \left[\left(b - r\frac{N(t)}{K}\right)N(t) - \mu S(t) - \eta f(S(t), I(t), R(t)) + \psi R(t)\right]dt \\ \qquad\quad - \sigma f(S(t), I(t), R(t))dB(t), \\ dI(t) = [\eta f(S(t), I(t), R(t)) - (\mu + \phi + \kappa)I(t)]dt + \sigma f(S(t), I(t), R(t))dB(t), \\ dR(t) = [\phi I(t) - (\mu + \psi)R(t)]dt, \end{cases} \tag{3}$$

where σ represents the intensity of white environmental noise and $B(t)$ is a one-dimensional standard Brownian motion. Obviously, model (2) is also a special case of model (3) for $\sigma = 0$. This paper aims to analyze the asymptotic properties of the stochastic model (3) by studying global stability, persistence, and stationary distribution.

In epidemiological studies, the basic reproduction number is an indicative factor in considering whether a disease is endemic or not. In this paper, we introduce a new threshold value through the Stratonovich stochastic differential equation and further analyze the dynamic properties of model (3).

2. Preliminaries

Denote $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, 1 \le i \le 3\}$, and $\Theta = \{(S, I, R) : S \ge 0, I \ge 0, R \ge 0, S + I + R > 0\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions ($\mathcal{F}_0$ is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). From [14], we give Itô's formula for general stochastic differential equations. Consider a 3-dimensional stochastic differential equation

$$dW(t) = f(W(t), t)dt + g(W(t), t)dB(t) \quad \text{for } t \ge t_0$$

with an initial value $W(0) = W_0 \in \mathbb{R}^3$, $B(t)$ denotes a 3-dimensional vector of standard Brownian motions defined on the complete probability space. Let the function $V(W, t) \in C^{2,1}(\mathbb{R}^3 \times [t_0, \infty); \mathbb{R}_+)$ be continuously twice differentiable in W and once in t. The differential operator L is defined as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{3} f_i(W, t)\frac{\partial}{\partial W_i} + \frac{1}{2}\sum_{i,j=1}^{3}\left[g^T(W, t)g(W, t)\right]_{ij}\frac{\partial^2}{\partial W_i \partial W_j}.$$

Let a function $V \in C^{2,1}(\mathbb{R}^3 \times [t_0, \infty); \mathbb{R}_+)$, we obtain

$$LV(W, t) = V_t(W, t) + V_W(W, t)f(W, t) + \frac{1}{2}\text{trace}\left[g^T(W, t)V_{WW}(W, t)g(W, t)\right],$$

where $V_t = \frac{\partial V}{\partial t}, V_W = (\frac{\partial V}{\partial W_1}, \frac{\partial V}{\partial W_2}, \frac{\partial V}{\partial W_3}), V_{WW} = (\frac{\partial^2 V}{\partial W_i \partial W_j})_{3\times 3}$. Then,

$$dV(W(t),t) = LV(W(t),t)dt + V_W(W(t),t)g(W(t),t)dB(t).$$

Lemma 1. *[15] The Itô stochastic differential equation (SDE)*

$$\begin{cases} dW_t = f(t, W(t))dt + g(t, W(t))dB(t), \\ W(0) = W_0, \end{cases}$$

is equivalent (has the same solution) as the Stratonovich SDE

$$\begin{cases} dW(t) = \left(f(t, W(t)) - \dfrac{1}{2} \dfrac{\partial g(t, W(t))}{dx} g(t, W(t)) \right) dt + g(t, W(t)) \circ dB(t), \\ W(0) = W_0. \end{cases}$$

In models (2) and (3), suppose that the function $f(S, I, R)$ satisfies the following conditions:

($\mathbf{H}_1$) $f(S, I, R)$ is nonnegative and twice continuously differentiable for all $(S, I, R) \in \Theta$, $\partial f(S, I, R)/\partial S > 0$ and $\partial f(S, I, R)/\partial R < 0$.

($\mathbf{H}_2$) $f(S, 0, R) = f(0, I, R) \equiv 0$, $f(S, I, R)/I \leq \partial f(S_0, 0, 0)/\partial I$ for any $(S, I, R) \in \Theta$, and $\partial f(S_0, 0, 0)/\partial I > 0$, where $S_0 = K$.

Under assumptions ($\mathbf{H}_1$) and ($\mathbf{H}_2$), it is obvious that

$$\frac{\partial f(0, I, R)}{\partial I} = \frac{\partial f(0, I, R)}{\partial R} = \frac{\partial f(S, 0, R)}{\partial S} = \frac{\partial f(S, 0, R)}{\partial R} \equiv 0, \quad (S, I, R) \in \Theta.$$

Lemma 2. *For any constants $p > q > 0$, let $D = \{(S, I, R) : S > 0, I > 0, R > 0, q \leq S + I + R \leq p\}$. Under ($\mathbf{H}_1$) and ($\mathbf{H}_2$), we have*

$$\max_{(S,I,R)\in D} \left\{ \frac{f(S, I, R)}{S}, \frac{f(S, I, R)}{I} \right\} < \infty,$$
$$\max_{(S,I,R)\in D} \left\{ \left| \frac{1}{I} \frac{\partial f(S, I, R)}{\partial I} - \frac{f(S, I, R)}{I^2} \right|, \left| \frac{1}{I} \frac{\partial f(S, I, R)}{\partial S} \right| \right\} < \infty.$$

The proof of Lemma 2 is similar to that of Lemma 2.1 in Ramziya et al. [16]. Denote $M_S \triangleq \max_\Gamma \{f(S, I, R)/S\}$. Define a bounded set

$$\Gamma = \left\{ (S, I, R) : S > 0, I > 0, R > 0, K\left(1 - \frac{\kappa}{r}\right) \leq S + I + R \leq K \right\}. \tag{4}$$

Lemma 3. *Suppose that ($\mathbf{H}_1$) and ($\mathbf{H}_2$) hold.*

(i) The region Γ is almost surely positive invariant in model (3).

(ii) For any initial value $(S(0), I(0), R(0)) \in R^3_+$, model (3) has a unique positive solution $(S(t), I(t), R(t))$ on $t > 0$. That is to say, the solution will remain in a compact subset of Γ. Furthermore,

$$K\left(1 - \frac{\kappa}{r}\right) \leq \liminf_{t\to\infty} (S(t) + I(t) + R(t)) \leq \limsup_{t\to\infty} (S(t) + I(t) + R(t)) \leq S_0.$$

The proof of Lemma 3 can be obtained using a similar method in Cai et al. [17]. The lemma shows that we can study the dynamic properties of model (3) in the bounded set Γ. To obtain the basic reproduction number of model (3), we define a C^2 function $V : V(S(t), I(t)) = \ln(S(t), I(t))$. By using Itô's formula, it follows that

$$\mathrm{d}\ln S = \left[\frac{1}{S} \left(\left(b - r\frac{N}{K} \right) N - \mu S - \eta f(S, I, R) + \psi R \right) - \frac{1}{2} \sigma^2 f^2(S, I, R) \frac{1}{S^2} \right] \mathrm{d}t$$

$$
\begin{aligned}
&-\sigma f(S,I,R)\frac{1}{S}\mathrm{d}B(t),\\
\mathrm{d}\ln I =&\left[\frac{1}{I}(\eta f(S,I,R)-(\mu+\phi+\kappa)I)-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{I^2}\right]\mathrm{d}t+\sigma f(S,I,R)\frac{1}{I}\mathrm{d}B(t).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathrm{d}S =&\left[\left(b-r\frac{N}{K}\right)N-\mu S-\eta f(S,I,R)+\psi R-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{S}\right]\mathrm{d}t-\sigma f(S,I,R)\mathrm{d}B(t),\\
\mathrm{d}I =&\left[\eta f(S,I,R)-(\mu+\phi+\kappa)I-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{I}\right]\mathrm{d}t+\sigma f(S,I,R)\mathrm{d}B(t).
\end{aligned}
\tag{5}
$$

We transform (5) into a Stratonovich SDE and obtain

$$
\begin{aligned}
\mathrm{d}S =&\bigg[\left(b-r\frac{N}{K}\right)N-\mu S-\eta f(S,I,R)+\psi R-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{S}\\
&-\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial S}\bigg]\mathrm{d}t-\sigma f(S,I,R)\circ\mathrm{d}B(t),\\
\mathrm{d}I =&\left[\eta f(S,I,R)-(\mu+\phi+\kappa)I-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{I}-\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial I}\right]\mathrm{d}t\\
&+\sigma f(S,I,R)\circ\mathrm{d}B(t).
\end{aligned}
$$

Taking the mean value of the above equations derives

$$
\begin{aligned}
E(\mathrm{d}S) =&\bigg[\left(b-r\frac{N}{K}\right)N-\mu S-\eta f(S,I,R)+\psi R-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{S}\\
&-\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial S}\bigg]\mathrm{d}t\\
E(\mathrm{d}I) =&\left[\eta f(S,I,R)-(\mu+\phi+\kappa)I-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{I}-\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial I}\right]\mathrm{d}t.
\end{aligned}
$$

As a result, the study of model (3) can be turned into

$$
\begin{cases}
E(\mathrm{d}S) =\bigg[\left(b-r\frac{N}{K}\right)N-\mu S-\eta f(S,I,R)+\psi R-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{S}\\
\qquad\qquad -\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial S}\bigg]\mathrm{d}t,\\
E(\mathrm{d}I) =\left[\eta f(S,I,R)-(\mu+\phi+\kappa)I-\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{I}-\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial I}\right]\mathrm{d}t,\\
\mathrm{d}R =[\phi I-(\mu+\psi)R]\mathrm{d}t,
\end{cases}
\tag{6}
$$

Denote $x=(I,S,R)^{\mathrm{T}}$. Then system (6) can be given as

$$
x'=\mathscr{F}(x)-\mathscr{V}(x),
$$

where

$$
\mathscr{F}(x)=\begin{pmatrix}\eta f(S,I,R)-\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial I}\\ 0\\ 0\end{pmatrix};
$$

$$
\mathscr{V}(x)=\begin{pmatrix}(\mu+\phi+\kappa)I+\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{I}\\ -\left(b-r\frac{N}{K}\right)N+\mu S+\eta f(S,I,R)-\psi R+\frac{1}{2}\sigma^2 f^2(S,I,R)\frac{1}{S}+\frac{1}{2}\sigma^2 f(S,I,R)\frac{\partial f(S,I,R)}{\partial S}\\ -\phi I(t)+(\mu+\psi)R(t)\end{pmatrix}.
$$

Through the calculation, model (3) has a unique disease-free equilibrium $E_0 = (S_0, 0, 0)$, which is also that of model (6). The Jacobian matrices of $\mathscr{F}(x)$ and $\mathscr{V}(x)$ at E_0 are denoted as F and V, respectively. Under the assumption $(\mathbf{H}_2)$, by the calculation, there are

$$F = \begin{pmatrix} \eta \frac{\partial f(S_0,0,0)}{\partial I} - \frac{1}{2}\sigma^2 \left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} \mu + \phi + \kappa & 0 & 0 \\ -b + 2r + \eta \frac{\partial f(S_0,0,0)}{\partial I} & r & -b + 2r - \psi \\ -\phi & 0 & \mu + \psi \end{pmatrix}.$$

Then the basic reproduction number of model (6) is

$$\begin{aligned} R_0^m = \rho(FV^{-1}) &= \frac{\partial f(S_0,0,0)}{\partial I} \frac{\eta}{\mu + \phi + \kappa} - \left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 \frac{\sigma^2}{2(\mu + \phi + \kappa)} \\ &= R_0 - \left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 \frac{\sigma^2}{2(\mu + \phi + \kappa)}, \end{aligned}$$

which is also that of model (3). In view of $\sigma = 0$, model (6) is then transformed into model (2). The basic reproduction number of model (2) is easily obtained as

$$R_0 = \frac{\partial f(S_0,0,0)}{\partial I} \frac{\eta}{\mu + \phi + \kappa}.$$

In the following, we present some properties of model (2).

Theorem 1. *The disease-free equilibrium $E_0 = (K, 0, 0)$ of model (2) is locally asymptotically stable if $R_0 < 1$.*

Proof. Given that assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold, the Jacobian matrix of model (2) evaluated at E_0 is

$$J(E_0) = \begin{pmatrix} -r & b - 2r & b - 2r + \psi \\ 0 & -(\mu + \kappa + \phi) & 0 \\ 0 & \phi & -(\mu + \psi) \end{pmatrix}.$$

Thus, the characteristic equation at E_0 is

$$(\lambda + r)(\lambda + \mu + \kappa + \phi)(\lambda + \mu + \psi) = 0,$$

which has eigenvalues $\lambda_1 = -r < 0, \lambda_2 = -(\mu + \kappa + \phi) < 0$ and $\lambda_3 = -(\mu + \psi) < 0$. Therefore, the disease-free equilibrium E_0 is locally asymptotically stable. The proof is complete. □

Theorem 2. *If $R_0 > 1$, model (2) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$, which is locally asymptotically stable.*

Proof. The Jacobian matrix of model (2) evaluated at E^* is

$$J(E^*) = \begin{pmatrix} A - \mu - \eta \frac{\partial f(S^*,I^*,R^*)}{\partial S} & A - \eta \frac{\partial f(S^*,I^*,R^*)}{\partial I} & A - \eta \frac{\partial f(S^*,I^*,R^*)}{\partial R} + \psi \\ \eta \frac{\partial f(S^*,I^*,R^*)}{\partial S} & \eta \frac{\partial f(S^*,I^*,R^*)}{\partial I} - (\mu + \kappa + \phi) & \eta \frac{\partial f(S^*,I^*,R^*)}{\partial R} \\ 0 & \phi & -(\mu + \psi) \end{pmatrix},$$

where $A = b - 2\frac{r}{K}(S^* + I^* + R^*)$. The characteristic polynomial of $J(E^*)$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$\begin{aligned}
a_1 =& A - \eta\frac{\partial f(S^*, I^*, R^*)}{\partial S} + \eta\frac{\partial f(S^*, I^*, R^*)}{\partial I} - 3\mu - \kappa - \phi - \psi > 0,\\
a_2 =& \left(A - \mu - \eta\frac{\partial f(S^*, I^*, R^*)}{\partial S}\right)\left(\eta\frac{\partial f(S^*, I^*, R^*)}{\partial I} - 2\mu - \kappa - \phi - \psi\right)\\
&+ \left(\eta\frac{\partial f(S^*, I^*, R^*)}{\partial I} - \mu - \kappa - \phi\right)(-\mu - \psi) - \eta\phi\frac{\partial f(S^*, I^*, R^*)}{\partial R}\\
&+ \left(-A + \eta\frac{\partial f(S^*, I^*, R^*)}{\partial I}\right)\eta\frac{\partial f(S^*, I^*, R^*)}{\partial S} > 0,\\
a_3 =& -\left(A - \mu - \eta\frac{\partial f(S^*, I^*, R^*)}{\partial S}\right)\left(\left(\eta\frac{\partial f(S^*, I^*, R^*)}{\partial I} - \mu - \kappa - \phi\right)(-\mu - \psi)\right.\\
&\left. - \eta\phi\frac{\partial f(S^*, I^*, R^*)}{\partial R}\right) + \eta\frac{\partial f(S^*, I^*, R^*)}{\partial S}\left(\left(-A + \eta\frac{\partial f(S^*, I^*, R^*)}{\partial I}\right)(\mu + \psi)\right.\\
&\left. + \left(-A + \eta\frac{\partial f(S^*, I^*, R^*)}{\partial R} - \psi\right)\phi\right) > 0.
\end{aligned}$$

We can verify that $a_1a_2 - a_3 > 0$. Therefore, E^* is locally asymptotically stable by employing the Routh–Hurwitz criterion. □

3. Existence and Uniqueness of the Global Positive Solution

Theorem 3. *There is a unique solution* $(S(t), I(t), R(t))$ *of model (3) on* $t \geq 0$ *for any given initial value* $(S(0), I(0), R(0)) \in R^3_+$*, and this solution remains in* R^3_+ *with probability one.*

Proof. The coefficients of model (3) are locally Lipschitz continuous. For any given initial value $(S(0), I(0), R(0))$, there exists a unique local solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ on $t \in [0, \tau_e)$, where τ_e denotes the explosion time. To prove the existence of a unique global solution to the stochastic model (3), it is only necessary to prove $\tau_e = \infty a.s.$.

Let $l_0 > 0$ be sufficiently large, such that $S(0)$, $I(0)$ and $R(0)$ all lie within the interval $[\frac{1}{l_0}, l_0]$. For each integer $l \geq l_0$, define a stopping time

$$\tau_l = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \leq \frac{1}{l} \text{ or } \max\{S(t), I(t), R(t)\} \geq l\right\},$$

where $\inf\emptyset = \infty$ (as usual $\emptyset$ = the empty set). Clearly, τ_l increases as $l \to \infty$. Let $\tau_\infty = \lim_{l\to\infty}\tau_l$, then $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. Otherwise, there exists a pair of constants $T > 0$ and $\epsilon \in (0, 1)$, such that $\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$. Hence, there exists an integer $l_1 \geq l_0$, such that

$$\mathbb{P}\{\tau_l \leq T\} \geq \epsilon, \quad \forall l \geq l_1. \tag{7}$$

Define a Lyapunov function by

$$V(S, I, R) = (S - 1 - \ln S) + (I - 1 - \ln I) + (R - 1 - \ln R).$$

For any $l \geq l_0$ and $t \in [0, \tau_l)$, applying Itô's formula to V, we obtain

$$dV(S, I, R) = LV(S, I, R)dt + \sigma f(S, I, R)\left(\frac{1}{S} - \frac{1}{I}\right)dB(t), \tag{8}$$

where

$$
\begin{aligned}
LV(S,I,R) =& \left(1-\frac{1}{S}\right)\left(\left(b-r\frac{N}{K}\right)N-\mu S-\eta f(S,I,R)+\psi R\right)+\frac{\sigma^2}{2}f^2(S,I,R)\frac{1}{S^2}\\
&+\left(1-\frac{1}{I}\right)(\eta f(S,I,R)-(\mu+\phi+\kappa)I)+\frac{\sigma^2}{2}f^2(S,I,R)\frac{1}{I^2}\\
&+\left(1-\frac{1}{R}\right)(\phi I-(\mu+\psi)R)\\
=& r\left(1-\frac{N}{K}\right)N-\frac{N}{S}\left(b-r\frac{N}{K}\right)+3\mu+\eta\frac{f(S,I,R)}{S}-\frac{\psi R}{S}+\frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{S}\right)^2\\
&-\kappa I-\eta\frac{f(S,I,R)}{I}+\phi+\kappa+\frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2-\frac{\phi I}{R}+\psi\\
\leq& r\left(1-\frac{N}{K}\right)N+3\mu+\eta M_S+\frac{1}{2}\sigma^2M_S^2+\phi+\kappa+\frac{1}{2}\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2+\psi\\
:=& K_1,
\end{aligned}
$$

where K_1 is a positive constant. Then according to (8), we have

$$dV(S,I,R) = K_1 dt+\sigma f(S,I,R)\left(\frac{1}{S}-\frac{1}{I}\right)dB(t). \tag{9}$$

Integrating both sides of (9) from 0 to $\tau_l \wedge T$ and taking the expectation, we have

$$
\begin{aligned}
&EV(S(\tau_l\wedge T),I(\tau_l\wedge T),R(\tau_l\wedge T))\\
&\leq EV(S(0),I(0),R(0))+K_1E(\tau_l\wedge T)+E\int_0^{\tau_l\wedge T}\sigma f(S,I,R)\left(\frac{1}{S}-\frac{1}{I}\right)dB(t)\\
&\leq V(S(0),I(0),R(0))+K_1T.
\end{aligned} \tag{10}
$$

Let $\Omega_l = \{\omega : \tau_l \leq T\}$ for $l \geq l_0$. From (7) we have $P(\Omega_l) \geq \varepsilon$. For every $\omega \in \Omega_l$, we learn $S(\tau_l,\omega)$ or $I(\tau_l,\omega)$ or $R(\tau_l,\omega)$ equals either l or $\frac{1}{l}$. Therefore, from (10), there is

$$
\begin{aligned}
&V(S(0),I(0),R(0))+K_1T\\
&\quad\geq E\big(I_{\Omega_l}V(S(\tau_l\wedge T),I(\tau_l\wedge T),R(\tau_l\wedge T))\big)\\
&\quad= P(\Omega_l)EV(S(\tau_l,\omega),I(\tau_l,\omega),R(\tau_l,\omega))\\
&\quad\geq \varepsilon\Big\{(l-1-\log l)\wedge\left(\tfrac{1}{l}-1-\log\tfrac{1}{l}\right)\wedge(l-1-\log l)\\
&\quad\wedge\left(\tfrac{1}{l}-1-\log\tfrac{1}{l}\right)\wedge(l-1-\log l)\wedge\left(\tfrac{1}{l}-1-\log\tfrac{1}{l}\right)\Big\},
\end{aligned}
$$

where I_{Ω_l} is the indicator function of Ω_l. Taking $l \to \infty$ results

$$\infty > V(S(0),I(0),R(0))+K_1T = \infty.$$

Hence, $\tau_\infty = \infty$ *a.s.*. It completes the proof. □

4. Global Stability of Disease-Free Equilibrium

Denote

$$\sigma_a = \sqrt{\frac{\eta}{\partial f(S_0,0,0)/\partial I}} \quad \text{and} \quad \sigma_b = \frac{\eta}{\sqrt{2(\mu+\phi+\kappa)}}.$$

The following result reveals the asymptotic property of E_0 in model (3).

Theorem 4. *Under* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *the disease-free equilibrium* E_0 *is globally asymptotically stable in probability one if one of the following conditions holds: (a)* $R_0^m < 1$ *and* $\sigma \leq \sigma_a$, *or (b)* $\sigma > \max\{\sigma_a, \sigma_b\}$.

Proof. Let k_1 and k_2 be two positive constants. Define a Lyapunov function

$$V(S,I,R) = \frac{1}{2}k_1(K-S)^2 + \frac{1}{k_2}I^{k_2} + \frac{1}{2}\frac{k_1\psi}{\phi}R^2.$$

Obviously, $V(S,I,R) \geq 0$ in the region Γ. Through Itô's formula and model (3), it follows that

$$dV(S,I,R) = LV(S,I,R)dt + \sigma f(S,I,R)\Big(k_1(K-S) + I^{k_2-1}\Big)dB(t),$$

where

$$\begin{aligned}
LV =& -k_1(K-S)\left(\left(b - r\frac{N}{K}\right)N - \mu S - \eta f(S,I,R) + \psi R\right) \\
&+ \frac{1}{2}k_1\sigma^2 f^2(S,I,R) + I^{k_2-1}(\eta f(S,I,R) - (\mu+\kappa+\phi)I) \\
&+ \frac{1}{2}(k_2-1)\sigma^2 f^2(S,I,R)I^{k_2-2} + \frac{k_1\psi}{\phi}R(\phi I - (\mu+\psi)R) \\
=& -k_1(K-S)\left(\left(b - r\frac{N}{K}\right)N - \mu S\right) + k_1(K-S)\eta I\frac{f(S,I,R)}{I} - k_1(K-S)\psi R \\
&+ \frac{1}{2}k_1\sigma^2 I^2\left(\frac{f(S,I,R)}{I}\right)^2 + \eta I^{k_2}\frac{f(S,I,R)}{I} - (\mu+\kappa+\phi)I^{k_2} \\
&+ \frac{1}{2}k_2\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2 I^{k_2} - \frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2 I^{k_2} + k_1\psi RI - \frac{k_1\psi(\mu+\psi)R^2}{\phi}.
\end{aligned}$$

For any constant $\rho > 0$ and $k_2 \leq 2$, we have

$$I(t) \leq K - S(t),\ \ I^2 \leq K^{2-k_2}I^{k_2},\ \text{and}\ I(K-S) \leq \frac{1}{4\rho}I^2 + \rho(K-S)^2 \leq \frac{1}{4\rho}K^{2-k_2}I^{k_2} + \rho(K-S)^2.$$

Based on the above inequalities and $(\mathbf{H}_2)$, it yields that

$$\begin{aligned}
LV \leq& -k_1(K-S)\left(\left(b - r\frac{N}{K}\right)N - \mu S - \eta\rho(K-S)\frac{\partial f(S_0,0,0)}{\partial I}\right) - k_1(K-S)\psi R \\
&+ \frac{1}{2}k_1\sigma^2 I^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 + \eta I^{k_2}\frac{f(S,I,R)}{I} - (\mu+\kappa+\phi)I^{k_2} \\
&+ \frac{1}{2}k_2\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 I^{k_2} - \frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2 I^{k_2} + k_1\psi R(K-S) - \frac{k_1\psi(\mu+\psi)R^2}{\phi} \\
&+ k_1\eta\frac{1}{4\rho}K^{2-k_2}I^{k_2}\frac{f(S,I,R)}{I} \\
\leq& -k_1(K-S)\left(\left(b - r\frac{N}{K}\right)N - \mu S - \eta\rho(K-S)\frac{\partial f(S_0,0,0)}{\partial I}\right) \\
&+ I^{k_2}\left(\eta\frac{f(S,I,R)}{I} + (\mu+\kappa+\phi) - \frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\right) + \frac{1}{2}k_2\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 I^{k_2} \\
&- \frac{k_1\psi(\mu+\psi)R^2}{\phi} + k_1\eta\frac{1}{4\rho}K^{2-k_2}I^{k_2}\frac{\partial f(S_0,0,0)}{\partial I} + \frac{1}{2}k_1\sigma^2 K^{2-k_2}I^{k_2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 \\
=& -k_1(K-S)\left(\left(b - r\frac{N}{K}\right)N - \mu S - \eta\rho(K-S)\frac{\partial f(S_0,0,0)}{\partial I}\right) \\
&+ I^{k_2}\left(\eta\frac{f(S,I,R)}{I} + (\mu+\kappa+\phi) - \frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\right) \\
&+ I^{k_2}\left(\frac{1}{2}k_1\sigma^2 K^{2-k_2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 + \frac{1}{2}k_2\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right.
\end{aligned}$$

$$+k_1\eta\frac{1}{4\rho}K^{2-k_2}\frac{\partial f(S_0,0,0)}{\partial I}\bigg)-\frac{k_1\psi(\mu+\psi)R^2}{\phi}.$$

Denote $h(x)=\eta x-(\mu+\phi+\kappa)-\frac{1}{2}\sigma^2x^2$. The function $h(x)$ increases for $x\in[0,\eta/\sigma^2]$. If $R_0^m<1$ and $\sigma\le\sigma_a$, then $f(S,I,R)/I\le\partial f(S_0,0,0)/\partial I\le\eta/\sigma^2$. We obtain

$$\begin{aligned}&\eta\frac{f(S,I,R)}{I}-(\mu+\phi+\kappa)-\frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\\&\le\eta\frac{\partial f(S_0,0,0)}{\partial I}-(\mu+\phi+\kappa)-\frac{1}{2}\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\triangleq A_1<0,\end{aligned}$$

If $\sigma>\max\{\sigma_a,\sigma_b\}$, we have

$$\begin{aligned}&\eta\frac{f(S,I,R)}{I}-(\mu+\phi+\kappa)-\frac{1}{2}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\\&\le\frac{\eta^2}{2\sigma^2}-(\mu+\phi+\kappa)\triangleq A_2<0.\end{aligned}$$

Suppose the condition (a) or (b) holds. Then,

$$\begin{aligned}LV\le&-k_1(K-S)\left(\left(b-r\frac{N}{K}\right)N-\mu S-\eta\rho(K-S)\frac{\partial f(S_0,0,0)}{\partial I}\right)\\&+I^{k_2}\Bigg(\max\{A_1,A_2\}+\frac{1}{2}k_1\sigma^2K^{2-k_2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2+\frac{1}{2}k_2\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\\&+k_1\eta\frac{1}{4\rho}K^{2-k_2}\frac{\partial f(S_0,0,0)}{\partial I}\Bigg)-\frac{k_1\psi(\mu+\psi)R^2}{\phi}.\end{aligned}$$

We choose the constant $\rho>0$, such that

$$\left(b-r\frac{N}{K}\right)N-\mu S-\eta\rho(K-S)\frac{\partial f(S_0,0,0)}{\partial I}>0,$$

and two positive constants (k_1,k_2) satisfy

$$\begin{aligned}&\max\{A_1,A_2\}+\frac{1}{2}k_1\sigma^2K^{2-k_2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2+\frac{1}{2}k_2\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\\&+k_1\eta\frac{1}{4\rho}K^{2-k_2}\frac{\partial f(S_0,0,0)}{\partial I}<0.\end{aligned}$$

Therefore, LV is negative definite in the region Γ. By applying the global asymptotic stability theorem [14,18], we can obtain that the disease-free equilibrium E_0 of model (3) is globally asymptotically stable in probability one. The proof is complete. □

For the stochastic model (3), Theorem 4 reflects the asymptotic property of the disease-free equilibrium E_0 under conditions (a) and (b). Denote

$$\frac{\sqrt{2(\mu+\phi+\kappa)(R_0-1)}}{R_0-1}\quad\text{for}\quad R_0>1.$$

Based on the relationship of R_0 and R_0^m, we analyze the properties under the ranges of R_0.

Corollary 1. *Under* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$*, the disease-free equilibrium* E_0 *is globally asymptotically stable in probability one if one of the following conditions holds: (a)* $R_0\le1$*, (b)* $1<R_0\le2$ *and* $\sigma>\sigma_S$*, or (c)* $R_0>2$ *and* $\sigma>\sigma_b$*.*

5. Permanence in the Mean of Disease

In this section, we discuss permanence in the mean of the disease. For any $t \geq 0$, the average value of a continuous function $u(t)$ is defined by $\langle u(t) \rangle = \int_0^t u(s)\mathrm{d}s/t$.

Theorem 5. *If $R_0^m > 1$ and $K \leq B/\mu$, then any solution $(S(t), I(t), R(t))$ of model (3) with initial values $(S(0), I(0), R(0)) \in R_+^3$ is permanent in the mean with probability one. That is, there exists a constant $m > 0$, such that*

$$\liminf_{t\to\infty}\langle S(t)\rangle \geq m, \quad \liminf_{t\to\infty}\langle I(t)\rangle \geq m, \quad \liminf_{t\to\infty}\langle R(t)\rangle \geq m \ a.s.,$$

where

$$B = \min\left\{K\mu, K\left(1 - \frac{\kappa}{r}\right)(\mu+\kappa), \frac{b^2K}{4r}\right\}.$$

Proof. Integrating model (3) from 0 to t and dividing by t, we have

$$\begin{aligned}
\varphi(t) &= \frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{\psi}{\mu+\psi}\frac{R(t)-R(0)}{t} \\
&= b\langle N(t)\rangle - \frac{r}{K}\langle N^2(t)\rangle - \mu\langle S(t)\rangle - \left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)\langle I(t)\rangle \\
&\geq B - \mu\langle S(t)\rangle - \left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)\langle I(t)\rangle.
\end{aligned}$$

That is,

$$\langle S(t)\rangle \geq \frac{B - \varphi(t) - \left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)\langle I(t)\rangle}{\mu}.$$

Thus,

$$S_0 - \langle S(t)\rangle \leq K - \frac{B}{\mu} + \frac{\varphi(t)}{\mu} + \frac{1}{\mu}\left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)\langle I(t)\rangle. \tag{11}$$

Through Lemma 3, $\lim_{t\to\infty}(\varphi(t)/\mu) = 0 \ \ a.s.$. Using Itô's formula, we have

$$\begin{aligned}
\mathrm{d}\ln I(t) = &\left(\eta\frac{f(S(t),I(t),R(t))}{I(t)} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\frac{f^2(S(t),I(t),R(t))}{I^2(t)}\right)\mathrm{d}t \\
&+ \sigma\frac{f(S(t),I(t),R(t))}{I(t)}\mathrm{d}B(t).
\end{aligned}$$

From $(\mathbf{H}_2)$, for any $t \geq 0$, it follows that

$$\begin{aligned}
\mathrm{d}\ln I(t) \geq &\left(\eta\frac{f(S(t),I(t),R(t))}{I(t)} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right)\mathrm{d}t \\
&+ \sigma\frac{f(S(t),I(t),R(t))}{I(t)}\mathrm{d}B(t) \\
= &\left(\eta\frac{\partial f(S_0,0,0)}{\partial I} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right. \\
&+ \eta\left(\frac{f(S(t),I(t),R(t))}{I(t)} - \frac{f(S_0,I(t),R(t))}{I(t)} + \frac{f(S_0,I(t),R(t))}{I(t)} - \frac{\partial f(S_0,0,R(t))}{\partial I}\right. \\
&\left.\left.+ \frac{\partial f(S_0,0,R(t))}{\partial I} - \frac{\partial f(S_0,0,0)}{\partial I}\right)\right)\mathrm{d}t + \sigma\frac{f(S(t),I(t),R(t))}{I(t)}\mathrm{d}B(t),
\end{aligned}$$

By Lagrange's mean value theorem, we have

$$\begin{aligned}
&\frac{f(S(t),I(t),R(t))}{I(t)}-\frac{f(S_0,I(t),R(t))}{I(t)}+\frac{f(S_0,I(t),R(t))}{I(t)}-\frac{\partial f(S_0,0,R(t))}{\partial I}\\
&+\frac{\partial f(S_0,0,R(t))}{\partial I}-\frac{\partial f(S_0,0,0)}{\partial I}\\
=&\frac{1}{I(t)}\frac{\partial f(\xi(t),I(t),R(t))}{\partial S}(S(t)-S_0)+\Big(\frac{1}{\zeta(t)}\frac{\partial f(S_0,\zeta(t),R(t))}{\partial I}-\frac{f(S_0,\zeta(t),R(t))}{\zeta^2(t)}\Big)I(t)\\
&+\frac{\partial^2 f(S_0,0,\theta(t))}{\partial I\partial R}R(t),
\end{aligned}$$

where $\xi(t)\in(S(t),S_0),\zeta(t)\in(0,I(t))$ and $\theta(t)\in(0,R(t))$. Hence,

$$\begin{aligned}
\mathrm{d}\ln I(t)\geq&\Big(\eta\frac{\partial f(S_0,0,0)}{\partial I}-(\mu+\phi+\kappa)-\frac{\sigma^2}{2}\Big(\frac{\partial f(S_0,0,0)}{\partial I}\Big)^2\\
&+\eta\frac{1}{I(t)}\frac{\partial f(\xi(t),I(t),R(t))}{\partial S}(S(t)-S_0)\\
&+\eta\frac{\frac{\partial f(S_0,\zeta(t),R(t))}{\partial I}\zeta(t)-f(S_0,\zeta(t),R(t))}{\zeta^2(t)}I(t)+\eta\frac{\partial^2 f(S_0,0,\theta(t))}{\partial I\partial R}R(t)\Big)\mathrm{d}t\\
&+\sigma\frac{f(S(t),I(t),R(t))}{I(t)}\mathrm{d}B(t).
\end{aligned}$$

From Lemma 2, it is obvious that

$$P_1=\max_{\Gamma}\Big\{\Big|\frac{1}{I}\frac{\partial f(S,I,R)}{\partial S}\Big|\Big\}<\infty,\quad P_2=\max_{\Gamma}\Big\{\Big|\frac{\partial^2 f(S_0,0,R)}{\partial I\partial R}\Big|\Big\}<\infty,$$

and

$$P_3=\max_{\Gamma}\Bigg\{\Bigg|\frac{f(S_0,I,R)-\frac{\partial f(S_0,I,R)}{\partial I}I}{I^2}\Bigg|\Bigg\}<\infty.$$

For any $t\geq 0$, $(\xi(t),I(t),R(t))\in\Gamma$, $(S_0,\zeta(t),R(t))\in\Gamma$, and $(S_0,0,\theta(t))\in\Gamma$ *a.s.* Thus,

$$\frac{1}{I(t)}\frac{\partial f(\xi(t),I(t),R(t))}{\partial S}\geq -P_1,\quad \frac{\partial^2 f(S_0,0,\theta(t))}{\partial I\partial R}\geq -P_2\text{ a.s.}$$

and

$$\frac{\frac{\partial f(S_0,\zeta(t),R(t))}{\partial I}\zeta(t)-f(S_0,\zeta(t),R(t))}{\zeta^2(t)}\geq -P_3\text{ a.s..}$$

Further, we have

$$\begin{aligned}
\mathrm{d}\ln I(t)\geq&\big(\eta(\partial f(S_0,0,0)/\partial I)-(\mu+\phi+\kappa)-\sigma^2(\partial f(S_0,0,0)/\partial I)^2/2\\
&-\eta(P_1(S_0-S(t))+P_3I(t)+P_2R(t))\big)\mathrm{d}t+\sigma(f(S(t),I(t),R(t))/I(t))\mathrm{d}B(t).
\end{aligned}$$

For the above inequality, we integrate from 0 to t and then divide by t into both sides. It follows that

$$\begin{aligned}
\mathrm{d}\ln I(t)\geq&\Big(\eta\frac{\partial f(S_0,0,0)}{\partial I}-(\mu+\phi+\kappa)-\frac{\sigma^2}{2}\Big(\frac{\partial f(S_0,0,0)}{\partial I}\Big)^2\\
&-\eta(P_1(S_0-S(t))+P_3I(t)+P_2R(t))\Big)\mathrm{d}t+\sigma\frac{f(S(t),I(t),R(t))}{I(t)}\mathrm{d}B(t).
\end{aligned}\tag{12}$$

Since

$$\langle R(t)\rangle=\frac{1}{\mu+\psi}\Big(\phi\langle I(t)\rangle+\frac{R(0)-R(t)}{t}\Big),$$

we substitute (11) into (12) and have

$$\begin{aligned}\frac{\ln I(t)}{t} \geq& \frac{\ln I(0)}{t} + \left(\eta\frac{\partial f(S_0,0,0)}{\partial I} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right) \\ &- \eta(P_1(S_0 - \langle S(t)\rangle) + P_3\langle I(t)\rangle + P_2\langle R(t)\rangle) + \sigma\frac{1}{t}\int_0^t \frac{f(S(r),I(r),R(r))}{I(r)}\mathrm{d}B(r) \\ =& \frac{\ln I(0)}{t} + \left(\eta\frac{\partial f(S_0,0,0)}{\partial I} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right) \\ &- \eta P_1\frac{1}{\mu}\left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)\langle I(t)\rangle - \eta P_2\frac{1}{\mu+\psi}\left(\phi\langle I(t)\rangle + \frac{R(0)-R(t)}{t}\right) - \eta P_3\langle I(t)\rangle \\ &+ \sigma\frac{1}{t}\int_0^t \frac{f(S(r),I(r),R(r))}{I(r)}\mathrm{d}B(r) - \eta P_1\left(K - \frac{B}{\mu} + \frac{\varphi(t)}{\mu}\right) \\ =& \left(\eta\frac{\partial f(S_0,0,0)}{\partial I} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right) - \eta P_1\left(K - \frac{B}{\mu}\right) \\ &- \eta\left(\frac{1}{\mu}\left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)P_1 + \frac{\phi}{\mu+\psi}P_2 + P_3\right)\langle I(t)\rangle + \Phi(t),\end{aligned}$$

where

$$\Phi(t) = \frac{\ln I(0)}{t} - \eta\left(\frac{P_1\varphi(t)}{\mu} + P_2\frac{R(0)-R(t)}{t(\mu+\psi)}\right) + \frac{1}{t}\int_0^t \sigma\frac{f(S(r),I(r),R(r))}{I(r)}\mathrm{d}B(r).$$

From the large number theorem for martingales [19], we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{f(S(s),I(s),R(s))}{I(s)}\mathrm{d}B(s) = 0.$$

Hence, $\lim_{t\to\infty}\Phi(t) = 0$ *a.s.*. That is to say, by $R_0^m > 1$, $\liminf_{t\to\infty}\langle I(t)\rangle \geq U/D \triangleq \tilde{I}_*$ a.s., where

$$\begin{aligned}U =& \left(\eta\frac{\partial f(S_0,0,0)}{\partial I} - (\mu+\phi+\kappa) - \frac{\sigma^2}{2}\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2\right) - \eta P_1\left(K - \frac{B}{\mu}\right) > 0, \\ D =& \eta\left(\frac{1}{\mu}\left(\mu+\kappa+\frac{\mu\phi}{\mu+\psi}\right)P_1 + \frac{\phi}{\mu+\psi}P_2 + P_3\right) > 0.\end{aligned}$$

Therefore, $I(t)$ is permanent in the mean with probability one. According to Lemma 2, $f(S(t),I(t),R(t)) \leq M_S S(t)$ for all $t \geq 0$. Integrating from 0 to t and dividing by t into both sides of the first equation of model (3), we have

$$\begin{aligned}\frac{S(t)-S(0)}{t} \geq& b\langle N(t)\rangle - \frac{r}{K}\langle N^2(t)\rangle - \frac{1}{t}\int_0^t(\eta M_S+\mu)S(s)\mathrm{d}s - \frac{\sigma}{t}\int_0^t f(S(s),I(s),R(s))\mathrm{d}B(s) \\ \geq& B - \frac{1}{t}\int_0^t(\eta M_S+\mu)S(s)\mathrm{d}s - \frac{\sigma}{t}\int_0^t f(S(s),I(s),R(s))\mathrm{d}B(s).\end{aligned}$$

Taking $t\to\infty$, we have $\liminf_{t\to\infty}\langle S(t)\rangle \geq B/(\mu+\eta M_S)$ a.s.. Similarly, from the third equation of model (3), we can obtain $\liminf_{t\to\infty}\langle R(t)\rangle \geq \phi/(\mu+\psi)\tilde{I}_*$ a.s.. □

6. Existence of Stationary Distribution

Theorem 6. *Suppose* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. If* $R_0^m > 1$, *then model (3) has a unique stationary distribution.*

Proof. From Lemma 2, let

$$M_1 = \max_{\Gamma}\left\{\left|\frac{\partial^2 f(S,0,0)}{\partial I \partial S}\right|\right\} < \infty, \quad M_2 = \max_{\Gamma}\left\{\left|\frac{f(S,I,0) - \frac{\partial f(S,I,0)}{\partial I} I}{I^2}\right|\right\} < \infty,$$
$$M_3 = \max_{\Gamma}\left\{\left|\frac{1}{I}\frac{\partial f(S,I,R)}{\partial R}\right|\right\} < \infty.$$

Since

$$\mu + \kappa + \phi + \frac{1}{2}\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 - \eta\frac{\partial f(S_0,0,0)}{\partial I} < 0,$$

we choose a constant $v \in (0,1)$ and make it small enough, such that

$$\mu + \kappa + \phi + \frac{1}{2}(1+v)\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2 - \eta\frac{\partial f(S_0,0,0)}{\partial I} < 0,$$

and $\eta M_3 - \psi/v < 0$. Let $a > 0$ be a large enough constant, and

$$\Gamma^* = \{(S,I,R) : S > 0, I > 0, R > 0, S + I + R \le S_0\},$$
$$D = \left\{(S,I,R) \in \Gamma : \frac{1}{a} < S < S_0 - \frac{1}{a}, \frac{1}{a} < I < S_0 - \frac{1}{a}, \frac{1}{a^2} < R < S_0 - \frac{1}{a^2}\right\}.$$

Next, we construct a nonnegative C^2-function V, such that $LV(S,I,R) \le -1$ for any $(S,I,R) \in \Gamma^* \backslash D$. For convenience, we can divide $\Gamma^* \backslash D$ into three domains, as follows

$$D_1 = \left\{(S,I,R) \in \Gamma^* : 0 < S < \frac{1}{a}\right\}, D_2 = \left\{(S,I,R) \in \Gamma^* : S \ge \frac{1}{a}, 0 < I < \frac{1}{a}\right\},$$
$$D_3 = \left\{(S,I,R) \in \Gamma^* : S \ge \frac{1}{a}, I \ge \frac{1}{a}, 0 < R < \frac{1}{a^2}\right\}.$$

Clearly, $D = D_1 \cup D_2 \cup D_3$. Next, we will prove that $LV(S,I,R) \le -1$ for any $(S,I,R) \in D_i (i = 1,2,3)$. Define a nonnegative Lyapunov function as follows

$$V(S,I,R) = Q(S,I,R) - Q(S_0,I_0,R_0) = V_1(I) + V_2(S,I) + V_3(S) + V_4(R) - Q(S_0,I_0,R_0),$$

where

$$V_1(I) = \frac{1}{v}I^{-v}, \quad V_2(S,I) = \frac{1}{v}I^{-v}(S_0 - S), \quad V_3(S) = \frac{1}{S}, \quad V_4(R) = R - 1 - \ln R,$$

and $Q(S_0,I_0,R_0)$ is the minimum value of $Q(S,I,R)$. Through Itô's formula and $(\mathbf{H}_2)$, we have

$$\begin{aligned} LV_1 &= -I^{-(v+1)}(\eta f(S,I,R) - (\mu+\kappa+\phi)I) + \frac{1}{2}(1+v)\sigma^2 I^{-(v+2)} f^2(S,I,R) \\ &= -I^{-v}\left(\eta\frac{f(S,I,R)}{I} - (\mu+\kappa+\phi)\right) + \frac{1}{2}(1+v)\sigma^2 I^{-v}\left(\frac{f(S,I,R)}{I}\right)^2. \end{aligned}$$

By Lagrange's mean value theorem, we have

$$\begin{aligned} \frac{\partial f(S_0,0,0)}{\partial I} - \frac{f(S,I,R)}{I} =& \frac{\partial f(S_0,0,0)}{\partial I} - \frac{\partial f(S,0,0)}{\partial I} + \frac{\partial f(S,0,0)}{\partial I} - \frac{f(S,I,0)}{I} \\ &+ \frac{f(S,I,0)}{I} - \frac{f(S,I,R)}{I} \\ =& \frac{\partial^2 f(\xi,0,0)}{\partial S \partial I}(S_0 - S) + \frac{f(S,\zeta,0) - \frac{\partial f(S,\zeta,0)}{\partial I}\zeta}{\zeta^2} I - \frac{1}{I}\frac{\partial f(S,I,\theta)}{\partial R} R \\ \le& M_1(S_0 - S) + M_2 I + M_3 R, \end{aligned}$$

where $\xi \in (S, S_0), \zeta \in (0, I)$ and $\theta \in (0, R)$. Hence,

$$
\begin{aligned}
LV_1 \leq & I^{-v}\left(\mu+\kappa+\phi+\frac{1}{2}(1+v)\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2-\eta\frac{\partial f(S_0,0,0)}{\partial I}\right) \\
& +\eta M_1(S_0-S)I^{-v}+\eta M_2 I^{1-v}+\eta M_3 I^{-v}R, \\
LV_2 = & -\frac{1}{v}I^{-v}\left((b-r\frac{N}{K})N-\mu S-\eta f(S,I,R)+\psi R\right) \\
& -I^{-v}(S_0-S)\left(\eta\frac{f(S,I,R)}{I}-(\mu+\kappa+\phi)\right) \\
& +\frac{1}{2}(1+v)\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2 I^{-v}(S_0-S)-\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2 I^{1-v} \\
= & -\frac{1}{v}I^{-v}\left((b-r\frac{N}{K})N-\mu S\right)+I^{1-v}\frac{\eta}{v}\frac{f(S,I,R)}{I}-\frac{\psi}{v}I^{-v}R \\
& +I^{-v}(S_0-S)\left(\mu+\kappa+\phi-\eta\frac{f(S,I,R)}{I}+\frac{1}{2}(1+v)\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\right) \\
& -I^{1-v}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2 \\
\leq & I^{-v}(S_0-S)\left(\mu+\kappa+\phi+\frac{1}{2}(1+v)\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\right) \\
& +I^{1-v}\frac{\eta}{v}\frac{f(S,I,R)}{I}-\frac{\psi}{v}I^{-v}R-I^{1-v}\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2, \\
LV_3 = & -\frac{1}{S^2}\left((b-r\frac{N}{K})N-\mu S-\eta f(S,I,R)+\psi R\right)+\sigma^2\left(\frac{f(S,I,R)}{S}\right)^2\frac{1}{S} \\
= & -\frac{1}{S^2}\left((b-r\frac{N}{K})N\right)+\frac{\mu}{S}+\eta\frac{f(S,I,R)}{S}\frac{1}{S}+\sigma^2\left(\frac{f(S,I,R)}{S}\right)^2\frac{1}{S}-\frac{\psi}{S^2}R \\
\leq & -\frac{(b-r\frac{N}{K})N}{S^2}+\frac{1}{S}\left(\mu+\eta M_S+\sigma^2 M_S^2\right)-\frac{\psi}{S^2}R \\
\leq & -\frac{(b-r\frac{N}{K})N}{2S^2}+\frac{1}{2(b-r\frac{N}{K})N}\left(\mu+\eta M_S+\sigma^2 M_S^2\right)^2-\frac{\psi}{S^2}R, \\
LV_4 = & \left(1-\frac{1}{R}\right)(\phi I-(\mu+\psi)R) \leq -\phi\frac{I}{R}+\phi S_0+\mu+\psi,
\end{aligned}
$$

where

$$\frac{1}{S}\left(\mu+\eta M_S+\sigma^2 M_S^2\right) \leq \frac{(b-r\frac{N}{K})N}{2S^2}+\frac{1}{2(b-r\frac{N}{K})N}\left(\mu+\eta M_S+\sigma^2 M_S^2\right)^2.$$

Based on the inequality of $LV_i (i=1,2,3,4)$ and $I^{1-v} \leq S_0^{1-v}$, for any $(S,I,R) \in \Gamma^* \backslash D$ we can obtain

$$LV \leq I^{-v}\left(\mu+\kappa+\phi+\frac{1}{2}(1+v)\sigma^2\left(\frac{\partial f(S_0,0,0)}{\partial I}\right)^2-\eta\frac{\partial f(S_0,0,0)}{\partial I}\right)-\frac{(b-r\frac{N}{K})N}{2S^2}-\phi\frac{I}{R}+Q,$$

where

$$
\begin{aligned}
Q = & S_0^{1-v}\left(\eta M_2+\frac{\eta}{v}\frac{\partial f(S_0,0,0)}{\partial I}\right)+\frac{1}{2(b-r\frac{N}{K})N}\left(\mu+\eta M_S+\sigma^2 M_S^2\right)^2+\phi S_0+\mu+\psi \\
& +S_0(S_0-S)\left(\eta M_1+\mu+\kappa+\phi+\frac{1}{2}(1+v)\sigma^2\left(\frac{f(S,I,R)}{I}\right)^2\right).
\end{aligned}
$$

For a larger enough $a > 0$, if $(S, I, R) \in D_1$, then

$$-\frac{(b - r\frac{N}{K})N}{2S^2} + Q < -1.$$

For $(S, I, R) \in D_2$, we have

$$I^{-v}\left(\mu + \kappa + \phi + \frac{1}{2}(1+v)\sigma^2\left(\frac{\partial f(S_0, 0, 0)}{\partial I}\right)^2 - \eta\frac{\partial f(S_0, 0, 0)}{\partial I}\right) + Q < -1,$$

and $-\phi I/R + Q < -1$ if $(S, I, R) \in D_3$. Therefore, we have $LV < -1$ for all $(S, I, R) \in \Gamma^* \backslash D$. Based on the theorem for a unique stationary distribution [14,18], we complete the proof. □

7. Numerical Simulations

In this section, the effect of white noise on model (3) can be determined by the Milstein method in Higham [20]. Thus, we obtain the discretization equations of model (3):

$$\begin{cases} S_{k+1} = S_k + \left[\left(b - r\frac{S(k) + I(k) + R(k)}{K}\right)(S(k) + I(k) + R(k)) - \eta f(S_k, I_k, R_k) + \psi R_k \right. \\ \qquad \left. - \mu S_k\right]\Delta t - f(S_k, I_k, R_k)\left[\sigma\xi_k\sqrt{\Delta t} + \frac{1}{2}\sigma^2\left(\xi_k^2 - 1\right)\Delta t\right], \\ I_{k+1} = I_k + [\eta f(S_k, I_k, R_k) - (\mu + \phi + \kappa)I_k]\Delta t + f(S_k, I_k, R_k)\left[\sigma\xi_k\sqrt{\Delta t} + \frac{1}{2}\sigma^2\left(\xi_k^2 - 1\right)\Delta t\right], \\ R_{k+1} = R_k + [\phi I_k - (\mu + \psi)R_k]\Delta t, \end{cases}$$

where ξ_k are the independent Gaussian random variables that follow the distribution $N(0, 1)$ for $k = 1, 2, \ldots, n$. Here, $f(S, I, R) = SI/N$.

Example 1. *Assume that $b = 0.3$, $r = 0.18$, $\psi = 0.11$, $K = 0.9$, $\eta = 0.86$, $\mu = 0.12$, $\phi = 0.13$, $\kappa = 0.21$, and $\Delta t = 0.01$. Obviously, $R_0 = 1.6087 > 1$. For model (2), the endemic equilibrium E^* is globally asymptotically stable. In model (3), if $\sigma = 0.84$, then $R_0^m = 0.8417 < 1$ and $\sigma^2 - \eta/\partial f(S_0, 0, 0)/\partial I = -0.0344 < 0$. Based on Theorem 4 (a), the disease-free equilibrium E_0 of model (3) is globally asymptotically stable. It reveals that $S(t)$ of the models (3) and (6) are close to the environmental capacity K, but $I(t)$ and $R(t)$ are close to 0. When $\sigma = 0.35$, and $R_0^m = 1.4755 > 1$, the disease is persistent. In addition, the intensity of white noise can suppress the disease outbreak. Figure 1 shows the curves of $(S(t), I(t), R(t))$ under these parameters.*

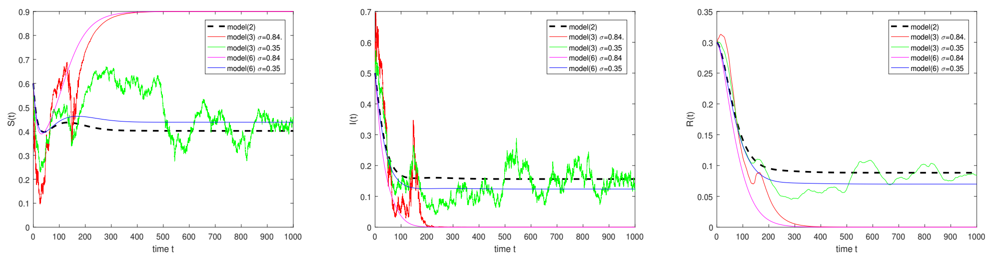

Figure 1. The paths of $S(t)$, $I(t)$, and $R(t)$ under the initial value $(S(0), I(0), R(0)) = (0.6, 0.5, 0.3)$.

Example 2. *Take the initial value $(S(0), I(0), R(0)) = (0.6, 0.3, 0.5)$, and parameters $\psi = 0.11$, $\eta = 0.54$, $\mu = 0.05$, $\phi = 0.2$, $\kappa = 0.09$ and $\sigma = 0.2$. The time step size $\Delta t = 0.01$. Through the calculation, we have $R_0 = 1.5882 > 1$, $R_0^m = 1.5294 > 1$, and $K - B/\mu = 0$. Based on Theorem 5, the disease is permanent. Next, we consider the effects of the environmental capacity K and the birth rate b according to two cases:*

(i) $K = 0.9, 1.1$, and $b = 0.2$. We observe that $S(t)$, $I(t)$, and $R(t)$ will increase when K increases. Figure 2 illustrates the different evolutions of $S(t)$, $I(t)$, and $R(t)$.

(ii) $b = 0.6, 0.2$, and $K = 0.9$. The nonlinear incidence initially becomes larger due to the increase of b, so that $S(t)$, $I(t)$, and $R(t)$ become smaller at the beginning. Moreover, the population slightly increases when the birth rate plays a major role (see Figure 3).

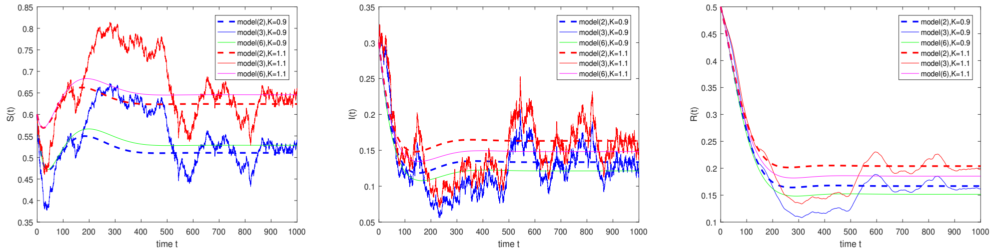

Figure 2. The paths of $S(t), I(t)$, and $R(t)$ under the environmental capacity $K = 0.9, 1.1$.

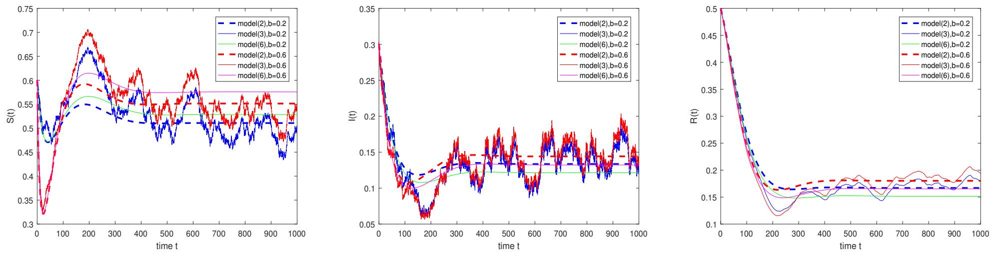

Figure 3. The paths of $S(t), I(t)$, and $R(t)$ under the birth rate $b = 0.6, 0.2$.

Example 3. *Take $b = 0.3$, $r = 0.25$, $K = 0.9$, $\eta = 0.5$, $\mu = 0.05$, $\phi = 0.2$, $\kappa = 0.09$, and $\sigma = 0.15$. Obviously, $R_0^m = 1.4375 > 1$. From Theorem 6, model (3) has a unique stationary distribution. Figure 4 provides the density functions of $S(t), I(t)$, and $R(t)$ in model (3).*

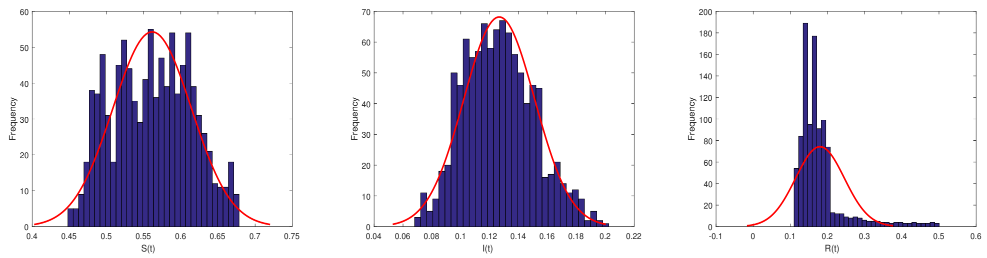

Figure 4. The histograms of $S(t), I(t)$, and $R(t)$ with the initial value $(S(0), I(0), R(0)) = (0.6, 0.2, 0.5)$.

8. Discussion

In this paper, we analyzed the dynamic properties of a stochastic SIRS epidemic model with Logistic birth and nonlinear incidence. Through a Stratonovich SDE, we obtained a new threshold value R_0^m to analyze the stability of model (3). Under the threshold value R_0^m, we derived some interesting results, including the global asymptotic stability of the disease-free equilibrium, permanent in the mean of the disease, and the existence of stationary distribution. We observed that environmental noise is a crucial influence in describing the dynamic behaviors of an epidemic. However, there are still some unsolved problems. For example, how to study the relationship of models (6) and (2), as well as (3). It is an exciting issue and will be the subject of our future work.

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