

Article

On Mathieu-Type Series with (p, ν) -Extended Hypergeometric Terms: Integral Representations and Upper Bounds

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Abstract: Integral form expressions are obtained for the Mathieu-type series and for their associated alternating versions, the terms of which contain a (p, ν) -extended Gauss hypergeometric function. Contiguous recurrence relations are found for the Mathieu-type series with respect to two parameters, and finally, particular cases and related bounding inequalities are established.

Keywords: (p, ν) -extended Beta function; (p, ν) -extended Gauss hypergeometric function; (p, ν) -extended confluent hypergeometric function; (p, ν) -extended Mathieu-type series; modified Bessel function of the second kind; bounding inequalities for (p, ν) -extended Mathieu-type series

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1. Introduction and Preliminaries

The series

$$S(x) = \sum_{n=0}^{\infty} \frac{2n}{(n^2 + x^2)^2}, \quad x > 0,$$

occurring in the classical literature in mathematical physics [1–4] and nowadays called the Mathieu series, was firstly considered by Émile Leonard Mathieu in his study of the clamped plate and membrane vibration models described by the fourth-order homogeneous and non-homogeneous differential equation $\Delta^2 U = g(x, y)$ associated with the Neumann boundary condition. He also studied the same-type Neumann problem for 3D prisms and other applied mathematical models, which occur in elasticity problems of rigid body motion, see, e.g., ([5], Section 8.3), Meleshko [6,7], and Meleshko and Gomilko [8]. The Mathieu-type series built with the help of a Gauss hypergeometric function was introduced by Pogány in ([9], pp. 309–310) in the following form:

$$S(x, \mu, \nu, a) = \sum_{n=0}^{\infty} \frac{{}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2} + 1; \nu + 1; -\frac{x^2}{a_n^2}\right)}{a_n^{\nu-\mu+1}(a_n^2 + x^2)^{\mu-\frac{1}{2}}}, \quad x > 0, \quad (1)$$

where $(x, \mu - \frac{1}{2}) \in \mathbb{R}_+^2$; $\nu < \mu - 1$, whilst the positive sequence $a = (a_n)$ monotonely increases and tends toward infinity. The Gauss hypergeometric function ([10], §15.2)

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (2)$$

completes the Mathieu-type series definition (1).

Later, Pogány [9], both alone and as a co-author, published a series of articles on more general Mathieu-type series and alternating Mathieu-type series, the terms of which

include other higher transcendental functions, such as the generalized hypergeometric function ${}_rF_s$ [11], the Fox–Wright generalized ${}_r\Psi_s$ function, Meijer’s G -function [12,13], the Fox H -function [14], the (p, q) -extended τ -hypergeometric function [15], V. P. Saxena’s I -function, and the \aleph -function [12]; see also [16,17].

For any $b, c \in \mathbb{C}$, $\Re(c) > \Re(b) > 0$, we can transform the ratio of Pochhammer symbols’ as follows:

$$\frac{(b)_n}{(c)_n} = \frac{B(b+n, c-b)}{B(b, c-b)}, \quad \Re(c-b) > 0, n \in \mathbb{N}_0,$$

which implies

$${}_2F_1(a, b; c; z) = \sum_{n \geq 0} (a)_n \frac{B(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}. \quad (3)$$

The same stands for Kummer’s confluent hypergeometric function:

$$\Phi(b; c; z) = \sum_{n \geq 0} \frac{(b)_n}{(c)_n} \frac{z^n}{n!} = \sum_{n \geq 0} \frac{B(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad (4)$$

where in both series, $-c \notin \mathbb{N}_0$ and $|z| < 1$.

The next main generalization direction concerns Euler’s Beta integral:

$$B(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx, \quad \min\{s, t\} > 0.$$

The Beta integral transform of some suitable input function h viz.

$$B[h](s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} h(x) dx,$$

was considered by Krattenthaler and Srinivasa Rao [18,19], assuming that this integral converges in a certain sense. When $h_1(x) = \exp\{-\frac{p}{x(1-x)}\}$; $\Re(p) \geq 0$, we arrive at the p -extended Beta function introduced by Chaudhry *et al.* ([20], p. 20, Equation (1.7))

$$B_p(x, y) := B[h_1](x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt, \quad \Re(p) \geq 0; \min\{\Re(x), \Re(y)\} > 0. \quad (5)$$

When we replace $B(x, y)$ with $B_p(x, y)$ in both (3) and (4), we obtain the p -variant of related p -Gauss hypergeometric and p -Kummer confluent hypergeometric functions (see below).

Recently, Parmar *et al.* [21] introduced the (p, ν) -extended Beta function by choosing

$$h_2(x) = \sqrt{\frac{2p}{\pi x(1-x)}} K_{\nu+\frac{1}{2}}\left(\frac{p}{x(1-x)}\right),$$

where $\Re(p) \geq 0$; $\min\{\Re(x), \Re(y)\} > 0$, \sqrt{p} takes its principal value, and $K_\mu(z)$ stands for the modified Bessel function of the second kind of the order μ ([10], p. 251, Equation (10.27.4))

$$K_\mu(x) = \frac{\pi}{2} \frac{I_{-\mu}(x) - I_\mu(x)}{\sin(\pi\mu)}.$$

If μ is not an integer, then $\lim_{\mu \rightarrow n}, n \in \mathbb{Z}$. Consequently, the related (p, ν) -extended Beta function reads as follows ([21], p. 93, Equation (13)):

$$B_{p,\nu}(x, y) := B[h_2](x, y) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}} (1-t)^{y-\frac{3}{2}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt. \quad (6)$$

The mentioned extensions of Euler's Beta functions have recently been studied by a number of authors (see [20,22,23]).

In view of the above, the (p, ν) -extended Gauss hypergeometric and (p, ν) -extended Kummer confluent hypergeometric functions are, respectively ([21], p. 98, Equations (40) and (41))

$$F_{p,\nu}(a, b; c; z) = \sum_{n \geq 0} (a)_n \frac{B_{p,\nu}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad p \geq 0; \Re(c) > \Re(b) > 0; |z| < 1, \quad (7)$$

$$\Phi_{p,\nu}(b; c; z) = \sum_{n \geq 0} \frac{B_{p,\nu}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad p \geq 0; \Re(c) > \Re(b) > 0, \quad (8)$$

when $\nu = 0$ and ([10], p. 254, Eq. 10.39.2)

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \quad (9)$$

The Bessel $K_{\frac{1}{2}}$ needs to be convoluted with the integrand of Euler's Beta function to obtain (5), from (7) and (8) and their special cases, the p -extended Gauss hypergeometric and the p -extended Kummer confluent hypergeometric functions ([22], pp. 591–592, Equations (2.2)–(2.3))

$$F_p(a, b; c; z) = \sum_{n \geq 0} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad p \geq 0; \Re(c) > \Re(b) > 0; |z| < 1,$$

$$\Phi_p(b; c; z) = \sum_{n \geq 0} \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad p \geq 0; \Re(c) > \Re(b) > 0.$$

Now, extending the Mathieu-type series studied in [9] by imposing the $F_{p,\nu}(a, b; c; z)$ building block function instead of the originally used ${}_2F_1$ in the summands, we define the Mathieu-type a -series $\mathcal{R}_{\mu,\zeta}$ and its alternating variant $\tilde{\mathcal{R}}_{\mu,\zeta}$ in the power series definition:

$$\mathcal{R}_{\mu,\zeta}(F_{p,\nu}; a; r) := \sum_{n \geq 1} \frac{F_{p,\nu}(\mu, b; c; -\frac{r^2}{a_n})}{a_n^\mu (a_n + r^2)^\zeta} \quad (p \geq 0; \mu, \zeta, r \in \mathbb{R}^+), \quad (10)$$

and

$$\tilde{\mathcal{R}}_{\mu,\zeta}(F_{p,\nu}; a; r) := \sum_{n \geq 1} \frac{(-1)^{n-1} F_{p,\nu}(\mu, b; c; -\frac{r^2}{a_n})}{a_n^\mu (a_n + r^2)^\zeta} \quad (p \geq 0; \mu, \zeta, r \in \mathbb{R}^+). \quad (11)$$

In this article, we provide integral representations for the Mathieu-type series and its alternating versions, the terms of which are constructed from the (p, ν) -extended Gauss hypergeometric function. The main achievements of the manuscript are the contiguous recurrence relations obtained for (p, ν) -extended Mathieu-type series with respect to both of their constituting parameters in Theorem 1 and Corollary 1. An upper bound is derived in Lemma 1. for the (p, ν) -extended Beta function $B_{p,\nu}$, which further implies the bounds for the moduli of the (p, ν) -extended hypergeometric function $F_{p,\nu}$ and the (p, ν) -extended Kummer's function $\Phi_{p,\nu}$ in Theorem 2. Finally, related bounding inequalities are given for the (p, ν) -extended Mathieu-type series $\mathcal{R}_{\mu,\zeta}(F_{p,\nu}; a; r)$ and for its alternating variant $\tilde{\mathcal{R}}_{\mu,\zeta}(F_{p,\nu}; a; r)$ in Theorem 3.

2. Contiguous Recurrence Integral Representations of $\mathcal{R}_{\mu,\zeta}(F_{p,\nu}; a; r)$ and $\tilde{\mathcal{R}}_{\mu,\zeta}(F_{p,\nu}; a; r)$

This section deals with integral expressions for the series $\mathcal{R}_{\mu,\zeta}(F_{p,\nu}; a; r)$ and $\tilde{\mathcal{R}}_{\mu,\zeta}(F_{p,\nu}; a; r)$. Now, we prove that there are first-order contiguous recurrence relations for both series with respect to the parameters μ and ζ . Some particular cases of our first main result are also considered.

Theorem 1. Let $\mu > 0$, $\zeta > 0$, $r > 0$, and the real positive sequence $\mathbf{a} = (a_n)_{n \geq 1}$ monotonely increases to ∞ . Then, for $\Re(p) > 0$, we have

$$\mathcal{R}_{\mu, \zeta}(F_{p, \nu}; \mathbf{a}; r) = \mu \mathcal{J}_{p, \nu}(\mu + 1, \zeta) + \zeta \mathcal{J}_{p, \nu}(\mu, \zeta + 1) \quad (12)$$

$$\tilde{\mathcal{R}}_{\mu, \zeta}(F_{p, \nu}; \mathbf{a}; r) = \mu \tilde{\mathcal{J}}_{p, \nu}(\mu + 1, \zeta) + \zeta \tilde{\mathcal{J}}_{p, \nu}(\mu, \zeta + 1), \quad (13)$$

where

$$\mathcal{J}_{p, \nu}(\mu, \zeta) = \int_{a_1}^{\infty} \frac{F_{p, \nu}(\mu, b; c; -\frac{r^2}{x})[a^{-1}(x)]}{x^{\mu}(x + r^2)^{\zeta}} dx \quad (14)$$

$$\tilde{\mathcal{J}}_{p, \nu}(\mu, \zeta) = \int_{a_1}^{\infty} \frac{F_{p, \nu}(\mu, b; c; -\frac{r^2}{x}) \sin^2(\frac{\pi}{2}[a^{-1}(x)])}{x^{\mu}(x + r^2)^{\zeta}} dx \quad (15)$$

and $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is an arbitrary increasing function restriction of which $a(x)|_{x \in \mathbb{N}} = \mathbf{a}$ and $[w]$ stands for the integer part of the quantity w .

Proof. Consider the Laplace transform formula of $t^{\mu-1} \Phi_{p, \nu}(b; c; z)$ by using the definition (7). For a real ω ,

$$F_{p, \nu}\left(\mu, b; c; \frac{\omega}{z}\right) = \frac{z^{\mu}}{\Gamma(\mu)} \int_0^{\infty} e^{-zt} t^{\mu-1} \Phi_{p, \nu}(b; c; \omega t) dt. \quad (16)$$

Inserting $\zeta = a_n + r^2$ into the Gamma function formula

$$\Gamma(\eta) \zeta^{-\eta} = \int_0^{\infty} e^{-\zeta t} t^{\eta-1} dt, \quad \Re(\zeta) > 0, \Re(\eta) > 0,$$

and, after rearrangement, by specifying $\omega = -r^2$, $z = a_n$, in (16), the auxiliary function in (14) becomes

$$\mathcal{J}_{p, \nu}(\mu, \zeta) = \frac{1}{\Gamma(\mu)\Gamma(\zeta)} \int_0^{\infty} \int_0^{\infty} e^{-r^2 s} t^{\mu-1} s^{\zeta-1} \left(\sum_{n \geq 1} e^{-a_n(t+s)} \right) \Phi_{p, \nu}(b; c; -r^2 t) dt ds.$$

Using the Cahen formula [24] for summing up the Dirichlet series with the method developed in ([9], p. 310, Equations (5) and (6)), we conclude

$$\mathcal{D}_a(t+s) = \sum_{n \geq 1} e^{-a_n(s+t)} = (s+t) \int_{a_1}^{\infty} e^{-(t+s)x} [a^{-1}(x)] dx.$$

This gives

$$\begin{aligned} \mathcal{J}_{p, \nu}(\mu, \zeta) &= \frac{1}{\Gamma(\mu)\Gamma(\zeta)} \int_0^{\infty} \int_0^{\infty} \int_{a_1}^{\infty} e^{-(r^2+x)s-tx} (t+s) t^{\mu-1} s^{\zeta-1} [a^{-1}(x)] \\ &\quad \cdot \Phi_{p, \nu}(b; c; -r^2 t) dt ds dx =: \mathcal{J}_t + \mathcal{J}_s, \end{aligned} \quad (17)$$

where

$$\begin{aligned} \mathcal{J}_t &= \frac{1}{\Gamma(\zeta)} \int_0^{\infty} \left(\int_{a_1}^{\infty} \left(\int_0^{\infty} \frac{e^{-xt} t^{\mu}}{\Gamma(\mu)} \Phi_{p, \nu}(b; c; -r^2 t) dt \right) e^{-xs} [a^{-1}(x)] dx \right) e^{-r^2 s} s^{\zeta-1} ds \\ &= \frac{\mu}{\Gamma(\zeta)} \int_{a_1}^{\infty} \left(\int_0^{\infty} e^{-(x+r^2)s} s^{\zeta-1} ds \right) \frac{[a^{-1}(x)]}{x^{\mu+1}} F_{p, \nu}\left(\mu + 1, b; c; -\frac{r^2}{x}\right) dx \\ &= \mu \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\mu+1}(x+r^2)^{\zeta}} F_{p, \nu}\left(\mu + 1, b; c; -\frac{r^2}{x}\right) dx = \mu \mathcal{J}(\mu + 1, \zeta). \end{aligned} \quad (18)$$

In a similar way, we obtain

$$\begin{aligned}\mathcal{J}_s &= \int_{a_1}^{\infty} \left(\int_0^{\infty} \left(\int_0^{\infty} \frac{s^{\zeta}}{\Gamma(\zeta)} e^{-(x+r^2)s} ds \right) \frac{e^{-xt} t^{\mu-1}}{\Gamma(\mu)} \Phi_{p,\nu}(b; c; -r^2 t) dt \right) [a^{-1}(x)] dx \\ &= \zeta \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{(x+r^2)^{\zeta+1}} \left(\int_0^{\infty} \frac{e^{-xt} t^{\mu-1}}{\Gamma(\mu)} \Phi_{p,\nu}(b; c; -r^2 t) dt \right) dx \\ &= \zeta \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\mu}(x+r^2)^{\zeta+1}} F_{p,\nu} \left(\mu, b; c; -\frac{r^2}{x} \right) dx = \zeta \mathcal{J}(\mu, \zeta + 1).\end{aligned}\quad (19)$$

By applying (18) and (19) to (17), we obtain the expression (12).

The derivation of (14) is similar to the exposed proving procedure. Having in mind again the Cahen formula, since the counting function equals

$$\tilde{A}(t) = \sum_{n: a_n \leq t} (-1)^{n-1} = \frac{1 - (-1)^{[a^{-1}(t)]}}{2} = \sin^2 \left(\frac{\pi}{2} [a^{-1}(t)] \right),$$

the integral form of the alternating Dirichlet series $\mathcal{D}_a(x)$ becomes ([25], p. 79)

$$\tilde{\mathcal{D}}_a(x) = \sum_{n \geq 1} (-1)^{n-1} e^{-a_n x} = x \int_{a_1}^{\infty} e^{-xt} \tilde{A}(t) dt,$$

and

$$\tilde{\mathcal{D}}_a(x) = x \int_{a_1}^{\infty} e^{-xt} \sin^2 \left(\frac{\pi}{2} [a^{-1}(t)] \right) dt.$$

Because

$$\tilde{\mathcal{D}}_a(t+s) = (t+s) \int_{a_1}^{\infty} e^{-(t+s)x} \sin^2 \left(\frac{\pi}{2} [a^{-1}(x)] \right) dx,$$

we conclude (15), and *a fortiori* (13) by obvious steps. \square

When $\nu = 0$ and by using (9), the results of Theorem 1 are simplified.

Corollary 1. Let $\mu > 0$, $\zeta > 0$, $r > 0$, and the real sequence \mathbf{a} monotonely increases and tends to ∞ . Then, for $\Re(p) \geq 0$, we have

$$\begin{aligned}\mathcal{R}_{\mu,\zeta}(F_p; \mathbf{a}; r) &= \mu \mathcal{J}_p(\mu + 1, \zeta) + \zeta \mathcal{J}_p(\mu, \zeta + 1) \\ \tilde{\mathcal{R}}_{\mu,\zeta}(F_p; \mathbf{a}; r) &= \mu \tilde{\mathcal{J}}_p(\mu + 1, \zeta) + \zeta \tilde{\mathcal{J}}_p(\mu, \zeta + 1),\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}_p(\mu, \zeta) &= \int_{a_1}^{\infty} \frac{F_p(\mu, b; c; -\frac{r^2}{x}) [a^{-1}(x)]}{x^{\mu}(x+r^2)^{\zeta}} dx, \\ \tilde{\mathcal{J}}_p(\mu, \zeta) &= \int_{a_1}^{\infty} \frac{F_p(\mu, b; c; -\frac{r^2}{x}) \sin^2 \left(\frac{\pi}{2} [a^{-1}(x)] \right)}{x^{\mu}(x+r^2)^{\zeta}} dx.\end{aligned}$$

Remark 1. The special case $\nu = p = 0$ immediately reduces the claim of Theorem 1 to the Gauss ${}_2F_1$ hypergeometric function's case studied in [9].

3. Bounding Inequalities for the (p, ν) -Extended Mathieu-Type Series

In this section, our main goal is to derive an upper bound for the (p, ν) -extended Beta function $B_{p,\nu}(x, y)$ in (6). By making use of this upper bound, we obtain bounds for the (p, ν) -extended Gauss hypergeometric $F_{p,\nu}$ and the (p, ν) -extended Kummer's confluent hypergeometric $\Phi_{p,\nu}$ via series representations (7) and (8). Finally, we obtain bounding inequalities for the (p, ν) -extended Mathieu-type series (10) and (11).

3.1. Upper Bound for (p, ν) -Extended Beta Function

Firstly, we establish the upper bound for the (p, ν) -extended hypergeometric function $F_{p,\nu}$ by applying the following result ([26], p. 17, Equation (5.3)):

$$|K_{\nu+\frac{1}{2}}(z)| < \frac{\sqrt{\pi} \left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)} \frac{\Gamma(2\nu+1, \Re(z))}{(\Re(z))^{2\nu+1}}, \quad \Re(z) > 0, \quad (20)$$

where the upper incomplete Gamma function is

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt, \quad \Re(a), \Re(x) > 0.$$

Consequently, since $\Gamma(a, x) < \Gamma(a)$, there holds ([26], p. 17)

$$\left|K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)\right| < \frac{1}{2} \left(\frac{2|p|t(1-t)}{\Re^2(p)}\right)^{\nu+\frac{1}{2}} \Gamma\left(\nu+\frac{1}{2}\right), \quad \Re(p) > 0, t \in (0, 1). \quad (21)$$

The immediate implication of (21) follows by means of (6).

Lemma 1. For all $\Re(p) > 0$, $\nu > 0$, $\min\{\Re(x), \Re(y)\} > 0$ and $t \in (0, 1)$, we have

$$|B_{p,\nu}(x, y)| \leq \frac{2^\nu |p|^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\Re(p))^{2\nu+1}} B(x + \nu, y + \nu). \quad (22)$$

This upper bound plays an important role in the whole section (indirectly) applied either for the sum or integral representations of the families of (p, ν) -extended special functions.

3.2. Bounds Obtained via Series Representations

The functional bound (21) will be used in proving our first set of the main bounding inequalities. More precisely, by applying (22) to all the series representations of the (p, ν) -extended special functions, which contain $B_{p,\nu}(x, y)$, such as the (p, ν) -extended Gauss hypergeometric $F_{p,\nu}$ and (p, ν) -extended Kummer's confluent hypergeometric $\Phi_{p,\nu}$, described in (7) and (8), we arrive at the results below.

Theorem 2. For all $\Re(p) > 0$, $\nu > 0$; $\Re(c) > \Re(b) > 0$ and for all $|z| < 1$, we have

$$\begin{aligned} |F_{p,\nu}(a, b; c; z)| &\leq \frac{2^\nu |p|^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\Re(p))^{2\nu+1}} \frac{B(b + \nu, c - b + \nu)}{B(b, c - b)} {}_2F_1(a, b + \nu; c + 2\nu; |z|), \\ |\Phi_{p,\nu}(b; c; z)| &\leq \frac{2^\nu |p|^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\Re(p))^{2\nu+1}} \frac{B(b + \nu, c - b + \nu)}{B(b, c - b)} \Phi(b + \nu; c + 2\nu; |z|). \end{aligned} \quad (23)$$

Proof. Regarding assertion (23), because all parameters and expressions involved are positive, by means of the series representation of the (p, ν) -extended Gauss hypergeometric function (7), and by Lemma 1, we conclude

$$\begin{aligned} F_{p,\nu}(a, b; c; z) &\leq \frac{2^\nu |p|^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\Re(p))^{2\nu+1} B(b, c - b)} \sum_{n \geq 0} \frac{(a)_n B(b + \nu + n, c - b + \nu)}{n!} \frac{|z|^n}{n!} \\ &= \frac{2^\nu |p|^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (\Re(p))^{2\nu+1} B(b, c - b)} \sum_{n \geq 0} \frac{(a)_n \Gamma(b + \nu + n) \Gamma(c - b + \nu)}{\Gamma(c + 2\nu + n)} \frac{|z|^n}{n!} \\ &= \frac{2^\nu |p|^{\nu+1} \Gamma(\nu + \frac{1}{2}) \Gamma(c - b + \nu) \Gamma(b + \nu)}{\sqrt{\pi} (\Re(p))^{2\nu+1} B(b, c - b) \Gamma(c + 2\nu)} \sum_{n \geq 0} \frac{(a)_n (b + \nu)_n}{(c + 2\nu)_n} \frac{|z|^n}{n!}. \end{aligned}$$

The rest is obvious. Moreover, by applying similar transformations *per definitionem*, we prove the second statement for the (p, ν) -extended confluent hypergeometric function (8). \square

Furthermore, we need a certain Luke's upper bound for the Gauss hypergeometric function. Precisely, for all $b \in (0, 1]$, $c \geq a > 0$ and $z > 0$, there holds ([27], p. 52, Equation (4.7))

$${}_2F_1(a, b; c; -z) < 1 - \frac{2ab(c+1)}{c(a+1)(b+1)} \left(1 - \frac{2(c+1)}{2(c+1) + (a+1)(b+1)z} \right). \quad (24)$$

Theorem 3. Let $\mu \in (0, 1]$, $\zeta > 0$, $\nu > 0$, and the positive real sequence $\mathbf{a} = (a_n)_{n \geq 1}$ monotonely increases and tends to ∞ . Then, for all $r \in (0, \sqrt{a_1})$, $\Re(p) > 0$ and $\Re(c) > \Re(b) > 0$, we have

$$\begin{aligned} \mathcal{R}_{\mu, \zeta}(F_{p, \nu}; \mathbf{a}; r) &\leq \mu \mathcal{Y}_{p, \nu} \left\{ \left(1 - \frac{2(\mu+1)(b+\nu)(c+2\nu+1)}{c(\mu+2)(b+\nu+1)} \right) \mathcal{X}_a(\mu+1, \zeta) \right. \\ &\quad \left. + \frac{4(\mu+1)(b+\nu)(c+2\nu+1)^2 \mathcal{X}_a(\mu, \zeta)}{(c+2\nu)(\mu+2)(b+\nu+1)[(\mu+2)(b+\nu+1)r^2 + 2(c+2\nu+1)a_1]} \right\} \\ &\quad + \zeta \mathcal{Y}_{p, \nu} \left\{ \left(1 - \frac{2\mu(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+1)(b+\nu+1)} \right) \mathcal{X}_a(\mu, \zeta+1) \right. \\ &\quad \left. + \frac{4\mu(b+\nu)(c+2\nu+1)^2 \mathcal{X}_a(\mu-1, \zeta+1)}{(c+2\nu)(\mu+1)(b+\nu+1)[(\mu+1)(b+\nu+1)r^2 + 2(c+2\nu+1)a_1]} \right\}. \end{aligned} \quad (25)$$

Moreover, for all $\mu + \zeta > 1$, $\nu > 0$; $r \in (0, \sqrt{a_1})$, $\Re(p) > 0$ and $\Re(c) > \Re(b) > 0$, we have

$$\begin{aligned} \tilde{\mathcal{R}}_{\mu, \zeta}(F_{p, \nu}; \mathbf{a}; r) &\leq \mu \mathcal{Y}_{p, \nu} \left\{ \left(1 - \frac{2(\mu+1)(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+2)(b+\nu+1)} \right) \frac{a_1^{-\mu-\zeta}}{\mu+\zeta} {}_2F_1\left(\zeta, \mu+\zeta; \zeta+1; -\frac{r^2}{a_1}\right) \right. \\ &\quad \left. + \frac{4(\mu+1)(b+\nu)(c+2\nu+1)^2}{(c+2\nu)(\mu+2)(b+\nu+1)} \frac{a_1^{1-\mu-\zeta} {}_2F_1\left(\zeta, \mu+\zeta-1; \zeta+1; -\frac{r^2}{a_1}\right)}{(\mu+\zeta-1)[(\mu+2)(b+\nu+1)r^2 + 2(c+2\nu+1)a_1]} \right\} \\ &\quad + \zeta \mathcal{Y}_{p, \nu} \left\{ \left(1 - \frac{2\mu(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+1)(b+\nu+1)} \right) \frac{a_1^{-\mu-\zeta}}{\mu+\zeta} {}_2F_1\left(\zeta+1, \mu+\zeta; \zeta+2; -\frac{r^2}{a_1}\right) \right. \\ &\quad \left. + \frac{4\mu(b+\nu)(c+2\nu+1)^2}{(c+2\nu)(\mu+1)(b+\nu+1)} \frac{a_1^{1-\mu-\zeta} {}_2F_1\left(\zeta+1, \mu+\zeta-1; \zeta+2; -\frac{r^2}{a_1}\right)}{(\mu+\zeta-1)[(\mu+1)(b+\nu+1)r^2 + 2(c+2\nu+1)a_1]} \right\}, \end{aligned} \quad (26)$$

where the integral's shorthand reads

$$\mathcal{X}_a(\mu, \zeta) := \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\mu}(x+r^2)^{\zeta}} dx; \quad \text{and} \quad \mathcal{Y}_{p, \nu} := \frac{2^{\nu}|p|^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}(\Re(p))^{2\nu+1}} \frac{\mathcal{B}(b+\nu, c-b+\nu)}{\mathcal{B}(b, c-b)}.$$

Proof. Firstly, consider the relation (10)

$$\mathcal{R}_{\mu, \zeta}(F_{p, \nu}; \mathbf{a}; r) = \mu \mathcal{J}_{p, \nu}(\mu+1, \zeta) + \zeta \mathcal{J}_{p, \nu}(\mu, \zeta+1),$$

in which we bound from above the auxiliary integral $\mathcal{J}_{p, \nu}$ described in (14). To do this, we quote that $F_{p, \nu}(a, b; c; z) > 0$ for all $a, b, c > 0$ and all non-positive values of z . Indeed, it is enough to consider the integral expression ([21], p. 99, Equation (42))

$$F_{p, \nu}(a, b; c; z) = \sqrt{\frac{2p}{\pi}} \frac{1}{\mathcal{B}(b, c-b)} \int_0^1 t^{b-\frac{3}{2}} (1-t)^{c-b-\frac{3}{2}} (1-zt)^{-a} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) dt,$$

where $p > 0$; $\Re(c) > \Re(b) > 0$; $|z| < 1$. Therefore, by virtue of (14) and (23),

$$\begin{aligned} \mathcal{J}_{p,\nu}(\mu, \zeta) &= \int_{a_1}^{\infty} \frac{F_{p,\nu}(\mu, b; c; -\frac{r^2}{x})[a^{-1}(x)]}{x^{\mu}(x+r^2)^{\zeta}} dx \leq \mathcal{Y}_{p,\nu} \int_{a_1}^{\infty} \frac{{}_2F_1(\mu, b+\nu; c+2\nu; -\frac{r^2}{x})[a^{-1}(x)]}{x^{\mu}(x+r^2)^{\zeta}} dx \\ &\leq \mathcal{Y}_{p,\nu} \left\{ \left(1 - \frac{2\mu(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+1)(b+\nu+1)} \right) \int_{a_1}^{\infty} \frac{[a^{-1}(x)]}{x^{\mu}(x+r^2)^{\zeta}} dx \right. \\ &\quad \left. + \frac{4\mu(b+\nu)(c+2\nu+1)^2}{(c+2\nu)(\mu+1)(b+\nu+1)} \int_{a_1}^{\infty} \frac{x^{1-\mu}(x+r^2)^{-\zeta} [a^{-1}(x)] dx}{(\mu+1)(b+\nu+1)r^2 + 2(c+2\nu+1)x} \right\} \\ &\leq \mathcal{Y}_{p,\nu} \left\{ \left(1 - \frac{2\mu(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+1)(b+\nu+1)} \right) \mathcal{X}_a(\mu, \epsilon) \right. \\ &\quad \left. + \frac{4\mu(b+\nu)(c+2\nu+1)^2 \mathcal{X}_a(\mu-1, \epsilon)}{(c+2\nu)(\mu+1)(b+\nu+1)((\mu+1)(b+\nu+1)r^2 + 2(c+2\nu+1)a_1)} \right\}. \end{aligned}$$

The rest in deriving (25) is straightforward.

Secondly, we recall (13), which reads as follows:

$$\tilde{\mathcal{R}}_{\mu,\zeta}(F_{p,\nu}; a; r) = \mu \tilde{\mathcal{J}}_{p,\nu}(\mu+1, \zeta) + \zeta \tilde{\mathcal{J}}_{p,\nu}(\mu, \zeta+1).$$

By the positivity of the integrand of (15), since (23), we conclude

$$\tilde{\mathcal{J}}_{p,\nu}(\mu, \zeta) \leq \int_{a_1}^{\infty} \frac{F_{p,\nu}(\mu, b; c; -\frac{r^2}{x})}{x^{\mu}(x+r^2)^{\zeta}} dx \leq \mathcal{Y}_{p,\nu} \int_{a_1}^{\infty} \frac{{}_2F_1(\mu, b+\nu; c+2\nu; -\frac{r^2}{x})}{x^{\mu}(x+r^2)^{\zeta}} dx.$$

In turn, with the aid of (24), we deduce that

$$\begin{aligned} \tilde{\mathcal{J}}_{p,\nu}(\mu, \zeta) &\leq \mathcal{Y}_{p,\nu} \left\{ \left(1 - \frac{2\mu(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+1)(b+\nu+1)} \right) \int_{a_1}^{\infty} \frac{dx}{x^{\mu}(x+r^2)^{\zeta}} \right. \\ &\quad \left. + \frac{4\mu(b+\nu)(c+2\nu+1)^2}{(c+2\nu)(\mu+1)(b+\nu+1)} \int_{a_1}^{\infty} \frac{x^{1-\mu}(x+r^2)^{-\zeta} dx}{(\mu+1)(b+\nu+1)r^2 + 2(c+2\nu+1)x} \right\}. \end{aligned}$$

If $\mu + \zeta > 2$, then ([28], p. 313, Equation (3.194 1)).

$$\int_{a_1}^{\infty} \frac{dx}{x^{\mu}(x+r^2)^{\zeta}} = \int_0^{\frac{1}{a_1}} \frac{t^{\mu+\zeta-2}}{(1+r^2t)^{\zeta}} dt = \frac{a_1^{1-\mu-\zeta}}{\mu+\zeta-1} {}_2F_1\left(\zeta, \mu+\zeta-1; \zeta+1; -\frac{r^2}{a_1}\right),$$

which implies

$$\int_{a_1}^{\infty} \frac{dx}{x^{\mu-1}(x+r^2)^{\zeta} \left((\mu+1)r^2 + 2\frac{c+2\nu+1}{b+\nu+1}x \right)} \leq \frac{a_1^{2-\mu-\zeta} {}_2F_1\left(\zeta, \mu+\zeta-2; \zeta+1; -\frac{r^2}{a_1}\right)}{(\mu+\zeta-2)((\mu+1)r^2 + 2\frac{c+2\nu+1}{b+\nu+1}a_1)}.$$

Collecting these formulae, we infer

$$\begin{aligned} \tilde{\mathcal{J}}_{p,\nu}(\mu, \zeta) &\leq \mathcal{Y}_{p,\nu} \left\{ \left(1 - \frac{2\mu(b+\nu)(c+2\nu+1)}{(c+2\nu)(\mu+1)(b+\nu+1)} \right) \frac{a_1^{1-\mu-\zeta}}{\mu+\zeta-1} {}_2F_1\left(\zeta, \mu+\zeta-1; \zeta+1; -\frac{r^2}{a_1}\right) \right. \\ &\quad \left. + \frac{4\mu(b+\nu)(c+2\nu+1)^2}{(c+2\nu)(\mu+1)(b+\nu+1)} \frac{a_1^{2-\mu-\zeta} {}_2F_1\left(\zeta, \mu+\zeta-2; \zeta+1; -\frac{r^2}{a_1}\right)}{(\mu+\zeta-2)[(\mu+1)(b+\nu+1)r^2 + 2(c+2\nu+1)a_1]} \right\}. \end{aligned}$$

Now, obvious steps lead to the asserted upper bound (26). \square

4. Concluding Remarks

- A The cited references for the Mathieu-type series are given concerning the integral representations, which are mainly obtained by virtue of Cahen's formula for the sum of Dirichlet series in the form of a Laplace integral. The contiguous recurrence relations exist for almost all already considered cases, together with the bounding inequalities for the studied general Mathieu-type series up to the related multiplicative constants.
- B It is worth mentioning that there is also another type of extended Beta function, which was introduced in [29], where the extended Beta function consists of the Beta integral transform of the Kummer confluent hypergeometric function, *viz.* ([29], p. 631, Definition 1.1.)

$$B_{p;\kappa,\lambda}^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha; \beta; -\frac{p}{t^\kappa(1-t)^\lambda}\right) dt, \quad (27)$$

where the parameters $\kappa, \lambda \geq 0$; $\min\{\Re(\alpha), \Re(\beta)\} > 0$; $\Re(x) > -\Re(\kappa\alpha)$, $\Re(y) > -\Re(\lambda\alpha)$. The special case $B_{p;1,1}^{(\alpha,\alpha)}(x,y) = B_p(x,y)$ coincides with the p -extended Beta function (5) introduced by Chaudhry *et al.* in [20].

- C The results for the extended hypergeometric functions $F_p^{(\alpha,\beta;\kappa,\mu)}$ relative to $B_{p;\kappa,\lambda}^{(\alpha,\beta)}(x,y)$ are published in ([30], pp. 140–143, Theorem 1 *et seq.*), together with the integral representations for the extended Mathieu-type series ([30], p. 140, Equations (1.3) and (1.4)).

$$\begin{aligned} \mathfrak{F}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)}; a; r) &= \sum_{n \geq 1} \frac{F_p^{(\alpha,\beta;\kappa,\mu)}(\lambda, b; c; -\frac{r^2}{a_n})}{a_n^\lambda (a_n + r^2)^\eta} \\ \widetilde{\mathfrak{F}}_{\lambda,\eta}(F_p^{(\alpha,\beta;\kappa,\mu)}; a; r) &= \sum_{n \geq 1} \frac{(-1)^{n-1} F_p^{(\alpha,\beta;\kappa,\mu)}(\lambda, b; c; -\frac{r^2}{a_n})}{a_n^\lambda (a_n + r^2)^\eta}, \end{aligned}$$

where $\lambda, \eta, r > 0$; $\Re(c) > \Re(b) > 0$.

- D Further research directions may include the asymptotic expansion of generalized Mathieu series [31,32], connections with the Riemann zeta and Dirichlet Beta functions [33], Mathieu series associated with the Mittag-Leffler function, harmonic Mathieu series, Fourier–Mathieu series and connections with the Butzer–Flocke–Hauss Omega function, the multiparameter variants of Mathieu-type series with reference to the recent monograph [34], and article [5]. Moreover, the probability distributions and allied topics defined in terms of Mathieu-type series are also studied, for instance in [35,36] and the appropriate references therein. These publications suggested some ideas for generalizing the Mathieu-type series studied here, e.g., new generalizations of the Beta functions related to (27), that can result in novel forms of the associated hypergeometric functions and the related Mathieu series.

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